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Rowley, Peter and Taylor, Paul

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Peter Rowley and Paul Taylor

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Abstract

In this paper we determine the suborbits of Janko's largest simple group in its conjugation action on each of its two conjugacy classes of involutions. We also provide matrix representatives of these suborbits in an accompanying computer file.

1 Introduction

Janko's simple group J_4 was the last sporadic simple group to be uncovered: in 1976 by Janko [10]. There he presented a vast amount of information relating to the local subgroups and conjugacy classes of this (possible) group. Only later was J_4 first constructed by D.J.Benson, J.H.Conway, S.P.Norton, R.A.Parker and J.G.Thackeray [14], making considerable use of machine calculations. More recently Ivanov and Meierfrankenfeld [9] gave a computer-free existence proof for J_4. As with all the sporadic simple groups, the maximal subgroups of J_4 have been extensively analysed and were eventually classified – see Kleidman and Wilson [11] and Lempken [13], [12].

It is the involutions of J_4 that will occupy our attention in this paper. For the rest of this paper, G denotes J_4 and t is some fixed involution of G. Set $X = t^G$, the G-conjugacy class of t. Of course, acting by conjugation yields a faithful transitive permutation representation of G on X, and it is this permutation action that we investigate here. Now G has two conjugacy classes of involutions, namely $2A$ and $2B$ – we shall employ Atlas [7] notation and conventions. Also the Atlas and its electronic sibling [16] as well as [11] and [13] will be our primary source for information about J_4. The suborbits of G on X will be our main focus; that is we wish to understand the $C_G(t)$-orbits of X. Apart from finding the sizes of $C_G(t)$-orbits of x we obtain representatives for each of these orbits. In order to carry out these calculations, we shall use the 112-dimensional representation of G over $GF(2)$ given in the electronic Atlas [16], in concert with the algebra packages Magma [5] and Gap [8]. Calculations using Gap reveals that the permutation rank of G on X is 20 when $t \in 2A$ and 119 when $t \in 2B$. So, not surprisingly, when $t \in 2B$ we face a much more challenging task. Similar investigations have been carried out for other sporadic finite simple groups – see [3] and [4].

Our main conclusions, contained in Section 2, appear in Table 1 (for $t \in 2A$) and Tables 2-9 (for $t \in 2B$). There we give not only the size of each $C_G(t)$-orbit of X but also include a number of its properties which in many cases aid speedy identification of the given orbit. For more details on this see Section 2. In order to facilitate further computation in J_4, we supply a file[1] – in the formats GAP and MAGMA [5] – containing t and for each $C_G(t)$-suborbit a 112×112 matrix (over $GF(2)$) representative. Additionally we also provide generators for $C_{C_G(t)}(x)$ for many of the representatives x. Accordingly there are two main folders in the file labelled `J4_2A_reps` and `J4_2B_reps`.

The folder `J4_2A_reps` contains a single file `J4_2A_reps.m`, containing a matrix `t`, and for each suborbit a representative `xCi`, where C is the ATLAS [7] name of the G-conjugacy class containing tx and i indicates that x is a representative for the i^{th} suborbit, with respect to Table 1 (if there is only one such orbit, the i is omitted). Note that we omit the so-called 'slave' class designations from all file and variable names, so that we have a representative called `x20B1` and not `x20B*1`. Since there are 119 suborbits in the $t \in 2B$ case, to avoid the files being too large the folder `J4_2B_reps` contains several files each containing about a dozen suborbit representatives, in rough correspondence with the tables in Section 2.2. Within these files, the representatives are named following the same scheme as above. For example, after loading the `J4_4AC_reps.m` file from the `J4_2B_reps` folder into the computer algebra package MAGMA, the element stored as `x4A3` corresponds to the suborbit representative x whose product tx is in class $4A$ and which lies in a suborbit of size $2^6.3^2.5.7.11$. (See Table 3.) The folders `J4_2A_reps` and `J4_2B_reps` also each contain a folder `Centralizers`, containing further files holding generators for the centralizers $C_{C_G(t)}(x)$ for many of the representatives x. The generators for the centralizer of a representative `xCi` are stored in an array called `CtxCi`. Centralizer generators are omitted where the orbit size is $2^{19}.3^2.5.7.11$ so that the centralizer is trivial, and where $z = tx$ has even order $2m$ and the centralizer has order 2, whence it is generated by z^m.

This paper is organized as follows. Section 2 begins with some general results and then, in two subsections, gives in addition to the tabulated data for $2A$ and $2B$ details of how the calculations were performed as well as introducing relevant notation. The following section, using the information in Subsection 2.2 and the computer files, probes the structure of the commuting graph on $X = 2B$. This graph, denoted $\mathcal{C}(G, X)$, is defined to be the graph whose vertex set is X with two distinct involutions in X joined by an edge if they commute. The commuting involution graph for $2A$ has already been described in [1].

[1] Available at http://personalpages.manchester.ac.uk/postgrad/p.taylor-5/J4_reps.rar

In Section 4 we gather together class constants for products of involutions in X and the dimension of the fixed point subspaces of elements of G acting on the 112-dimensional $GF(2)$-module for G. This information will aid our determination of $C_G(t)$-orbits of X and the identification of which class a given element of G belongs to.

Finally, the authors acknowledge the contribution made to the project by Chris Bates.

2 Suborbits of $X = t^G$

When investigating centralizers of involutions in finite groups computationally, the Bray algorithm [6] is often a vital tool. So it is here, and we recall the essential part of it with a minor variation.

Lemma 2.1 *[6] Suppose H is a finite group, K a subgroup of H and s an involution of H. Let $k \in K$ and let n be the order of $[s, k]$. If n is even, then $[s, k]^{n/2}, [s, k^{-1}]^{n/2} \in C_H(s)$ and if n is odd, then $k[s, k]^{(n-1)/2} \in C_H(s)$.*

Suppose C is a conjugacy class of G. Then we define the following subset of X

$$X_C = \{x \in X | tx \in C\}.$$

Plainly X_C is a union of certain $C_G(t)$-orbits of X. Our usual strategy is to examine X_C for each conjugacy class of G, hunting for $C_G(t)$-orbit representatives with $tx \in C$. Since for each C $|X_C|$ may be (and has been, see Tables 12 and 13 in Section 4) calculated we can tell when we have found all such representatives. We note that of the 62 classes of G, $|X_C|$ is non-zero for 14 classes when $t \in 2A$ and 47 classes when $t \in 2B$.

As intimated in the introduction our calculations will be carried out using the 112-dimensional $GF(2)$ representation for G supplied by [16]. So $G = \langle a, b\rangle$ where a and b are Type 1 generators (see [16]); that is, $a \in 2A$, $b \in 4A$, ab has order 37 and $ababb$ has order 10. Throughout V will denote the 112-dimensional $GF(2)G$-module.

Lemma 2.2 *Suppose H is a finite group, $N \trianglelefteq H$ and $K \leq H$ Put $\overline{H} = H/N$. Let $\mathcal{N}$ be a complete set of right coset representatives for $K \cap N$ in N and let $\mathcal{H} \subseteq G$ be such that $\overline{\mathcal{H}} = \{\overline{h} \mid h \in \mathcal{H}\}$ is a complete set of right coset representatives for $\overline{K}$ in $\overline{H}$ with $|\overline{\mathcal{H}}| = |\mathcal{H}|$. Then $\{nh \mid n \in \mathcal{N}, h \in \mathcal{H}\}$ is a complete set of right coset representatives for K in H.*

Proof Suppose that $Kn_1h_1 = Kn_2h_2$ where $n_1, n_2 \in \mathcal{N}$ and $h_1, h_2 \in \mathcal{H}$. Then $n_1h_1h_2^{-1}h_1^{-1} \in K$. Also $\overline{h_1} = \overline{h_2}$ and hence, as $|\overline{\mathcal{H}}| = |\mathcal{H}|$, $h_1 = h_2$. Thus $n_1n_2^{-1} \in K \cap N$ which gives $n_1 = n_2$, so verifying the lemma. □

Lemma 2.3 *Suppose H is a finite group, s is an involution in H and $Y = s^H$. Let $w = sy$ where $y \in Y$ and $w' \in \langle w \rangle$. Set $N = N_H(\langle w \rangle)$. If w is $C_H(s)$-conjugate to w', then w is $C_N(s)$-conjugate to w'.*

Proof By assumption $w^c = w'$ for some $c \in C_H(s)$. Since w and w' have the same order $\langle w \rangle = \langle w' \rangle$ and so $c \in N$. Therefore w and w' are $C_N(s)$-conjugate. □

Here is an example of how we use Lemma 2.3. Assume that $t \in 2B$, $x \in X$ and $z = tx \in 15A$. Now $C_G(z) \cong \mathbb{Z}_2 \times \mathbb{Z}_{15}$ and so $C_{C_G(t)}(x) = C_{C_G(z)}(t) \cong \mathbb{Z}_2$. From the size of X_{15A} (see Table 13, Section 4) we see that X_{15A} consists of two $C_G(t)$-orbits. Set $N = N_G(\langle z \rangle)$. Since $N \leq N_G(\langle z^3 \rangle)$ with the latter group of shape $D_{10} \times 2^3 : L_3(2)$ (see [11] or [13]), we see that $N \cong D_{10} \times S_3 \times 2$. Because t must invert both z^3 and z^5, t must project non-trivially into the D_{10} and S_3 direct factors. Thus $C_N(t) \cong 2^3$ with $C_{C_N(t)}(z) \cong 2$. Therefore the elements in $\langle z \rangle$ of order 15 have two orbits (under conjugation by $C_N(t)$ namely $\{z, z^4, z^{11}, z^{14}\}$ and $\{z^2, z^7, z^8, z^{13}\}$. Now suppose that x and x^{z^3} are in the same $C_G(t)$-orbit. Then $x^c = x^{z^3}$ for some $c \in C_G(t)$. Consequently

$$z^c = (tx)^c = tx^c = tx^{z^3} = tz^{-3}xz^3 = z^3txz^3 = z^7$$

which, by Lemma 2.3, means z and z^7 are $C_N(t)$-conjugate, but they are not. Therefore x and x^{z^3} are not in the same $C_G(t)$-orbit and so we may take x and x^{z^3} as our $C_G(t)$-orbit representatives.

2.1 $t \in 2A$

For an involution $t \in 2A$, the group $C_G(t)$ has structure $2^{1+12}.3.M_{22} : 2$, this being the second maximal subgroup of G listed in [7]. Beginning with this group and following [11], we construct $Q = O_{2,3}(C_G(t)) \cong 2^{1+12}.3$ by randomly finding elements in $C_G(t)$ having order 21 or 33 and taking respectively their 7^{th} or 11^{th} powers. We identify t as the unique non-trivial central element of Q.

Given an element $x \in 2A$, the size of $C_Q(x)$ is a $C_G(t)$-orbit invariant, as are the values q_{2A} and q_{2B}, being respectively the number of $2A$-, $2B$-elements in $C_Q(x)$. (See Section 2.2 for more details.) As it transpires, these invariants, along with the order of $z = tx$ and the dimension of its fixed space on V, are enough to distinguish all twenty $C_G(t)$-orbits. Therefore it is simple to find representative elements for the suborbits by random searching. We list these in Table 1.

The only remaining difficulty is calculating $C_{C_G(t)}(x)$, and hence the sizes of the orbits. Considering the action of $C_G(t)$ on Q by conjugation gives us a homomorphism ϕ from $C_G(t)$ to a subgroup S of $Sym(24576)$. In Magma we can (just) construct this homomorphism explicitly. Now, suppose $z = tx$ has even order $2m$. Then $z^m \in C_{C_G(t)}(x)$, and $C_{C_G(t)}(x) \leq C_G(z) \leq C_G(z^m)$. In particular,

$C_{C_G(t)}(x) \leq C_{C_G(t)}(z^m)$. So we may compute $C_S(\phi(z^m))$, and if it is sufficiently small, compute $C_{C_G(t)}(x)$ in its inverse image. Similarly, if $C_Q(x)$ is non-trivial, we may compute its stabilizer in S, and compute $C_{C_G(t)}(x)$ in the inverse image of this group. Note that this second approach could work where z has odd order, but we discover that $C_Q(x) = 1$ in all cases when z has odd order: 3, 5 and 11. However, since we have already found 17 $C_G(t)$-orbits and the permutation rank of G on X is 20 we infer that X_{3A}, X_{5A} and X_{11B} are each single $C_G(t)$-orbits. To compute their centralizers we work in $C^\star_G(z)$ (further details of this strategy are given in Section 2.2).

We note finally that, although $4A$ and $4B$ elements cannot be distinguished by the dimensions of their fixed spaces, one $C_G(t)$-orbit in $X_{4A} \cup X_{4B}$ has the same size as X_{4A} and all the others are larger, so it is trivial to separate these orbits.

Representative	$\lvert\mathcal{O}_x\rvert$	$\lvert C_Q(x)\rvert$	q_{2A}	q_{2B}
t	1	$2^{13}.3$	1387	2772
$x2A_1$	$2^5.3^2.5.7.11$	2^8	107	84
$x2A_2$	$2.3^2.7.11$	2^{12}	747	1364
$x2B_1$	$2^7.3^2.5.11$	2^7	71	56
$x2B_2$	$2^4.3.5.7.11$	$2^9.3$	139	180
$x3A$	$2^{14}.3^2.5.11$	1	0	0
$x4A$	$2^8.3^2.5.7.11$	2^6	33	30
$x4B_1$	$2^{12}.3^3.5.7.11$	2^2	1	2
$x4B_2$	$2^{11}.3^3.5.7.11$	2^2	3	0
$x4B_3$	$2^{10}.3^2.5.7.11$	2^3	3	4
$x4B_4$	$2^{10}.3^2.5.7.11$	2^4	9	6
$x4C_1$	$2^{11}.3^2.5.7.11$	2^3	3	4
$x4C_2$	$2^{11}.3^2.5.7.11$	2^3	7	0
$x5A$	$2^{15}.3^2.5.7.11$	1	0	0
$x6B$	$2^{15}.3^3.5.7.11$	1	0	0
$x6C$	$2^{14}.3^2.5.7.11$	1	0	0
$x8C$	$2^{15}.3^3.5.7.11$	1	0	0
$x10A$	$2^{16}.3^3.5.7.11$	1	0	0
$x11B$	$2^{20}.3^3.5.7$	1	0	0
$x12B$	$2^{17}.3^2.5.7.11$	1	0	0

$C_G(t)$-orbits of X when $t \in 2A$

Table 1

2.2 $t \in 2B$

Just as in Subsection 2.1 we begin by summarizing the structure of $C_G(t)$. Set $Q = O_2(C_G(t))$. Then $Q \cong 2^{11}$ and $C_G(t)/Q \cong M_{22} : 2$ $(= Aut(M_{22}))$. Also $M_1 = N_G(Q)$ is a maximal subgroup of G with $M_1/Q \cong M_{24}$ and is the first maximal subgroup (as listed in [16]) of G. We begin our calculations starting with M_1 as given in [16]. After determining Q, we choose $t \in Q \cap 2B$ (and of course then fix it). Using Lemma 2.1 (with $H = K = G, s = t$) we calculate $C_G(t)$ (generators for $C_G(t)$ are given in the file).

For $R \leq Q$ we define

$$q_{2A}(R) = |R \cap 2A| \quad \text{and}$$
$$q_{2B}(R) = |R \cap 2B|.$$

Then $q_{2A}(Q) = 1771$ and $q_{2B}(Q) = 276$. In Tables 2-9 for $x \in X$ we write q_{2A} for $q_{2A}(C_Q(x))$ and q_{2B} for $q_{2B}(C_Q(x))$. Furthermore, we have that $Q \cap 2A$ splits into two $C_G(t)$-orbits of sizes 231 and 1540 while $Q \cap 2B$ splits into three $C_G(t)$-orbits of sizes 1, 44 and 231. Such $C_G(t)$-orbits, from time-to-time, play a useful role in discriminating between certain $C_G(t)$-orbits of X.

When trying to find new $C_G(t)$-orbits (and representatives of such orbits), it is useful to quickly discover whether a (usually randomly chosen) $x \in X$ is in one of the $C_G(t)$-orbits already catalogued at that point. Our first step for a given $x \in X$ is to calculate $C_Q(x)$ (computationally this is relatively quick as $|Q| = 2^{11}$). Then we determine $q_{2A}(C_Q(x))$ and $q_{2B}(C_Q(x))$ (by calculating $\dim(C_V(\xi))$, $\xi \in C_Q(x)$ – for $2A$-elements it is 62 and for $2B$-elements it is 56). A further straightforward calculation is to determine $\dim(C_V(t) \cap C_V(x))$, which we denote by d_x in the following tables. Put $z = tx$. If z has even order, say $2m$, then z^m is an involution which commutes with both t and x. So $z^m \in C_G(t)$. Hence, we can ask where $w = z^m$ is in $C_G(t)$. Set $\overline{C_G(t)} = C_G(t)/Q$. Then either $\overline{w} = \overline{1}$ or $\overline{w}$ belongs to one of the three involutions conjugacy classes in $\overline{C_G(t)} \cong M_{22} : 2$, which we label by $\overline{2A}$, $\overline{2B}$, $\overline{2C}$. We have that $\overline{2A}$ is in the derived subgroup of $\overline{C_G(t)}$ while $\overline{2B}$ and $\overline{2C}$ are not, and we choose our notation so it agrees with [11]. So when z has even order, column 3 in Tables 2-9 has entries $\overline{1A}$,$\overline{2A}$,$\overline{2B}$ or $\overline{2C}$ if, respectively, $\overline{w}$ is in $\overline{1A}$,$\overline{2A}$,$\overline{2B}$ or $\overline{2C}$.

We dwell a little longer on the case when $z = tx$, $x \in X$ has even order. Then $z^2 = txtx = tt^x$. Set $y = t^x$. So $z^2 = ty$ and $y \in X$. Assume that $x_1, x_2 \in X$ and $x_1^c = x_2$ for some $c \in C_G(t)$. Then $(tx_1)^c = t^c x_1^c = tx_2$. Hence, $((tx_1)^2)^c = (tx_2)^2$. That is, $ty_1^c = ty_2$ where $y_1 = t_1^x$ and $y_2 = t_2^x$. Therefore, $y_1^c = y_2$. This observation provides us with a possible way of discerning whether $x_1, x_2 \in X$ where $tx_1, tx_2 \in C = z^G$ are in different $C_G(t)$-orbits. If we can see that $y_1 = t_1^x$ and $y_2 = t_2^x$ are in different $C_G(t)$-orbits, then so must x_1 and x_2 be. Note that the $C_G(t)$-orbits of y_1 and y_2 will be subsets of X_D where $D = (z^2)^G$. As our strategy is to analyze X_E for class $E = w^G$ of G, starting with w of small order and working up, we will have to hand data about the $C_G(t)$-orbits of y_1 and y_2. In Tables 2-9, where relevant we have a final column giving the $C_G(t)$-orbit of $y = t^x$.

There is a further invariant of a $C_G(t)$-orbit which we mention after outlining our routine for calculating $C_{C_G(t)}(x)$, $x \in X$. Since G is a large matrix group whose elements are 112×112 matrices and $|C_G(t)| = 2^{19}.3^2.5.7.11 = 1,816,657,920$ using standard MAGMA commands will not (unless you are very lucky) produce $C_{C_G(t)}(x)$. (While calculating $C_G(x)$ and then trying to work out $C_G(t) \cap C_G(x)$ brings you up against the membership problem.) In our initial set up we define a permutation action of $C_G(t)$ upon $\Omega = Q$ (given by conjugation of $C_G(t)$ on Q). This gives a homomorphism

$$\varphi : C_G(t) \to \overline{C_G(t)} = M(\cong M_{22} : 2) \leq Sym(2048).$$

Calculations within $Sym(2048)$ are quick. Also we have an accompanying map

$$\psi : \Omega \to \{1, \ldots, 2048\}$$

which commutes with the conjugation action of $C_G(t)$ on Ω and the permutation action of M on $\{1, \ldots, 2048\}$. Thus we may look at

$$S_x = \mathrm{Stab}_M(\psi(C_Q(x))),$$

and this will be a further invariant of the $C_G(t)$-orbit of x. Let K_x be the inverse image in $C_G(t)$ of S_x. Since $C_Q(x) \trianglelefteq C_{C_G(t)}(x)$, it follows that $C_{C_G(t)}(x) \leq K_x$. So this restricts the location of $C_{C_G(t)}(x)$. Now we use the Bray algorithm as in Lemma 2.1 where we take $H = G$, $K = K_x$ and $s = x$ to produce elements which commute with x. Suppose L is the group generated by these elements. So $L \leq C_G(x)$. However, we are investigating $C_{C_G(t)}(x)$ and we have no guarantee that L is contained in $C_G(t)$. Therefore we must take $C_L(t)$ $(\leq C_{C_G(t)}(x))$. We repeat this procedure until we obtain a subgroup $L_\infty \leq C_{C_G(t)}(x)$ which has 'small index' in K_x. We may also suppose $C_Q(x) \leq L_\infty$. If S_x is of reasonable size this process, so far, has always been successful. Our next step is to obtain a complete set of right coset representatives $\overline{\mathcal{K}_\infty}$ for $\overline{L_\infty}$ in S_x $(= \overline{K_x})$. Let $\mathcal{K}$ be the set consisting of one arbitrary element k from each $\varphi^{-1}(\overline{k})$, $\overline{k} \in \overline{\mathcal{K}}$. Also let $\mathcal{Q}$ be a complete set of right coset representatives for $C_Q(x)$ in Q. Since $|Q| = 2^{11}$ and L_∞ has 'small index' in K_x this can be achieved using standard MAGMA commands. With $H = K_x$, $N = C_Q(x)$ and $K = L_\infty$, Lemma 2.2 implies that $\mathcal{R} = \{qk | q \in \mathcal{Q}, k \in \mathcal{K}\}$ is a complete set of right coset representatives for $L_\infty \in K_x$ (recall $C_Q(x) \trianglelefteq C_{C_G(t)}(x)$ and $C_Q(x) \leq L_\infty \leq C_{C_G(t)}(x)$). Usually the size of $\mathcal{R}$ is at most $3,000$ and so it is straightforward to check through and find which ones commute with t, giving

$$\mathcal{R}_t = \{r | r \in \mathcal{R}, [t, r] = 1\}.$$

Then $C_{C_G(t)}(x) = \langle L_\infty, \mathcal{R}_t \rangle$ with $|C_{C_G(t)}(x)| = |L_\infty||\mathcal{R}_t|$ and very importantly we now have the size of the $C_G(t)$-orbit of x.

We illustrate how the above pans out in a concrete example. Let $x = x4B_6$. Then $|C_Q(x)| = 2^2$. Calculating in the 2048 degree permutation representation gives $|S_x| = 768 = 2^8.3$. So $|K_x| = 2^{11}.2^8.3 = 2^{19}.3$ and hence $|C_{C_G(t)}(x)|$ divides $2^{10}.3$. Using the Bray algorithm we arrive at an L_∞ with $C_Q(x) \leq L_\infty$ and $|L_\infty| = 2^9$. Then $|\mathcal{Q}| = 2^9$, $|\overline{\mathcal{K}}| = |\mathcal{K}| = 2.3$ and so $|\mathcal{R}| = 2^9.2.3 = 2^{10}.3 = 3,072$. Checking reveals that $|\mathcal{R}_t| = 2$. Consequently $|C_{C_G(t)}(x)| = 2^{10}$ and the $C_G(t)$-orbit of x has size $2^9.3^2.5.7.11 = 1,774,080$.

The approach just outlined works very well when $z = tx$ has order 4 for then we always have $C_Q(x) \neq 1$, and consequently $S_x \neq M$. When $z = tx$ has order 8, then $C_Q(x) = 1$ in the majority of cases. In order to determine $C_{C_G(t)}(x)$ when $z = tx$ has order 8, we calculate $C_{C_G(t)}(y)$ where $y = t^x$ in the manner described above. Since $C_{C_G(t)}(x) \leq C_{C_G(t)}(y)$ and the order of $C_{C_G(t)}(y)$ is not too large we can then determine $C_{C_G(t)}(x)$.

We now give details of breaking X_C into $C_G(t)$-orbits for various classes C of G. So set $z = tx$, where $x \in X$. For certain classes C there are some difficulties which must be overcome. However the case when z has order 2 has been much studied.

2.2.1 Order of z is 2

The $C_G(t)$-orbits (and $C_{C_G(t)}(x)$) for $C = 2A$ or $2B$ may be read off Table 1 of [11].

2.2.2 Order of z is 3

We find one representative $x = x3A_2$ for which $C_Q(x)$ has order 2^3 and so it is easy to compute $C_{C_G(t)}(x)$. A second representative can easily be found having $C_Q(x)$ trivial. To compute its centralizer we first find $C_G(z)$ for $z = tx$ using [2], since z is a strongly real element inverted by t. Computing $C_{C_G(t)}(x)$ in this smaller group we determine that $X_{3A} = x3A_1^{C_G(t)} \cup x^{C_G(t)}$, so we take this x as our representative $x3A_1$ and we are done.

2.2.3 Order of z is 4

Finding randomly elements $x \in 2B$ with $z = tx$ having order 4, and calculating the various invariants described above, we arrive at a collection of 19 representatives known to be in different $C_G(t)$ orbits (we also know the sizes of these orbits). Since $4C$-elements can be distinguished by the dimensions of their fixed spaces on V, it is easy to partition these orbits into those in $X_{4A} \cup X_{4B}$ and those in X_{4C}. This method does not allow us to distinguish between $4A$- and $4B$-elements. However, we know that $4A$-elements can be found by powering down from elements of order 20, 40 or 44. So we find elements $x' \in 2B$ with tx' having order 20, 40 or 44, and then $x = t(tx')^n$ ($n = 5, 8$ or 11 respectively) is a $2B$-element with $tx = z \in 4A$. In this way we find representatives matching in their invariants two of our already-encountered representatives. These are the representatives $x4A_1$ and $x4A_2$ in Table

3. We know from the structure constants that X_{4A} has size not divisible by 5, and only one of our representatives is from an orbit $\mathcal{O}$ with $5 \nmid |\mathcal{O}|$, so we conclude this orbit must lie in X_{4A}. This is representative $x4A_4$ in Table 3. Now we have that $|X_{4A}| - |x4A_1^{C_G(t)}| - |x4A_2^{C_G(t)}| - |x4A_4^{C_G(t)}| = 2^6.3^2.5.7.11$, which is the size of the smallest remaining orbit for which we have a representative, so we conclude it too lies in X_{4A}. Then the remaining orbits lie in X_{4B}.

Unfortunately, the sizes of the orbits we have lying in X_{4B} do not total $|X_{4B}|$. Continued random searching yields no new representatives, so we conclude that one or more orbits exist in X_{4B} that match in all the invariants we calculate and so are invisible to our search. Eventually we determine this is indeed the case and that two orbits of size $2^{10}.3^2.5.7.11$ are the culprits. We describe the procedure by which we arrive at this conclusion at the end of this section.

2.2.4 Order of z is 5

In the case when $z = tx \in 5A$ it turns out that X_{5A} is a union of two $C_G(t)$-orbits. We first locate $x = x5A_2$ which has $|C_Q(x)| = 2^3$. Proceeding as above we then calculate that $C_{C_G(t)}(x)$ has order $2^6.3.7$ and so $x^{C_G(t)}$ is a $C_G(t)$-orbit of size $2^{13}.3.5.11$. As a byproduct, by orders, $C_G^*(z) = \langle t, z, C_{C_G(t)}(x)\rangle$. Now searching within $C_G^*(z)$ we were able to find x_1 and zx_1 in $2B$ and calculate (directly) that $|C_{C_G(z)}(x_1)| = 2^4$. Let $g \in G$ be such that $(zx_1)^g = t$. Then $tx_1^g = z^g \in 5A$ and $C_{C_G(t)}(x_1^g) = C_{C_G(z)}(x_1)^g$, so x_1^g is in a different $C_G(t)$-orbit than $x5A_2$, and $|(x_1^g)^{C_G(t)}| = 2^{15}.3^2.5.7.11$. To find our representative we need to obtain such a g. To do this, we search for an involution $r \in G$ such that tr and zx_1r both have odd order. Then the groups $\langle t, r\rangle$ and $\langle r, zx_1\rangle$ are dihedral groups with their respective involutions conjugate, and so we can find an element g conjugating zx_1 to t by multiplying suitably chosen elements from $\langle t, r\rangle$ and $\langle r, zx_1\rangle$. We then take our representative $x5A_1 = x_1^g$.

2.2.5 Order of z is 8

We make some further comments on breaking X_C into $C_G(t)$-orbits when $tx = z \in C$ has order 8. There are three G-conjugacy classes of elements of order 8 which, unfortunately, are not distinguished by the dimension of their fixed space on V (see Table 11, Section 4.1). However, elements in $8A$, respectively $8B$ and $8C$, can be obtained by powering down from any element of G of order 40, respectively order 24 and order 16. Thus to find $x \in X$ so as $z = tx \in 8B$ we first find elements of order 24. In more detail we choose random $x' \in X$ and check whether tx' has order 24. On obtaining such an element we then set $z = (tx')^3$ ($\in 8B$) and $x = tz$. Observe that $x \in X$. Hunting for $C_G(t)$-representatives in this manner we find that X_{8B} consists precisely of two $C_G(t)$-orbits. Note that (see Table 4) we need to examine the class of $\overline{w}$, or $y = x^t$ in order to tell these two orbits apart.

In investigating X_{8C} we proceed as above except we require tx' to have order 16. On looking through a number of such x and calculating $C_G(t)$-orbit invariants such as $|C_Q(x)|$, d_x and the $C_G(t)$-orbit to which $y = t^x$ belongs we were able to pin down the two $C_G(t)$-orbits $x8C_1^{C_G(t)}$ and $x8C_4^{C_G(t)}$. However despite extensive searching as described above we were unable to find further representatives for $C_G(t)$-orbits (though, because of the structure constants and the sizes of the known $C_G(t)$-orbits, we knew they were there). It appears that the powering down from elements of order 16 was giving us a skewed view in that we were not encountering any elements in (what turn out to be) two $C_G(t)$-orbits of size 56,770,560 (with representatives $x8C_2, x8C_3$). So we employ a different strategy, as follows.

We know that for $z \in 8B \cup 8C$, $z^2 \in 4B$, while for $z \in 8A$, $z^2 \in 4A$. However, $4A$- and-$4B$ elements are not distinguished by the dimensions of their fixed spaces. Fortunately we have the element $y = t^x$ giving $ty = z^2$, and since we have catalogued all of the order 4 orbits we can determine the conjugacy class of z^2 and hence of z by examining y. So we can now determine whether a z is in $8A$ or in $8B \cup 8C$, and we have already found both orbits with $z \in 8B$ so we may place every orbit. Two orbits in X_{8A} have identical invariants, and we describe the procedure for determining this and for finding representatives at the end of this subsection.

2.2.6 Order of z is 10

From calculations performed for X_{5A}, we have $C_G^*(z_1)$ explicitly where $z_1 = tx_1$ and $x_1 = x5A_2$. Also $t \in C_G^*(z_1)$. Now suppose $z = tx \in 10A$ and, after conjugating, that $z^2 = z_1$. So $z \in C_G^*(z_1)$. Clearly $C_G^*(z) \leq C_G^*(z_1)$ and working directly in the latter group we find $x10A_1^{C_G(t)} \cup x10A_2^{C_G(t)} = X_{10A}$ (and can distinguish the orbits by d_x). Moving onto the case $z = tx \in 10B$ we encounter four $C_G(t)$-orbits, two of which (having size $2^{16}.3^2.5.7.11$) don't appear to have any properties which would otherwise allow us to conclude that they are in fact in different $C_G(t)$-orbits. We give the details of how this unfolds.

With $C_G^*(z_1)$ as above we choose $t_1 \in C_G^*(z_1) \backslash C_G(z_1)$ (t_1 not in the same $C_G^*(z_1)$-conjugacy class as t) and find s_i $(i = 1, 2, 3)$ such that $s_i \in X \cap C_G(z_1) \cap C_G(t_1)$ and $t_1 z_1 s_i \in X$ $(i = 1, 2, 3)$. Then $z_1 s_i \in 10B$ $(i = 1, 2, 3)$. Set $x_i' = t_1 z_1 s_i$ $(\in X)$. Calculating directly in $C_G^*(z_1)$ gives that $|C_{C_G(t_1)}(x_i')| = 8$ for $i = 1, 2$ and that $|C_{C_G(t_1)}(x_3')| = 16$. Using the odd order Dihedral trick from Section 2.2.4 we find $g \in G$ such that $t_1^g = t$. Now set $x_i = (x_i')^g$. Then we find that $C_Q(x_i) = 1$ for $i = 1, 2, 3$. By normal random searching we can also find $x_4 \in X$ with $|C_Q(x_4)| = 4$ and $|x_4^{C_G(t)}| = 2^{15}.3^2.5.7.11$. By the sizes of $C_Q(x)$ we know that $x_3^{C_G(t)} \neq x_4^{C_G(t)}$, and by the orbit sizes that neither of these is equal to $x_1^{C_G(t)}$ or $x_2^{C_G(t)}$.

Now we look at $x_1^{C_G(t)}$ and $x_2^{C_G(t)}$. Suppose that x_1 and x_2 are $C_G(t)$-conjugate. Then x_1' and x_2' must be $C_G(t_1)$-conjugate and so $x_1'^h = x_2'$ for some $h \in C_G(t_1)$.

Therefore $(t_1x_1')^h = t_1x_2'$. We also know from our earlier calculations that $(t_1x_1')^2 = (t_1x_2')^2 \in \langle z_1\rangle$ and thus $h \in C_G(t_1) \cap C_G((t_1x_1')^2) = C_G(t_1) \cap C_G(\langle z_1\rangle)$. Looking at $C_G(\langle z_1\rangle)$ we see that no such h exists. Consequently x_1 and x_2 are not $C_G(t)$-conjugate. Since $|x_1^{C_G(t)}| + |x_2^{C_G(t)}| + |x_3^{C_G(t)}| + |x_4^{C_G(t)}| = |X_{10B}|$, we are finished with X_{10B}.

2.2.7 Order of z is 11

Now we outline how X_{11A} was studied. Suppose $z = tx \in 11A, x \in X$. Recall from [7] or [11] that $N_G(\langle z\rangle)$ is a maximal subgroup of G of order $2^4.3.5.11^3$. A G-conjugate of this subgroup is available from [16]. Call this subgroup N_1. Within N_1 we found $z_1 \in 11A$ with $\langle z_1\rangle \trianglelefteq N_1$. Then $t_1, x_1 \in 2B \cap N_1$ were obtained (randomly) so as $t_1x_1 = z_1$. On calculating we found that $|C_{C_G(z_1)}(t_1)| = 22$ and thus, by the size of X_{11A}, X_{11A} is a $C_G(t)$-orbit. Now lady luck was with us as tt_1 had order 33 and so, by conjugating with a suitable $g \in \langle t, t_1\rangle$ we obtained $x = x_1^g$ so as $tx \in 11A$ and $C_{C_G(t)}(x) = (C_{C_G(z_1)}(x_1))^g$. For $x_2 \in 2B \cap N_1$ with $z_2 = t_1x_2 \in 11B \cap N_1$ we have $|C_G(z)| = 2.11^2$ (see [7]). Clearly 2 divides $|C_{C_G(z_2)}(t_1)|$ and, calculating in N_1 we find that $|C_{C_G(z_2)}(t_1)| = 2$. So X_{11B} is a $C_G(t)$-orbit and conjugating by g gives $x11B_1$.

2.2.8 Order of z is 12

Elements g of G of order 12 for which $\dim(C_V(g)) = 10$ must be in class $12C$. Such considerations do not distinguish between the classes $12A$ and $12B$ although elements in G of order 24 always square to $12B$-elements. As a consequence it is easy to break X_{12C} into $C_G(t)$-orbits.

We describe how we deal with X_{12A} and X_{12B}. First we find an $x \in X$ such that $z = tx \in 12A \cup 12B$ by checking that $\dim(C_V(z)) = 12$. We next calculate $C_G(z)$ (which has order $2^6.3$). Since $z^4 \in 3A$, $C_G(z^4) \equiv 6^{\cdot}M_{22}$. Employing [2] (since z^4 is a strongly real element inverted by t) and using the Meataxe [15] to check that the order is correct, quickly deliver $C_G(z^4)$. We note that when using [2] here, we already have generators for $C_G(t)$ and so can take random elements directly without having to use Bray's algorithm [6]. We generally find the full centralizer of $z^4 \in 3A$ after around 500 loops of the procedure. Within this smaller group we then calculate $C_G(z)$. Then we get $C_G^*(z) = C_G(z)\langle t\rangle$. Define the following subset of $C_G^*(z)$

$$\mathcal{I} = \{w \mid w \in 2B, wz \in 2B, w \in C_G^*(z) \setminus C_G(z)\}.$$

Since $|C_G^*(z) \setminus C_G(z)| = 2^6.3$ it is straightforward to enumerate $\mathcal{I}$. We discover that $|\mathcal{I}| = 72$ when $z \in 12A$ and $|\mathcal{I}| = 84$ when $z \in 12B$ (recall we can identify a $12B$-element as the square of an element of order 24). The size of $\mathcal{I}$ is what we use to distinguish between $12A$ and $12B$-elements. So now suppose that we have chosen $x \in X$ such that $z = tx \in 12A$ – the strategy we now follow works just as well for $12B$. So next we determine the $C_G^*(z)$-orbits (under conjugation) of $\mathcal{I}$. It turns out that there are four such orbits $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$, $\mathcal{O}_4$ with $|\mathcal{O}_1| = |\mathcal{O}_2| = 24$ and $|\mathcal{O}_3| = |\mathcal{O}_4| = 12$. Note that what we are doing is making z the subject and letting t 'vary'. So, at this stage, we see that X_{12A} consists of at most four $C_G(t)$-orbits. Running through random elements of $C_G(z^4)$ we find $h \in C_G(z^4)$ for which $t^h = s \in \mathcal{O}_1$. So we take $x_1 = (sz)^{(h^{-1})}$. Then $x_1 \in X$ and $tx_1 \in 12A$. Calculations then reveal that $C_Q(x_1) = 1$, $|C_{C_G(t)}(x_1)| = 8$ and $t^{x_1} \in X_{6B}$. We find we may similarly conjugate with elements from $C_G(z^4)$ to make t the 'subject' for $\mathcal{O}_3$ and $\mathcal{O}_4$. Though we didn't find an $h \in C_G(z^4)$ such that $t^h \in \mathcal{O}_2$ (for different z this situation may, and can, vary). To deal with the case of $\mathcal{O}_2$ we take $s_1 \in \mathcal{O}_1$ with $t^h = s_1$ where $h \in C_G(z^4)$ and choose some $s_2 \in \mathcal{O}_2$. Then hunt for an involution $r \in G$ such that both $s_1 r$ and $s_2 r$ have odd order - this is quickly done. Hence t can be conjugated to s_2 by using h multiplied by suitably chosen elements from $\langle s_1, r\rangle$ and $\langle s_2, r\rangle$.

2.2.9 Order of z is 20

In $C_G(z_1)$ (z_1 as in Section 2.2.6) we find an element f of order 4. Set $z = fz_1$ and take this to define $20A$. Then we calculate $C_G^*(z) = C^*_{C_G^*(z_1)}(z)$ and working in this group we discover that $20A$ breaks into 2 $C_G(t)$-orbits of different sizes. As a representative for $20B$ we use z^3 and repeat the above process.

2.2.10 Order of z is 40

Again calculating within $C_G^*(z_1)$ reveals that $|C_{C_G(t)}(x)| = 2$ for all $x \in X$ such that $tx \in 40A$. Thus, on account of $|X_{40A}|$, X_{40A} splits into two $C_G(t)$-orbits. To locate representatives we start with $v \in C_G^*(z_1)$ such that $vz \in X$ and look at $e = \dim(C_V(x) \cap C_V(v^{vz}))$. We quickly observe e being 4 and 5 and this serves to distinguish orbits. We deal with $40B*$ similarly.

We end this section dealing with the following conundrum. Occasionally, we find that two $C_G(t)$-orbits agree on every invariant we consider. In these cases, we use the following method to find representatives of such orbits.

Suppose we have elements $x_1, x_2 \in X$ that produce the same invariants but that we suspect may lie in different orbits. We aim to find a subset $Y \subseteq C_G(t)$ of manageable size such that any element of $C_G(t)$ conjugating x_1 to x_2 must lie in Y. We may then simply test whether $x_1^y = x_2$ for all $y \in Y$.

Let H_1, H_2 be the images of $C_{C_G(t)}(x_1), C_{C_G(t)}(x_2)$ in the factor group $M \cong C_G(t)/Q$. (Recall $M \leq Sym(2048)$ where the action is given by conjugation on Q, so computation in M is straightforward.) If H_1, H_2 are not M-conjugate then clearly x_1, x_2 are in different $C_G(t)$-orbits. Suppose they are conjugate by an element g. We compute $N = N_M(H_1)$, and so form the coset Ng consisting of all elements in M conjugating H_1 to H_2. Note that any element of $C_G(t)$ conjugating x_1 to x_2 must have its image in Ng. Any such element must also conjugate $C_Q(x_1)$ to $C_Q(x_2)$, and since our group action in M corresponds to conjugacy on Q we can use this fact to narrow down the search further, since we need consider only those elements of Ng that map $\psi(C_Q(x_1))$ to $\psi(C_Q(x_2))$. Let Z be the set of elements of Ng satisfying this condition. (Of course if $C_Q(x_1) = 1$ then $Z = Ng$.) Then our set Y is the inverse image of Z, this set having size $2^{11}.|Z|$. We generally find that, either by virtue of the size of N, or the restriction added when $C_Q(x_1)$ is nontrivial, that the set Y has size not more than a few hundred thousand, and a search through all these elements is feasible.

For example, the representatives $x4B_3$ and $x4B_4$ have precisely the same invariants (see Table 3). Let these elements act as x_1, x_2 above. Then we find that the normalizer N has size 768. In this case, the groups $C_Q(x_1), C_Q(x_2)$ are non-trivial, having size 2^2, so we can pare down the set Ng to just those elements that conjugate $C_Q(x_1)$ to $C_Q(x_2)$. This gives us a set Z having size 128, and so $|Y| = 2^{11}.|Z| = 262,144$. We check each of the elements of Y in turn and, discovering that none of them conjugates $x_1 = x4B_3$ to $x_2 = x4B_4$, we conclude that they indeed lie in different $C_G(t)$-orbits.

Representative, x	$\|\mathcal{O}_x\|$	Class of $\overline{w}$	$\|C_Q(x)\|$	q_{2A}	q_{2B}	d_x
t	1	$\overline{1A}$	2^{11}	1771	276	56
$x2A_1$	$2^6.3^2.7.11$	$\overline{2C}$	2^6	51	12	31
$x2A_2$	$2^4.3.5.7.11$	$\overline{2A}$	2^7	91	36	32
$x2A_3$	$3.7.11$	$\overline{1A}$	2^{11}	1771	276	36
$x2B_1$	$2^5.3^2.5.7.11$	$\overline{2A}$	2^7	91	36	28
$x2B_2$	$2^5.3.5.7.11$	$\overline{2B}$	2^7	91	36	28
$x2B_3$	$2^2.11$	$\overline{1A}$	2^{11}	1771	276	36
$x3A_1$	$2^{13}.3^2.7.11$	-	1	0	0	20
$x3A_2$	$2^{12}.3.5.11$	-	2^3	7	0	20
$x5A_1$	$2^{15}.3^2.5.7.11$	-	1	0	0	12
$x5A_2$	$2^{13}.3.5.11$	-	2^3	7	0	12
$x11A$	$2^{18}.3^2.5.7$	-	1	0	0	1
$x11B$	$2^{18}.3^2.5.7.11$	-	1	0	0	6
$x23A$	$2^{19}.3^2.5.7.11$	-	1	0	0	1
$x29A$	$2^{19}.3^2.5.7.11$	-	1	0	0	0
$x31A$	$2^{19}.3^2.5.7.11$	-	1	0	0	1
$x31B*5$	$2^{19}.3^2.5.7.11$	-	1	0	0	1
$x31C*6$	$2^{19}.3^2.5.7.11$	-	1	0	0	1
$x37A$	$2^{19}.3^2.5.7.11$	-	1	0	0	2
$x37B*2$	$2^{19}.3^2.5.7.11$	-	1	0	0	2
$x37C*4$	$2^{19}.3^2.5.7.11$	-	1	0	0	2
$x43A$	$2^{19}.3^2.5.7.11$	-	1	0	0	0
$x43B*6$	$2^{19}.3^2.5.7.11$	-	1	0	0	0
$x43C*7$	$2^{19}.3^2.5.7.11$	-	1	0	0	0

$z = tx$; $z = 1$ or z of prime order
Table 2

Representative, x	$\|\mathcal{O}_x\|$	Class of $\overline{w}$	$\|C_Q(x)\|$	q_{2A}	q_{2B}	$\|S_x\|$	d_x
$x4A_1$	$2^{12}.3^2.7.11$	$\overline{2C}$	2	1	0	3840	16
$x4A_2$	$2^{10}.3.5.7.11$	$\overline{2A}$	2^3	7	0	768	17
$x4A_3$	$2^6.3^2.5.7.11$	$\overline{1A}$	2^7	91	36	192	20
$x4A_4$	$2^7.3^2.7.11$	$\overline{1A}$	2^6	51	12	320	20
$x4B_1$	$2^{12}.3^2.5.7.11$	$\overline{2C}$	2	1	0	3840	16
$x4B_2$	$2^{10}.3^2.5.7.11$	$\overline{2A}$	2^2	3	0	768	16
$x4B_3$	$2^{10}.3^2.5.7.11$	$\overline{2A}$	2^2	3	0	128	17
$x4B_4$	$2^{10}.3^2.5.7.11$	$\overline{2A}$	2^2	3	0	128	17
$x4B_5$	$2^{10}.3^2.5.7.11$	$\overline{2A}$	2^2	3	0	128	16
$x4B_6$	$2^9.3^2.5.7.11$	$\overline{2A}$	2^2	3	0	768	18
$x4B_7$	$2^8.3^2.5.7.11$	$\overline{1A}$	2^6	51	12	32	19
$x4B_8$	$2^8.3^2.5.7.11$	$\overline{2A}$	2^3	7	0	768	19
$x4B_9$	$2^8.3.5.7.11$	$\overline{1A}$	2^7	91	36	192	18
$x4B_{10}$	$2^8.3.5.7.11$	$\overline{2A}$	2^3	7	0	768	18
$x4C_1$	$2^{12}.3^2.5.7.11$	$\overline{2A}$	2^2	3	0	128	14
$x4C_2$	$2^{12}.3^2.5.7.11$	$\overline{2B}$	2^2	3	0	128	14
$x4C_3$	$2^{11}.3^2.5.7.11$	$\overline{2A}$	2^2	3	0	128	14
$x4C_4$	$2^{11}.3^2.5.7.11$	$\overline{2B}$	2^2	3	0	768	14
$x4C_5$	$2^9.3.5.7.11$	$\overline{1A}$	2^7	91	36	168	18
$x4C_6$	$2^9.3.5.7.11$	$\overline{2B}$	2^3	7	0	768	18

$z = tx$ of order 4
Table 3

Representative, x	$\lvert\mathcal{O}_x\rvert$	Class of $\overline{w}$	$\lvert C_Q(x)\rvert$	q_{2A}	q_{2B}	d_x	y
$x8A_1$	$2^{15}.3^2.5.7.11$	$\overline{2A}$	2	1	0	9	$x4A_2$
$x8A_2$	$2^{15}.3^2.5.7.11$	$\overline{2C}$	1	0	0	9	$x4A_1$
$x8A_3$	$2^{13}.3^2.5.7.11$	$\overline{1A}$	2	1	0	10	$x4A_4$
$x8A_4$	$2^{13}.3^2.5.7.11$	$\overline{1A}$	2^2	3	0	10	$x4A_3$
$x8A_5$	$2^{13}.3^2.5.7.11$	$\overline{2C}$	1	0	0	10	$x4A_1$
$x8A_6$	$2^{13}.3^2.5.7.11$	$\overline{2C}$	1	0	0	10	$x4A_1$
$x8A_7$	$2^{13}.3^2.7.11$	$\overline{1A}$	2	1	0	10	$x4A_4$
$x8A_8$	$2^{13}.3^2.7.11$	$\overline{2C}$	1	0	0	10	$x4A_1$
$x8B_1$	$2^{15}.3^2.5.7.11$	$\overline{2A}$	1	0	0	8	$x4B_5$
$x8B_2$	$2^{15}.3^2.5.7.11$	$\overline{2C}$	1	0	0	8	$x4B_1$
$x8C_1$	$2^{16}.3^2.5.7.11$	$\overline{2C}$	1	0	0	8	$x4B_1$
$x8C_2$	$2^{14}.3^2.5.7.11$	$\overline{2A}$	1	0	0	9	$x4B_2$
$x8C_3$	$2^{14}.3^2.5.7.11$	$\overline{2A}$	1	0	0	9	$x4B_6$
$x8C_4$	$2^{13}.3^2.5.7.11$	$\overline{2A}$	1	0	0	10	$x4B_6$

$z = tx$ of order 8
Table 4

Representative, x	$\lvert\mathcal{O}_x\rvert$	Class of $\overline{w}$	$\lvert C_Q(x)\rvert$	q_{2A}	q_{2B}	d_x	y
$x6A$	$2^{13}.3^2.7.11$	$\overline{2C}$	1	0	0	10	$x3A_1$
$x6B_1$	$2^{15}.3^2.5.7.11$	$\overline{2C}$	1	0	0	11	$x3A_1$
$x6B_2$	$2^{13}.3^2.5.7.11$	$\overline{2A}$	1	0	0	12	$x3A_1$
$x6B_3$	$2^{13}.3^2.5.7.11$	$\overline{2A}$	2^2	3	0	12	$x3A_2$
$x6B_4$	$2^{13}.3^2.5.7.11$	$\overline{2C}$	2^2	3	0	11	$x3A_2$
$x6B_5$	$2^{13}.3^2.5.7.11$	$\overline{2C}$	1	0	0	11	$x3A_1$
$x6C_1$	$2^{15}.3^2.5.7.11$	$\overline{2A}$	1	0	0	10	$x3A_1$
$x6C_2$	$2^{13}.3^2.5.7.11$	$\overline{2A}$	2^2	3	0	10	$x3A_2$
$x6C_3$	$2^{13}.3^2.5.7.11$	$\overline{2A}$	1	0	0	10	$x3A_1$
$x6C_4$	$2^{13}.3^2.5.7.11$	$\overline{2B}$	1	0	0	10	$x3A_1$
$x6C_5$	$2^{12}.3.5.7.11$	$\overline{2B}$	2^3	7	0	10	$x3A_2$

$z = tx$ of order 6
Table 5

Representative, x	$\lvert\mathcal{O}_x\rvert$	Class of $\overline{w}$	$\lvert C_Q(x)\rvert$	q_{2A}	q_{2B}	d_x	y
$x12A_1$	$2^{16}.3^2.5.7.11$	$\overline{2C}$	1	0	0	6	$x6B_1$
$x12A_2$	$2^{16}.3^2.5.7.11$	$\overline{2C}$	1	0	0	6	$x6B_1$
$x12A_3$	$2^{15}.3^2.5.7.11$	$\overline{2A}$	1	0	0	7	$x6B_2$
$x12A_4$	$2^{15}.3^2.5.7.11$	$\overline{2A}$	1	0	0	7	$x6B_3$
$x12B_1$	$2^{16}.3^2.5.7.11$	$\overline{2C}$	1	0	0	6	$x6B_1$
$x12B_2$	$2^{16}.3^2.5.7.11$	$\overline{2C}$	1	0	0	6	$x6B_1$
$x12B_3$	$2^{15}.3^2.5.7.11$	$\overline{2A}$	1	0	0	7	$x6B_3$
$x12B_4$	$2^{15}.3^2.5.7.11$	$\overline{2A}$	1	0	0	7	$x6B_2$
$x12B_5$	$2^{15}.3^2.5.7.11$	$\overline{2A}$	1	0	0	6	$x6B_3$
$x12C_1$	$2^{16}.3^2.5.7.11$	$\overline{2A}$	1	0	0	5	$x6C_1$
$x12C_2$	$2^{16}.3^2.5.7.11$	$\overline{2A}$	1	0	0	5	$x6C_1$
$x12C_3$	$2^{16}.3^2.5.7.11$	$\overline{2A}$	1	0	0	5	$x6C_1$
$x12C_4$	$2^{16}.3^2.5.7.11$	$\overline{2B}$	1	0	0	5	$x6C_4$
$x12C_5$	$2^{16}.3^2.5.7.11$	$\overline{2B}$	1	0	0	5	$x6C_4$
$x12C_6$	$2^{16}.3^2.5.7.11$	$\overline{2B}$	2	1	0	5	$x6C_3$

$z = tx$ of order 12
Table 6

Representative, x	$\|\mathcal{O}_x\|$	Class of $\overline{w}$	$\|C_Q(x)\|$	q_{2A}	q_{2B}	$\|S_x\|$	d_x	y
$x10A_1$	$2^{16}.3^2.5.7.11$	$\overline{2C}$	1	0	0	887040	7	$x5A_1$
$x10A_2$	$2^{15}.3^2.5.7.11$	$\overline{2A}$	1	0	0	887040	8	$x5A_1$
$x10B_1$	$2^{16}.3^2.5.7.11$	$\overline{2A}$	1	0	0	887040	6	$x5A_1$
$x10B_2$	$2^{16}.3^2.5.7.11$	$\overline{2A}$	1	0	0	887040	6	$x5A_1$
$x10B_3$	$2^{15}.3^2.5.7.11$	$\overline{2A}$	1	0	0	887040	6	$x5A_1$
$x10B_4$	$2^{15}.3^2.5.7.11$	$\overline{2A}$	4	3	0	48	6	$x5A_2$
$x20A_1$	$2^{17}.3^2.5.7.11$	$\overline{2C}$	1	0	0	887040	4	$x10A_1$
$x20A_2$	$2^{16}.3^2.5.7.11$	$\overline{2A}$	1	0	0	887040	5	$x10A_2$
$x20B{*}_1$	$2^{17}.3^2.5.7.11$	$\overline{2C}$	1	0	0	887040	4	$x10A_1$
$x20B{*}_2$	$2^{16}.3^2.5.7.11$	$\overline{2A}$	1	0	0	887040	5	$x10A_2$
$x40A_1$	$2^{18}.3^2.5.7.11$	$\overline{2C}$	1	0	0	887040	3	$x20B{*}_1$
$x40A_2$	$2^{18}.3^2.5.7.11$	$\overline{2A}$	1	0	0	887040	3	$x20B{*}_2$
$x40B{*}_1$	$2^{18}.3^2.5.7.11$	$\overline{2C}$	1	0	0	887040	3	$x20A_1$
$x40B{*}_2$	$2^{18}.3^2.5.7.11$	$\overline{2A}$	1	0	0	887040	3	$x20A_2$

$z = tx$ of orders 10,20,40
Table 7

Representative, x	$\lvert\mathcal{O}_x\rvert$	Class of $\overline{w}$	$\lvert C_Q(x)\rvert$	q_{2A}	q_{2B}	$\lvert S_x\rvert$	d_x	y
$x22A$	$2^{18}.3^2.5.7.11$	$\overline{2C}$	1	0	0	887040	1	$x11A$
$x33A_1$	$2^{18}.3^2.5.7.11$	-	1	0	0	887040	0	-
$x33A_2$	$2^{18}.3^2.5.7.11$	-	1	0	0	887040	0	-
$x33B*_1$	$2^{18}.3^2.5.7.11$	-	1	0	0	887040	0	-
$x33B*_2$	$2^{18}.3^2.5.7.11$	-	1	0	0	887040	0	-
$x44A_1$	$2^{18}.3^2.5.7.11$	$\overline{2C}$	1	0	0	887040	1	$x22A$
$x44A_2$	$2^{18}.3^2.5.7.11$	$\overline{2C}$	1	0	0	887040	1	$x22A$
$x66A_1$	$2^{18}.3^2.5.7.11$	$\overline{2C}$	1	0	0	887040	0	$x33B*_1$
$x66A_2$	$2^{18}.3^2.5.7.11$	$\overline{2C}$	1	0	0	887040	0	$x33B*_2$
$x66B*_1$	$2^{18}.3^2.5.7.11$	$\overline{2C}$	1	0	0	887040	0	$x33A_1$
$x66B*_2$	$2^{18}.3^2.5.7.11$	$\overline{2C}$	1	0	0	887040	0	$x33A_2$

$z = tx$ of orders 22,33,44,66
Table 8

Representative, x	$\lvert\mathcal{O}_x\rvert$	Class of $\overline{w}$	$\lvert C_Q(x)\rvert$	q_{2A}	q_{2B}	$\lvert S_x\rvert$	d_x	y
$x15A_1$	$2^{18}.3^2.5.7.11$	-	1	0	0	887040	4	-
$x15A_2$	$2^{18}.3^2.5.7.11$	-	1	0	0	887040	4	-
$x16A_1$	$2^{17}.3^2.5.7.11$	$\overline{2C}$	1	0	0	887040	5	$x8C_1$
$x16A_2$	$2^{17}.3^2.5.7.11$	$\overline{2A}$	1	0	0	887040	5	$x8C_4$
$x24A_1$	$2^{17}.3^2.5.7.11$	$\overline{2A}$	1	0	0	887040	3	$x12B_5$
$x24A_2$	$2^{17}.3^2.5.7.11$	$\overline{2C}$	1	0	0	887040	3	$x12B_1$
$x24B*_1$	$2^{17}.3^2.5.7.11$	$\overline{2A}$	1	0	0	887040	3	$x12B_5$
$x24B*_2$	$2^{17}.3^2.5.7.11$	$\overline{2C}$	1	0	0	887040	3	$x12B_1$
$x30A_1$	$2^{18}.3^2.5.7.11$	$\overline{2C}$	1	0	0	887040	2	$x15A_1$
$x30A_2$	$2^{18}.3^2.5.7.11$	$\overline{2C}$	1	0	0	887040	2	$x15A_2$

$z = tx$ of orders 15,16,24,30
Table 9

3 Commuting Involution Graph for Class $2B$

Recall that for a group H and an involution conjugacy class Y of H, the commuting involution graph $\mathcal{C}(H,Y)$ is the graph with Y as its vertex set and two distinct vertices joined by an edge if and only if they commute. For $s \in Y$, we define $\Delta_i(s) = \{y \in Y \mid d(s,y) = i\}$, the i^{th} disc of $\mathcal{C}(H,Y)$ relative to s. Each disc is a union of $C_G(s)$-orbits.

We take $G = J_4$, $t \in 2B$ and $X = t^G$ as in Section 2.2, and examine $\mathcal{C}(G,X)$. Clearly $\Delta_1(t) = X_{2A} \cup X_{2B}$. Lemma 2.2 of [1] allows us to determine quickly from the power maps in [16] the locations in the graph of several of the sets X_C. Part (ii) implies that all suborbits contained in $X_{4C} \cup X_{6C} \cup X_{10B} \cup X_{12C}$ are in $\Delta_2(t)$, while part (iv) allows us to determine that some suborbits have distance 3 or greater from t, in this case those X_C with C a class of elements of odd order greater than 10.

For each $C_G(t)$-orbit representative $x \notin \Delta_1(t)$ we have the invariant q_{2B} being the number of $2B$-elements in $C_Q(x)$. Since $C_Q(x) \leq C_{C_G(t)}(x)$, if $q_{2B} \neq 0$ then clearly $x^{C_G(t)} \in \Delta_2(t)$. Omitting those covered above, this is the case for suborbits with representatives $x4A_3, x4A_4, x4B_7, x4B_9$. For the remaining orbits, we can check if a representative is in $\Delta_2(t)$ by checking whether any of the elements in $C_{C_G(t)}(x)$ are $2B$-elements. This allows us to find the remaining suborbits in disc 2.

Finally, we must determine whether the orbits not in $\Delta_1(t) \cup \Delta_2(t)$ comprise a single third disc, or whether the graph has diameter greater than 3. Our strategy is as follows. Taking a representative $x \notin \Delta_1(t) \cup \Delta_2(t)$, we find elements $y \in C_G(x)$ using [6] until we find a $2B$-element for which the order of ty is 3, 4, 5, 6, 8, 10, 12 or 16. Then we know from the above that y is in $\Delta_2(t)$ and so x is in $\Delta_3(t)$. For all remaining orbits we were in fact able to find such a y, so we conclude that the graph has diameter 3.

These results are summarized in Table 10. We note that only X_{20A} and X_{20B} straddle two discs in the graph.

Disc	Orbits
$\Delta_0(t)$	t
$\Delta_1(t)$	X_{2A}, X_{2B}
$\Delta_2(t)$	X_{3A}, X_{4A}, X_{4B}, X_{4C}, X_{5A}, X_{6A}, X_{6B}, X_{6C}, X_{8A}, X_{8B}, X_{8C}, X_{10A}, X_{10B}, X_{11B}, X_{12A}, X_{12B}, X_{12C}, X_{16A}, X_{24A}, X_{24B*}, $x20A_1^{C_G(t)}$, $x20B*_1^{C_G(t)}$
$\Delta_3(t)$	X_{11A}, X_{15A}, X_{22A}, X_{23A}, X_{29A}, X_{30A}, X_{31A}, X_{31B*5}, X_{31C*6}, X_{33A}, X_{33B*}, X_{37A}, X_{37B*2}, X_{37C*4}, X_{40A}, X_{40B}, X_{43A}, X_{43B*6}, X_{43C*7}, X_{44A}, X_{66A}, X_{66B*}, $x20A_2^{C_G(t)}$, $x20B*_2^{C_G(t)}$

Discs in the commuting involution graph $\mathcal{C}(G, X)$ for $G = J_4, X = 2B$.

Table 10

4 Structure Constants and Fixed Spaces

4.1 Dimensions of $C_V(g), g \in G$

$g \in C$	$\dim(C_V(g))$
$1A$	112
$2A$	62
$2B$	56
$3A$	40
$4A$	32
$4B$	32
$4C$	28
$5A$	24
$6A$	20
$6B$	22
$6C$	20
$7A$	16
$7B**$	16
$8A$	16
$8B$	16
$8C$	16
$10A$	14
$10B$	12
$11A$	2
$11B$	12
$12A$	12
$12B$	12
$12C$	10
$14A$	8
$14B**$	8
$14C$	8
$14D**$	8
$15A$	8
$16A$	8
$20A$	8
$20B*$	8

$g \in C$	$\dim(C_V(g))$
$21A$	4
$21B**$	4
$22A$	2
$22B$	6
$23A$	2
$24A$	6
$24B*$	6
$28A$	4
$28B**$	4
$29A$	0
$30A$	4
$31A$	2
$31B*5$	2
$31C*6$	2
$33A$	0
$33B*$	0
$35A$	0
$35B**$	0
$37A$	4
$37B*2$	4
$37C*4$	4
$40A$	4
$40B*$	4
$42A$	2
$42B**$	2
$43A$	0
$43B*6$	0
$43C*7$	0
$44A$	2
$66A$	0
$66B*$	0

Table 11

4.2 The Structure Constants for $G = J_4$ and $t \in 2A$

C	$\lvert X_C \rvert$	
$1A$	1	1
$2A$	112266	$2.3^6.7.11$
$2B$	81840	$2^4.3.5.11.31$
$3A$	8110080	$2^{14}.3^2.5.11$
$4A$	887040	$2^8.3^2.5.7.11$
$4B$	70963200	$2^{12}.3^2.5^2.7.11$
$4C$	14192640	$2^{12}.3^2.5.7.11$
$5A$	113541120	$2^{15}.3^2.5.7.11$
$6B$	340623360	$2^{15}.3^3.5.7.11$
$6C$	56770560	$2^{14}.3^2.5.7.11$
$8C$	340623360	$2^{15}.3^3.5.7.11$
$10A$	681246720	$2^{16}.3^3.5.7.11$
$11B$	990904320	$2^{20}.3^3.5.7$
$12B$	1362493440	$2^{17}.3^3.5.7.11$

Table 12

4.3 The Structure Constants for $G = J_4$ and $t \in 2B$

C	$\lvert X_C \rvert$	
$1A$	1	1
$2A$	63063	$3^2.7^2.11.13$
$2B$	147884	$2^2.11.3361$
$3A$	6352896	$2^{12}.3.11.47$
$4A$	4331712	$2^6.3.7.11.293$
$4B$	32524800	$2^9.3.5^2.7.11^2$
$4C$	43760640	$2^{10}.3.5.7.11.37$
$5A$	114892800	$2^{13}.3.5^2.11.17$
$6A$	5677056	$2^{13}.3^2.7.11$
$6B$	227082240	$2^{16}.3^2.5.7.11$
$6C$	203427840	$2^{12}.3.5.7.11.43$
$8A$	351977472	$2^{14}.3^2.7.11.31$
$8B$	227082240	$2^{16}.3^2.5.7.11$
$8C$	369008640	$2^{13}.3^2.5.7.11.13$
$10A$	340623360	$2^{15}.3^3.5.7.11$
$10B$	681246720	$2^{16}.3^3.5.7.11$
$11A$	2575360	$2^{18}.3^2.5.7$
$11B$	908328960	$2^{18}.3^2.5.7.11$
$12A$	681246720	$2^{16}.3^3.5.7.11$
$12B$	794787840	$2^{15}.3^2.5.7^2.11$
$12C$	1362493440	$2^{17}.3^3.5.7.11$
$15A$	1816657920	$2^{19}.3^2.5.7.11$
$16A$	908328960	$2^{18}.3^2.5.7.11$
$20A$	681246720	$2^{16}.3^3.5.7.11$
$20B*$	681246720	$2^{16}.3^3.5.7.11$
$22A$	908328960	$2^{18}.3^2.5.7.11$
$23A$	1816657920	$2^{19}.3^2.5.7.11$
$24A$	908328960	$2^{18}.3^2.5.7.11$
$24B*$	908328960	$2^{18}.3^2.5.7.11$
$29A$	1816657920	$2^{19}.3^2.5.7.11$
$30A$	1816657920	$2^{19}.3^2.5.7.11$
$31A$	1816657920	$2^{19}.3^2.5.7.11$
$31B*5$	1816657920	$2^{19}.3^2.5.7.11$
$31C*6$	1816657920	$2^{19}.3^2.5.7.11$
$33A$	1816657920	$2^{19}.3^2.5.7.11$
$33B*$	1816657920	$2^{19}.3^2.5.7.11$
$37A$	1816657920	$2^{19}.3^2.5.7.11$
$37B*2$	1816657920	$2^{19}.3^2.5.7.11$
$37C*4$	1816657920	$2^{19}.3^2.5.7.11$
$40A$	1816657920	$2^{19}.3^2.5.7.11$
$40B*$	1816657920	$2^{19}.3^2.5.7.11$
$43A$	1816657920	$2^{19}.3^2.5.7.11$
$43B*6$	1816657920	$2^{19}.3^2.5.7.11$
$43C*7$	1816657920	$2^{19}.3^2.5.7.11$
$44A$	1816657920	$2^{19}.3^2.5.7.11$
$66A$	1816657920	$2^{19}.3^2.5.7.11$
$66B*$	1816657920	$2^{19}.3^2.5.7.11$

Table 13

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