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De Teran, Fernando and Dopico, Froilan M. and Mackey, D. Steven

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FIEDLER COMPANION LINEARIZATIONS AND THE RECOVERY OF MINIMAL INDICES *

FERNANDO DE TERÁN†, FROILÁN M. DOPICO‡, AND D. STEVEN MACKEY §

Abstract. A standard way of dealing with a matrix polynomial $P(\lambda)$ is to convert it into an equivalent matrix pencil – a process known as linearization. For any regular matrix polynomial, a new family of linearizations generalizing the classical first and second Frobenius companion forms has recently been introduced by Antoniou and Vologiannidis, extending some linearizations previously defined by Fiedler for scalar polynomials. We prove that these pencils are linearizations even when $P(\lambda)$ is a singular square matrix polynomial, and show explicitly how to recover the left and right minimal indices and minimal bases of the polynomial $P(\lambda)$ from the minimal indices and bases of these linearizations. In addition, we provide a simple way to recover the eigenvectors of a regular polynomial from those of any of these linearizations, without any computational cost. The existence of an eigenvector recovery procedure is essential for a linearization to be relevant for applications.

Key words. singular matrix polynomials, matrix pencils, minimal indices, minimal bases, linearization, recovery of eigenvectors, Fiedler pencils, companion forms

AMS subject classifications. 65F15, 15A18, 15A21, 15A22

1. Introduction. Throughout this work we consider $n \times n$ matrix polynomials with degree $k \geq 2$ of the form

$$P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i \,, \qquad A_0, \ldots, A_k \in \mathbb{F}^{n \times n}, \quad A_k \neq 0 \,, \tag{1.1}$$

where $\mathbb{F}$ is the field of real or complex numbers and λ is a scalar variable in $\mathbb{F}$. Our main focus is on singular matrix polynomials, although new results are also obtained for regular polynomials. An $n \times n$ matrix polynomial $P(\lambda)$ is said to be *singular* if $\det P(\lambda)$ is identically zero, and it is said to be *regular* otherwise. Square singular polynomials appear in practice, although not as frequently as regular polynomials. One well-known example is the study of differential-algebraic equations (see for instance [7] and the references therein). Other sources of problems involving singular matrix polynomials are control and linear systems theory [27], where the problem of computing minimal polynomial bases of null spaces of singular matrix polynomials continues to be the subject of intense research (see [3] and the references therein for an updated bibliography). In this context, it should be noted that the matrix polynomials arising in control problems are often full-rank rectangular polynomials. However, square singular polynomials are also present in applications connected with linear systems [36].

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†Departamento de Matemáticas, Universidad Carlos III de Madrid, Avda. Universidad 30, 28911 Leganés, Spain (`fteran@math.uc3m.es`).

‡Instituto de Ciencias Matemáticas CSIC-UAM-UC3M-UCM and Departamento de Matemáticas, Universidad Carlos III de Madrid, Avda. Universidad 30, 28911 Leganés, Spain (`dopico@math.uc3m.es`).

§Department of Mathematics, Western Michigan University, Kalamazoo, MI 49008, USA (`steve.mackey@wmich.edu`).

The standard way to numerically solve polynomial eigenvalue problems for *regular* polynomials $P(\lambda)$ is to first linearize $P(\lambda)$ into a matrix pencil $L(\lambda) = \lambda X + Y$ with $X, Y \in \mathbb{F}^{nk \times nk}$, and then compute the eigenvalues and eigenvectors of $L(\lambda)$ using well-established algorithms for matrix pencils [20]. The classical approach uses the *first and second companion forms* [19], sometimes known as the *Frobenius companion forms* of $P(\lambda)$, as linearizations. However, these companion forms usually do not share any algebraic structure that $P(\lambda)$ might have. For example, if $P(\lambda)$ is symmetric, Hermitian, alternating, or palindromic, then the companion forms won't retain any of these structures. Consequently, the rounding errors inherent to numerical computations may destroy qualitative aspects of the spectrum. This has motivated intense activity towards the development of new classes of linearizations. Several classes have been introduced in [4, 5] and [30], generalizing the Frobenius companion forms in a number of different ways. Other classes of linearizations were introduced and studied in [1, 2], motivated by the use of non-monomial bases for the space of polynomials. The numerical properties of the linearizations in [30] have been analyzed in [22, 23, 26], while the exploitation of these linearizations for the preservation of structure in a wide variety of contexts has been extensively developed in [14, 24, 25, 29, 31].

The linearizations introduced in [2, 4, 30] were originally studied only for *regular* matrix polynomials. Very recently, though, linearizations of square *singular* matrix polynomials have been considered in [11]. It has been shown in [11] that for any square matrix polynomial $P(\lambda)$, singular or regular, almost all the elements in the vector spaces of pencils $\mathbb{L}_1(P)$ and $\mathbb{L}_2(P)$ defined in [30] are linearizations for $P(\lambda)$. Furthermore, these linearizations allow us to easily recover the *complete eigenstructure* of $P(\lambda)$, i.e., the finite and infinite elementary divisors together with the left and right minimal indices [16, 27], and also to recover the corresponding minimal bases. We remark that the results in [11] can be applied to the important cases of the first and second companion forms of $P(\lambda)$, since these pencils belong to $\mathbb{L}_1(P)$ and $\mathbb{L}_2(P)$, respectively. Moreover, [11] includes similar results for those pencils in [2, Sections 2 and 3] that are defined in terms of degree-graded polynomial bases, because each of these pencils is strictly equivalent to some pencil in $\mathbb{L}_2(P)$.

There are *three main results* in this work. The first is to show that the family of linearizations introduced in [4] for regular matrix polynomials $P(\lambda)$ are still linearizations when $P(\lambda)$ is a *singular* square matrix polynomial. This requires very different techniques from those used in [4] for the regular case. Second we show how these linearizations can be used to immediately recover the complete eigenstructure of $P(\lambda)$. Finally, we develop simple procedures to recover the eigenvectors of a regular polynomial $P(\lambda)$ from those of any linearization in [4], without any computational cost. Recovery procedures for eigenvectors were not addressed in [4], but are very important for practical applications, as well as in any numerical algorithm for polynomial eigenvalue problems based on linearizations.

It will be convenient to have a simple name for the pencils introduced in [4], since they play such a central role in this paper. Note that these pencils arise from the companion matrices for scalar polynomials introduced by Fiedler [15], in the same way that the classical first and second companion forms arise from the companion matrices of Frobenius. Hence we will refer to the pencils introduced in [4] as the *Fiedler companion pencils*, or *Fiedler pencils* for short.

The results in this work expand the arena in which to look for linearizations of singular square matrix polynomials with additional useful properties. In particular, for finding structured linearizations of structured singular polynomials, this expansion is essential. Indeed, for singular polynomials P that are symmetric, Hermitian, alternating, or palindromic, it was shown in [11] that *none* of the pencils in $\mathbb{L}_1(P)$ or $\mathbb{L}_2(P)$ with structure corresponding to that of P (see [25, 31]) is *ever* a linearization when $P(\lambda)$ is singular. However, using pencils closely related to the Fiedler pencils, it is possible to develop structured linearizations for at least some large classes of structured singular matrix polynomials [12, 32].

Apart from the preservation of structure, there is another property that may be potentially useful; some Fiedler pencils have a much smaller bandwidth than the classical Frobenius companion forms [15] (see also an example in Section 3 below). It may be possible to exploit this band structure to develop fast algorithms to compute the complete eigenstructure of high degree matrix polynomials. As far as we know, though, this has not yet been addressed either for regular or for singular polynomials.

As we have already mentioned, minimal indices and bases arise in many problems in control [16, 27], and their numerical computation is a difficult problem that can be addressed in several different ways [3]. Among them, the Frobenius companion linearization approach is one of the most reliable methods from a numerical point of view [6, 35]. This method has only been considered in the case of rectangular matrix polynomials with full row (or column) rank. The results in this paper, together with those in [11], open up the possibility of dealing with non-full rank matrix polynomials, as well as of using linearizations different from the classical Frobenius companion forms in the numerical computation of minimal indices and bases of matrix polynomials.

We begin in Section 2 by recalling some basic concepts that are used throughout the paper, followed in Section 3 by the fundamental definitions and notation needed for working effectively with Fiedler pencils. Section 4 then proves that Fiedler pencils are always strong linearizations, even for singular matrix polynomials. In Section 5 we show how to recover the minimal indices and bases of a singular square matrix polynomial from those of any Fiedler pencil; as a consequence, we are then able to characterize which Fiedler pencils are strictly equivalent and which are not. Section 6 provides a very simple recipe for recovering, without any computational cost, the eigenvectors of a regular matrix polynomial from the eigenvectors of any of its Fiedler companion linearizations. Finally, we wrap up in Section 7 with some conclusions and discussion of ongoing related work.

2. Basic concepts.

We present some basic concepts related to matrix polynomials (singular or not), referring the reader to [11, Section 2] for a more complete treatment. We adopt the following notation: 0_d is the $d \times d$ zero matrix and I_d is the $d \times d$ identity matrix.

We denote by $\mathbb{F}(\lambda)$ the field of rational functions with coefficients in $\mathbb{F}$, and by $\mathbb{F}(\lambda)^n$ the vector space of column n-tuples with entries in $\mathbb{F}(\lambda)$. The *normal rank* of a matrix polynomial $P(\lambda)$, denoted $\operatorname{nrank} P(\lambda)$, is the rank of $P(\lambda)$ considered as a matrix with entries in $\mathbb{F}(\lambda)$, or, equivalently, the size of the largest non-identically zero minor of $P(\lambda)$ [17]. A *finite eigenvalue* of $P(\lambda)$ is a complex number λ_0 such that

$$\operatorname{rank} P(\lambda_0) < \operatorname{nrank} P(\lambda)\,.$$

We say that $P(\lambda)$ with degree k has an *infinite eigenvalue* if the *reversal polynomial*

$$\operatorname{rev} P(\lambda) := \lambda^k P(1/\lambda) = \sum_{i=0}^{k} \lambda^i A_{k-i} \tag{2.1}$$

has zero as eigenvalue.

An $n \times n$ singular matrix polynomial $P(\lambda)$ has *right* (*column*) and *left* (*row*) *null vectors*, that is, vectors $x(\lambda) \in \mathbb{F}(\lambda)^{n\times 1}$ and $y(\lambda)^T \in \mathbb{F}(\lambda)^{1\times n}$ such that $P(\lambda)x(\lambda) \equiv 0$ and $y(\lambda)^T P(\lambda) \equiv 0$, where $y(\lambda)^T$ denotes the transpose of $y(\lambda)$. This leads to the following definition.

Definition 2.1. *The* right *and* left nullspaces *of the $n \times n$ matrix polynomial $P(\lambda)$, denoted by $\mathcal{N}_r(P)$ and $\mathcal{N}_\ell(P)$ respectively, are the following subspaces:*

$$\begin{aligned}\mathcal{N}_r(P) &:= \left\{x(\lambda) \in \mathbb{F}(\lambda)^{n\times 1} \,:\, P(\lambda)x(\lambda) \equiv 0\right\}\,, \\ \mathcal{N}_\ell(P) &:= \left\{y(\lambda)^T \in \mathbb{F}(\lambda)^{1\times n} \,:\, y(\lambda)^T P(\lambda) \equiv 0^T\right\}\,.\end{aligned}$$

Note that we have the identity

$$\operatorname{nrank}(P) = n - \dim \mathcal{N}_r(P) = n - \dim \mathcal{N}_\ell(P), \tag{2.2}$$

and, in particular, $\dim \mathcal{N}_r(P) = \dim \mathcal{N}_\ell(P)$.

It is well known that the *elementary divisors* of $P(\lambda)$ (see definition in [17]) corresponding to its finite eigenvalues, as well as the dimensions of $\mathcal{N}_r(P)$ and $\mathcal{N}_\ell(P)$, are invariant under *equivalence* with respect to *unimodular* matrices, i.e., under pre- and post-multiplication by matrix polynomials with nonzero constant determinant [17]. The elementary divisors of $P(\lambda)$ corresponding to the infinite eigenvalue are defined as the elementary divisors corresponding to the zero eigenvalue of the reversal polynomial [21, Definition 1].

Next we recall the definition of linearization as introduced in [19], and also the related notion of strong linearization introduced in [18] and named in [28].

Definition 2.2. *A matrix pencil $L(\lambda) = \lambda X + Y$ with $X, Y \in \mathbb{F}^{nk\times nk}$ is a* linearization *of an $n\times n$ matrix polynomial $P(\lambda)$ of degree k if there exist two unimodular $nk \times nk$ matrices $U(\lambda)$ and $V(\lambda)$ such that*

$$U(\lambda)L(\lambda)V(\lambda) = \begin{bmatrix} I_{(k-1)n} & 0 \\ 0 & P(\lambda) \end{bmatrix}, \tag{2.3}$$

or, in other words, if $L(\lambda)$ is equivalent *to* $\operatorname{diag}(I_{(k-1)n}, P(\lambda))$. *A linearization $L(\lambda)$ is called a* strong linearization *if* $\operatorname{rev} L(\lambda)$ *is also a linearization of* $\operatorname{rev} P(\lambda)$.

These definitions were introduced in [18, 19] only for regular polynomials, and were extended in [11, Section 2] to square singular matrix polynomials. Lemma 2.3 shows why linearizations and strong linearizations are relevant in the study of both regular and singular matrix polynomials.

Lemma 2.3. [11, Lemma 2.3] *Let $P(\lambda)$ be an $n \times n$ matrix polynomial of degree k and $L(\lambda)$ an $nk \times nk$ matrix pencil, and consider the following conditions on $L(\lambda)$ and $P(\lambda)$:*

(a) $\dim \mathcal{N}_r(L) = \dim \mathcal{N}_r(P)$,

(b) *the finite elementary divisors of $L(\lambda)$ and $P(\lambda)$ are identical*,

(c) *the infinite elementary divisors of $L(\lambda)$ and $P(\lambda)$ are identical*.

Then $L(\lambda)$ is

- *a linearization of $P(\lambda)$ if and only if conditions* (a) *and* (b) *hold,*
- *a strong linearization of $P(\lambda)$ if and only if conditions* (a), (b) *and* (c) *hold.*

We mention briefly that linearizations with smaller size than the ones in Definition 2.2 have been introduced recently in [7], and that their minimal possible size has been determined in [10].

A *vector polynomial* is a vector whose entries are polynomials in the variable λ. For any subspace of $\mathbb{F}(\lambda)^n$, it is always possible to find a basis consisting entirely of vector polynomials; simply take an arbitrary basis and multiply each vector by the denominators of its entries. The *degree* of a vector polynomial is the greatest degree of its components, and the *order* of a polynomial basis is defined as the sum of the degrees of its vectors [16, p. 494]. Then the following definition makes sense.

DEFINITION 2.4. [16] *Let $\mathcal{V}$ be a subspace of $\mathbb{F}(\lambda)^n$. A* minimal basis *of $\mathcal{V}$ is any polynomial basis of $\mathcal{V}$ with least order among all polynomial bases of $\mathcal{V}$.*

It can be shown [16] that for any given subspace $\mathcal{V}$ of $\mathbb{F}(\lambda)^n$, the ordered list of degrees of the vector polynomials in any minimal basis of $\mathcal{V}$ is always the same. These degrees are then called the *minimal indices* of $\mathcal{V}$. Specializing $\mathcal{V}$ to be the left and right nullspaces of a singular matrix polynomial gives Definition 2.5; here $\deg(p(\lambda))$ denotes the degree of the vector polynomial $p(\lambda)$.

DEFINITION 2.5. *Let $P(\lambda)$ be a square singular matrix polynomial, and let the sets $\{y_1(\lambda)^T, \ldots, y_p(\lambda)^T\}$ and $\{x_1(\lambda), \ldots, x_p(\lambda)\}$ be minimal bases of, respectively, the left and right nullspaces of $P(\lambda)$, ordered such that $\deg(y_1) \leq \deg(y_2) \leq \cdots \leq \deg(y_p)$ and $\deg(x_1) \leq \deg(x_2) \leq \cdots \leq \deg(x_p)$. Let $\eta_i = \deg(y_i)$ and $\varepsilon_i = \deg(x_i)$ for $i = 1, \ldots, p$. Then $\eta_1 \leq \eta_2 \leq \cdots \leq \eta_p$ and $\varepsilon_1 \leq \varepsilon_2 \leq \cdots \leq \varepsilon_p$ are, respectively, the* left *and* right minimal indices *of $P(\lambda)$.*

For the sake of brevity, we will call minimal bases of the left and right nullspaces of $P(\lambda)$ simply *left and right minimal bases of $P(\lambda)$.*

In the case of matrix pencils, the left (right) minimal indices can be read off from the sizes of the left (right) singular blocks of the Kronecker canonical form of the pencil [17, Chap. XII]. Due to this fact, the minimal indices of a pencil can be stably computed through unitary transformations that lead to the GUPTRI form [33, 8, 9, 13]. Therefore it is natural to look for relationships between the minimal indices of a singular matrix polynomial P and the minimal indices of a given linearization of P, since this would provide a numerical method for computing the minimal indices of P. This was done in [11] for the pencils introduced in [30] and will be accomplished in this work for the *Fiedler Companion Pencils* introduced in [4]. Note in this context that Lemma 2.3 only implies that linearizations of P have the same *number* of minimal indices as P, but this lemma does not provide the values of the minimal indices of P in terms of the minimal indices of a linearization. In fact, it is known [11] that different linearizations of the same polynomial P may have different minimal indices. This is the reason why each different family of potential linearizations of singular polynomials requires a separate study to establish the relationships (if any) between the minimal indices of the polynomial and those of the linearizations in that family.

In this paper, we adopt the following definition, which was introduced in [11, Section 2] as an extension to matrix polynomials of the one introduced in [34] for pencils.

DEFINITION 2.6. *The* complete eigenstructure *of a matrix polynomial $P(\lambda)$ consists of*

1. *its finite and infinite elementary divisors, and*
2. *its left and right minimal indices.*

3. Definition of Fiedler companion pencils. Let $P(\lambda)$ be the matrix polynomial in (1.1). By using $P(\lambda)$, we define first the following $nk \times nk$ matrices

$$M_k := \begin{bmatrix} A_k & \\ & I_{(k-1)n} \end{bmatrix}, \quad M_0 := \begin{bmatrix} I_{(k-1)n} & \\ & -A_0 \end{bmatrix} \tag{3.1}$$

and

$$M_i := \begin{bmatrix} I_{(k-i-1)n} & & & \\ & -A_i & I_n & \\ & I_n & 0 & \\ & & & I_{(i-1)n} \end{bmatrix}, \quad i = 1, \ldots, k-1, \tag{3.2}$$

that are the building factors needed to define the Fiedler pencils. These are constructed in [4] as the pencils

$$\lambda M_k - M_{i_0} M_{i_1} \cdots M_{i_{k-1}},$$

where $(i_0, i_1 \ldots, i_{k-1})$ is any possible permutation of the n-tuple $(0, 1, \ldots, k-1)$. We will use very often in this paper some properties of the permutations above. These properties are better expressed if we introduce Definition 3.1.

Definition 3.1 (Fiedler Pencils). *Let $P(\lambda)$ be the matrix polynomial in* (1.1) *and M_i, $i = 0, \ldots, k$, be the matrices defined in* (3.1) *and* (3.2). *Given any bijection $\sigma : \{0, 1, \ldots, k-1\} \to \{1, \ldots, k\}$, the* Fiedler pencil of $P(\lambda)$ associated with σ *is the $nk \times nk$ matrix pencil*

$$F_\sigma(\lambda) := \lambda M_k - M_{\sigma^{-1}(1)} \cdots M_{\sigma^{-1}(k)}. \tag{3.3}$$

Note that $\sigma(i)$ is the position of the factor M_i in the product $M_{\sigma^{-1}(1)} \cdots M_{\sigma^{-1}(k)}$ giving the zero-degree term in (3.3). *For brevity, we will denote this zero-degree term by M_σ, that is*

$$M_\sigma := M_{\sigma^{-1}(1)} \cdots M_{\sigma^{-1}(k)}. \tag{3.4}$$

Sometimes we will write the bijection σ using the array notation $\sigma = (\sigma(0), \sigma(1), \ldots, \sigma(k-1))$. Unless otherwise stated, the matrices M_i, $i = 0, \ldots, k$, M_σ and the Fiedler pencil $F_\sigma(\lambda)$ refer to the matrix polynomial $P(\lambda)$ in (1.1). When necessary, we will indicate explicitly the dependence of these matrices on a certain matrix polynomial $Q(\lambda)$ as follows: $M_i(Q)$, $M_\sigma(Q)$ and $F_\sigma(Q)$. In this situation, the dependence on λ is dropped in the Fiedler pencil $F_\sigma(Q)$ for simplicity. Since matrix polynomials will be always denoted by capital letters, there is no risk of confusion between $F_\sigma(\lambda)$ and $F_\sigma(Q)$.

The set of Fiedler pencils includes the well-known *first* and *second companion forms* [19] of the polynomial in (1.1). They are

$$C_1(\lambda) := \lambda \begin{bmatrix} A_k & & & \\ & I_n & & \\ & & \ddots & \\ & & & I_n \end{bmatrix} + \begin{bmatrix} A_{k-1} & A_{k-2} & \cdots & A_0 \\ -I_n & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & -I_n & 0 \end{bmatrix}$$

and

$$C_2(\lambda) := \lambda \begin{bmatrix} A_k & & & \\ & I_n & & \\ & & \ddots & \\ & & & I_n \end{bmatrix} + \begin{bmatrix} A_{k-1} & -I_n & \cdots & 0 \\ A_{k-2} & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & -I_n \\ A_0 & 0 & \cdots & 0 \end{bmatrix}.$$

More precisely, $C_1(\lambda) = F_{\sigma_1}(\lambda)$ and $C_2(\lambda) = F_{\sigma_2}(\lambda)$, where $\sigma_1 = (k, k-1, \ldots, 2, 1)$ and $\sigma_2 = (1, 2, \ldots, k-1, k)$. These companion forms are always strong linearizations of $P(\lambda)$ [18, Prop. 1.1].

The set of Fiedler pencils also includes block pentadiagonal pencils [15] that, so, have a much smaller bandwidth than the first and second companion forms if the degree k of the polynomial is high. For these pencils the M_σ matrix (3.4) is constructed as follows: let $B = M_1M_3\cdots$ be the product of the odd M_i factors and let $C = M_2M_4\cdots$ be the product of the even M_i factors with the exception of M_0. Then, it is immediate to check that the product of M_0, B and C in any order yields a pentadiagonal M_σ. For instance for degree $k = 6$, $M_\sigma = M_0BC$ is

$$M_\sigma = \begin{bmatrix} -A_5 & -A_4 & I_n & 0 & 0 & 0 \\ I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & -A_3 & 0 & -A_2 & I_n & 0 \\ 0 & I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -A_1 & 0 & I_n \\ 0 & 0 & 0 & -A_0 & 0 & 0 \end{bmatrix}.$$

Note that the matrices B and C previously described are nonsingular and that their inverses are block tridiagonal matrices, so pre or post-multiplying a block pentadiagonal Fiedler pencil by B^{-1} or C^{-1} we get block tridiagonal pencils strictly equivalent to some Fiedler pencils. For instance: $F_\sigma(\lambda) = \lambda M_k - M_0BC$ is strictly equivalent to the block tridiagonal pencil $\lambda M_k C^{-1} - M_0B$.

The commutativity relations

$$M_iM_j = M_jM_i, \quad \text{for } |i-j| > 1, \tag{3.5}$$

can be easily checked. They imply that some Fiedler pencils associated with different bijections σ are equal. For instance, for $k = 3$, the Fiedler pencils $\lambda M_3 - M_0M_2M_1$ and $\lambda M_3 - M_2M_0M_1$ are equal. These relations suggest that the relative positions of the matrices M_i and M_{i+1}, for $i = 0, \ldots, k-2$, in the product M_σ are of fundamental interest in studying Fiedler pencils. This motivates Definition 3.2.

DEFINITION 3.2. *Let $\sigma : \{0, 1, \ldots, k-1\} \to \{1, \ldots, k\}$ be a bijection.*

(a) *For $i = 0, \ldots, k-2$, we say that σ has a* consecution *at i if $\sigma(i) < \sigma(i+1)$ and that σ has an* inversion *at i if $\sigma(i) > \sigma(i+1)$.*
We denote by $c(\sigma)$ (resp. $i(\sigma)$) the number of consecutions (resp. inversions) of σ.

(b) *The consecution-inversion structure sequence of σ, denoted by* $\mathrm{CISS}(\sigma)$*, is the tuple $(c_1, i_1, c_2, i_2, \ldots, c_\ell, i_\ell)$, where σ has c_1 consecutive consecutions at $0, 1, \ldots, c_1 - 1$; i_1 consecutive inversions at $c_1, c_1+1, \ldots, c_1+i_1-1$ and so on, up to i_ℓ inversions at $k-1-i_\ell, \ldots, k-2$.*

REMARK 1. We will often use the following simple observations on the concepts introduced in Definition 3.2 without being explicitly referred to.

1. Part (a) is related to the matrix M_σ in (3.4) as follows: σ has a consecution at i if and only if M_i is to the left of M_{i+1} in M_σ, and σ has an inversion at i if and only if M_i is to the right of M_{i+1} in M_σ.
2. Note that c_1 and i_ℓ in $\mathrm{CISS}(\sigma)$ may be zero (in the first case, σ has an inversion at 0 and in the second one it has a consecution at $k-2$) but $i_1, c_2, i_2, \ldots, i_{\ell-1}, c_\ell$ are strictly positive. These conditions uniquely determine $\mathrm{CISS}(\sigma)$ and, in particular, the parameter ℓ.
Note also that $c(\sigma) = \sum_{j=1}^{\ell} c_j$, $i(\sigma) = \sum_{j=1}^{\ell} i_j$, and $c(\sigma) + i(\sigma) = k-1$.

In Section 5 we will use the concept of *reversal bijection*: the reversal bijection of a given bijection $\sigma : \{0,1,\dots,k-1\} \to \{1,2,\dots,k\}$ is another bijection from $\{0,1,\dots,k-1\}$ into $\{1,2,\dots,k\}$ that is denoted by $\operatorname{rev}\sigma$ and is defined as $\operatorname{rev}\sigma(i) = k+1-\sigma(i)$, for $0 \le i \le k-1$. Note that $\operatorname{rev}\sigma$ reverses the order of the building factors M_j in the zero degree term M_σ of the Fiedler pencil (3.3) of $P(\lambda)$. More precisely, the pencil of $P(\lambda)$

$$F_{\operatorname{rev}\sigma}(\lambda) = \lambda M_k - M_{\operatorname{rev}\sigma}$$

satisfies

$$M_{\operatorname{rev}\sigma} = M_{\sigma^{-1}(k)} M_{\sigma^{-1}(k-1)} \cdots M_{\sigma^{-1}(1)}\,.$$

In Section 5 we will also use the block-transpose operation. More information on this operation can be found in [29, Chapter 3]. Here, we simply recall the definition.

Definition 3.3. *Let $A = (A_{ij})$ be a block $r \times s$ matrix with $m \times n$ blocks A_{ij}. The block transpose of A is the block $s \times r$ matrix $A^{\mathcal{B}}$ with $m \times n$ blocks defined by $\left(A^{\mathcal{B}}\right)_{ij} = A_{ji}$.*

4. Fiedler pencils are strong linearizations. We prove in this section that every Fiedler pencil $F_\sigma(\lambda)$ of a square matrix polynomial $P(\lambda)$ (regular or singular) is a strong linearization for $P(\lambda)$. This fact was proved *only for regular polynomials* in [4]. A general proof including the singular case requires different techniques, in particular, the use of the *Horner shifts* of $P(\lambda)$ introduced in Definition 4.1. The *Horner shifts* are fundamental in the rest of the paper.

Definition 4.1. *Let $P(\lambda) = A_0 + \lambda A_1 + \cdots + \lambda^k A_k$ be a matrix polynomial of degree k. For $d = 0,\dots,k$, the* degree d Horner shift of $P(\lambda)$ *is the matrix polynomial $P_d(\lambda) := A_{k-d} + \lambda A_{k-d+1} + \cdots + \lambda^d A_k$.*

Observe that the Horner shifts of $P(\lambda)$ satisfy the following recurrence relation

$$P_0(\lambda) = A_k, \;\; P_{d+1}(\lambda) = \lambda P_d(\lambda) + A_{k-d-1}, \text{ for } 0 \le d \le k-1, \;\text{ and }\; P_k(\lambda) = P(\lambda).$$

4.1. Auxiliary matrices and equivalences. In Definition 4.2 some unimodular matrices are defined based on Horner shifts. They will be used to construct the matrices $U(\lambda)$ and $V(\lambda)$ in the linearization transformation (2.3).

Definition 4.2. *Let $P(\lambda) = \sum_{i=0}^{k} \lambda^i A_i$ be an $n \times n$ matrix polynomial and $P_i(\lambda)$ the degree i Horner shift of $P(\lambda)$. We define the following unimodular matrix polynomials of dimension $nk \times nk$. For $i = 1,\dots,k-1$,*

$$Q_i := \begin{bmatrix} I_{(i-1)n} & & & \\ & I_n & 0 & \\ & \lambda I_n & I_n & \\ & & & I_{(k-i-1)n} \end{bmatrix}, \quad R_i := \begin{bmatrix} I_{(i-1)n} & & & \\ & 0 & I_n & \\ & -I_n & P_i(\lambda) & \\ & & & I_{(k-i-1)n} \end{bmatrix},$$

$$S_i := \begin{bmatrix} I_{(i-1)n} & & & \\ & 0 & -I_n & \\ & I_n & P_i(\lambda) & \\ & & & I_{(k-i-1)n} \end{bmatrix}, \quad T_i := \begin{bmatrix} I_{(i-1)n} & & & \\ & I_n & \lambda I_n & \\ & 0 & I_n & \\ & & & I_{(k-i-1)n} \end{bmatrix}.$$

For brevity, we have omitted the dependence on λ in the matrices Q_i, R_i, S_i and T_i. Associated with the matrix polynomial (1.1), we will also use the matrices

$$\widetilde{M}_0 := \begin{bmatrix} -I_{(k-1)n} & \\ & -A_0 \end{bmatrix} \quad \text{and} \quad \widetilde{M}_i := \begin{bmatrix} -I_{(k-i-1)n} & & & \\ & -A_i & I_n & \\ & I_n & 0 & \\ & & & I_{(i-1)n} \end{bmatrix}, \tag{4.1}$$

for $i = 1, \ldots, k-1$. Note that the only difference between these matrices $\widetilde{M}_i's$ and the matrices $M_i's$ defined in (3.1) and (3.2) are the minus signs in the upper-left identity blocks. In particular, $\widetilde{M}_{k-1} = M_{k-1}$.

The matrices Q_i, R_i, S_i and T_i induce the unimodular equivalences established in Lemma 4.3. The proof is immediate and is omitted.

LEMMA 4.3. *Let Q_i, R_i, S_i and T_i, for $i = 1, \ldots, k-1$, be the matrices introduced in* Definition 4.2. *Then*

(a) $$Q_i \begin{bmatrix} 0_{(i-1)n} & & & \\ & \lambda P_{i-1}(\lambda) & 0 & \\ & 0 & \lambda I_n & \\ & & & \lambda I_{(k-i-1)n} \end{bmatrix} R_i =$$

$$= \begin{bmatrix} 0_{(i-1)n} & & & \\ & 0_n & & \\ & & \lambda P_i(\lambda) & \\ & & & \lambda I_{(k-i-1)n} \end{bmatrix} + \begin{bmatrix} 0_{(i-1)n} & & & \\ & 0 & \lambda P_{i-1}(\lambda) & \\ & -\lambda I_n & \lambda^2 P_{i-1}(\lambda) & \\ & & & 0_{(k-i-1)n} \end{bmatrix}.$$

(b) $$Q_i M_{k-(i+1)} \widetilde{M}_{k-i} R_i = \widetilde{M}_{k-(i+1)} + \begin{bmatrix} 0_{(i-1)n} & & & \\ & 0 & \lambda P_{i-1}(\lambda) & \\ & -\lambda I_n & \lambda^2 P_{i-1}(\lambda) & \\ & & & 0_{(k-i-1)n} \end{bmatrix}.$$

(c) $$S_i \begin{bmatrix} 0_{(i-1)n} & & & \\ & \lambda P_{i-1}(\lambda) & 0 & \\ & 0 & \lambda I_n & \\ & & & \lambda I_{(k-i-1)n} \end{bmatrix} T_i =$$

$$= \begin{bmatrix} 0_{(i-1)n} & & & \\ & 0_n & & \\ & & \lambda P_i(\lambda) & \\ & & & \lambda I_{(k-i-1)n} \end{bmatrix} + \begin{bmatrix} 0_{(i-1)n} & & & \\ & 0 & -\lambda I_n & \\ & \lambda P_{i-1}(\lambda) & \lambda^2 P_{i-1}(\lambda) & \\ & & & 0_{(k-i-1)n} \end{bmatrix}.$$

(d) $$S_i \widetilde{M}_{k-i} M_{k-(i+1)} T_i = \widetilde{M}_{k-(i+1)} + \begin{bmatrix} 0_{(i-1)n} & & & \\ & 0 & -\lambda I_n & \\ & \lambda P_{i-1}(\lambda) & \lambda^2 P_{i-1}(\lambda) & \\ & & & 0_{(k-i-1)n} \end{bmatrix}.$$

Lemma 4.4 is the key technical result that allows us to prove the main results in this section, i.e., Theorem 4.5 and Corollary 4.6.

LEMMA 4.4. *Let $P(\lambda)$ be the matrix polynomial in* (1.1)*, let $F_\sigma(\lambda) = \lambda M_k - M_\sigma$ be given by* (3.3)*, i.e., it is the Fiedler pencil of $P(\lambda)$ associated with the bijection σ, and let $\widetilde{M}_0, \ldots, \widetilde{M}_{k-1}$ be the matrices in* (4.1)*. For $i = 1, \ldots, k$, define $M_\sigma^{(k)} := \widetilde{M}_0$*

and if $i \neq k$

$$M_\sigma^{(i)} := \begin{cases} \left(\prod_{\sigma^{-1}(j)<k-i} M_{\sigma^{-1}(j)}\right) \cdot \widetilde{M}_{k-i}\,, & \text{if } \sigma \text{ has a consecution at } k-i-1 \\ \widetilde{M}_{k-i} \cdot \left(\prod_{\sigma^{-1}(j)<k-i} M_{\sigma^{-1}(j)}\right)\,, & \text{if } \sigma \text{ has an inversion at } k-i-1 \end{cases}\,,$$

where the relative order of the M_j*'s factors is the same as in* M_σ*, and*

$$F_\sigma^{(i)}(\lambda) := \begin{bmatrix} 0_{(i-1)n} & & \\ & \lambda P_{i-1}(\lambda) & \\ & & \lambda I_{(k-i)n} \end{bmatrix} - M_\sigma^{(i)}\,.$$

Observe that $F_\sigma^{(1)}(\lambda) = F_\sigma(\lambda)$ *and* $F_\sigma^{(k)}(\lambda) = \operatorname{diag}(I_{(k-1)n}, P(\lambda))$*. Then:*

(a) *If* σ *has a consecution at* $k-i-1$ *then* $F_\sigma^{(i+1)}(\lambda) = Q_i F_\sigma^{(i)}(\lambda) R_i$*, where* Q_i *and* R_i *are the matrices introduced in Definition* 4.2*.*

(b) *If* σ *has an inversion at* $k-i-1$ *then* $F_\sigma^{(i+1)}(\lambda) = S_i F_\sigma^{(i)}(\lambda) T_i$*, where* S_i *and* T_i *are the matrices introduced in Definition* 4.2*.*

Proof. We will only prove part (a) because part (b) is similar. So, let us assume that σ has a consecution at $k-i-1$. Then, by using trivial commutativity relations of the type (3.5), the factors of $M_\sigma^{(i)}$ can be rearranged until $\widetilde{M}_{k-i}$ is adjacent to M_{k-i-1}, that is $M_\sigma^{(i)} = \cdots M_{k-i-1}\widetilde{M}_{k-i}\cdots$. Now, since Q_i and R_i commute with M_s, for $s < k-i-1$, we have

$$\begin{aligned} Q_i M_\sigma^{(i)} R_i &= \cdots (Q_i M_{k-i-1} \widetilde{M}_{k-i} R_i) \cdots \\ &= \cdots \left(\widetilde{M}_{k-i-1} + \begin{bmatrix} 0_{(i-1)n} & & & \\ & 0 & \lambda P_{i-1}(\lambda) & \\ & -\lambda I_n & \lambda^2 P_{i-1}(\lambda) & \\ & & & 0_{(k-i-1)n} \end{bmatrix} \right) \cdots \\ &= M_\sigma^{(i+1)} + \begin{bmatrix} 0_{(i-1)n} & & & \\ & 0 & \lambda P_{i-1}(\lambda) & \\ & -\lambda I_n & \lambda^2 P_{i-1}(\lambda) & \\ & & & 0_{(k-i-1)n} \end{bmatrix}, \end{aligned} \tag{4.2}$$

where we have used Lemma 4.3 (b) and the fact that a multiplication of the matrix in the second term of the second line by M_s, with $s \leq k-i-2$, keeps this matrix unchanged. Observe that if $i = k-1$ then $M_\sigma^{(i+1)} = \widetilde{M}_0$. The result is obtained from equation (4.2) and Lemma 4.3 (a) as follows:

$$\begin{aligned} Q_i F_\sigma^{(i)}(\lambda) R_i = &\begin{bmatrix} 0_{(i-1)n} & & & \\ & 0_n & & \\ & & \lambda P_i & \\ & & & \lambda I_{(k-i-1)n} \end{bmatrix} + \begin{bmatrix} 0_{(i-1)n} & & & \\ & 0 & \lambda P_{i-1} & \\ & -\lambda I_n & \lambda^2 P_{i-1} & \\ & & & 0_{(k-i-1)n} \end{bmatrix} \\ &- M_\sigma^{(i+1)} - \begin{bmatrix} 0_{(i-1)n} & & & \\ & 0 & \lambda P_{i-1} & \\ & -\lambda I_n & \lambda^2 P_{i-1} & \\ & & & 0_{(k-i-1)n} \end{bmatrix} = F_\sigma^{(i+1)}(\lambda)\,, \end{aligned}$$

where, for brevity, we have dropped the dependence on λ in the Horner shifts. □

4.2. Strong linearizations and transforming unimodular matrices. We finally prove in Theorem 4.5 that every Fiedler pencil of a square matrix polynomial (regular or singular) is a strong linearization for this polynomial. In the proof, the unimodular matrices transforming a Fiedler pencil into $\mathrm{diag}(I_{(k-1)n}, P(\lambda))$ are constructed as products of the matrices introduced in Definition 4.2. These constructions are presented in Corollary 4.6 and will be used to deduce the minimal bases and eigenvector recovery formulas.

THEOREM 4.5. *Let $P(\lambda)$ be an $n\times n$ matrix polynomial (singular or regular) with degree $k \geq 2$ and let $F_\sigma(\lambda)$ be any Fiedler companion pencil of $P(\lambda)$. Then $F_\sigma(\lambda)$ is a strong linearization for $P(\lambda)$.*

Proof. We will first show that $F_\sigma(\lambda)$ is a linearization of $P(\lambda)$ and after that we will prove that it is also a strong linearization.

Assume that $P(\lambda)$ is the polynomial in (1.1) and that $F_\sigma(\lambda)$ is given by (3.3). To prove that $F_\sigma(\lambda)$ is a linearization of $P(\lambda)$ we use Lemma 4.4 to construct a sequence of $k-1$ unimodular equivalences

$$F_\sigma(\lambda) = F_\sigma^{(1)}(\lambda) \longrightarrow F_\sigma^{(2)}(\lambda) \longrightarrow \cdots \longrightarrow F_\sigma^{(k)}(\lambda) = \mathrm{diag}(I_{(k-1)n}, P(\lambda)), \qquad (4.3)$$

where $F_\sigma^{(i+1)}(\lambda) = Q_i F_\sigma^{(i)}(\lambda) R_i$ if σ has a consecution at $k-i-1$ and $F_\sigma^{(i+1)}(\lambda) = S_i F_\sigma^{(i)}(\lambda) T_i$ if σ has an inversion at $k-i-1$. The product of several unimodular matrices is again unimodular, which completes the proof that $F_\sigma(\lambda)$ is a linearization for $P(\lambda)$.

To prove that $F_\sigma(\lambda)$ is a strong linearization of $P(\lambda)$, it remains to prove that $\mathrm{rev}\, F_\sigma(\lambda)$ is a linearization for $\mathrm{rev}\, P(\lambda)$. For this purpose we will use the following strategy: we will prove that $-\mathrm{rev}\, F_\sigma(\lambda)$ is strictly equivalent to one of the Fiedler pencils of $-\mathrm{rev}\, P(\lambda)$, i.e., equivalent in the sense of Definition 2.2 but with nonsingular constant matrices U and V. We have already proved that each of these pencils is a linearization of $-\mathrm{rev}\, P(\lambda)$ and, so, unimodularly equivalent to $\mathrm{diag}(I_{(k-1)n}, -\mathrm{rev}\, P(\lambda))$. Hence $\mathrm{rev}\, F_\sigma(\lambda)$ is unimodularly equivalent to $\mathrm{diag}(I_{(k-1)n}, \mathrm{rev}\, P(\lambda))$, and thus is a linearization for $\mathrm{rev}\, P(\lambda)$.

Let us prove that $-\mathrm{rev}\, F_\sigma(\lambda)$ is strictly equivalent to one of the Fiedler pencils of $-\mathrm{rev}\, P(\lambda)$. If $F_\sigma(\lambda)$ is given by (3.3) then

$$-\mathrm{rev}\, F_\sigma(\lambda) = \lambda(M_{\sigma^{-1}(1)} \cdots M_{\sigma^{-1}(s-1)} M_0 M_{\sigma^{-1}(s+1)} \cdots M_{\sigma^{-1}(k)}) - M_k,$$

where $s = \sigma(0)$. Pre and post-multiplying in the appropriate order by the inverses of $M_1, M_2, \ldots, M_{k-1}$, we get that $-\mathrm{rev}\, F_\sigma(\lambda)$ is strictly equivalent to

$$\lambda M_0 - (M_{\sigma^{-1}(s-1)}^{-1} \cdots M_{\sigma^{-1}(1)}^{-1} M_k M_{\sigma^{-1}(k)}^{-1} \cdots M_{\sigma^{-1}(s+1)}^{-1}),$$

which in turn is strictly equivalent to

$$\lambda(RM_0R) - R(M_{\sigma^{-1}(s-1)}^{-1} \cdots M_{\sigma^{-1}(1)}^{-1} M_k M_{\sigma^{-1}(k)}^{-1} \cdots M_{\sigma^{-1}(s+1)}^{-1})R, \qquad (4.4)$$

where

$$R := \begin{bmatrix} & & I_n \\ & \iddots & \\ I_n & & \end{bmatrix}$$

is the $k \times k$ block backwards "identity" matrix. Now define

$$\widehat{M}_0 := RM_0R = \begin{bmatrix} -A_0 & \\ & I_{(k-1)n} \end{bmatrix}, \qquad \widehat{M}_k := RM_kR = \begin{bmatrix} I_{(k-1)n} & \\ & A_k \end{bmatrix},$$

and

$$\widehat{M}_i := RM_i^{-1}R = \begin{bmatrix} I_{(i-1)n} & & & \\ & A_i & I_n & \\ & I_n & 0 & \\ & & & I_{(k-i-1)n} \end{bmatrix}, \quad i = 1, \ldots, k-1 .$$

With these definitions, the pencil (4.4), that is strictly equivalent to $-\mathrm{rev}\, F_\sigma(\lambda)$, can be written as

$$\lambda \widehat{M}_0 - (\widehat{M}_{\sigma^{-1}(s-1)} \cdots \widehat{M}_{\sigma^{-1}(1)} \widehat{M}_k \widehat{M}_{\sigma^{-1}(k)} \cdots \widehat{M}_{\sigma^{-1}(s+1)}) ,$$

which is a Fiedler pencil for the polynomial $-\mathrm{rev}\, P(\lambda)$. This completes the proof. □

Corollary 4.6 is an immediate consequence of the first part of the proof of Theorem 4.5.

Corollary 4.6. *Let $P(\lambda)$ be the matrix polynomial in* (1.1)*, let $F_\sigma(\lambda)$ be the Fiedler pencil of $P(\lambda)$ associated with the bijection σ, and let Q_i, R_i, S_i and T_i, for $i = 1, \ldots, k-1$, be the matrices introduced in* Definition 4.2*. Then*

$$U(\lambda) F_\sigma(\lambda) V(\lambda) = \begin{bmatrix} I_{(k-1)n} & 0 \\ 0 & P(\lambda) \end{bmatrix},$$

where $V(\lambda)$ and $U(\lambda)$ are the following $nk \times nk$ unimodular matrix polynomials:

$$V(\lambda) := V_{k-2} V_{k-3} \cdots V_1 V_0 , \quad \textit{with} \quad V_i = \begin{cases} R_{k-(i+1)} , & \textit{if } \sigma \textit{ has a consecution at } i \\ T_{k-(i+1)} , & \textit{if } \sigma \textit{ has an inversion at } i \end{cases}$$

and

$$U(\lambda) := U_0 U_1 \cdots U_{k-3} U_{k-2} , \quad \textit{with} \quad U_i = \begin{cases} Q_{k-(i+1)} , & \textit{if } \sigma \textit{ has a consecution at } i \\ S_{k-(i+1)} , & \textit{if } \sigma \textit{ has an inversion at } i \end{cases}$$

5. The recovery of minimal indices and bases. In this section, we deal with singular square matrix polynomials because regular polynomials do not have minimal indices and bases. The recovery of the minimal indices and bases of a polynomial from those of a Fiedler pencil is based on Lemma 5.1, which is *valid for any linearization* and not only for Fiedler pencils.

Lemma 5.1. *Let the $nk \times nk$ pencil $L(\lambda)$ be a linearization of an $n \times n$ matrix polynomial $P(\lambda)$ of degree $k \geq 2$, and let $U(\lambda)$ and $V(\lambda)$ be two unimodular matrix polynomials such that*

$$U(\lambda) L(\lambda) V(\lambda) = \begin{bmatrix} I_{(k-1)n} & 0 \\ 0 & P(\lambda) \end{bmatrix}. \tag{5.1}$$

Consider $U(\lambda)$ and $V(\lambda)$ as block $k \times k$ matrices with $n \times n$ blocks, and let U^L and V^R be, respectively, the last block-row of $U(\lambda)$ and the last block-column of $V(\lambda)$. Then:

(a) *The linear map*

$$\begin{array}{rccc} \mathcal{R} : & \mathcal{N}_r(P) & \longrightarrow & \mathcal{N}_r(L) \\ & v & \longmapsto & V^R v \end{array}$$

is an isomorphism of $\mathbb{F}(\lambda)$-vector spaces.

(b) *The linear map*

$$\begin{array}{rccc} \mathcal{L}: & \mathcal{N}_\ell(P) & \longrightarrow & \mathcal{N}_\ell(L) \\ & w^T & \longmapsto & w^T U^L \end{array}$$

is an isomorphism of $\mathbb{F}(\lambda)$*-vector spaces.*

Proof. We only prove part (a) because part (b) is similar. Assume that $v \in \mathcal{N}_r(P)$, i.e., $P(\lambda)v = 0$. By multiplying on the right both sides of (5.1) by $[0^T v^T]^T$, where the 0^T vector has $(k-1)n$ entries, we get $U(\lambda)L(\lambda)(V^R v) = 0$, that implies $L(\lambda)(V^R v) = 0$ because $U(\lambda)$ is nonsingular as an $\mathbb{F}(\lambda)$-matrix. Therefore we have proved that $v \in \mathcal{N}_r(P) \Rightarrow \mathcal{R}(v) = V^R v \in \mathcal{N}_r(L)$ and that the linear map in part (a) is well defined. To prove that $\mathcal{R}$ is an isomorphism it suffices to prove that $\mathcal{R}$ is injective because $\dim \mathcal{N}_r(P) = \dim \mathcal{N}_r(L)$ from (5.1). To this purpose simply note that $\mathcal{R}(v) = V^R v = 0$ implies $v = 0$, because the columns of V^R are linearly independent in $\mathbb{F}(\lambda)^{nk \times 1}$ since $V(\lambda)$ is unimodular. □

Lemma 5.1 implies that every basis of $\mathcal{N}_r(L)$ is of the form $\mathcal{B}_r = \{V^R v_1, \ldots, V^R v_p\}$, where $\mathcal{E}_r = \{v_1, \ldots, v_p\}$ is a basis of $\mathcal{N}_r(P)$ that is uniquely determined by $\mathcal{B}_r$. However, this does not mean that $\mathcal{E}_r$ may be easily obtained from $\mathcal{B}_r$, that $\mathcal{E}_r$ is a minimal basis of $P(\lambda)$ whenever $\mathcal{B}_r$ is a minimal basis of $L(\lambda)$, or that the minimal indices of $P(\lambda)$ are simply related to those of $L(\lambda)$. In the particular case of Fiedler linearizations, we will see that $\mathcal{E}_r$ is immediately recovered from $\mathcal{B}_r$ because one of the blocks of V^R will always be equal to I_n, we will prove that $\mathcal{E}_r$ is a minimal basis whenever $\mathcal{B}_r$ is, and we will show that the minimal indices of $P(\lambda)$ are obtained from the ones of $F_\sigma(\lambda)$ by substraction of a constant quantity easily determined by σ. This will require a careful analysis of the last block column of the matrix $V(\lambda)$ we have determined in Corollary 4.6. Analogous results hold for left minimal indices and bases.

5.1. The last block column of $V(\lambda)$ and other technical results. In this section I_n and 0_n blocks of size precisely $n \times n$ are simply denoted by I and 0. The expression of $V(\lambda)$ in Corollary 4.6 indicates that products of T_i's and R_i's matrices have to be used to get the last block column of $V(\lambda)$. Lemma 5.2 is a first step in this direction. The proof is omitted because it is a simple induction on the number of factors.

LEMMA 5.2. *Let $P_d(\lambda)$, for $d = 0, \ldots, k$, be the Horner shifts of the matrix polynomial $P(\lambda)$ in* (1.1) *and let R_i and T_i, for $i = 1, \ldots, k-1$, be the matrices introduced in* Definition 4.2. *Then, for each $i = 1, \ldots, k-1$ and $j = 1, \ldots, k-i-1$:*

$$\text{(a)}\quad T_i T_{i+1} \cdots T_{i+j} = \left[\begin{array}{c|ccccc|c} I_{(i-1)n} & & & & & & \\ \hline & I & \lambda I & \ldots & & \lambda^{j+1} I & \\ & & \ddots & \ddots & & \vdots & \\ & & & & \ddots & \lambda I & \\ & & & & & I & \\ \hline & & & & & & I_{(k-(i+j+1))n} \end{array}\right].$$

$$\text{(b)}\quad R_i R_{i+1} \cdots R_{i+j} = \left[\begin{array}{c|ccccc|c} I_{(i-1)n} & & & & & & \\ \hline & 0 & 0 & \ldots & 0 & I & \\ & -I & 0 & \ldots & 0 & P_i(\lambda) & \\ & & \ddots & \ddots & \vdots & \vdots & \\ & & & \ddots & 0 & P_{i+j-1}(\lambda) & \\ & & & & -I & P_{i+j}(\lambda) & \\ \hline & & & & & & I_{(k-(i+j+1))n} \end{array}\right].$$

Given a bijection $\sigma : \{0, 1, \dots, k-1\} \to \{1, \dots, k\}$, we will use frequently in this section the consecution-inversion structure sequence introduced in Definition 3.2, in particular, the entries of $\mathrm{CISS}(\sigma) = (c_1, i_1, \dots, c_\ell, i_\ell)$ and the summations

$$s_0 := 0, \quad s_j := \sum_{p=1}^{j} (c_p + i_p) \quad \text{for } j = 1, \dots, \ell.$$

Recall also that $s_\ell = k - 1$. We still need to introduce the following additional quantities:

$$m_0 := 0, \quad m_j := i_1 + i_2 + \cdots + i_j\,, \ \text{ for } j = 1, \dots, \ell\,, \tag{5.2}$$

so $m_\ell = i(\sigma)$, that is, the number of inversions of σ.

In order to write down in Lemma 5.3 a reasonably simple formula for the last block column of the matrix $V(\lambda)$ in Corollary 4.6, we need to define some block column matrices associated with the matrix polynomial $P(\lambda)$ in (1.1) and the bijection σ. These matrices are denoted as $\Lambda_{\sigma,j}(P)$, for $j = 1, \dots, \ell$, and $\widehat{\Lambda}_{\sigma,j}(P)$, for $j = 1, \dots, \ell - 1$, and they are defined in terms of the Horner shifts of $P(\lambda)$ and $\mathrm{CISS}(\sigma) = (c_1, i_1, \dots, c_\ell, i_\ell)$ as follows

$$\Lambda_{\sigma,j}(P) := \begin{bmatrix} \lambda^{i_j} I \\ \vdots \\ \lambda I \\ I \\ P_{k-s_{j-1}-c_j} \\ \vdots \\ P_{k-s_{j-1}-2} \\ P_{k-s_{j-1}-1} \end{bmatrix} \quad \text{and} \quad \widehat{\Lambda}_{\sigma,j}(P) := \begin{bmatrix} \lambda^{i_j - 1} I \\ \vdots \\ \lambda I \\ I \\ P_{k-s_{j-1}-c_j} \\ \vdots \\ P_{k-s_{j-1}-2} \\ P_{k-s_{j-1}-1} \end{bmatrix}, \tag{5.3}$$

where for simplicity we omit the dependence on λ of the Horner shifts P_d. Note that $\Lambda_{\sigma,j}(P)$ and $\widehat{\Lambda}_{\sigma,j}(P)$ are associated with the pair (c_j, i_j) of entries of $\mathrm{CISS}(\sigma)$, and that $\widehat{\Lambda}_{\sigma,j}(P)$ is just a "truncated" version of $\Lambda_{\sigma,j}(P)$, with one less block at the top. Note also that $\widehat{\Lambda}_{\sigma,j}(P)$ is defined only if $j < \ell$, so that $i_j > 0$ and $i_j - 1 \geq 0$, and that if $c_1 = 0$ then $\Lambda_{\sigma,1}(P) = [\lambda^{i_1} I, \dots, \lambda I, I]^T$.

LEMMA 5.3. *Let $P(\lambda)$ be the matrix polynomial in* (1.1)*, let $F_\sigma(\lambda)$ be the Fiedler pencil of $P(\lambda)$ associated with the bijection σ, and let $V(\lambda)$ be the $nk \times nk$ unimodular matrix polynomial in Corollary* 4.6*. Consider $V(\lambda)$ as a $k \times k$ block matrix with $n \times n$ blocks. If* $\mathrm{CISS}(\sigma) = (c_1, i_1, \dots, c_\ell, i_\ell)$ *then the last block-column of $V(\lambda)$ is*

$$\Lambda_\sigma^R(P) := \begin{bmatrix} \lambda^{m_{\ell-1}} \Lambda_{\sigma,\ell}(P) \\ \lambda^{m_{\ell-2}} \widehat{\Lambda}_{\sigma,\ell-1}(P) \\ \vdots \\ \lambda^{m_1} \widehat{\Lambda}_{\sigma,2}(P) \\ \widehat{\Lambda}_{\sigma,1}(P) \end{bmatrix} \quad \text{if } \ell > 1, \tag{5.4}$$

and $\Lambda_\sigma^R(P) := \Lambda_{\sigma,1}(P)$ if $\ell = 1$.

Proof. Note, in the first place, that, according to $\mathrm{CISS}(\sigma) = (c_1, i_1, \ldots, c_\ell, i_\ell)$, the factors defining $V(\lambda)$ in Corollary 4.6 can be grouped in the form

$$V(\lambda) = \widetilde{V}_\ell \cdots \widetilde{V}_2 \widetilde{V}_1 \,, \quad \text{with} \quad \widetilde{V}_j = \widetilde{V}_j^{inv} \widetilde{V}_j^{con} \,, \quad \text{for } j = 1, \ldots, \ell,$$

where $\widetilde{V}_j^{inv}$ is the product of the following i_j matrices

$$\widetilde{V}_j^{inv} = T_{k-s_j} T_{k-s_j+1} \cdots T_{k-s_j+(i_j-1)}$$

defined in Definition 4.2, and $\widetilde{V}_j^{con}$ is the product of the following c_j matrices

$$\widetilde{V}_j^{con} = R_{k-s_{j-1}-c_j} R_{k-s_{j-1}-c_j+1} \cdots R_{k-s_{j-1}-1} \,,$$

also defined in Definition 4.2. By Lemma 5.2, these matrices are given by

$$\widetilde{V}_j^{inv} = \left[\begin{array}{c|cccc|c} I_{(k-s_j-1)n} & & & & & \\ \hline & I & \lambda I & \ldots & \lambda^{i_j} I & \\ & & \ddots & \ddots & \vdots & \\ & & & \ddots & \lambda I & \\ & & & & I & \\ \hline & & & & & I_{(s_{j-1}+c_j)n} \end{array}\right]$$

and

$$\widetilde{V}_j^{con} = \left[\begin{array}{c|ccccc|c} I_{(k-s_{j-1}-c_j-1)n} & & & & & & \\ \hline & 0 & \ldots & \ldots & 0 & I & \\ & -I & \ddots & & 0 & P_{k-s_{j-1}-c_j}(\lambda) & \\ & & \ddots & \ddots & \vdots & \vdots & \\ & & & \ddots & 0 & P_{k-s_{j-1}-2}(\lambda) & \\ & & & & -I & P_{k-s_{j-1}-1}(\lambda) & \\ \hline & & & & & & I_{s_{j-1}\,n} \end{array}\right].$$

A direct multiplication gives

$$\widetilde{V}_j = \widetilde{V}_j^{inv} \widetilde{V}_j^{con} = \left[\begin{array}{c|c|c} I_{(k-s_j-1)n} & & \\ \hline & * \ * \ \cdots \ * \ \Lambda_{\sigma,j}(P) & \\ \hline & & I_{s_{j-1}\,n} \end{array}\right], \tag{5.5}$$

where the central diagonal block is a $(c_j + i_j + 1) \times (c_j + i_j + 1)$ block matrix with $n \times n$ blocks whose first $c_j + i_j$ block-columns are denoted by $*$ and are of no interest in our argument. Now, note that the central diagonal nonidentity block of $\widetilde{V}_j$ in (5.5), i.e., $[* \ * \ \cdots \ * \ \Lambda_{\sigma,j}(P)]$, overlaps the central diagonal nonidentity block of the adjacent factor $\widetilde{V}_{j+1}$ (resp. $\widetilde{V}_{j-1}$) in the upper-left (resp. lower-right) $n \times n$ block of the central diagonal nonidentity block of $\widetilde{V}_j$. In particular, the last block row and the last block column of the central diagonal nonidentity block of $\widetilde{V}_j$ have the same block index as the first block row and the first block column of the central diagonal nonidentity block of $\widetilde{V}_{j-1}$. This overlap causes some nontrivial interaction when multiplying all the $\widetilde{V}_j$

factors together to get the last block column of $V(\lambda)$. If we multiply out this product from right to left, i.e.,

$$V(\lambda) = (\widetilde{V}_\ell(\widetilde{V}_{\ell-1}\cdots(\widetilde{V}_3(\widetilde{V}_2\widetilde{V}_1))\cdots)),$$

use (5.5) and take into account the discussed overlap, it is easy to see (inductively) that, for $j = 1, \ldots, \ell$,

$$\widetilde{V}_j\widetilde{V}_{j-1}\cdots\widetilde{V}_1 = \left[\begin{array}{c|cccc} I_{(k-s_j-1)n} & & & & \\ \hline & * & \ldots & * & \lambda^{m_{j-1}}\Lambda_{\sigma,j}(P) \\ & * & \ldots & * & \lambda^{m_{j-2}}\widehat{\Lambda}_{\sigma,j-1}(P) \\ & \vdots & & \vdots & \vdots \\ & * & \ldots & * & \lambda^{m_1}\widehat{\Lambda}_{\sigma,2}(P) \\ & * & \ldots & * & \widehat{\Lambda}_{\sigma,1}(P) \end{array}\right],$$

where the blocks with $*$ have no relevance in our argument. Now the result follows by taking $j = \ell$ in the previous identity. □

We want to point out two key features of the matrix $\Lambda_\sigma^R(P)$ in (5.4) that will be used in the rest of the paper and that are essential to recover right minimal bases of $P(\lambda)$ from the ones of $F_\sigma(\lambda)$:

(a) $\Lambda_\sigma^R(P)$ has exactly one block equal to I_n residing in $\widehat{\Lambda}_{\sigma,1}(P)$ at block index $k - c_1$, i.e., I_n is the $(c_1 + 1)$th block of $\Lambda_\sigma^R(P)$ from the bottom.

(b) The topmost block of $\Lambda_\sigma^R(P)$ is the topmost block of $\lambda^{m_{\ell-1}}\Lambda_{\sigma,\ell}(P)$, which is equal to $\lambda^{m_{\ell-1}}\lambda^{i_\ell}I_n = \lambda^{m_\ell}I_n = \lambda^{i(\sigma)}I_n$.

The last result in this subsection is Lemma 5.4. It has nothing to do with linearizations and is valid for any singular matrix polynomial. This lemma will be used to establish the shift relationship between the right minimal indices of $P(\lambda)$ and $F_\sigma(\lambda)$.

LEMMA 5.4. *Let $P(\lambda)$ be an $n \times n$ matrix polynomial with degree $k \geq 2$ and let $P_j(\lambda)$, for $j = 0, \ldots, k-1$, be its degree j Horner shift. Let $v(\lambda) \in \mathbb{F}(\lambda)^n$ be a vector polynomial such that $v(\lambda) \in \mathcal{N}_r(P)$. If $P_j(\lambda)v(\lambda) \neq 0$, then*

$$\deg\left(P_j(\lambda)v(\lambda)\right) \leq -1 + \deg\left(v(\lambda)\right). \tag{5.6}$$

Proof. Assume that $P(\lambda)$ is the matrix polynomial in (1.1). First observe that $P_j(\lambda)v(\lambda) \neq 0$ implies that $\deg(v(\lambda)) \geq 1$, because otherwise, since $v(\lambda) \in \mathcal{N}_r(P)$, we should have $v \in \mathcal{N}_r(A_i)$, for all $0 \leq i \leq k$, and then $P_j(\lambda)v = 0$ for all $0 \leq j \leq k$. Hence the right-hand side of (5.6) is non-negative.

By definition of the Horner shift $P_j(\lambda)$ we have

$$P_j(\lambda)v(\lambda) = \left(\lambda^j A_k + \lambda^{j-1}A_{k-1} + \cdots + A_{k-j}\right)v(\lambda),$$

and so

$$\lambda^{k-j}P_j(\lambda)v(\lambda) = \left(\lambda^{k-j}P_j(\lambda) - P(\lambda)\right)v(\lambda) = -\left(\lambda^{k-j-1}A_{k-j-1} + \cdots + \lambda A_1 + A_0\right)v(\lambda).$$

Thus, taking into account that $P_j(\lambda)v(\lambda) \neq 0$,

$$\begin{aligned}(k-j) + \deg\left(P_j(\lambda)v(\lambda)\right) &= \deg\left(\lambda^{k-j}P_j(\lambda)v(\lambda)\right) \\ &= \deg\left((\lambda^{k-j-1}A_{k-j-1} + \cdots + \lambda A_1 + A_0)v(\lambda)\right) \\ &\leq (k-j-1) + \deg\left(v(\lambda)\right),\end{aligned}$$

and hence $\deg\left(P_j(\lambda)v(\lambda)\right) \leq -1 + \deg\left(v(\lambda)\right)$. □

5.2. The recovery of right minimal indices and bases. We define first a degree-shift isomorphism between $\mathcal{N}_r(P)$ and $\mathcal{N}_r(F_\sigma)$ in Theorem 5.5 and, as an immediate consequence, the recovery of right minimal indices and bases is established in Corollary 5.6, which is one of the main results in this paper.

THEOREM 5.5. *Let $P(\lambda)$ be an $n \times n$ matrix polynomial as in* (1.1)*, let $F_\sigma(\lambda)$ be the Fiedler pencil of $P(\lambda)$ associated with the bijection σ, let $i(\sigma)$ be the number of inversions of σ, and let $\Lambda_\sigma^R(P)$ be the $nk \times n$ matrix defined in Lemma* 5.3*. Then the linear map*

$$\begin{array}{rccc} \mathcal{R}_\sigma: & \mathcal{N}_r(P) & \longrightarrow & \mathcal{N}_r(F_\sigma) \\ & v & \longmapsto & \Lambda_\sigma^R(P)v \end{array}$$

is an isomorphism of $\mathbb{F}(\lambda)$-vector spaces with degree shift $i(\sigma)$ on the vector polynomials in $\mathcal{N}_r(P)$. In other words, $\mathcal{R}_\sigma$ is a bijection between the subsets of vector polynomials in $\mathcal{N}_r(P)$ and $\mathcal{N}_r(F_\sigma)$ with the property that

$$\deg \mathcal{R}_\sigma(v) = i(\sigma) + \deg v \tag{5.7}$$

for every nonzero vector polynomial $v \in \mathcal{N}_r(P)$. Furthermore, for any nonzero vector polynomial v, $\deg \mathcal{R}_\sigma(v)$ is attained only in the topmost $n \times 1$ block of $\mathcal{R}_\sigma(v)$.

Proof. The fact that $\mathcal{R}_\sigma$ is an isomorphism follows from Lemma 5.1, because $\Lambda_\sigma^R(P)$ is the last block-column of the unimodular matrix polynomial $V(\lambda)$ in Corollary 4.6.

The form of $\Lambda_\sigma^R(P)$ guarantees that $\mathcal{R}_\sigma(v)$ is a vector polynomial whenever v is, and, because of the identity block in $\Lambda_\sigma^R(P)$ at block index $k - c_1$, that $\mathcal{R}_\sigma(v)$ is a vector rational (non-polynomial) function whenever v is non-polynomial. Thus $\mathcal{R}_\sigma$ also establishes a bijection between the vector polynomials in $\mathcal{N}_r(P)$ and those in $\mathcal{N}_r(F_\sigma)$.

To see why the degree shifting property (5.7) holds, first observe that there are only two different types of blocks in $\Lambda_\sigma^R(P)$:

$$\lambda^p I \quad \text{with } 0 \le p \le i(\sigma), \qquad \text{and} \qquad \lambda^q P_j(\lambda) \quad \text{with } \left\{ \begin{array}{l} k - s_{\ell-1} - c_\ell \le j \le k-1 \\ 0 \le q \le m_{\ell-1} \le i(\sigma) \end{array} \right. .$$

Thus $\mathcal{R}_\sigma(v)$ is made up of blocks of the form $\lambda^p v$ and $\lambda^q P_j(\lambda)v$. Clearly for a nonzero vector polynomial $v \in \mathcal{N}_r(P)$ the maximum degree among all the blocks $\lambda^p v$ is $i(\sigma) + \deg v$, attained only in the topmost block of $\mathcal{R}_\sigma(v)$. Blocks of the form $\lambda^q P_j(\lambda)v$ are either 0 (if $P_j(\lambda)v = 0$) or, by Lemma 5.4, have degree bounded by

$$\deg\left(\lambda^q P_j(\lambda)v\right) \le m_{\ell-1} - 1 + \deg v < i(\sigma) + \deg v \,.$$

Thus $\deg \mathcal{R}_\sigma(v) = i(\sigma) + \deg v$, with equality attained only in the topmost block of $\mathcal{R}_\sigma(v)$. □

We stress the fact that the recovery procedure for right minimal indices and bases that we present in Corollary 5.6 below is very simple and does not use at all the rather complicated structure of $\Lambda_\sigma^R(P)$. In plain words, one could say that a hard work is needed to get and prove the final "recovery recipe", but that this recipe can be used without effort, except that of determining c_1 and $i(\sigma)$.

COROLLARY 5.6 (**Right minimal indices and bases recovery from Fiedler pencils**). *Let $P(\lambda)$ be an $n \times n$ singular matrix polynomial with degree $k \ge 2$, let $F_\sigma(\lambda) \in \mathbb{F}(\lambda)^{nk \times nk}$ be the Fiedler pencil of $P(\lambda)$ associated with the bijection σ, and let $i(\sigma)$ be the number of inversions of σ and* $\mathrm{CISS}(\sigma) = (c_1, i_1, \ldots, c_\ell, i_\ell)$*. Suppose that $nk \times 1$ vectors are partitioned as $k \times 1$ block vectors with $n \times 1$ blocks.*

(a) *If* $z(\lambda) \in \mathbb{F}(\lambda)^{nk\times 1}$ *is a right null vector of* $F_\sigma(\lambda)$ *and* $x(\lambda)$ *is the* $(k-c_1)$th *block of* $z(\lambda)$*, then* $x(\lambda)$ *is a right null vector of* $P(\lambda)$*.*

(b) *If* $\{z_1(\lambda),\dots,z_p(\lambda)\}$ *is a right minimal basis of* $F_\sigma(\lambda)$ *and* $x_j(\lambda)$ *is the* $(k-c_1)$th *block of* $z_j(\lambda)$*, for* $j=1,\dots,p$*, then* $\{x_1(\lambda),\dots,x_p(\lambda)\}$ *is a right minimal basis of* $P(\lambda)$*.*

(c) *If* $0 \le \varepsilon_1 \le \varepsilon_2 \le \cdots \le \varepsilon_p$ *are the right minimal indices of* $P(\lambda)$*, then*

$$\varepsilon_1 + i(\sigma) \le \varepsilon_2 + i(\sigma) \le \cdots \le \varepsilon_p + i(\sigma)\,,$$

are the right minimal indices of $F_\sigma(\lambda)$*.*

Note that these results hold for the first (resp. second) companion form of $P(\lambda)$ *by taking* $c_1 = 0$ *(resp.* $c_1 = k-1$*) and* $i(\sigma) = k-1$ *(resp.* $i(\sigma) = 0$*).*

Proof. Part (a) follows from the fact that the linear map $\mathcal{R}_\sigma$ in Theorem 5.5 is an isomorphism and that $x(\lambda) = \mathcal{R}_\sigma^{-1}(z(\lambda))$, because the $(k-c_1)$th block of $\Lambda_\sigma^R(P)$ is I_n. To prove part (b), note first that isomorphisms transform any basis into another basis, so the fact that $\{z_1(\lambda),\dots,z_p(\lambda)\}$ is a basis of $\mathcal{N}_r(F_\sigma)$ implies directly that $\{x_1(\lambda),\dots,x_p(\lambda)\}$ is a basis of $\mathcal{N}_r(P)$. To see that $\{x_1(\lambda),\dots,x_p(\lambda)\}$ is minimal, we proceed by contradiction: assume that there exists another polynomial basis $\{\widetilde{x}_1(\lambda),\dots,\widetilde{x}_p(\lambda)\}$ of $\mathcal{N}_r(P)$ with less order, then the degree shift property (5.7) implies that the basis $\{\mathcal{R}_\sigma(\widetilde{x}_1(\lambda)),\dots,\mathcal{R}_\sigma(\widetilde{x}_p(\lambda))\}$ of $\mathcal{N}_r(F_\sigma)$ has less order than $\{\mathcal{R}_\sigma(x_1(\lambda)),\dots,\mathcal{R}_\sigma(x_p(\lambda))\} = \{z_1(\lambda),\dots,z_p(\lambda)\}$, which is a contradiction. Part (c) follows from part (b) and equation (5.7). □

5.3. The recovery of left minimal indices and bases. For the recovery of the left minimal indices and bases of $P(\lambda)$, we will develop in Theorem 5.7 and Corollary 5.9 analogous results to Theorem 5.5 and Corollary 5.6. A possible strategy to prove Theorem 5.7 is based on part (b) of Lemma 5.1 and consists in finding directly an expression for the last block-row of the unimodular matrix $U(\lambda)$ in Corollary 4.6, according to the spirit of Lemma 5.3 for the last block-column of $V(\lambda)$. We adopt here a different strategy that requires less algebraic effort and is based on applying Theorem 5.5 to the Fiedler pencil $F_{\operatorname{rev}\sigma}(P^T)$ of the polynomial $P(\lambda)^T$, where $\operatorname{rev}\sigma$ is the reversal bijection of σ introduced in Section 3. Note that the block-transpose operation introduced in Definition 3.3 is used in Theorem 5.7.

THEOREM 5.7. *Let* $P(\lambda)$ *be an* $n\times n$ *matrix polynomial as in* (1.1)*, let* $F_\sigma(\lambda)$ *be the Fiedler pencil of* $P(\lambda)$ *associated to the bijection* σ*, let* $c(\sigma)$ *be the number of consecutions of* σ*, and let* $\Lambda^R_{\operatorname{rev}\sigma}(P)$ *be, for the reversal bijection of* σ*, the* $nk\times n$ *matrix defined in Lemma* 5.3*. Then the linear map*

$$\begin{array}{rccc} \mathcal{L}_\sigma: & \mathcal{N}_\ell(P) & \longrightarrow & \mathcal{N}_\ell(F_\sigma) \\ & u^T & \longmapsto & u^T\Lambda_\sigma^L(P)\,, \end{array}$$

where $\Lambda_\sigma^L(P) := \left[\Lambda^R_{\operatorname{rev}\sigma}(P)\right]^{\mathcal{B}}$*, is an isomorphism of* $\mathbb{F}(\lambda)$*-vector spaces with degree shift* $c(\sigma)$ *on the vector polynomials in* $\mathcal{N}_\ell(P)$*. In other words,* $\mathcal{L}_\sigma$ *is a bijection between the subsets of vector polynomials in* $\mathcal{N}_\ell(P)$ *and* $\mathcal{N}_\ell(F_\sigma)$ *with the property that*

$$\deg \mathcal{L}_\sigma(u^T) = c(\sigma) + \deg(u^T) \tag{5.8}$$

for every nonzero vector polynomial $u^T \in \mathcal{N}_\ell(P)$*. Furthermore, for any nonzero vector polynomial* u^T*,* $\deg \mathcal{L}_\sigma(u^T)$ *is attained only in the leftmost* $1\times n$ *block of* $\mathcal{L}_\sigma(u^T)$*.*

Proof. We will obtain the linear map $\mathcal{L}_\sigma$ as a composition of three linear maps, each of which is a $\mathbb{F}(\lambda)$-vector space isomorphism inducing a bijection with specific degree shift properties on vector polynomial elements; the second of these three maps will come from the one in Theorem 5.5 between right null spaces. The three maps are the following.

(a) The first map is

$$\begin{array}{rccc} \Psi_1 : & \mathcal{N}_\ell(P) & \longrightarrow & \mathcal{N}_r(P^T) \\ & u^T & \longmapsto & u \end{array} .$$

That this is a vector space isomorphism follows immediately from the fact that $u^T P(\lambda) = 0^T$ if and only if $P(\lambda)^T u = 0$. It is also clear that this map induces a degree-preserving bijection between the vector polynomials in $\mathcal{N}_\ell(P)$ and $\mathcal{N}_r(P^T)$.

(b) The second map is obtained from the one in Theorem 5.5 applied to the transpose polynomial $P(\lambda)^T$ and the reversal bijection, $\operatorname{rev}\sigma$, of σ:

$$\begin{array}{rccc} \Psi_2 : & \mathcal{N}_r(P^T) & \longrightarrow & \mathcal{N}_r(F_{\operatorname{rev}\sigma}(P^T)) \\ & v & \longmapsto & \Lambda^R_{\operatorname{rev}\sigma}(P^T)v\,. \end{array}$$

This linear map is an isomorphism by Theorem 5.5 with degree shift $i(\operatorname{rev}\sigma) = c(\sigma)$ on the vector polynomials in $\mathcal{N}_r(P^T)$, and such that, for any nonzero vector polynomial v, $\deg \Psi_2(v)$ is attained only in the topmost block of $\Psi_2(v)$.

(c) The third map is Ψ_3^{-1}, where

$$\begin{array}{rccc} \Psi_3 : & \mathcal{N}_\ell(F_\sigma(P)) & \longrightarrow & \mathcal{N}_r(F_{\operatorname{rev}\sigma}(P^T)) \\ & u^T & \longmapsto & u \end{array} .$$

To see that this is well-defined, note, in the first place, that $u^T \in \mathcal{N}_\ell(F_\sigma(P))$ if and only if $u^T F_\sigma(P) = 0^T$, and this is equivalent to $[F_\sigma(P)]^T u = 0$. On the other hand, by (3.3),

$$F_\sigma(P) = \lambda M_k(P) - M_{\sigma^{-1}(1)}(P)\, M_{\sigma^{-1}(2)}(P) \cdots M_{\sigma^{-1}(k)}(P)\,,$$

so

$$\begin{aligned} [F_\sigma(P)]^T &= \lambda\,[M_k(P)]^T - \left[M_{\sigma^{-1}(k)}(P)\right]^T \cdots \left[M_{\sigma^{-1}(2)}(P)\right]^T \left[M_{\sigma^{-1}(1)}(P)\right]^T \\ &= \lambda M_k(P^T) - M_{\sigma^{-1}(k)}(P^T) \cdots M_{\sigma^{-1}(2)}(P^T)\, M_{\sigma^{-1}(1)}(P^T) = F_{\operatorname{rev}\sigma}(P^T)\,. \end{aligned}$$

Thus $u^T \in \mathcal{N}_\ell(F_\sigma(P))$ if and only if $u \in \mathcal{N}_r(F_{\operatorname{rev}\sigma}(P^T))$, so Ψ_3 is a well-defined bijection, and clearly linear, hence a vector space isomorphism. That this map induces a degree-preserving bijection on vector polynomials is again obvious.

Now, the composition $(\Psi_3^{-1})\Psi_2\Psi_1$ gives

$$\begin{array}{rcccl} (\Psi_3^{-1})\Psi_2\Psi_1 : & \mathcal{N}_\ell(P) & \longrightarrow & \mathcal{N}_\ell(F_\sigma) & \\ & u^T & \longmapsto & \left[\Lambda^R_{\operatorname{rev}\sigma}(P^T)u\right]^T & = u^T \left[\Lambda^R_{\operatorname{rev}\sigma}(P^T)\right]^T , \end{array}$$

and this is the desired isomorphism $\mathcal{L}_\sigma$, because $\left[\Lambda^R_{\operatorname{rev}\sigma}(P^T)\right]^T = \left[\Lambda^R_{\operatorname{rev}\sigma}(P)\right]^{\mathcal{B}}$. □

To establish Corollary 5.9, i.e., the counterpart for left minimal indices and bases of Corollary 5.6, we need first to find the position of the unique I_n block in $\Lambda^L_\sigma(P) := \left[\Lambda^R_{\operatorname{rev}\sigma}(P)\right]^{\mathcal{B}}$. This is elementary but simultaneously requires some care. It is accomplished in Lemma 5.8.

LEMMA 5.8. *Let $\Lambda_\sigma^L(P) = \left[\Lambda_{\operatorname{rev}\sigma}^R(P)\right]^{\mathcal{B}}$ be the $n \times nk$ matrix in Theorem 5.7, and consider it as a block $1 \times k$ matrix with $n \times n$ blocks. Then $\Lambda_\sigma^L(P)$ has exactly one block equal to I_n residing at block index*

$$\begin{cases} k & \text{if } c_1 > 0 \\ (k - i_1) & \text{if } c_1 = 0 \end{cases} .$$

Proof. Let $\mathrm{CISS}(\sigma) = (c_1, i_1, \ldots, c_\ell, i_\ell)$ and recall that $\Lambda_\sigma^R(P)$ has exactly one block equal to I_n at block index $k - c_1$. Note that σ has a consecution (resp. an inversion) at i if and only if $\operatorname{rev}\sigma$ has an inversion (resp. a consecution) at i for $i = 0, \ldots, k-2$. Therefore, if $c_1 > 0$ then the c_1 initial consecutions of σ correspond to c_1 initial inversions in $\operatorname{rev}\sigma$, which implies $\mathrm{CISS}(\operatorname{rev}\sigma) = (0, c_1, \ldots)$ and that $\Lambda_{\operatorname{rev}\sigma}^R(P)$ has exactly one block equal to I_n at block index k. On the other hand, if $c_1 = 0$ then the i_1 initial inversions of σ correspond to i_1 initial consecutions in $\operatorname{rev}\sigma$, which implies $\mathrm{CISS}(\operatorname{rev}\sigma) = (i_1, \ldots)$ and that $\Lambda_{\operatorname{rev}\sigma}^R(P)$ has exactly one block equal to I_n at block index $k - i_1$. □

Theorem 5.7 combined with Lemma 5.8 allow us to state Corollary 5.9, whose proof is similar to that of Corollary 5.6 and is omitted.

COROLLARY 5.9 (**Left minimal indices and bases recovery from Fiedler pencils**). *Let $P(\lambda)$ be an $n \times n$ singular matrix polynomial with degree $k \geq 2$, let $F_\sigma(\lambda) \in \mathbb{F}(\lambda)^{nk \times nk}$ be the Fiedler pencil of $P(\lambda)$ associated with the bijection σ, and let $c(\sigma)$ be the number of consecutions of σ and $\mathrm{CISS}(\sigma) = (c_1, i_1, \ldots, c_\ell, i_\ell)$. Suppose that $1 \times nk$ vectors are partitioned as $1 \times k$ block vectors with $1 \times n$ blocks.*

(a) *If $z(\lambda)^T \in \mathbb{F}(\lambda)^{1 \times nk}$ is a left null vector of $F_\sigma(\lambda)$ and*

$$y(\lambda)^T \;\text{ is the }\; \begin{cases} k\text{th block of } z(\lambda)^T \text{ if } c_1 > 0 \\ (k - i_1)\text{th block of } z(\lambda)^T \text{ if } c_1 = 0 \end{cases} ,$$

then $y(\lambda)^T$ is a left null vector of $P(\lambda)$.

(b) *If $\{z_1(\lambda)^T, \ldots, z_p(\lambda)^T\}$ is a left minimal basis of $F_\sigma(\lambda)$ and*

$$y_j(\lambda)^T \;\text{ is the }\; \begin{cases} k\text{th block of } z_j(\lambda)^T \text{ if } c_1 > 0 \\ (k - i_1)\text{th block of } z_j(\lambda)^T \text{ if } c_1 = 0 \end{cases} ,$$

for $j = 1, \ldots, p$, then $\{y_1(\lambda)^T, \ldots, y_p(\lambda)^T\}$ is a left minimal basis of $P(\lambda)$.

(c) *If $0 \leq \eta_1 \leq \eta_2 \leq \cdots \leq \eta_p$ are the left minimal indices of $P(\lambda)$, then*

$$\eta_1 + c(\sigma) \leq \eta_2 + c(\sigma) \leq \cdots \leq \eta_p + c(\sigma) ,$$

are the left minimal indices of $F_\sigma(\lambda)$.

Note that these results hold for the first (resp. second) companion form of $P(\lambda)$ by taking $(c_1, i_1) = (0, k-1)$ (resp. $(c_1, i_1) = (k-1, 0)$) and $c(\sigma) = 0$ (resp. $c(\sigma) = k-1$).

5.4. Strictly equivalent Fiedler pencils. It is very simple to determine which Fiedler pencils of the same square *singular* matrix polynomial are strictly equivalent[1] and which are not, as a consequence of the recovery formulas we have established for the right and left minimal indices of a polynomial from its Fiedler pencils.

COROLLARY 5.10. *Let $P(\lambda)$ be a singular square matrix polynomial of degree $k \geq 2$. Then two Fiedler pencils $F_{\sigma_1}(\lambda)$ and $F_{\sigma_2}(\lambda)$ of $P(\lambda)$ are strictly equivalent if and only if $c(\sigma_1) = c(\sigma_2)$ (or, equivalently, $i(\sigma_1) = i(\sigma_2)$).*

[1] Recall that two matrix pencils $L_1(\lambda)$ and $L_2(\lambda)$ are strictly equivalent if $L_1(\lambda) = W L_2(\lambda) Z$, where W and Z are constant square nonsingular matrices [17, Chapter XII].

Proof. It is well known [17, Chapter XII, Th. 5] that two pencils are strictly equivalent if and only if they have the same finite and infinite elementary divisors and the same left and right minimal indices. By Corollary 5.6 (resp. Corollary 5.9) the right (resp. left) minimal indices of $F_{\sigma_1}(\lambda)$ and $F_{\sigma_2}(\lambda)$ are equal if and only if $i(\sigma_1) = i(\sigma_2)$ (resp. $c(\sigma_1) = c(\sigma_2)$). But $c(\sigma_1)+i(\sigma_1) = k-1$ and $c(\sigma_2)+i(\sigma_2) = k-1$, and, so, $c(\sigma_1) = c(\sigma_2)$ is a necessary and sufficient condition for the equality of both the left and the right minimal indices of $F_{\sigma_1}(\lambda)$ and $F_{\sigma_2}(\lambda)$. In addition, since any Fiedler pencil is a strong linearization of $P(\lambda)$ by Theorem 4.5, $F_{\sigma_1}(\lambda)$ and $F_{\sigma_2}(\lambda)$ have the same finite and infinite elementary divisors (see Lemma 2.3). Therefore, $c(\sigma_1) = c(\sigma_2)$ is a necessary and sufficient condition for the strict equivalence of $F_{\sigma_1}(\lambda)$ and $F_{\sigma_2}(\lambda)$. $\square$

Obviously, there are Fiedler pencils of the same singular square matrix polynomial $P(\lambda)$ that are not strictly equivalent, which is in stark contrast with the case of $P(\lambda)$ being regular. In this case, any two strong linearizations (not necessarily Fiedler pencils) of $P(\lambda)$ are always strictly equivalent because they are regular pencils and have the same finite and infinite elementary divisors as the polynomial [17, Chapter XII, Th. 2] (see also [18, Prop. 1.2]).

We finish this section by showing that there are no Fiedler pencils strictly equivalent to the first (or the second) companion form of a square singular matrix polynomial other than itself.

COROLLARY 5.11. *For a singular square matrix polynomial $P(\lambda)$ of degree $k \geq 2$, the first companion form is never strictly equivalent to any other Fiedler pencil of $P(\lambda)$. The same holds for the second companion form. In particular, the first and second companion forms of $P(\lambda)$ are never strictly equivalent.*

Proof. We only prove the result for the first companion form $C_1(\lambda)$. Recall from Section 3 that $C_1(\lambda) = F_{\sigma_1}(\lambda)$, with $\sigma_1 = (k, k-1, \ldots, 2, 1)$. Note that $c(\sigma_1) = 0$. So, from Corollary 5.10, we know that a Fiedler pencil $F_\sigma(\lambda)$ of $P(\lambda)$ is strictly equivalent to $C_1(\lambda)$ if and only if $c(\sigma) = 0$, and this happens if and only if $\sigma = (k, k-1, \ldots, 1)$. So $F_\sigma(\lambda) = C_1(\lambda)$. $\square$

Note that Corollary 5.11 shows that the proof presented in [4, Theorem 2.3] that Fiedler pencils are strong linearizations of any *regular* matrix polynomial cannot be extended to singular polynomials. Another proof that the first and second companion forms of square singular matrix polynomials are never strictly equivalent was presented in [11, Corollary 5.11].

6. Recovery of eigenvectors of regular matrix polynomials. If the polynomial $P(\lambda)$ in (1.1) is *regular* then it does have neither minimal indices nor minimal bases, and eigenvalues and eigenvectors are the most interesting spectral magnitudes associated with $P(\lambda)$. In the regular case, the characterization of a *finite eigenvalue* reduces to the classical one, i.e., a number $\lambda_0 \in \mathbb{C}$ is a finite eigenvalue of $P(\lambda)$ if and only if $\det P(\lambda_0) = 0$. Besides, $0 \neq x_0 \in \mathbb{C}^{n\times 1}$ (resp. $0^T \neq y_0^T \in \mathbb{C}^{1\times n}$) is a right (resp. left) eigenvector of $P(\lambda)$ corresponding to λ_0 if $P(\lambda_0)x_0 = 0$ (resp. $y_0^T P(\lambda_0) = 0$), i.e., right (resp. left) eigenvectors are the nonzero vectors in the right (resp. left) null space $\mathcal{N}_r(P(\lambda_0)) \subset \mathbb{C}^{n\times 1}$ (resp. $\mathcal{N}_\ell(P(\lambda_0)) \subset \mathbb{C}^{1\times n}$). The definition of the *infinite eigenvalue* and its corresponding eigenvectors of $P(\lambda)$ is based on the one in Section 2, i.e., $P(\lambda)$ has an infinite eigenvalue with left eigenvector $y_\infty^T \in \mathbb{C}^{1\times n}$ and right eigenvector $x_\infty \in \mathbb{C}^{n\times 1}$ if $\operatorname{rev} P(\lambda)$ has the eigenvalue 0 with left eigenvector $y_\infty^T \in \mathbb{C}^{1\times n}$ and right eigenvector $x_\infty \in \mathbb{C}^{n\times 1}$.

In this section, we show how to recover the eigenvectors of a regular polynomial from those of its Fiedler pencils. These recovery procedures are direct consequences of the results presented in Sections 4 and 5, that are valid both for regular and singular polynomials[2]. Therefore, for brevity, we do not provide the proofs of the presented results and only sketch the main ideas and state the results. The reader can easily complete the details.

The first key idea is that if equation (2.3), i.e., the unimodular transformation defining linearization, is evaluated at a finite eigenvalue λ_0 of $P(\lambda)$, then the matrices $U(\lambda_0)$ and $V(\lambda_0)$ are nonsingular, and a counterpart of Lemma 5.1 can be proved in exactly the same way by replacing $\mathbb{F}(\lambda)$-vector spaces by $\mathbb{C}$-vector spaces. This is Lemma 6.1, that is valid *for any linearization* and not only for Fiedler pencils.

LEMMA 6.1. *Let the $nk \times nk$ pencil $L(\lambda)$ be a linearization of an $n \times n$ regular matrix polynomial $P(\lambda)$ of degree $k \geq 2$ and $U(\lambda)$ and $V(\lambda)$ be two unimodular matrix polynomials such that*

$$U(\lambda)L(\lambda)V(\lambda) = \begin{bmatrix} I_{(k-1)n} & 0 \\ 0 & P(\lambda) \end{bmatrix}. \tag{6.1}$$

Suppose that $\lambda_0 \in \mathbb{C}$ is a finite eigenvalue of $P(\lambda)$, consider $U(\lambda_0)$ and $V(\lambda_0)$ as block $k \times k$ matrices with $n \times n$ blocks, and let U_0^L and V_0^R be, respectively, the last block-row of $U(\lambda_0)$ and the last block-column of $V(\lambda_0)$. Then:

(a) *The linear map*

$$\begin{array}{rccc} \mathcal{R}_0 : & \mathcal{N}_r(P(\lambda_0)) & \longrightarrow & \mathcal{N}_r(L(\lambda_0)) \\ & v & \longmapsto & V_0^R v \end{array}$$

is an isomorphism of $\mathbb{C}$-vector spaces.

(b) *The linear map*

$$\begin{array}{rccc} \mathcal{L}_0 : & \mathcal{N}_\ell(P(\lambda_0)) & \longrightarrow & \mathcal{N}_\ell(L(\lambda_0)) \\ & w^T & \longmapsto & w^T U_0^L \end{array}$$

is an isomorphism of $\mathbb{C}$-vector spaces.

Lemma 6.1 (a) can be applied to Fiedler pencils by using as $V(\lambda)$ the matrix appearing in Corollary 4.6 and, therefore, taking as V_0^R the matrix $\Lambda_\sigma^R(P)$ in Lemma 5.3 evaluated at λ_0. The point to be remarked is that the unique I_n block of $\Lambda_\sigma^R(P)$ is also a I_n block in $\Lambda_\sigma^R(P)$ evaluated at λ_0. Therefore, we can state an analogous of Corollary 5.6 for the recovery of right eigenvectors.

COROLLARY 6.2 (Right eigenvector recovery from Fiedler pencils for finite eigenvalues). *Let $P(\lambda)$ be an $n \times n$ regular matrix polynomial with degree $k \geq 2$, let $F_\sigma(\lambda) \in \mathbb{F}(\lambda)^{nk \times nk}$ be the Fiedler pencil of $P(\lambda)$ associated with the bijection σ, and let* $\mathrm{CISS}(\sigma) = (c_1, i_1, \ldots, c_\ell, i_\ell)$. *Suppose that λ_0 is a finite eigenvalue of $P(\lambda)$ and that $nk \times 1$ vectors are partitioned as $k \times 1$ block vectors with $n \times 1$ blocks.*

(a) *If $z \in \mathbb{C}^{nk \times 1}$ is a right eigenvector of $F_\sigma(\lambda)$ with finite eigenvalue $\lambda_0 \in \mathbb{C}$ and x is the $(k - c_1)$*th *block of z, then x is a right eigenvector of $P(\lambda)$ with finite eigenvalue λ_0.*

(b) *If $\{z_1, \ldots, z_p\}$ is a basis of $\mathcal{N}_r(F_\sigma(\lambda_0))$ and x_j is the $(k - c_1)$*th *block of z_j, for $j = 1, \ldots, p$, then $\{x_1, \ldots, x_p\}$ is a basis of $\mathcal{N}_r(P(\lambda_0))$.*

Note that these results hold for the first (*resp. second*) *companion form of $P(\lambda)$ by taking $c_1 = 0$* (*resp. $c_1 = k - 1$*).

[2]Of course, results like Theorems 5.5 and 5.7 give trivial information if $P(\lambda)$ is regular because, in this case, the $\mathbb{F}(\lambda)$-null spaces $\mathcal{N}_r(P)$, $\mathcal{N}_\ell(P)$, $\mathcal{N}_r(F_\sigma)$ and $\mathcal{N}_\ell(F_\sigma)$ are trivial and only contain the corresponding zero vector.

For the recovery of left eigenvectors, one gets the following counterpart of Corollary 5.9.

COROLLARY 6.3 (Left eigenvector recovery from Fiedler pencils for finite eigenvalues). *Under the same assumptions and notation as in Corollary* 6.2. *Suppose that* $1 \times nk$ *vectors are partitioned as* $1 \times k$ *block vectors with* $1 \times n$ *blocks.*

(a) *If* $z^T \in \mathbb{C}^{1\times nk}$ *is a left eigenvector of* $F_\sigma(\lambda)$ *with finite eigenvalue* $\lambda_0 \in \mathbb{C}$ *and*

$$y^T \ \text{ is the } \begin{cases} k\text{th block of } z^T \text{ if } c_1 > 0 \\ (k-i_1)\text{th block of } z^T \text{ if } c_1 = 0 \end{cases},$$

then y^T *is a left eigenvector of* $P(\lambda)$ *with finite eigenvalue* λ_0.

(b) *If* $\{z_1^T, \dots, z_p^T\}$ *is a basis of* $\mathcal{N}_\ell(F_\sigma(\lambda_0))$ *and*

$$y_j^T \ \text{ is the } \begin{cases} k\text{th block of } z_j^T \text{ if } c_1 > 0 \\ (k-i_1)\text{th block of } z_j^T \text{ if } c_1 = 0 \end{cases},$$

for $j = 1, \dots, p$, *then* $\{y_1^T, \dots, y_p^T\}$ *is a basis of* $\mathcal{N}_\ell(P(\lambda_0))$.

Note that these results hold for the first (*resp. second*) *companion form of* $P(\lambda)$ *by taking* $(c_1, i_1) = (0, k-1)$ (*resp.* $(c_1, i_1) = (k-1, 0)$).

Our last result deals with the recovery of left and right eigenvectors corresponding to the infinite eigenvalue, which is very simple in the case of Fiedler pencils. Note that if the matrix polynomial $P(\lambda)$ in (1.1) is regular and has infinite as one of its eigenvalues, then the corresponding right (resp. left) eigenvectors are the right (resp. left) null vectors of $(\operatorname{rev} P)(0) = A_k$. For every Fiedler pencil $F_\sigma(\lambda)$ of $P(\lambda)$ its right (resp. left) eigenvectors for the infinite eigenvalue are the right (resp. left) null vectors of $(\operatorname{rev} F_\sigma)(0) = M_k = \operatorname{diag}(A_k, I_{(k-1)n})$. Based on these relationships, one can prove immediately that $z \in \mathbb{F}^{nk\times 1}$ (resp. $w^T \in \mathbb{F}^{1\times nk}$) is a right (resp. left) null vector of M_k if and only if $z = [x^T \ 0^T_{(k-1)n\times 1}]^T$ (resp. $w^T = [y^T \ 0_{1\times(k-1)n}]$), where x (resp. y^T) is a right (resp. left) null vector of A_k. From this, it follows easily that the map $x \longmapsto [x^T \ 0^T_{(k-1)n\times 1}]^T$ (resp. $y^T \longmapsto [y^T \ 0_{1\times(k-1)n}]$) from $\mathcal{N}_r(A_k)$ to $\mathcal{N}_r(M_k)$ (resp. from $\mathcal{N}_\ell(A_k)$ to $\mathcal{N}_\ell(M_k)$) is an isomorphism. This discussion allows us to state Theorem 6.4, where we show that eigenvectors associated to the infinite eigenvalue are recovered in the same way for any Fiedler pencil.

THEOREM 6.4 (Left and right eigenvector recovery from Fiedler pencils for the infinite eigenvalue). *Let* $P(\lambda)$ *be an* $n\times n$ *regular matrix polynomial with degree* $k \geq 2$, *let* $F_\sigma(\lambda) \in \mathbb{F}(\lambda)^{nk\times nk}$ *be the Fiedler pencil of* $P(\lambda)$ *associated with the bijection* σ. *Then:*

1. $z \in \mathbb{F}^{nk\times 1}$ (*resp.* $w^T \in \mathbb{F}^{1\times nk}$) *is a right* (*resp. left*) *eigenvector of* $F_\sigma(\lambda)$ *for the infinite eigenvalue if and only if* $z = [x^T \ 0^T_{(k-1)n\times 1}]^T$ (*resp.* $w^T = [y^T \ 0_{1\times(k-1)n}]$), *where* x (*resp.* y^T) *is a right* (*resp. left*) *eigenvector of* $P(\lambda)$ *for the infinite eigenvalue.*
2. $\{z_1, \dots, z_p\} \subset \mathbb{F}^{nk\times 1}$ (*resp.* $\{w_1^T, \dots, w_p^T\} \subset \mathbb{F}^{1\times nk}$) *is a basis of the right* (*resp. left*) *null space of* $(\operatorname{rev} F_\sigma)(0)$ *if and only if* $z_j = [x_j^T \ 0^T_{(k-1)n\times 1}]^T$ (*resp.* $w_j^T = [y_j^T \ 0_{1\times(k-1)n}]$), *for* $j = 1, \dots, p$, *where* $\{x_1, \dots, x_p\}$ (*resp.* $\{y_1^T, \dots, y_p^T\}$) *is a basis of the right* (*resp. left*) *null space of* $(\operatorname{rev} P)(0)$.

7. Conclusions and future work. We have proved that every Fiedler pencil of a given square matrix polynomial $P(\lambda)$ is always a strong linearization of $P(\lambda)$, even in the case that $P(\lambda)$ is singular. In addition, we have shown that the minimal indices and bases of a singular square matrix polynomial are very easily recovered from the minimal indices and bases of any of its Fiedler pencils, without any computational cost. These simple recovery procedures have been extended to the recovery of the eigenvectors of a regular matrix polynomial from the eigenvectors of its Fiedler pencils. These results now make it possible to use well-established numerical algorithms on any Fiedler pencil [33, 6, 8, 9, 13] to obtain the complete eigenstructure of a square matrix polynomial (regular or singular).

This paper continues the work initiated by the authors in [11] with the aim of creating a wide arena of linearizations for *singular* matrix polynomials that allow us to easily recover the complete eigenstructure of a matrix polynomial. Note that the mere definition of strong linearization does not guarantee that the minimal indices and bases of $P(\lambda)$ can be easily recovered, or even that the minimal indices and bases of $P(\lambda)$ have any simple relationship at all to those of a given linearization. Consequently each family of linearizations requires a separate study in order to establish convenient recovery procedures.

Another goal of our continuing work on linearizations for singular polynomials is to create more possibilities for preserving any structure that a matrix polynomial might possess in a linearization. The results of [11] show that for many types of structure, this cannot be achieved using any of the linearizations defined in [30]. Thus the next steps in our investigation are to modify Fiedler pencils with the aim of finding structured linearizations for singular structured matrix polynomials [12], and also to extend Fiedler pencils to try to deal with the very important case of (non-square) rectangular matrix polynomials.

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