Noncommutative geometry: illustrations from the representation theory of GL(n)

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Non Commutative Geometry: Illustrations from the Representation Theory of $GL(n)$

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Introduction

In these lectures we will cover the following material.
• An elementary introduction to C^*-algebras, leading up to a statement of the Richness Lemma for liminal C^*-algebras.
• A concise account of C^*-modules and strong Morita equivalence. Here we follow closely the account by Alain Connes in his forthcoming book "Non Commutative Geometry".
• A concise account of equivalence bimodules in the representation theory of reductive groups.
• Complete proof of the following new result:

Theorem 1. The reduced Iwahori-Hecke C^*-algebra for $GL(n)$ is strongly Morita equivalent to the Brylinski quotient

$$\mathrm{Bryl}\,(\mathbf{T}^n; S_n)$$

where the symmetric group S_n acts on the compact torus $\mathbf{T}^n$ by permuting co-ordinates. □

To explain Theorem 1 and its background, we shall give an elementary account of the Brylinski quotient.

A partition of a number n is a representation of n as the sum of any number of positive integral parts. Thus

$$5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1 = 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1$$

has 7 partitions. The order of the parts is irrelevant. If α is a partition of n then $d(\alpha)$ will denote the number of *distinct* parts in α. Thus, with $n = 5$, we have

α	$d(\alpha)$
5	1
4+1	2
3+2	2
3+1+1	2
2+2+1	2
2+1+1+1	2
1+1+1+1+1	1

The Brylinski quotient is naturally defined in great generality – we shall need only the special case

$$Bryl(n) = Bryl(\mathbf{T}^n; S_n)$$

where $\mathbf{T}^n$ is the compact torus of dimension n and S_n is the symmetric group acting on $\mathbf{T}^n$ by permuting co-ordinates. Then

$$Bryl(n) = \bigsqcup_{\alpha} (\mathbf{T}^n)^{\gamma}/Z(\gamma)$$

where α is a partition of n, γ has cycle type $\alpha, (\mathbf{T}^n)^\gamma$ is the γ-fixed set, and $Z(\gamma)$ the centralizer of γ. Each cycle provides us with one circle, and cycles of equal length provide us with a symmetric product of circles. With $n = 5$, the space $Bryl(5)$ is a compact space whose connected components are the following 7 compact orbifolds, one for each partition α:

α	orbifold
5	$\mathbf{T}$
$4+1$	$\mathbf{T}^2$
$3+2$	$\mathbf{T}^2$
$3+1+1$	$\mathbf{T} \times Sym^2\mathbf{T}$
$2+2+1$	$(Sym^2\mathbf{T}) \times \mathbf{T}$
$2+1+1+1$	$\mathbf{T} \times Sym^3\mathbf{T}$
$1+1+1+1+1$	$Sym^5\mathbf{T}$

Now $Sym^n\mathbf{T}$ is the space of unordered n-tuples $\{z_1, \ldots, z_n\}$ with each $z_j \in \mathbf{T}, 1 \leq j \leq n$. The map

$$\begin{array}{rcl} Sym^n\mathbf{T} & \longrightarrow & \mathbf{T} \\ \{z_1, \ldots, z_n\} & \longmapsto & z_1 \ldots z_n \end{array}$$

is a deformation retract. This implies that the orbifold associated to the partition α is homotopy equivalent to $\mathbf{T}^{d(\alpha)}$. Up to homotopy equivalence, $Bryl(5)$ therefore has the following connected components.

α	component
5	$\mathbf{T}$
$4+1$	$\mathbf{T}^2$
$3+2$	$\mathbf{T}^2$
$3+1+1$	$\mathbf{T}^2$
$2+2+1$	$\mathbf{T}^2$
$2+1+1+1$	$\mathbf{T}^2$
$1+1+1+1+1$	$\mathbf{T}$.

For the torus of dimension d, we have

$$\begin{aligned} K_0(\mathbf{T}^d) &= \mathbf{Z}^{2^{d-1}} \\ K_1(\mathbf{T}^d) &= \mathbf{Z}^{2^{d-1}} \end{aligned}$$

So for $Bryl(5)$ we have

$$\begin{aligned} K_0 &= \mathbf{Z} \oplus \mathbf{Z}^2 \oplus \mathbf{Z}^2 \oplus \mathbf{Z}^2 \oplus \mathbf{Z}^2 \oplus \mathbf{Z}^2 \oplus \mathbf{Z} = \mathbf{Z}^{12} \\ K_1 &= \mathbf{Z}^{12}. \end{aligned}$$

Theorem 1 implies that, for $GL(5)$ we have

$$K_j(C_r^*(G//I)) \cong \mathbf{Z}^{12}. \qquad j = 0, 1.$$

In general, the K-groups will be finitely generated free abelian groups. The formula for the number of generators is

$$\sum_\alpha 2^{d(\alpha)-1}$$

the sum taken over all partitions of n. Summarizing this elementary discussion, we have

Corollary 1. Let $G = GL(n)$. The K-groups K_0 and K_1 of the reduced Iwahori-Hecke C^*-algebra are finitely generated free abelian groups: the number of generators is

$$\sum_\alpha 2^{d(\alpha)-1}$$

where α is a partition of n and $d(\alpha)$ is the number of distinct parts in α. □

Theorem 1 has an important application, which we explain in more detail towards the end of these lectures. Suffice to say that if we successively use Theorem 1, the Chern character, the inverse of the equivariant Chern character, the inverse Fourier Transform, and the $B - C$ conjecture (a provable theorem) for the affine Weyl group W, then we get:

$$\begin{aligned}
K_0(C_r^*(G//I)) \otimes_{\mathbf{Z}} \mathbf{C} &\cong K_0(Bryl) \otimes_{\mathbf{Z}} \mathbf{C} \\
&\cong H^{ev}(Bryl; \mathbf{C}) \\
&\cong K_0(C(\mathbf{T}^n) \rtimes S_n) \otimes_{\mathbf{Z}} \mathbf{C} \\
&\cong K_0(C^*(W)) \otimes_{\mathbf{Z}} \mathbf{C} \\
&\cong K_0^W(\Sigma) \otimes_{\mathbf{Z}} \mathbf{C}
\end{aligned}$$

where Σ denotes a single apartment in the affine building βG of G. There is of course a similar statement for K_1. Since

- Σ embeds isometrically in βG

- $C_r^*(G//I)$ is a direct summand of $C_r^*(G)$

we conclude with the following

Theorem 2. For $j = 0, 1$ we have

$$K_j(C_r^*(G//I)) \otimes_{\mathbf{Z}} \mathbf{C} \cong K_j^W \text{ (apartment) } \otimes_{\mathbf{Z}} \mathbf{C}$$

which is the $B - C$ conjecture for $GL(n)$ (modulo torsion) localized to a single apartment. □

We would like to make an important general point. Let $G = GL(n)$ and consider the C^*-algebras

- $C_r^*(G)$ the reduced C^*-algebra of G
- $C_r^*(G//I)$ the reduced Iwahori-Hecke C^*-algebra.

These are non-commutative C^*-algebras. In accord with the view forcefully presented in Connes' forthcoming book "Non Commutative Geometry", these two C^*-algebras are "non-commutative spaces". It is a striking fact that two equivalence bimodules exist which wash away the non-commutativity. In fact

- $C_r^*(G)$ is strongly Morita equivalent to the tempered dual of G
- $C_r^*(G//I)$ is strongly Morita equivalent to the unramified tempered dual of G.

The tempered dual, and the unramified tempered dual, are geometric objects. For example, the unramified tempered dual of $GL(5)$ is the disjoint union of 7 orbifolds:

$$\begin{array}{l} \mathbb{T} \\ \mathbb{T}^2 \\ \mathbb{T}^2 \\ \mathbb{T} \times Sym^2\mathbb{T} \\ \mathbb{T} \times Sym^2\mathbb{T} \\ \mathbb{T} \times Sym^3\mathbb{T} \\ Sym^5\mathbb{T} \end{array}$$

We can say therefore that for the general linear group $GL(n)$, the "non commutative geometry" becomes "commutative geometry".

We must emphasize that for the special linear group $SL(n)$ the non-commutativity is a more permanent feature and can *not* be washed away by equivalence bimodules.

This non-commutative geometry first appears for the reduced Iwahori-Hecke C^*-algebra of $SL(2)$. This unital C^*-algebra is strongly Morita equivalent to $\mathbb{C} \oplus D$ when

$$D = \{f \in C([0,1], M_2(\mathbb{C})) : f(1) \text{ is diagonal}\}$$

whose dual is

The "K-theory" of such a space cannot be directly defined in terms of vector bundles. It is defined indirectly as K-theory of the above non-commutative C^*-algebra. As such, we have

$$\begin{aligned} K_0 &= \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \\ K_1 &= 0 \end{aligned}$$

We see therefore that

- the Steinberg representation of $SL(2)$ contributes one generator to K_0
- K_0 detects reducibility in the unramified unitary principal series of $SL(2)$.

The affine Weyl group of $SL(2)$ is the infinite dihedral group W. For $C^*(W)$ we also have

$$\begin{aligned} K_0(C^*(W)) &= \mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z} \\ K_1(C^*(W)) &= 0 \end{aligned}$$

even though

$$C^*(W) \text{ is not isomorphic to } \mathbf{C} \oplus \mathbf{D}.$$

One reason is that

- the dual of $C^*(W)$ is connected quasi-compact
- the dual of $D \oplus \mathbf{C}$ is disconnected quasi-compact.

1 C^*-algebras

1.1. $*$ algebras. Let A be an algebra over the field C of complex numbers. An *involution* in A is a map $x \longmapsto x^*$ of A into itself such that

(i) $(x^*)^* = x$

(ii) $(x+y)^* = x^* + y^*$

(iii) $(\lambda x)^* = \overline{\lambda} x^*$

(iv) $(xy)^* = y^* x^*$

for any $x, y \in A$ and $\lambda \in \mathbb{C}$. An algebra over C endowed with an involution is called a $*$ algebra. The element x^* is often called the *adjoint* of A. A subset of A which is closed under the involution operation is said to be *self-adjoint.*

1.2. A normed $*$ algebra is a normed algebra A together with an involution $x \longmapsto x^*$ such that $||x^*|| = ||x||$ for each $x \in A$. If, in addition, A is complete, A is called a Banach $*$ algebra.

1.3 A C^* algebra is a Banach $*$ algebra such that $||x||^2 = ||x^*x||$ for every $x \in A$. The condition $||x||^2 = ||x^*x||$ is called the C^* condition. A C^* algebra with unit is called a unital C^* algebra.

The C^* condition hides an absolutely crucial feature by letting one believe that, as in a Banach algebra, there is freedom in the choice of the norm. In fact if a $*$ algebra is a C^* algebra it is so for a *unique* norm, given for any x by the equality:

$$\begin{aligned} ||x||^2 &= \text{Spectral radius of } x^*x \\ &= \sup\{|\lambda| : x^*x - \lambda \text{ not invertible}\}. \end{aligned}$$

1.4. Examples

1.4.1. Let X be a locally compact Hausdorff space, and A the algebra of complex-valued continous functions vanishing at infinity on X. Set

$$\begin{aligned} f^*(x) &= \overline{f(x)} \qquad f \in A, x \in X \\ ||f|| &= \sup\{|f(x)| : x \in X\}. \\ (fg)(x) &= f(x)g(x) \end{aligned}$$

Then A is a commutative C^* algebra denoted $C_0(X)$.
The C^* algebra $C_0(X)$ is unital if and only if X is compact.

1.4.2. Let H be a complex Hilbert space and $A = \mathcal{L}(H)$ the algebra of continous endomorphisms of H. Set

$$\begin{aligned}(T^*\xi|\eta) &= (\xi|T\eta) \qquad \xi,\eta \in H\\(ST)(\xi) &= S(T\xi) \qquad \xi \in H\\||T|| &= \sup\{||T\xi|| : ||\xi|| \leq 1\}\end{aligned}$$

Then $\mathcal{L}(H)$ is a unital C^* algebra.

1.4.3. Let $k(H)$ be the algebra of compact operators on H. This is the closure in $\mathcal{L}(H)$ of the finite-rank operators on H. Then $k(H)$ is a C^*-algebra. This C^*-algebra is unital if and only if H is finite-dimensional.

1.5. Automatic Continuity. Let A be a Banach $*$ algebra, B a C^* algebra and π a morphism of A into B; this means that π is a morphism of the underlying $*$ algebras, without any condition on the norms. Then $||\pi(x)|| \leq ||x||$ for every $x \in A$. See [**D**, 1.3.7.] It follows that an isomorphism of C^*-algebras is automatically isometric.

1.6. Let A and B be C^*-algebras, ϕ a morphism of A into B. Then the image $\phi(A)$ is a sub-C^*-algebra of B.

1.7. Let A be a $*$ algebra and H a Hilbert space. A representation of A in H is a morphism of the $*$ algebra A into the $*$ algebra $\mathcal{L}(H)$. In other words, a representation of A in H is a map π of A into $\mathcal{L}(H)$ such that

$$\begin{aligned}\pi(x+y) &= \pi(x)+\pi(y) \qquad & \pi(\lambda x) &= \lambda\pi(x)\\ \pi(xy) &= \pi(x)\pi(y) & \pi(x^*) &= \pi(x)^*\end{aligned}$$

for $x, y \in A, \lambda \in \mathbb{C}$.

1.8. Two representations π and π' of A in H and H' are said to be equivalent, and we write $\pi \cong \pi'$ if there is an isomorphism U of the Hilbert space H onto the Hilbert space H' which transforms $\pi(x)$ into $\pi'(x)$ for each $x \in A$. In other words, $U\pi(x) = \pi'(x)U$ for any $x \in A$. Hence the definition of a class of representations. The operator U is an *intertwining operator* for π and π'.

1.9. Let Γ be a finite group acting as automorphisms of the C^*-algebra A. Let A^Γ be the fixed-point set. Then A^Γ is a sub-C^*-algebra of A.

1.10. Irreducible representations. The representation π of the C^*-algebra A in H is *ireducible* if H admits no invariant closed subspaces except O and H.

1.11. A C^*-algebra A is *liminal* if, for every irreducible representation π of A and each $x \in A, \pi(x)$ is compact.

1.12. Let A be a C^*-algebra, and B a sub-C^*-algebra of A. Then B is said to be a *rich* sub-C^*-algebra of A if the following conditions are satisfied.

(i) For every irreducible representation π of $A, \pi|_B$ is irreducible;

(ii) If π and π' are inequivalent irreducible representation of A, then $\pi|_B$ and $\pi'|_B$ are inequivalent.

1.13. The Richness Lemma. Let A be a liminal C^*-algebra with Hausdorff dual, and B a rich sub-C^*-algebra of A. Then $B = A$.

This result is proved in [**D**, 11.1.4].

1.14. Non-commutative topology. The conventional wisdom is that C^*-algebra theory may be viewed as "non-commutative topology". Each property concerning a locally compact Hausdorff space X can in principle be formulated in terms of the function algebra $C_0(X)$ and will then usually make sense (and hopefully be true) for any non-commutative C^*-algebra. Here is a list of some of the "dualities"

topology	$\longleftrightarrow$	algebra
$C_0(X)$	$\longleftrightarrow$	C^*-algebra A
proper map	$\longleftrightarrow$	morphism
homeomorphism	$\longleftrightarrow$	automorphism
measure	$\longleftrightarrow$	positive functional
disjoint union	$\longleftrightarrow$	direct sum
compact	$\longleftrightarrow$	unital
σ-compact	$\longleftrightarrow$	σ-unital
open subset	$\longleftrightarrow$	ideal
open dense subset	$\longleftrightarrow$	essential ideal
closed subset	$\longleftrightarrow$	quotient
compactifications	$\longleftrightarrow$	unitizations
connected	$\longleftrightarrow$	projectionless
2nd countable	$\longleftrightarrow$	separable

The idea is that since an algebra isomorphism of $C_0(X)$ onto $C_0(Y)$ induces a homomorphism of X with Y, *all* topological information about X is stored algebraically in $C_0(X)$. See [W O, p.24].

product $\longleftrightarrow$ tensor product

§5. C^*-modules and strong Morita equivalence

5.1. In this section we give a concise account of C^*-modules over a C^*-algebra and of strong Morita equivalence of C^*-algebras. These concepts are mainly due to Rieffel [**R**]. We shall follow very closely the exposition by Connes in his forthcoming book "Non commutative geometry", Chapter II, Appendix A.

Let B be a C^*-algebra. By a B-valued inner product on a right B-module $\mathcal{E}$ we mean a B-valued sesquilinear form $<,>$, conjugate linear in the first variable, and such that:

i) $<\xi,\xi>$ is a positive element of B for any $\xi\in\mathcal{E}$.

ii) $<\xi,\eta>^*=<\eta,\xi>$ for all $\xi,\eta\in\mathcal{E}$.

iii) $<\xi,\eta b>=<\xi,\eta>b$ for all $b\in B,\xi\in\mathcal{E},\eta\in\mathcal{E}$.

By a *pre-C^*-module* over B we mean a right B-module $\mathcal{E}$ endowed with a B-valued inner product. The following equality then defines a semi-norm on $\mathcal{E}$:

$$||\xi|| = ||<\xi,\xi>||^{\frac{1}{2}} \qquad \xi\in E.$$

(where $||<\xi,\xi>||$ is the C^*-algebra norm of $<\xi,\xi>\in B$.

2.1.1 Definition. A C^*-module $\mathcal{E}$ over B is a pre-C^*-module $\mathcal{E}$ for which $||\quad||$ is a complete norm.

By completion any pre-C^*-module yields an associated C^*-module. Given a C^*-module $\mathcal{E}$ over B, an endomorphism T of $\mathcal{E}$ is by definition a continuous endomorphism of the right B-module $\mathcal{E}$ which admits an adjoint T^*, that is an endomorphism of the right B-module $\mathcal{E}$ such that:

$$<\xi,T\eta>=<T^*\xi,\eta> \qquad \forall\xi,\eta\in\mathcal{E}.$$

One checks that T^* is uniquely determined by T and that equipped with this involution the algebra $End_B(\mathcal{E})$ of endomorphisms of $\mathcal{E}$ is a C^*-algebra. One has

$$<T\xi,T\xi>\leq ||T||^2<\xi,\xi> \qquad \forall\xi\in\mathcal{E},T\in End_B\mathcal{E}$$

where $||T||$ is the C^*-algebra norm of T.

Of particular importance are the compact endomorphisms obtained from the norm closure of endomorphisms of finite rank:

2.1.2 Proposition. [**R**]. Let $\mathcal{E}$ be a C^*-module over B.

a) For any $\xi,\eta\in\mathcal{E}$ the following equality defines an endomorphism $|\xi><\eta|\in End_B(\mathcal{E})$:

$$(|\xi><\eta|)(\alpha)=\xi<\eta,\alpha> \qquad \forall\alpha\in\mathcal{E}.$$

b) The linear span of the above endomorphisms is a self-adjoint two-sided ideal of $End_B(\mathcal{E})$.

The usual properties of the Dirac bra-ket notation hold in this framework, so that for instance:

$$(|\xi><\eta|)^* = |\eta><\xi| \qquad \forall \xi,\eta \in \mathcal{E}.$$

$$\begin{aligned}(|\xi><\eta|)(|\xi'><\eta'|) &= |\xi<\eta,\xi'>><\eta'| \\ &= |\xi><(|\eta><\xi'|)\eta'| \ \forall \xi,\xi',\eta,\eta' \in \mathcal{E}\end{aligned}$$

We let $End_B^0(\mathcal{E})$ be the norm closure in $End_B(\mathcal{E})$ of the above 2-sided ideal (prop. 2.1.2b). An element of $End_B^0(\mathcal{E})$ is called a *compact endomorphism* of $\mathcal{E}$. There are obvious corresponding notions and notations: $Hom_B(\mathcal{E}_1,\mathcal{E}_2), Hom_B^0(\mathcal{E}_1,\mathcal{E}_2)$ for pairs of C^*-modules over B.

Consider the speical case when B is a commutative C^*-algebra, so that B is the C^*-algebra $C_0(X)$ of continuous functions vanishing at ∞ on the locally compact Hausdorff space X. Then a complex Hermitian vector bundle E on X gives rise to a C^*-module: $\mathcal{E} = C_0(X,E)$ is the $C_0(X)$ module of continuous sections of E vanishing at ∞ and the $C_0(X)$-valued inner product is given by

$$<\xi,\eta>(x) = <\xi(x),\eta(x)> \qquad \forall \xi,\eta \in \mathcal{E}, x \in X.$$

2.1.3 Proposition. ([**R**]) Let B, C be C^*-algebras, $\mathcal{E}'$ (resp. $\mathcal{E}''$) be a C^*-module over B (resp. C) and ρ a $*$-homomorphism $B \to End_C(\mathcal{E}'')$. Then the following equality yields a structure of pre-C^*-module over C on the algebraic tensor product $\mathcal{E} = \mathcal{E}' \otimes_B \mathcal{E}''$:

$$<\xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2> = <\rho(<\xi_2,\xi_1>)\eta_1,\eta_2> \in C$$

$\forall \xi_j \in \mathcal{E}', \eta_j \in \mathcal{E}''$.

We shall still denote by $\mathcal{E}' \otimes_B \mathcal{E}''$ the associated C^*-module over C. Given $T \in End_B(\mathcal{E}')$ the following equality defines an endomorphism $T \otimes 1 \in End_C(\mathcal{E}' \otimes_B \mathcal{E}'')$:

$$(T\otimes 1)(\xi \otimes \eta) = T\xi \otimes \eta \qquad \forall \xi \in \mathcal{E}', \eta \in \mathcal{E}''.$$

By a $(B-C)$ C^*-bimodule we shall mean a C^*-module $\mathcal{E}$ over C together with a $*$ homomorphism from B to $End_C(\mathcal{E})$. In particular given a C^*-algebra B, we denote by 1_B the $B-B$ C^*-bimodule given by: $\mathcal{E} = B$, the actions of B by left and right multiplications, and the B-valued inner product: $<b_1,b_2> = b_1^* b_2 \quad \forall b_1,b_2 \in B$.

2.1.4 Definition. Let B, C be C^*-algebras. A strong Morita equivalence $B \simeq C$ is given by a pair $\mathcal{E}_1, \mathcal{E}_2$ of C^*-bimodules such that:

$$\mathcal{E}_1 \otimes_C \mathcal{E}_2 = 1_B, \ \ \mathcal{E}_2 \otimes_B \mathcal{E}_1 = 1_C.$$

Once can then show that the linear span in C of the inner products $<\xi,\eta>; \xi,\eta \in \mathcal{E}_1$, is a dense two-sided ideal and that the left action $\rho: B \to End_C(\mathcal{E}_1)$ is an isomorphism of B with $End_C^0(\mathcal{E}_1)$. It follows thus that $\overline{\mathcal{E}}_1$, the complex conjugate of the vector space $\mathcal{E}_1$, which is in a natural way a $C-B$-bimodule:

$$c \cdot \overline{\xi} \cdot b \overset{\text{def}}{=} (b^* \xi c^*)^- \qquad \forall \xi \in \mathcal{E}_1$$

is also endowed with a B-valued inner product;

$$< \bar{\xi}, \bar{\eta} >= \rho^{-1}(|\eta >< \xi|) \in B.$$

Endowed with this inner product, $\overline{\mathcal{E}}_1$ is a $C - B$ C^*-bimodule. The bimodule $\mathcal{E}_1$ is then a $B - C$-equivalence bimodule in the sense of [**R**] and one checks that the above definition 2.1.4 is equivalent to the existence of a $B - C$ equivalence bimodule. (One can then take $\mathcal{E}_2 = \overline{\mathcal{E}}_1$.)

Let B, C be C^*-algebras and $\mathcal{E}_1$ an equivalence $B - C - C^*$-bimodule. One obtains a functor from the category of unitary representations of C to that of B by:

$$\mathcal{H} \in Rep\ C \longrightarrow \mathcal{E}_1 \otimes_C \mathcal{H} \in Rep\ B,$$

and using $\mathcal{E}_2 = \overline{\mathcal{E}}_1$ as the inverse of $\mathcal{E}_1$ one gets a natural equivalence between the two categories of representations.

It follows in particular that two strongly Morita equivalent C^*-algebras have the same space of classes of irreducible representations. In particular if a C^*-algebra B is strongly Morita equivalent to some commutative C^*-algebra then the latter is unique and is the C^*-algebra of continuous functions vanishing at ∞ on the space of irreducible representations of B.

Strong Morita equivalence preserves many other properties. An equivalence $B - C - C^*$-bimodule determines an isomorphism between the lattices of two-sided ideals of B and C, and hence a homeomorphism between the primitive ideal spaces of B and C. It does also give a canonical isomorphism of the K-theory groups $K_\bullet(B) \simeq K_\bullet(C)$.

2.2. We shall now give some striking examples of equivalence bimodules in the representation theory of reductive groups. Let G be a linear connected reductive group. We shall consider real or p-adic groups. The example to bear in mind is the general linear group $GL(n)$.

2.3. The parameter space Y. The construction of the parameter space Y is due to Harish-Chandra [**H**]. Choose one Levi subgroup M in each conjugacy class in G. The subgroup ${}^\circ M$ of M is defined as follows: ${}^\circ M$ is the intersection of the kernels of all maps $x \mapsto |\chi(x)|$ where χ is a rational character of M and $|.|$ is the norm of the underlying field; this norm may be archimedean or nonarchimedean. Let now $\Psi(M)$ be the group of all unitary characters of M which are trivial on ${}^\circ M$. In the p-adic case, such characters are called unramified. In the real case, we have the Langlands splitting

$$(*) \qquad M = {}^\circ M.A$$

where A is the split component of M. The subgroup A is a vector group, and $\Psi(M)$ may be identified with the unitary dual $\hat{A}$ which is also a vector group. In the p-adic case $\Psi(M)$ has the structure of a compact torus of dimension equal to the parabolic rank of M. In fact, if G is a p-adic Chevalley group and M is minimal, then M is a maximal torus in G; the splitting $(*)$ is valid but A now has the structure of a finitely generated free abelian group whose rank is the parabolic rank of M. Again, the group $\Psi(M)$ may be identified with the unitary dual $\hat{A}$ of A so that $\Psi(M)$ is a compact torus.

Let now $E_2(M)$ be the discrete series of M. The discrete series of M comprises equivalence classes of irreducible unitary representations of M whose restrictions to ${}^\circ M$

have matrix coefficients which are absolutely square integrable. The set $E_2(M)$ admits a natural action of the group $\Psi(M)$ defined as follows:

$$(\lambda\sigma)(x) = \lambda(x)\sigma(x)$$

for all x in M, λ in $\Psi(M), \sigma$ in $E_2(M)$. So the group $\Psi(M)$ acts by twisting each representation in the discrete series of M by a unitary character which is trivial on ${}^\circ M$.

The action of $\Psi(M)$ on $E_2(M)$ partitions $E_2(M)$ into disjoint orbits. We make $E_2(M)$ into a topological space by saying that each orbit D in $E_2(M)$ is a connected component in $E_2(M)$ and that each orbit D inherits its topology from $\Psi(M)$. Then $E_2(M)$ is a locally compact Hausdorff space: in the real case, each component is a finite-dimensional vector space; in the p-adic case, each component is a compact torus whose dimension is the parabolic rank of M. The appropriate Weyl group $W(M)$ is defined to be $N_G(M)/M$; the subgroup of $W(M)$ which stabilizes the orbit D is denoted $W(M, D)$. The parameter space in then defined as

(*) $$Y = \bigsqcup D/W(M, D).$$

It is a disjoint union of orbifolds: in the real case, each orbifold has the structure of a closed simplicial cone; in the p-adic case, each orbifold is the quotient of a compact torus by the finite group $W(M, D)$.

2.4. Each point in D is an equivalence class of irreducible unitary representations in the discrete series of M. Choose a point in D and then choose an element σ in this equivalence class. Extend σ across a parabolic subgroup P with Levi factor M and then unitarily induce from P to G. The resulting induced representation is denoted $(H(\sigma), \pi(\sigma))$. The group $\Psi(M)$ acts by twisting with certain unitary characters; and we can take the family of Hilbert spaces $\{H(\lambda\sigma) : \lambda \in \Psi(M)\}$ to be constant, i.e., independent of λ. In this way, the parameter space Y becomes a Hilbert bundle which is trivialized on each component. In other words, Y admits a continuous field of Hilbert spaces which is constant on each orbifold. Owing to the choices which we have made, this continuous field of Hilbert spaces is not quite canonical.

DEFINITION. $A =$ reduced C^*-algebra $C_r^*(G)$.

DEFINITION. $B = C_0(Y)$.

DEFINITION. $\mathcal{E} =$ all continuous sections of the field $\{H(\sigma) : \sigma \in Y\}$ which vanish at infinity.

2.5. The representations $\pi(\sigma)$ are generically irreducible. This statement may be made a little more precise in the following way. Equip the space D with a $W(M)$-invariant cell-structure. This determines a cell-structure on the orbifold $D/W(M, D)$. Reducibility of $\pi(\sigma)$ can occur only if the isotropy subgroup W_σ of $W(M)$ is not the trivial group: so that in each cell of top dimension the representation $\pi(\sigma)$ is necessarily irreducible. Reducibility can occur only in the $\ell - 1$ skeleton of the orbifold, where ℓ is its dimension.

We now make the *temporary assumption that all representations* $\pi(\sigma)$ *are irreducible*, and proceed as follows. The B-valued inner product and the $A - B$-bimodule structure on $\mathcal{E}$ are determined by

(*) Choose one D in each $W(M)$-orbit; then take $D/W(M,D)$

$$
\begin{aligned}
<\xi,\eta>(\sigma) &= <\xi(\sigma),\eta(\sigma)> \\
(\xi b)(\sigma) &= \xi(\sigma).b(\sigma) \\
(\varphi\xi)(\sigma) &= \left(\int \varphi(g).(\pi(\sigma)g)dg\right).\xi(\sigma)
\end{aligned}
$$

for all ξ,η in $\mathcal{E}, b$ in B, σ in Y, and all test-functions φ in $L^1(G)$.

The bimodule $\mathcal{E}$ is full and

$$\mathcal{E} \cong C_0(Y, H)$$

where H is the standard Hilbert space.

The fact that A is liminal, together with the left A-module structure, determines a map

$$A \to K(\mathcal{E}).$$

Injectivity. Let $x \in A, x \neq 0$. Then there exists π in $\hat{A}$ such that $||\pi(x)|| = ||x||$ by [**D**, 3.3.6]. Now π is a tempered representation of G and so $\pi = \pi(\sigma)$ for some $\sigma \in X$ by the Plancherel theorem [**H**]. Therefore $\pi(\sigma)x \neq 0$. So the map $\tau \longmapsto \pi(\tau)x$ is nonzero at $\tau = \sigma$. Therefore the map $A \longrightarrow k(\mathcal{E}), x \longmapsto \hat{x}$, is injective.

2.6. The Richness Lemma (1.13) is now used to prove that the map $A \to K(\mathcal{E})$ is surjective. This secures an isomorphism of C^*-algebras:

$$A \cong K(\mathcal{E})$$

and shows that $\mathcal{E}$ is an equivalence bimodule and that A is strongly Morita equivalent to the abelian C^*-algebra B. This implies that the locally compact Hausdorff space Y is a model of the reduced dual $\hat{A}$ in its hull-kernel topology.

2.6.1. EXAMPLE [**PP**]. G is any connected complex semisimple Lie group.

2.6.2. EXAMPLE [**P**]. G is the general linear group over a local field.

3 The Brylinski Quotient

3.1. Let $G = GL(n) = GL(n, F)$ where F is a non-archimedean local field. By the Harish-Chandra Plancherel Theorem [**H**], an irreducible unitary representation π is tempered if and only if it comes off the discrete series of some Levi factor. That is, there exists a block-diagonal subgroup

$$M = GL(n_1) \times \ldots \times GL(n_k)$$

such that

$$\pi = \pi_1 \times \ldots \times \pi_k$$

where $\pi_1, \ldots, \pi_k$ are discrete series. Note that π is automatically irreducible, by a result of Bernstein.

3.2. Let I be the Iwahori subgroup of $GL(n)$. Now $\pi_1 \times \ldots \times \pi_k$ will admit I-fixed vectors iff the supercuspidal support of $\pi_1 \times \ldots \times \pi_k$ comprises unramified quasi-characters of $GL(1)$ by [**BK**, §7][**Bo**]. Therefore each of $\pi_1, \ldots, \pi_k$ must
- be discrete series
- have support comprising unramified quasi-characters of $GL(1)$.

But each discrete series representation comes off a segment [**Zel**]. Since this segment comprises unramified quasi-characters of $GL(1)$, each of $\pi_1, \ldots, \pi_k$ must be a Steinberg, with twisting allowed by unramified unitary characters $\chi_1, \ldots, \chi_k$. That is, we must have

$$\begin{aligned} M &= GL(n_1) \times \ldots \times GL(n_k) \\ \pi &= \pi_1 \times \ldots \times \pi_k \\ \pi_j &= (\chi_j \circ \det) St(n_j) \qquad j = 1, \ldots, k \end{aligned}$$

This proves

3.3. Lemma. The representation

$$(\chi_1 \circ \det) St(n_1) \times \ldots \times (\chi_k \circ \det) St(n_k)$$

is unitary, irreducible, tempered and admits I-fixed vectors. Moreover, all such representations are accounted for in this way.

3.4. The next step is to delve into the combinatorics of the representations which feature in Lemma 3.3.

Suppose that there are r_1 blocks of size $n_1, \ldots, r_\ell$ blocks of size n_ℓ. Then the Weyl group of the Levi factor M is

$$W(M) = S_{r_1} \times \ldots \times S_{r_\ell}.$$

This Weyl group permutes blocks of the same size. By standard Bruhat Theory, the Weyl group controls equivalences of parabolically induced representations. It follows that the parameter space for the tempered representations which admit I-fixed vectors is

$$X = \bigsqcup \mathbf{T}^{r_1}/S_{r_1} \times \ldots \times \mathbf{T}^{r_\ell}/S_{r_\ell}$$

The disjoint union is over all partitions

$$n_1 + \ldots n_1 + \ldots + n_\ell + \ldots + n_\ell = r_1.n_1 + \ldots + r_\ell.n_\ell = n.$$

3.5. Let now γ be an element in S_n whose cycle type is the partition $p(n)$ given by

$$n_1 + \ldots + n_1 + \ldots + n_\ell + \ldots + n_\ell = n.$$

The centralizer $Z(\gamma)$ is the product of wreath products:

$$Z(\gamma) = (\mathbf{Z}/n_1 \wr S_{n_1}) \quad \times \ldots \times \quad (\mathbf{Z}/n_\ell \wr S_{n_\ell}).$$

The Brylinski quotient Bryl $(\mathbf{T}^n; S_n)$ is by definition

$$\bigsqcup (\mathbf{T}^n)^\gamma / Z(\gamma)$$

where $(\mathbf{T}^n)^\gamma$ is the fixed point set

$$\{t \in \mathbf{T}^n : \gamma t = t\}.$$

In the disjoint union, one γ is taken in each conjugacy class. That is, the disjoint union is over all partitions $p(n)$ of n.

Now

$$\begin{aligned}
Bryl(\mathbf{T}^n; S_n) &= \bigsqcup (\mathbf{T}^n)^\gamma / Z(\gamma) \\
&= \bigsqcup \frac{\{(a,\ldots,a,\ldots,b,\ldots,b,\ldots,c,\ldots,c,\ldots,d,\ldots,d)\}}{(\mathbf{Z}/n_1 \wr S_{r_1}) \quad \times \ldots \times \quad (\mathbf{Z}/n_\ell \wr S_{r_\ell})} \\
&= \bigsqcup \frac{\{(a,\ldots,a,\ldots,b,\ldots,b,\ldots,c,\ldots,c,\ldots,d,\ldots,d)\}}{S_{r_1} \quad \times \ldots \times \quad S_{r_\ell}} \\
&\cong \bigsqcup Sym^{r_1}\mathbf{T} \quad \times \ldots \times \quad Sym^{r_\ell}\mathbf{T} \\
&= X.
\end{aligned}$$

We summarize the present discussion in

3.6. Lemma. The parameter space for the tempered representations of $GL(n)$ which admit I-fixed vectors is the Brylinski quotient $Bryl(\mathbf{T}^n; S_n)$. $\square$

4 The Reduced Iwahori-Hecke C^*-algebra.

4.1. Let $G = GL(n)$ and let I be the Iwahori subgroup. Choose left-invariant Haar measure such that I has volume 1. The left regular representation λ of $L^1(G)$ on $L^2(G)$ is defined as follows:

$$(\lambda(f))(h) = f * h \qquad f \in L^1(G), h \in L^2(G)$$

where $*$ denotes convolution. The reduced C^*-algebra is the closure (in the norm topology) of the image of λ:

$$A = C_r^*(G) = \overline{\lambda(L^1(G))} \subset \mathcal{L}(L^2(G)).$$

The dual of A is homeomorphic to the tempered dual of G.

4.2. Let $e : G \longrightarrow \mathbf{R}$ be defined as follows:

$$e(x) = \begin{cases} 1 & x \in I \\ 0 & \text{otherwise} \end{cases}$$

Then e is a projection in A. It determines a (2-sided) ideal AeA. This is a non-unital ideal in A. The main result of [**P**], together with the theory of the trivial type, makes it clear that AeA admits a complementary 2-sided ideal in A, that is

$$A = AeA \oplus D.$$

4.3. Consider the corner eAe. This is a unital corner with unit e. Now eAe is strongly Morita equivalent to AeA with equivalence bimodule $\mathcal{E} = eA$:

$$eAe \quad \overset{eA}{-} \quad AeA$$

Let

$$\begin{aligned} B &= eAe \\ C &= AeA \\ \mathcal{E} &= eA \end{aligned}$$

We have to check 3 points:
(1) $\mathcal{E}$ admits a C-valued inner product given by

$$< x, y >_{\mathcal{E}} = x^* y$$

Then $||x||_{\mathcal{E}} = ||x^*x||_C^{1/2} = ||x||$. Since $C = AeA$ is a closed 2-sided ideal, C is $||.||$-complete and $\mathcal{E}$ is a C^*-module.
The set $\{< x, y >_{\mathcal{E}} : x, y \in \mathcal{E}\}$ is

$$\{x^* y : xy \in \mathcal{E}\} = \{a^* eb : a, b \in A\} = AeA = C$$

hence $\mathcal{E}$ is a full C^*-module.
(2) $\mathcal{E}$ is a right C-module given by

$$\mathcal{E} \times C \to \mathcal{E}$$

$$(ea, c) \mapsto eac$$

(3) The standard rank 1 operators are given by

$$\begin{aligned} \Theta_{x,y}(z) &= x < y, z > \\ &= xy^* z \end{aligned}$$

with $x, y, z \in \mathcal{E}$. Thus

$$\Theta_{x,y} : z \longmapsto (eab^*e)z$$

with $x = ea, y = eb$. In particular

$$\Theta_{x,x} : z \mapsto (eaa^*e)z$$

These rank 1 operators on $\mathcal{E}$ are realized by positive elements in eAe. The linear span of such positive elements is eAe itself. So the linear span of the $\Theta_{x,y}$ is isomorphic to eAe. Since eAe is complete, the closure of the linear span of the $\Theta_{x,y}$ is isomorphic to eAe. Hence eAe is isomorphic to $k(\mathcal{E})$ the compact endomorphisms of $\mathcal{E}$. Hence $\mathcal{E}$ is an equivalence bimodule

$$eAe \quad \overset{\mathcal{E}}{-} \quad AeA$$

4.4. The equivalence bimodule $\mathcal{E}$ determines a homeomorphism of dual spaces:

$$(eAe)^\wedge \cong (AeA)^\wedge.$$

Now AeA is a closed 2-sided ideal in A:

$$A = AeA \oplus D$$

Therefore $\hat{A} = (AeA)^\wedge \bigsqcup D^\wedge$.

Also

$$\begin{aligned} \pi(e) &= \int e(x)\pi(x)dx \\ &= \int_I \pi(x)dx \end{aligned}$$

which is projection onto the subspace of I-fixed vectors occurring in π.

So

$$\pi(e) \neq 0 \Longleftrightarrow \pi \text{ admits nonzero } I-\text{fixed vectors.}$$

Since

$$\pi(xey) = \pi(x)\pi(e)\pi(y)$$

it follows that the dual of AeA is precisely those tempered representations of A which admit nonzero I-fixed vectors. We shall summarize this by saying that

- $(AeA)^\wedge$ is the unramified tempered dual.

We emphasize that the unramified tempered dual comprises unitary, irreducible, tempered representation of $GL(n)$ which admit non zero I-fixed vectors.

Combining the present discussion, section §3, and the main result in **[P]** we conclude the following.

4.5. Lemma. The dual of eAe is the unramified tempered dual of $GL(n)$. The Brylinski quotient $Bryl(\mathbf{T}^n; S_n)$ is a model of the unramified tempered dual. □

5 C^*-Plancherel Theorem

5.1. Let $\mathcal{H}(G//I)$ be the Iwahori-Hecke algebra. This comprises all complex-valued functions ϕ on G which are compactly supported, locally constant and bi-invariant with respect to I:

$$\phi(i_1 x i_2) = \phi(x)$$

for all $i_1, i_2 \in I, x \in G$. The product in $\mathcal{H}(G//I)$ is the convolution product.

Since

$$\phi = e * \phi * e$$

it is immediate that $\mathcal{H}(G//I) \subset eAe$. In fact $\mathcal{H}(G//I)$ is dense in eAe in the reduced C^* algebra norm, and we refer to eAe as the reduced Iwahori-Hecke C^* algebra. The notation is

$$C_r^*(G//I) = eAe.$$

5.2. An Hermitian vector bundle now presents itself.

- The base space X is the dual of eAe. By Lemma 4.5, the base space is the Brylinski quotient $Bryl(\mathbf{T}^n; S_n)$.
- The total space S is the set of all I-fixed vectors. Of course, only those representations π in the dual of eAe admit non zero I-fixed vectors.
- The fibre S_π comprises all I-fixed vectors in the representation π. We have

$$\begin{array}{ccc} S_\pi & \subset & S \\ \downarrow & & \downarrow \\ \pi & \in & X \end{array}$$

5.3. The vector bundle S admits an endomorphism bundle

$$End(S) \cong S \otimes S^*$$

Since S is Hermitian, the continuous sections of $End(S)$ form a unital C^* algebra whose dual is homeomorphic to X.

Note that all unramified representations of G can be realized on a fixed Hilbert space. In consequence, the bundle S is a trivialized vector bundle. Let $\phi \in \mathcal{H}(G//I)$.

5.4. Definition. The Fourier Transform $\hat{\phi}$ is defined as

$$\hat{\phi}(\pi) = \pi(\phi) = \int \phi(g)\pi(g)dg.$$

Since $\phi = e\phi e$ it follows that

$$\hat{\phi}(\pi) = \pi(e\phi e) = \pi(e)\pi(\phi)\pi(e)$$

where $\pi(e)$ projects onto the I-fixed subspace of π. Therefore

$$\hat{\phi}(\pi) \in End(S_\pi).$$

Define

$$\begin{aligned} \alpha : \mathcal{H}(G//I) &\longrightarrow C(EndS) \\ \phi &\longmapsto \hat{\phi} \end{aligned}$$

So

$$(\alpha(\phi))(\pi) = \hat{\phi}(\pi) \in End(S_\pi).$$

The following result was conjectured by Nigel Higson in discussions between Nigel Higson and myself, and subsequently proved by myself.

5.5. Theorem. (C^*-Plancherel Theorem for the reduced Iwahori-Hecke C^*-algebra of $GL(n)$). The Fourier Transform extends uniquely to an isomorphism of unital C^* algebras:

$$C_r^*(G//I) \cong C(EndS).$$

5.5.1. Proof. We have already shown that the Fourier Transform determines a map

$$\mathcal{H}(G//I) \longrightarrow C(EndS).$$

5.5.2. Injectivity. Let $y \in eAe$. Then there exists π in $(eAe)^\wedge$ such that $||y|| = \sup ||\pi(y)||$ by [**D**, 3.3.6]. Therefore

$$y \neq 0 \Longrightarrow (\exists \pi)\pi(y) \neq 0$$

and

$$y_1 \neq y_2 \Longrightarrow (\exists \pi)\pi(y_1) \neq \pi(y_2)$$

so that

$$\alpha : \mathcal{H}(G//I) \longrightarrow C(EndS)$$

is injective.

5.5.3. Surjectivity. The image $\alpha(eAe)$ is a sub-C^*-algebra of $C(EndS)$ by 1.6. Let

$$B = \alpha(eAe).$$

Note that $C(EndS)$ is a liminal C^*-algebra with compact Hausdorff dual X. Let

$$C = C(EndS).$$

We shall now apply the Richness Lemma 1.13.

- Let $\pi \in \hat{C}$. Then $\pi \in X$.

Consider the restriction $\pi|_B$. This is irreducible because

$$\pi : \mathcal{H}(G//I) \longrightarrow End(S_\pi)$$

is a simple $\mathcal{H}(G//I)$-module, $\mathcal{H}(G//I)$ is dense in eAe, and B is isomorphic to eAe. The simplicity of the $\mathcal{H}(G//I)$-module of I-fixed vectors in π is a classical result of Borel [**Bo**].

• Let $\pi, \psi \in \hat{C}$. Then $\phi, \psi \in X$. Restrict to B. We obtain

$$\left.\begin{array}{l} \pi : \mathcal{H}(G//I) \longrightarrow End S_\pi \\ \psi : \mathcal{H}(G//I) \longrightarrow End S_\psi \end{array}\right\}$$

Suppose $\pi \neq \psi$. Then the simple $\mathcal{H}$-modules S_π and S_ψ are distinct owing to the bijection between unramified irreducible representations of G and simple $\mathcal{H}$-modules. This again is a classical result of Borel [**Bo**]. Once again $\mathcal{H}(G//I)$ is dense in eAe which is isomorphic to B. So we conclude that $\pi|_B$ and $\psi|_B$ are distinct irreducible representations of B. By the Richness Lemma, $B = C$. $\square$

6 Localized $B - C$ Conjecture

6.1. In these lectures we have tried to maintain the theme of strong Morita equivalence. Our conclusion so far is that the reduced Iwahori-Hecke C^*-algebra $C_r^*(G//I)$ is strongly Morita equivalent to the commutative C^* algebra $C(X)$ where X is the Brylinski quotient

$$X = Bryl(\mathbf{T}^n; S_n).$$

The equivalence bimodule $\mathcal{E}$ which effects this strong Morita equivalence comprises continuous sections of the vector bundle S of all Iwahori-fixed vectors. This bundle S is a Hermitian vector bundle with base X. The picture is:

$$C_r^*(G//I) \quad \overset{C(S)}{-} \quad C(X).$$

6.2. As in [**BK**]§5.4, the algebra $\mathcal{H}(G//I)$ can be described in purely combinatorial terms. It is canonically isomorphic to the affine Hecke algebra $\mathcal{H}(n,q)$ where q is the cardinality of the residue field of the local field F. Recall that $G = GL(n) = GL(n, F)$. This algebra has a standard basis $\{[w]\}$ with w ranging over the affine Weyl group

$$W = \mathbf{Z}^n \rtimes S_n$$

where S_n acts on $\mathbf{Z}^n$ by permuting co-ordinates.

The canonical inner product on $\mathcal{H}(G//I)$ which is

$$< f, h >= \int f \ \overline{g}$$

corresponds to the combinational inner product on $\mathcal{H}(n,q)$: the elements $[w]$ form an orthogonal basis for this inner product, and

$$< [w], [w] >= q^{\ell(w)}$$

where ℓ is the length function on W (giving the fundamental involutions length 1).

6.3. The algebra $\mathcal{H}(n,q)$ acts by convolution on the ℓ^2-completion of the pre-Hilbert space $\mathcal{H}(n,q)$. This is the left regular representation of the affine Hecke algebra $\mathcal{H}(n,q)$. The norm closure of the image of the left regular representation defines the reduced C^*-algebra $\mathcal{H}_r^*(n,q)$.

Let f_w be the indicator function (characteristic function) of the double coset IwI:

$$f_w(x) = \begin{cases} 1 & x \in IwI \\ 0 & \text{otherwise.} \end{cases}$$

The map

$$f_w \longmapsto [w]$$

extends to an isomorphism of unital C^*-algebras

$$C_r^*(G//I) \cong \mathcal{H}_r^*(n,q).$$

6.4. Combining section 6.3 with Lemma 5. x we have a strong Morita equivalence

$$\mathcal{H}_r^*(n,q) \simeq C(X).$$

Notice one striking fact: the right-hand-side is independent of q. Now q, as cardinality of a residue field, is any prime power. So we have isomorphic C^*-algebras

$$\mathcal{H}_r^*(n,2) \qquad \mathcal{H}_r^*(n,3) \qquad \mathcal{H}_r^*(n,4) \qquad \mathcal{H}_r^*(n,5) \qquad\qquad \mathcal{H}_r^*(n,7)$$

The algebra $\mathcal{H}_r^*(n,q)$ is defined for all $q \geq 1$.
Note also that

$$\mathcal{H}_r^*(n,1) = C^*(W)$$

the C^* algebra of the discrete group W. Set

$$\begin{aligned} A(1) &= \mathcal{H}_r^*(n,1) = C^*(W) \\ A(q) &= \mathcal{H}_r^*(n,q) \qquad\qquad q > 1. \end{aligned}$$

Conjecture 1. (Higson-Plymen). The field of C^*-algebras $\{A(q) : q \geq 1\}$ has constant K-theory.

Conjecture 2. (Higson-Plymen). Same as Conjecture 1, with $GL(n)$ replaced by any split reductive p-adic group G.

6.5. Back to Conjecture 1. In the E-theory framework of Connes-Higson [**CH**] this is a *deformation* of $C^*(W)$ into the constant C^*-algebra $\mathcal{H}_r^*(n,q)$ where $q > 1$.

Let $n \geq 2$. The C^*-algebras $C^*(W)$ and $\mathcal{H}_r^*(n,q)$ where q is a prime power, are definitely not isomorphic. For example:
• The C^*-algebra $C^*(W)$ has quasi-compact connected non-Hausdorff dual. The dual of $C^*(W)$ is homeomorphic to the unitary dual of the discrete group W. The result now follows from Mackey orbital analysis of the semidirect product $\mathbf{Z}^n \rtimes S_n$.
• The C^*-algebra $\mathcal{H}_r^*(n,q)$ has compact disconnected Hausdorff dual X.

The Brylinski quotient $X = Bryl(n)$ can be regarded as a *reconstruction* of the dual of W which is true to K-theory.

These conjectures are analogues of the Connes-Kasparov conjecture for Lie groups, as formulated by [**CH**]. For let G be a Lie group. Let

$$\begin{array}{ll} C(q) = C_r^*(G) & \qquad q > 1 \\ C(1) = C^*(\mathfrak{p} \rtimes K) & \end{array}$$

where $\mathfrak{p}$ is the tangent space at K (maximal compact) of the symmetric space G/K. According to the conjecture, the field $\{C(q) : q \geq 1\}$ has constant K-theory. In this analogy, we have

	Lie Groups	Hecke Algebras
$q > 1$	$C_r^*(G)$	$\mathcal{H}_r^*(n,q)$
$q = 1$	$C^*(\mathfrak{p} \rtimes K)$	$C^*(W)$

In other words

• The affine Weyl group is a discrete *motion* group.

6.6. Theorem.

$$K_j\mathcal{H}_r^*(n,q) \cong K^j(X)$$

for all $q = p^m > 1$.

Proof. This follows from 5.5 and 2.1. □

6.7. We may take an apartment Σ in the affine building βG of G as model for the universal example $\underline{E}W$. So we have

$$\Sigma = \underline{E}W.$$

Of course Σ has the structure of affine Euclidean space [**BT**]. The B-C conjecture for the discrete group W is the provable statement

$$K_j^W(\Sigma) \cong K_jC^*(W).$$

The left-hand-side comprises W-invariant "Dirac operators" on Σ, organized into two equivariant K-homology groups, $j = 0,1$.

6.8. Combining 6.6-6.7 we obtain an isomorphism

$$K_j^W(\Sigma) \cong K_j C_r^*(G//I)$$

Now

- Σ embeds isometrically in the affine building βG
- $C_r^*(G//I)$ embeds as direct summand of $C_r^* G$.

6.9. So the isomorphism in 6.8 is a localized version of the isomorphism (conjectured by Baum-Connes) for $GL(n)$

$$K_j^G(\beta G) \cong K_j C_r^*(G).$$

6.10. The isomorphism in 6.8 is a consequence of Conjecture 1. Let's see how close we can get to 6.8 without assuming Conjecture 1.
Consider the K-theory of the C^*-algebra of the discrete group W. Now W is an amenable group so $C^*W = C_r^*W$. Fourier Transform determines an isomorphism

$$C^*(W) \cong C(\mathbf{T}^n) \rtimes S_n$$

the crossed product of an abelian C^* algebra by the symmetric group S_n.
The equivariant Chern character [**BC**] determines isomorphisms

$$\begin{aligned} ch : K_0(C(\mathbf{T}^n) \rtimes S_n) \otimes_{\mathbf{Z}} \mathbf{C} &\cong H^{ev}(X; \mathbf{C}) \\ ch : K_1(C(\mathbf{T}^n) \rtimes S_n) \otimes_{\mathbf{Z}} \mathbf{C} &\cong H^{odd}(X; \mathbf{C}) \end{aligned}$$

6.11. The classical Chern character gives isomorphisms

$$\begin{aligned} K^0(X) \otimes_{\mathbf{Z}} \mathbf{Q} &\cong H^{ev}(X; \mathbf{Q}). \\ K^1(X) \otimes_{\mathbf{Z}} \mathbf{Q} &\cong H^{odd}(X; \mathbf{Q}). \end{aligned}$$

We now have

$$\begin{aligned} K_0(C^*(W)) \otimes_{\mathbf{Z}} \mathbf{C} &\cong K_0(C(\mathbf{T}^n) \rtimes S_n) \otimes_{\mathbf{Z}} \mathbf{C} && (1) \\ &\cong H^{ev}(X; \mathbf{C}) && (2) \\ &\cong K^0(X) \otimes_{\mathbf{Z}} \mathbf{C} && (3) \\ &\cong K_0(\mathcal{H}_r^*(n,q)) \otimes_{\mathbf{Z}} \mathbf{C} && (4) \end{aligned}$$

whenever $q = p^m > 1$.
The reasons are

1. Fourier Transform
2. Equivariant Chern character

3. Chern character

4. Theorem 6.6

Exactly similar conclusion for K_1. Summarizing we have

Theorem 2. We have

- $K_j(\mathcal{H}_r^*(n,q)) \otimes_{\mathbf{Z}} \mathbb{C} \cong K_j(C^*(W)) \otimes_{\mathbf{Z}} \mathbb{C}$ for all $q = p^m > 1$ and $j = 0, 1$.

- The Baum-Connes conjecture for $GL(n)$, localized to an apartment, is true modulo torsion:

$$(K_j^W(\Sigma)) \otimes_{\mathbf{Z}} \mathbb{C} \cong K_j(C_r^*(G//I)) \otimes_{\mathbf{Z}} \mathbb{C}.$$

$\square$

7 Satake Isomorphism

7.1. The strong Morita equivalence

$$C_r^*(G//I) \quad \overset{C(S)}{-} \quad C(X)$$

has been proved by Fourier Transform methods. This depends crucially on the Zelevinsky theory of segments [**Z**]; note that this can be reproved in the Hecke algebra framework of Howe-Bushnell-Kutzko [**H**][**BK**]. The question arises whether the result under discussion may succumb to more intrinsic methods, i.e. avoiding Fourier Transform.

7.2. A purely algebraic approach *cannot* work, owing to the difference between the reduced C^*-algebra $C_r^*(G//I)$ and the full C^*-algebra $C^*(G//I)$. This is manifest even for $GL(2)$. For $GL(2)$ the Brylinski quotient is

$$X = (\mathbf{T}^2)^e/Z(e) \bigsqcup (\mathbf{T}^2)^\gamma/Z(\gamma)$$

where

$$\mathbf{Z}/2 = \{e, \gamma\}.$$

So we have

$$X = Sym^2\mathbf{T} \bigsqcup \mathbf{T}$$

But the full Iwahori-Hecke C^*-algebra is strongly Morita equivalent to

$$Y = Sym^2\mathbf{T} \bigsqcup \mathbf{T} \bigsqcup \mathbf{T}.$$

This is because $GL(2)$ admits a "circle" of representations which are
- unramified
- non-tempered.

These representations are very easy to describe explicitly. They are

$$\begin{array}{ccc} GL(2) & \longrightarrow & U(1) \\ x & \longmapsto & z^{val(detx)} \end{array}$$

where the parameter z is a complex number of modulus 1, $z \in U(1)$. If $x \in I$ then $\det x \in \mathcal{O}^\times$ so that

$$x \in I \Longrightarrow z^{val(detx)} = 1.$$

The trivial representation of $GL(2)$ is of course given by $z = 1$: it is non-tempered.

7.3. Let K be a maximal compact subgroup of $GL(n)$. They are all conjugate. We may take $K = GL(n, \mathcal{O})$. We have

$$C_r^*(G//K) \subset C_r^*(G//I).$$

The Brylinski quotient X contains the ordinary quotient

$$\begin{aligned} X &= (\mathbb{T}^n)^e/Z(e) \bigsqcup \cdots \\ &= \mathbb{T}^n/S_n \bigsqcup \cdots \\ &= Sym^n\mathbb{T} \bigsqcup \cdots \end{aligned}$$

7.4. In the standard strong Morita equivalence

$$C_r^*(G//I) \quad \overset{C(S)}{-} \quad C(X)$$

the sub-C^*-algebra $C_r^*(G//K)$ determines a strong Morita equivalence

$$C_r^*(G//K) \quad \overset{C(L)}{-} \quad C(BbbT^n/S_n)$$

where L is the line bundle of all K-fixed vectors. This of course means that

$$C_r^*(G//K) \cong C(\mathbb{T}^n/S_n).$$

This is the C^*-algebra version of the Satake isomorphism [**Car**, p.147]

$$\mathcal{H}(G//K) \cong \mathbb{C}[\Lambda]^W$$

where the lattice $\Lambda = \mathbb{Z}^n$. The group algebra $\mathbb{C}[\Lambda]$ will Fourier Transform to a dense subalgebra of $C(\mathbb{T}^n/S_n)$. The intermediate compact subgroups J

$$I \subset J \subset K$$

and the associated bi-invariant C^*-algebras $C_r^*(G//J)$ may account for the various components in the Brylinski quotient: this is currently under investigation [**HP**].

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