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Multiplicative structure of 2×2 tropical matrices

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Abstract

We study the algebraic structure of the semigroup of all 2×2 tropical matrices under multiplication. Using ideas from tropical geometry, we give a complete description of Green's relations and the idempotents and maximal subgroups of this semigroup.

1 Introduction

Tropical algebra (also known as max-plus algebra) is the linear algebra of the real numbers augmented with $-\infty$ when equipped with the binary operations of addition and maximum. Interest in this branch of mathematics is motivated by a wide range of applications in numerous subject areas including combinatorial optimisation and scheduling problems [5], analysis of discrete event systems [17], control theory [7], formal language and automata theory [22, 26], phylogenetics [13], statistical inference [21], algebraic geometry [2, 19, 25] and combinatorial/geometric group theory [3]. Tropical algebra and many of its basic properties have been independently rediscovered many times by researchers in these fields. The first detailed axiomatic study of "max-plus algebra" was conducted by Cuninghame-Green [8] and this theory has been developed further by a number of researchers (see [1, 15] for surveys).

Many problems arising from application areas are naturally expressed as tropical matrix algebra problems, and much of the theory of tropical algebra is concerned with matrices. An important aspect is the algebraic structure of tropical matrices under multiplication; many authors have proved a number of interesting *ad hoc* results (see for example [10, 14, 22, 26]) but until recently there has been no systematic study in this area. This surprising omission is

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due largely to the difficulty, both conceptual and technical, of the subject. Even the case of 2×2 matrices, which is the main object of study in this paper, demonstrates a number of interesting phenomena. We believe that the development of a coherent and comprehensive theory of tropical matrix semigroups of arbitrary finite dimension is a major challenge.

The aim of this paper is to initiate the systematic study of the semigroup-theoretic structure of tropical matrices under multiplication, by considering the most natural starting point: the monoid of all 2×2 tropical matrices. We give a complete geometric description of Green's relations in this semigroup, from which we are also able to deduce that the semigroup is regular, and to describe all of its maximal subgroups. Since conducting this research, we have learned that an independent study of some of these topics has recently been conducted by Izhakian and Margolis [16]. Another concurrent study of tropical matrix semigroups, with the emphasis more on geometric than on algebraic properties, has been conducted by Merlet [18].

In addition to this introduction, this paper comprises three sections. In Section 2 we give a brief expository introduction to the tropical semiring and tropical matrix algebra, including a summary of known results about tropical matrix semigroups. Section 3 is devoted to an examination of the ideal structure of the monoid of all 2×2 tropical matrices, obtaining in particular geometric descriptions of Green's relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, $\mathcal{D}$ and $\mathcal{J}$, and of the associated partial orders. Finally, in Section 4 we consider the idempotent elements of this monoid; combined with the results of the previous section, this allows us to prove that the monoid is *regular*, and to describe completely its maximal subgroups.

2 Preliminaries

Let $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$. We extend the addition and order on $\mathbb{R}$ to $\bar{\mathbb{R}}$ in the obvious way, and define operations multiplication $\otimes$ and addition $\oplus$ on $\bar{\mathbb{R}}$ by $a \otimes b = a + b$ and $a \oplus b = \max\{a, b\}$ for all $a, b \in \bar{\mathbb{R}}$. Then $\bar{\mathbb{R}}$ is a semiring with multiplicative identity 0 and additive identity $-\infty$. In fact, $\bar{\mathbb{R}}$ is an idempotent semifield, since $a \oplus a = a$ for all $a \in \bar{\mathbb{R}}$ and $a \otimes -a = 0$ for all $a \in \mathbb{R}$. We call $(\bar{\mathbb{R}}, \otimes, \oplus)$ the *tropical semiring*; some authors refer to it as the *max-plus* semiring.

For each positive integer n let $M_n(\bar{\mathbb{R}})$ denote the set of $n \times n$ matrices with entries in $\bar{\mathbb{R}}$. The $\otimes$ and $\oplus$ operations on $\bar{\mathbb{R}}$ induce corresponding operations on $M_n(\bar{\mathbb{R}})$ in the obvious way. Indeed, if $A, B \in M_n(\bar{\mathbb{R}})$ then we have

$$\begin{aligned} (A \otimes B)_{ij} &= \bigoplus_{k=1}^{n} A_{ik} \otimes B_{kj}, \text{ and} \\ (A \oplus B)_{ij} &= A_{ij} \oplus B_{ij}, \end{aligned}$$

for all $1 \leq i, j \leq n$, where $X_{i,j}$ denotes the (i, j)th entry of the matrix X. For brevity, we shall usually write AB in place of $A \otimes B$ for a product of matrices. It is then easy to check that $M_n(\bar{\mathbb{R}})$ is an idempotent semiring, with multiplicative identity

$$\begin{pmatrix} 0 & -\infty & \cdots & -\infty \\ -\infty & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\infty \\ -\infty & \cdots & -\infty & 0 \end{pmatrix}$$

and additive identity

$$\begin{pmatrix} -\infty & \cdots & -\infty \\ \vdots & \ddots & \vdots \\ -\infty & \cdots & -\infty \end{pmatrix}.$$

We call $(M_n(\bar{\mathbb{R}}), \otimes, \oplus)$ the $n \times n$ *tropical matrix semiring*. The main object of study in this paper is the multiplicative monoid of this semiring, which we shall refer to simply as $M_n(\bar{\mathbb{R}})$.

We summarise some known results about this semigroup. It is readily verified (see for example [12]) that the invertible elements of $M_n(\bar{\mathbb{R}})$ (the *units* in the terminology of ring theory or semigroup theory) are exactly the *monomial matrices*, that is, matrices with exactly one entry in each row and column not equal to $-\infty$. It follows easily that the group of units in $M_n(\bar{\mathbb{R}})$ is isomorphic to the permutation wreath product $\mathbb{R} \wr (S_n, \{1, \dots, n\})$ of the additive group of real numbers with the symmetric group on n points.

It is known [10] that the semigroup $M_n(\bar{\mathbb{R}})$ is *weakly permutable*, in the sense that there is a positive integer k such that every sequence of k elements admits two distinct permutations such that the corresponding products of elements are equal in the semigroup. It is clear from the definition that weak permutability is inherited by subsemigroups. It is also known [4, 9] that a group is weakly permutable if and only if it has an abelian subgroup of finite index. It follows that every subgroup of $M_n(\bar{\mathbb{R}})$ (including those whose identity element is an idempotent other than the identity of $M_n(\bar{\mathbb{R}})$) has an abelian subgroup of finite index. Moreover, it is also shown in [10] that finitely generated subsemigroups of $M_n(\bar{\mathbb{R}})$ have polynomial growth.

The semigroup $M_n(\bar{\mathbb{R}})$ acts naturally on the left and right of the space of n-vectors over $\bar{\mathbb{R}}$, known as *affine tropical n-space*. Notice that a tropical multiple of a vector $(x_1, \ldots, x_n) \in \bar{\mathbb{R}}^n$ has the form $(x_1 + \lambda, \ldots, x_n + \lambda)$ for some $\lambda \in \bar{\mathbb{R}}$. From affine tropical n-space we obtain *projective tropical $(n-1)$-space* by discarding the zero vector $(-\infty, \ldots, -\infty)$ and identifying two non-zero vectors if one is a tropical multiple of the other.

We can represent affine tropical 2-space (or the *tropical plane*) pictorially as a quadrant of the Euclidean plane with two sets of axes as shown in Figure 1. The set of tropical multiples of $v \in \bar{\mathbb{R}}^2$ is then equal to the line of gradient 1 which passes through v, as shown in Figure 2(a); notice that this line includes the zero vector. Vector addition in $\bar{\mathbb{R}}^2$ may also be described pictorially as follows. For $u, v \in \bar{\mathbb{R}}^2$ the sum $u \oplus v$ is given by the upper right-most vertex of the unique rectangle with u and v as vertices and edges parallel to the axes, see Figure 2(b). Note that the sides of this rectangle may have infinite length.

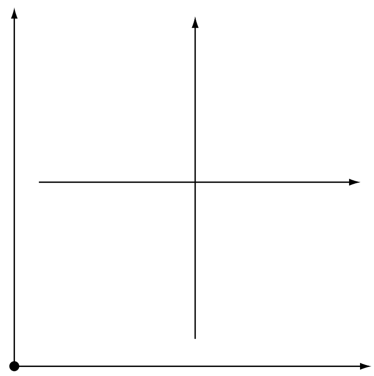


Figure 1: The tropical axes.

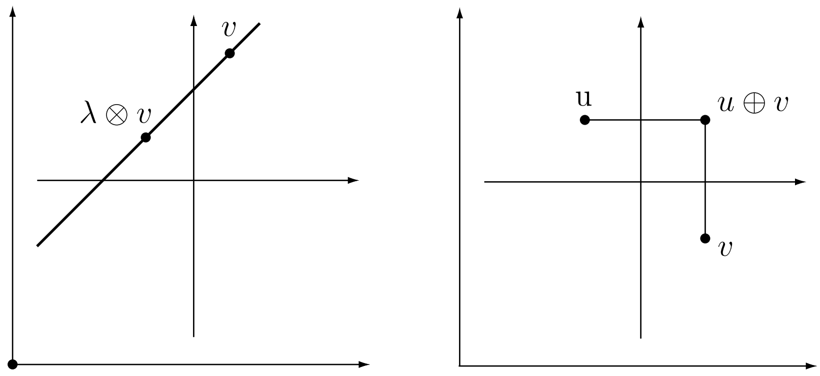
Projective tropical 1-space can be conveniently identified with the two point compactification of the real line

$$\hat{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$$

via the map which takes the equivalence class of a non-zero vector $(a, b) \in \bar{\mathbb{R}}^2$ to $b - a$ if a and b are real, ∞ if $a = -\infty$ and $-\infty$ if $b = -\infty$[3]. In pictorial terms, the image of a point (a, b) with real coordinates under this projection may be thought of as the intercept of the line of gradient 1 through the point (a, b) with the vertical[4] axis through $(0, 0)$.

[3]In fact, if we extend subtraction in the obvious way to $\bar{\mathbb{R}} \times \bar{\mathbb{R}} \setminus \{(-\infty, -\infty)\}$, we have that the projection of (a, b) corresponds to $b - a$ for all non-zero points (a, b).

[4]The choice of the vertical axis here is of course arbitrary. One could instead take



(a) Tropical vector scaling in $\bar{\mathbb{R}}^2$.

(b) Tropical vector addition in $\bar{\mathbb{R}}^2$.

Figure 2: Tropical linear combinations of vectors.

3 Green's Relations

We begin by briefly recalling the definitions of a number of binary relations which are used to analyse the structure of a monoid. For further reference and examples we refer the reader to [6].

Let S be a monoid and let $A, B \in S$. We define a binary relation $\leq_{\mathcal{R}}$ on S by $A \leq_{\mathcal{R}} B$ exactly if $AS \subseteq BS$, or equivalently, if $A = BX$ for some $X \in S$. Similarly, we define $A \leq_{\mathcal{L}} B$ if $SA \subseteq SB$, and $A \leq_{\mathcal{J}} B$ if $SAS \subseteq SBS$. The relations $\leq_{\mathcal{R}}$, $\leq_{\mathcal{L}}$ and $\leq_{\mathcal{J}}$ are *preorders* (reflexive, transitive binary relations) on the monoid S.

Next, we define a binary relation $\mathcal{R}$ on S by $A\mathcal{R}B$ if A and B generate the same principal right ideal in S, or equivalently, if $A \leq_{\mathcal{R}} B$ and $B \leq_{\mathcal{R}} A$. Similarly, we define $A\mathcal{L}B$ if A and B generate the same principal left ideal in S, and $A\mathcal{J}B$ if A and B generate the same principal two-sided ideal in S. The relations $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{J}$ are all equivalence relations. In fact they are the largest equivalence relations contained in the preorders $\leq_{\mathcal{R}}$, $\leq_{\mathcal{L}}$ and $\leq_{\mathcal{J}}$ respectively, from which it follows that these preorders induce partial orders on the equivalence classes of the respective equivalence relations.

We let $\mathcal{H}$ denote the intersection $\mathcal{L} \cap \mathcal{R}$, and $\mathcal{D}$ be the intersection of all equivalence relations containing $\mathcal{L}$ and $\mathcal{R}$. Both are equivalence relations,

signed perpendicular distance of the given line from the point $(0, 0)$; this is arguably conceptually cleaner but makes no practical difference and introduces an extra factor of $\sqrt{2}$ into computations.

and it is well known and easy to show that we have $A\mathcal{D}B$ if and only if there exists $Z \in S$ such that $A\mathcal{R}Z$ and $Z\mathcal{L}B$.

We shall also need some basic ideas from tropical geometry. For each positive integer k we define a *(k-generated) convex cone* in $\bar{\mathbb{R}}^n$ to be a non-empty set which is the set of all tropical linear combinations of vectors from some given subset (of cardinality k or less) of $\bar{\mathbb{R}}^n$. Convex cones are the tropical analogue of linear subspaces in classical linear algebra. However, we shall refrain from terming them *(tropical linear) subspaces*, since this term is generally applied to a distinct concept which in tropical geometry plays the role of affine linear subspaces in classical algebraic geometry [11].

Since convex cones are closed under scaling, each convex cone V in affine tropical n-space is naturally associated with a subset in projective $(n-1)$-space, which we call the *projectivisation* of V. We define a *(k-generated) convex set* in projective tropical $(n-1)$-space to be the projectivisation of a (k-generated) convex cone in affine tropical n-space. In the case that $n = 2$, so that the projective space is $\hat{\mathbb{R}}$, it is easily seen that the only convex sets are the empty set, the singleton sets and *intervals* (open, closed, half-open and half-closed) where the latter are defined in the obvious way using the order on $\hat{\mathbb{R}}$. The 2-generated convex sets are the empty set, singleton sets, and *closed* intervals of $\hat{\mathbb{R}}$; we call these the *closed* convex sets.

Now let $A \in M_n(\bar{\mathbb{R}})$. We define the *column space* $C(A)$ of A to be the convex cone which is the set of tropical linear combinations of the columns of A. We shall also be interested in the projectivisation of $C(A)$, which we call the *projective column space* of A and denote $PC(A)$. Dually, the *row space* $R(A)$ of A is the convex cone given by the set of tropical linear combinations of the rows of A, and its projectivisation is called the *projective row space* of A, denoted $PR(A)$.

The following characterisation of the $\mathcal{R}$ and $\mathcal{L}$ preorders is well known at least in the case of matrices over fields (see for example [20, Lemma 2.1]) and extends without difficulty to matrices over the tropical semiring. For completeness, we include a brief proof.

Lemma 3.1. *Let $A, B \in M_n(\bar{\mathbb{R}})$. Then the following are equivalent:*

(i) $A \leq_{\mathcal{R}} B$ *[respectively,* $A \leq_{\mathcal{L}} B$*];*

(ii) $C(A) \subseteq C(B)$ *[respectively,* $R(A) \subseteq R(B)$*] in affine tropical n-space;*

(iii) $PC(A) \subseteq PC(B)$ *[respectively,* $PR(A) \subseteq PR(B)$*] in projective tropical $(n-1)$-space.*

Proof. We prove the equivalence of the statements involving $\leq_{\mathcal{R}}$ and column spaces, the equivalence of the statements involving $\leq_{\mathcal{L}}$ and row spaces being dual. The equivalence of (ii) and (iii) follows from the fact that convex cones, and hence column spaces, are closed under scaling, so it will suffice to show that (i) and (ii) are equivalent.

If (i) holds, that is, $A \leq_{\mathcal{R}} B$, then by definition there is a matrix $X \in M_n(\bar{\mathbb{R}})$ such that $BX = A$. Now, since the columns of BX are contained in $C(B)$ it follows that $C(BX) = C(A) \subseteq C(B)$ so that (ii) holds. Conversely, suppose that (ii) holds. Since the tropical semiring has a multiplicative identity, the columns of A are contained in $C(A)$, and hence in $C(B)$. Thus, every column of A can be written as a linear combination of the columns of B, which means exactly that there exists $X \in M_n(\bar{\mathbb{R}})$ such that $A = BX$. Thus (i) holds. $\square$

Corollary 3.2. *Let $A, B \in M_n(\bar{\mathbb{R}})$. Then the following are equivalent:*

(i) $A\mathcal{R}B$ *[respectively,* $A\mathcal{L}B$*];*

(ii) $C(A) = C(B)$ *[respectively,* $R(A) = R(B)$*] in affine tropical n-space;*

(iii) $PC(A) = PC(B)$ *[respectively,* $PR(A) = PR(B)$*] in projective tropical* $(n-1)$*-space.*

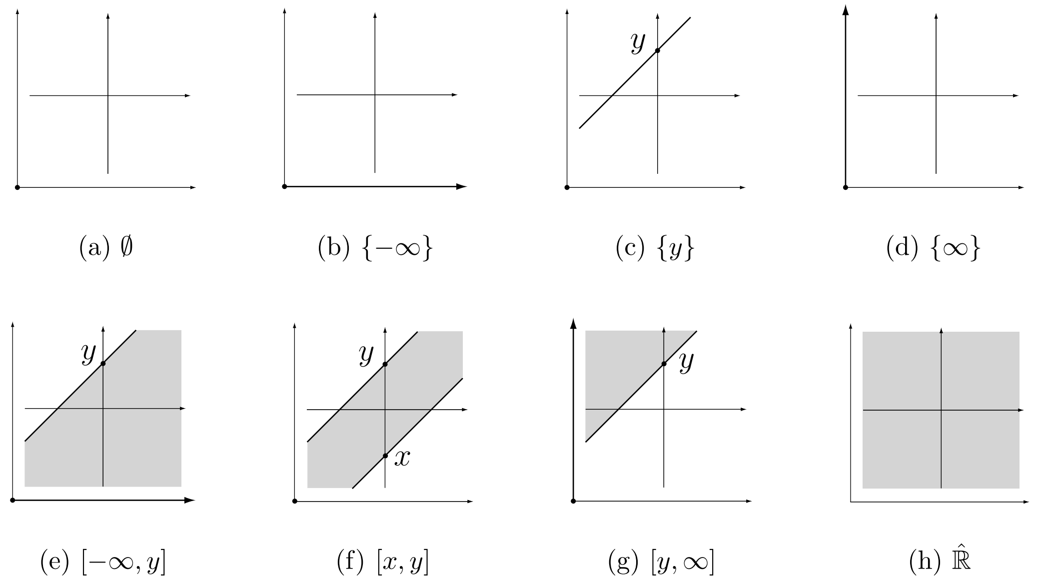
By Corollary 3.2, the $\mathcal{R}$-classes of $M_2(\bar{\mathbb{R}})$ are in a natural bijective correspondence with the 2-generated tropical convex cones in the tropical plane, and hence also with the with the closed convex sets in $\hat{\mathbb{R}}$. For such a set $M \subseteq \hat{\mathbb{R}}$ we denote by R_M the corresponding $\mathcal{R}$-class. Since $\hat{\mathbb{R}}$ is order isomorphic to the closed unit interval, and the closed intervals are definable topologically, combining with Lemma 3.1 yields the following natural description of the natural partial order on the $\mathcal{R}$-classes, or equivalently, on the intersection lattice of principal right ideals.

Corollary 3.3. *The lattices of principal right ideals and of principal left ideals in $M_2(\bar{\mathbb{R}})$ are isomorphic to the intersection lattice generated by the closed subintervals of the closed unit interval.*

It follows from the description of tropical vector scaling and addition given in Section 2 that the 2-generated convex cones in the affine tropical plane can take 8 essentially distinct forms. Figure 3 shows these in affine space, the captions giving the associated subsets of projective space $\hat{\mathbb{R}}$.

Using the geometric description of tropical vector operations given in Figure 2, it is easily seen that for a non-zero matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$



(a) $\emptyset$ (b) $\{-\infty\}$ (c) $\{y\}$ (d) $\{\infty\}$

(e) $[-\infty, y]$ (f) $[x, y]$ (g) $[y, \infty]$ (h) $\hat{\mathbb{R}}$

Figure 3: The 2-generated tropical convex cones of $\bar{\mathbb{R}}^2$, which correspond to closed intervals in $\hat{\mathbb{R}}$ and to the $\mathcal{R}$-classes of $M_2(\bar{\mathbb{R}})$.

the (affine) column space $C(A)$ is exactly the region of the quadrant bounded by the lines

$$\{(a+\lambda, c+\lambda) \mid \lambda \in \bar{\mathbb{R}}\} \text{ and } \{(b+\lambda, d+\lambda) \mid \lambda \in \bar{\mathbb{R}}\}.$$

If A has a zero column, $a = c = -\infty$, say, then the projective column space of A is the singleton $\{d - b\}$ (using the natural extension of subtraction to $\bar{\mathbb{R}} \times \bar{\mathbb{R}} \setminus \{(-\infty, -\infty)\}$ as described in Section 2). Otherwise, the projective column space of A is the closed interval (or singleton if $c - a = d - b$) with endpoints $c - a$ and $d - b$. Explicit descriptions of the $\mathcal{R}$-classes as sets of matrices are given in Figure 4.

For $U \subseteq M_2(\bar{\mathbb{R}})$ we define the transpose of U to be the set U^T of all transposes of matrices in U, $U^T = \{A^T : A \in U\}$. It follows easily from Corollary 3.2 that each $\mathcal{L}$-class is the transpose of an $\mathcal{R}$-class; for each closed convex subset M of $\hat{\mathbb{R}}$ we therefore define $L_M = R_M^T$.

Our next objective is to describe the $\mathcal{D}$ and $\mathcal{J}$ relations and the $\mathcal{J}$-preorder on $M_2(\bar{\mathbb{R}})$. Recall that every $\mathcal{D}$-class and every $\mathcal{J}$-class is a union of $\mathcal{R}$-classes, and that the $\mathcal{R}$-class of a matrix is determined by its projective column space. It therefore follows that the $\mathcal{D}$ and $\mathcal{J}$ relations can be described in terms of projective column spaces (or symmetrically, of projective

$$
\begin{aligned}
R_{\emptyset} &= \left\{ \begin{pmatrix} -\infty & -\infty \\ -\infty & -\infty \end{pmatrix} \right\}, \\
R_{\{-\infty\}} &= \left\{ \begin{pmatrix} a & b \\ -\infty & -\infty \end{pmatrix} \mid a, b \in \bar{\mathbb{R}}, a \oplus b \in \mathbb{R} \right\}, \\
R_{\{y\}} &= \left\{ \begin{pmatrix} a & b \\ a+y & b+y \end{pmatrix} \mid a, b \in \bar{\mathbb{R}}, a \oplus b \in \mathbb{R} \right\}, \\
R_{\{\infty\}} &= \left\{ \begin{pmatrix} -\infty & -\infty \\ a & b \end{pmatrix} \mid a, b \in \bar{\mathbb{R}}, a \oplus b \in \mathbb{R} \right\}, \\
R_{[-\infty,y]} &= \left\{ \begin{pmatrix} a & b \\ -\infty & b+y \end{pmatrix}, \begin{pmatrix} b & a \\ b+y & -\infty \end{pmatrix} \mid a, b \in \mathbb{R} \right\}, \\
R_{[x,y]} &= \left\{ \begin{pmatrix} a & b \\ a+x & b+y \end{pmatrix}, \begin{pmatrix} b & a \\ b+y & a+x \end{pmatrix} \mid a, b \in \mathbb{R} \right\}, \\
R_{[y,\infty]} &= \left\{ \begin{pmatrix} -\infty & a \\ b & a+y \end{pmatrix}, \begin{pmatrix} a & -\infty \\ a+y & b \end{pmatrix} \mid a, b \in \mathbb{R} \right\}, \\
R_{\hat{\mathbb{R}}} &= \left\{ \begin{pmatrix} a & -\infty \\ -\infty & b \end{pmatrix}, \begin{pmatrix} -\infty & a \\ b & -\infty \end{pmatrix} \mid a, b \in \mathbb{R} \right\}.
\end{aligned}
$$

Figure 4: The $\mathcal{R}$-classes of $M_2(\bar{\mathbb{R}})$. The parameters x and y run through all values in $\mathbb{R}$ with $x < y$.

row spaces). To obtain such a description, we consider the natural distance function $\delta : \hat{\mathbb{R}} \times \hat{\mathbb{R}} \to \mathbb{R} \cup \{\infty\}$ defined by

$$
\delta(x, y) = \begin{cases} |y - x| & \text{if } x, y \in \mathbb{R} \\ 0 & \text{if } x = y = -\infty \text{ or } x = y = \infty \\ \infty & \text{otherwise.} \end{cases}
$$

The function δ satisfies $\delta(x, y) = 0$ if and only if $x = y$. It is also symmetric and satisfies a triangle inequality when the usual order on $\mathbb{R}$ is extended to $\mathbb{R} \cup \{\infty\}$ in the obvious way. Thus, it is an *extended metric*, and induces obvious notions of *isometric embedding* and *isometry* between subsets of $\hat{\mathbb{R}}$. For $M, N \subseteq \hat{\mathbb{R}}$ we write $M \cong N$ to denote that M and N are isometric. Note that we do *not* require isometries or isometric embeddings to preserve the orientation of $\hat{\mathbb{R}}$, so for example $[-\infty, 0] \cong [0, \infty]$.

We define the *diameter* $d(S)$ of a subset $S \subseteq \hat{\mathbb{R}}$ (or of an isometry type of subsets of $\hat{\mathbb{R}}$) to be

$$
d(S) = \sup_{x,y \in S} \delta(x, y)
$$

where we take 0 to be the supremum of the empty set, and ∞ to be the supremum of any set not bounded above by a real number.

We shall be particularly interested in isometries and isometric embeddings between closed convex subsets of $\hat{\mathbb{R}}$, where a simple combinatorial characterisation applies. It is readily verified that two distinct such sets are isometric if and only if (i) they are both singletons, (ii) they are both closed intervals of the same finite diameter, or (iii) they are both closed intervals with one real endpoint and one endpoint at ∞ or $-\infty$. It is also easy to check that isometric embedding induces a partial order on the closed convex subsets (the only non-trivial part of this claim being that the order is antisymmetric, that is, that two such sets which embed isometrically into each other are necessarily isometric).

Proposition 3.4. *Let $A \in M_2(\bar{\mathbb{R}})$. Then $PC(A) \cong PR(A)$.*

Proof. We proceed by case analysis, considering each possible form of $PC(A)$. If $PC(A) = \emptyset$ then A is the zero matrix so $PR(A) = \emptyset$. If $PC(A) = \hat{\mathbb{R}}$ then A is a unit and so $PR(A) = \hat{\mathbb{R}}$.

If $PC(A) = \{y\}$ is a singleton then $A \in R_{\{y\}}$ for some $y \in \hat{\mathbb{R}}$. By reference to Figure 4 we see that A has at least one non-zero row (a, b). It is then easy to verify (for example, by locating A^T in Figure 4) that in each case $A^T \in R_{\{b-a\}}$, where we again use the extended subtraction defined in Section 2. Thus, $PR(A) = PC(A^T) = \{b - a\}$ is isometric to $PC(A)$.

If $PC(A) = [x, y]$ is a closed interval with real endpoints then using Figure 4 once again we see that either

$$A = \begin{pmatrix} a & b \\ a+x & b+y \end{pmatrix} \text{ or } A = \begin{pmatrix} b & a \\ b+y & a+x \end{pmatrix},$$

where $a, b \in \mathbb{R}$. In the former case we have

$$A^T = \begin{pmatrix} a & (a+x) \\ a+(b-a) & (a+x)+(b-a+y-x) \end{pmatrix}$$

from which it follows that $A^T \in R_{[b-a,b-a+y-x]}$ and $PR(A) = [b-a, b-a+y-x]$ is again a closed interval of diameter $y - x$ and hence isometric to $PC(A)$. The latter case is similar, as are the cases where one end of the interval is ∞ or $-\infty$. $\square$

Proposition 3.5. *Let M and N be closed convex subsets in $\hat{\mathbb{R}}$, and suppose $M \cong N$. Then there exists a matrix $Z \in M_2(\bar{\mathbb{R}})$ such that $PC(Z) = M$ and $PR(Z) = N$*

Proof. Once again, the proof is by case analysis with reference to Figure 4. If $M = \emptyset$ then $N = \emptyset$ and it suffices to take Z to be the zero matrix, while if $M = \hat{\mathbb{R}}$ then $N = \hat{\mathbb{R}}$ and we may take Z to be the identity matrix.

Suppose now that $M = \{x\}$ is a singleton (with $x \in \hat{\mathbb{R}}$ either real or infinite). Then $N = \{y\}$ must be a singleton too and by reference to Figure 4 it is seen that the matrices

$$A = \begin{pmatrix} 0 & y \\ x & x+y \end{pmatrix} \text{ and } B = \begin{pmatrix} -(x+y) & -x \\ -y & 0 \end{pmatrix}$$

satisfy $A \in R_{\{x\}}$, $A^T \in R_{\{y\}}$, for $x, y \neq \infty$ and $B \in R_{\{x\}}$, $B^T \in R_{\{y\}}$, for $x, y \neq -\infty$. Similarly, the matrices

$$X = \begin{pmatrix} -\infty & -\infty \\ 0 & -\infty \end{pmatrix}, Y = \begin{pmatrix} -\infty & 0 \\ -\infty & -\infty \end{pmatrix}$$

satisfy $X \in R_{\{\infty\}}$, $X^T \in R_{\{-\infty\}}$ and $Y \in R_{\{-\infty\}}$, $Y^T \in R_{\{\infty\}}$. Thus, for every pair $(x, y) \in \hat{\mathbb{R}} \times \hat{\mathbb{R}}$ there exists a matrix Z satisfying $PC(Z) = \{x\}$ and $PR(Z) = PC(Z^T) = \{y\}$ as required.

Next suppose $M = [x, y]$ is an interval with real endpoints. Then $N = [w, z]$ must be an interval with real endpoints satisfying $z - w = y - x$ so that $w + y = x + z$. Now consider the matrix

$$Z = \begin{pmatrix} 0 & w \\ x & w+y \end{pmatrix} = \begin{pmatrix} 0 & w \\ x & x+z \end{pmatrix}.$$

Referring once more to Figure 4 we see that $Z \in R_{[x,y]}$ while $Z^T \in R_{[w,z]}$ so that $PC(Z) = M$ and $PR(Z) = PC(Z^T) = N$ as required.

Now consider the case that $M = [-\infty, y]$ with y real. Then either $N = [-\infty, z]$ with z real, or $N = [x, \infty]$ with x real. In the former case it suffices to take the matrix

$$Z = \begin{pmatrix} 0 & z \\ y & -\infty \end{pmatrix},$$

while in the latter case one considers

$$Z = \begin{pmatrix} 0 & x \\ -\infty & x+y \end{pmatrix}.$$

In both cases, reference to Figure 4 once more establishes that the given matrix has the correct column and row spaces.

Finally, an argument entirely similar to the previous one applies in the case that $M = [y, \infty]$ with y real, and hence completes the proof. $\square$

Theorem 3.6. *Let $A, B \in M_2(\bar{\mathbb{R}})$. Then the following are equivalent:*

(i) $A \leq_{\mathcal{J}} B$*;*

(ii) $PC(A)$ *embeds isometrically in* $PC(B)$*;*

(iii) $PR(A)$ *embeds isometrically in* $PR(B)$*.*

Proof. The equivalence of (ii) and (iii) follows from Proposition 3.4.

Suppose next that (i) holds, and let $X, Y \in M_2(\bar{\mathbb{R}})$ be such that $A = XBY$. Then $A = XBY \leq_{\mathcal{R}} XB$ so by Lemma 3.1, $PC(A) \subseteq PC(XB)$, and in particular $PC(A)$ embeds isometrically in $PC(XB)$. Similarly, $XB \leq_{\mathcal{L}} B$ so by Lemma 3.1, $PR(XB)$ embeds isometrically in $PR(B)$. Now, by Proposition 3.4, $PC(XB) \cong PR(XB)$ and $PR(B) \cong PC(B)$, so by transitivity of isometric embedding we conclude that $PC(A)$ embeds isometrically in $PC(B)$ and (ii) holds.

Finally, suppose (ii) holds. Let $M \subseteq PC(B)$ be the image of an isometric embedding of $PC(A)$ into $PC(B)$. Then M is clearly a closed convex set isometric to $PC(A)$ which by Proposition 3.4 is also isometric to $PR(A)$. Hence, by Proposition 3.5, there is a matrix $Z \in M_2(\bar{\mathbb{R}})$ such that $PC(Z) = M \subseteq PC(B)$ and $PR(Z) = PR(A)$. But now by Corollary 3.2 and Lemma 3.1 we have $A\mathcal{L}Z$ and $Z \leq_{\mathcal{R}} B$, from which it follows that $A \leq_{\mathcal{J}} B$. □

Theorem 3.7. *Let $A, B \in M_2(\bar{\mathbb{R}})$. Then the following are equivalent:*

(i) $A\mathcal{D}B$*;*

(ii) $A\mathcal{J}B$*;*

(iii) $PC(A) \cong PC(B)$*;*

(iv) $PR(A) \cong PR(B)$*.*

Proof. The equivalence of (iii) and (iv) follows from Proposition 3.4. That (i) implies (ii) follows from general facts about semigroups (see for example [6]), while the fact that (ii) implies (iii) is a corollary of Theorem 3.6 and our observation that the isometric embeddability order on closed convex subsets of $\hat{\mathbb{R}}$ is antisymmetric.

Finally, if (iii) holds then by Proposition 3.4 we have $PR(A) \cong PC(A) \cong PC(B)$, so by Proposition 3.5 there is a matrix $Z \in M_2(\bar{\mathbb{R}})$ such that $PC(Z) = PC(B)$ and $PR(Z) = PR(A)$. By Corollary 3.2 it follows that $B\mathcal{R}Z$ and $Z\mathcal{L}A$. Since $\mathcal{D}$ is an equivalence relation containing $\mathcal{L}$ and $\mathcal{R}$ we conclude that $A\mathcal{D}B$ so that (i) holds. □

Theorems 3.6 and 3.7 allows us to deduce a great deal about the two-sided ideal structure of $M_2(\bar{\mathbb{R}})$. An immediate corollary is a description of the lattice order on the two-sided principal ideals (or equivalently, on the $\mathcal{J}$-classes).

Corollary 3.8. *The lattice of principal two-sided ideals in $M_2(\bar{\mathbb{R}})$ is isomorphic to the lattice of isometry types of closed convex subsets of $\hat{\mathbb{R}}$ under the partial order given by isometric embedding.*

We now turn our attention to non-principal ideals, which it transpires can also be characterized by certain isometry types of convex sets in $\hat{\mathbb{R}}$. Let $\mathbb{S}$ be the set of convex sets in $\hat{\mathbb{R}}$ consisting of all the closed convex sets, all the open intervals of finite diameter, and the open interval $(-\infty, \infty)$. Note that we exclude the half-infinite open intervals. Once again, it is easily seen that isometric embedding induces a partial order on the isometry types of sets in $\mathbb{S}$. Note also that no two isometry types of sets in $\mathbb{S}$ admit isometric embeddings of exactly the same collection of closed convex sets.

Theorem 3.9. *Let I be an ideal of $M_2(\bar{\mathbb{R}})$. Then there exists a subset $I' \in \mathbb{S}$ such that for all $X \in M_2(\bar{\mathbb{R}})$ we have $X \in I$ if and only if the projective column space of X embeds isometrically into I'. Moreover, the set I' is unique up to isometry.*

Proof. Let I be an ideal of $M_2(\bar{\mathbb{R}})$, and let T be the set of all isometry types of closed convex sets in $\hat{\mathbb{R}}$ which arise as projective column spaces (or equivalently, projective row spaces) of matrices in I. If T has a maximal element under the isometric embedding order, then it follows from Theorem 3.6 that it suffices to take I' to be this convex set.

Suppose, then, that T has no maximal element. Then clearly it cannot contain the isometry type of a convex set of infinite diameter (since there are only finitely many such up to isometry, and they are above all other convex sets in the isometric embedding order), but must contain infinitely many intervals of finite diameter. If the diameters of these intervals are bounded above by a real number, then we let w be the supremum of the diameters. Since T has no maximal element, this supremum is not attained in T. It follows from Theorem 3.6 that a matrix lies in I if and only if its projective column space has diameter strictly less than w. This is the case exactly if the projective column space embeds isometrically in an open interval of diameter w, so it suffices to take I' to be such an interval.

On the other hand, if the diameters of the intervals are not bounded above then, by Theorem 3.6 again, we see that I contains every matrix with projective column space of finite diameter, and it follows that we may take I' to be the open interval $(-\infty, \infty)$.

Finally, the uniqueness up to isometry of I' follows from Theorem 3.6 and the fact that no two distinct isometry types of sets in $\mathbb{S}$ embed exactly the same closed convex sets. $\square$

For I an ideal of $M_2(\bar{\mathbb{R}})$, we denote by $S(I)$ the unique convex subset $S(I) \in \mathbb{S}$ such that I consists of those matrices with projective column space which embeds isometrically in $S(I)$.

Corollary 3.10. *The two-sided ideals of $M_2(\bar{\mathbb{R}})$ are totally ordered under inclusion.*

Proof. The claim follows immediately from Theorem 3.9, and the obvious fact that the isometry types of sets in $\mathbb{S}$ are totally ordered under isometric embedding. $\square$

Corollary 3.11. *Let I be an ideal in $M_2(\bar{\mathbb{R}})$. Then the following are equivalent:*

(i) $S(I)$ is closed;

(ii) I is principal;

(iii) I is finitely generated.

Proof. By Proposition 3.5, every closed convex set is the projective column space of some matrix in $M_2(\bar{\mathbb{R}})$, so that (i) implies (ii) follows from Theorem 3.6. That (ii) implies (iii) is by definition. Finally, suppose (iii) holds, let G be a finite generating set for I, and let $S = \{PC(X) \mid X \in G\}$. By Corollary 3.10, S is totally ordered under isometric embedding, and since it is finite, it must contain a maximum element. This maximum element is a closed convex set, and an easy argument now shows that it must be equal to $S(I)$. $\square$

The equivalence of (ii) and (iii) in Corollary 3.11 may be viewed as an algebraic manifestation of the fact that every finitely generated tropical convex cone in $\bar{\mathbb{R}}^2$ is 2-generated.

Corollary 3.12. *Every ideal in $M_2(\bar{\mathbb{R}})$ is either principal, or the difference between a principal ideal and its generating $\mathcal{J}$-class.*

Proof. Let I be an ideal and consider the convex set $S(I) \in \mathbb{S}$. If $S(I)$ is closed then by Corollary 3.11, I is principal. Otherwise, $S(I)$ is an open interval. Let J be the the smallest closed interval in $\hat{\mathbb{R}}$ containing $S(I)$. Clearly, a given closed convex set K embeds isometrically into $S(I)$ if and only if it embeds isometrically into J but is not isometric to J. Hence, by

Theorems 3.6 and 3.7, a matrix is in I if and only if it lies in the ideal corresponding to J (which by Corollary 3.11 is principal) but not in the $\mathcal{J}$-class corresponding to J. $\square$

4 Idempotents and Subgroups

Our aim in this section is to identify the idempotent elements of $M_2(\bar{\mathbb{R}})$, and draw some conclusions about both its semigroup-theoretic structure and its maximal subgroups. Recall that an element e in a semigroup is called *idempotent* if $e^2 = e$.

Proposition 4.1. *The idempotents of $M_2(\bar{\mathbb{R}})$ are exactly the matrices of the form*

$$\begin{pmatrix} 0 & x \\ y & x+y \end{pmatrix}, \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}, \begin{pmatrix} x+y & x \\ y & 0 \end{pmatrix} \text{ and } \begin{pmatrix} -\infty & -\infty \\ -\infty & -\infty \end{pmatrix}$$

where $x, y \in \bar{\mathbb{R}}$ with $x + y \leq 0$.

Proof. It is readily verified by direct computation that these matrices are idempotent. Conversely, suppose that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then we have

$$\begin{aligned} \max(a+a, b+c) &= a, \quad (1) & \max(b+c, d+d) &= d, \quad (2) \\ \max(a+c, c+d) &= c, \quad (3) & \max(a+b, b+d) &= b. \quad (4) \end{aligned}$$

By (1) and (2) we see that $a + a \leq a$ and $d + d \leq d$ giving $-\infty \leq a, d \leq 0$. First suppose that $a < 0$. Then $a+a < a$ and by (1) it follows that $a = b+c$. If $d = 0$ then we obtain a matrix of the form

$$\begin{pmatrix} b+c & b \\ c & 0 \end{pmatrix},$$

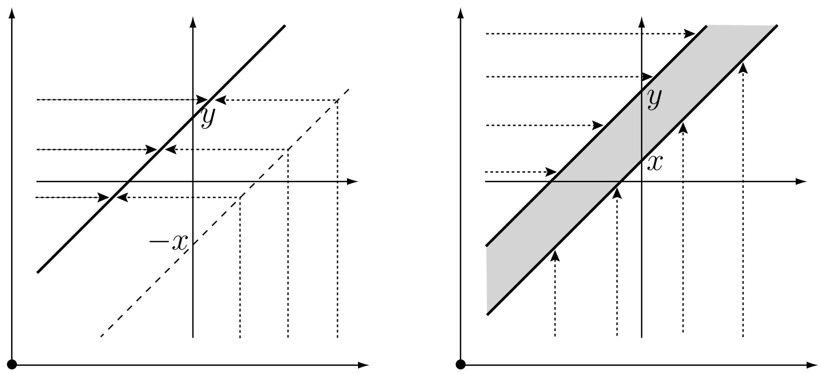
where $b, c \in \bar{\mathbb{R}}$ with $b + c < 0$. On the other hand, if $d \neq 0$ then by (2) we find $-\infty \leq a = b + c = d < 0$. Since $a, d < 0$, by (3) and (4) we see that $b = c = -\infty$. Now, since $a = d = b + c$ this yields the zero matrix.

Next suppose that $a = 0$. By (1) we have that $b + c \leq 0$. Arguing as before, by (2) we have either $d = 0$ or $d = b + c$, giving matrices of the form

$$\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & b \\ c & b+c \end{pmatrix}$$

where $b, c \in \bar{\mathbb{R}}$ with $b + c \leq 0$. $\square$

While the purely computational approach to finding idempotents employed in the proof of Proposition 4.1 is straightforward in the 2×2 case, it is conceptually unenlightening and quickly becomes intractable in higher dimensions. In any semigroup of functions, the idempotents are exactly the *projections*, that is, those functions which fix their images pointwise. In $M_2(\bar{\mathbb{R}})$, then, an idempotent element is a matrix which (viewed as acting from the left on column vectors) fixes pointwise the tropical convex cone generated by its own columns. Figure 5 illustrates the geometric action of some typical idempotents. In higher dimensions, the complex structure of tropical convex cones [11] makes it a delicate task to locate the idempotents by geometric arguments, but nevertheless we believe that only this approach is feasible.



(a) A projection onto the column space corresponding to $R_{\{y\}}$.

(b) A projection onto the column space corresponding to $R_{[x,y]}$.

Figure 5: Examples of projections.

Cross-referencing Proposition 4.1 with Figure 4, we quickly see that $M_2(\bar{\mathbb{R}})$ has an idempotent in every $\mathcal{R}$-class. Recall that a semigroup S is called *regular* if for every element $X \in S$ there is an element $Y \in S$ such that $XYX = X$ (*von Neumann regularity* in the terminology of ring theory). It is well known that a semigroup is regular if and only if every $\mathcal{R}$-class contains an idempotent, so we have established the following theorem.

Theorem 4.2. *The semigroup $M_2(\bar{\mathbb{R}})$ of all 2×2 tropical matrices is regular.*

We now turn our attention to maximal subgroups of $M_2(\bar{\mathbb{R}})$. It is a foundational result of semigroup theory (see for example [6]) that every subgroup of a semigroup lies in a unique maximal subgroup, and that the maximal subgroups are exactly the $\mathcal{H}$-classes of idempotent elements. We thus begin by describing those $\mathcal{H}$-classes which contain idempotents.

Theorem 4.3. *Let M and N be closed convex subsets of $\hat{\mathbb{R}}$. Then the $\mathcal{H}$-class $R_M \cap L_N$ contains an idempotent if and only if one of the following conditions holds:*

(i) $M = \{x\}$ and $N = \{y\}$ with $\{x, y\} \neq \{-\infty, \infty\}$;

(ii) $M = -N = \{-x \mid x \in N\}$ where $|N| \neq 1$.

Proof. Suppose first that $R_M \cap L_N$ contains an idempotent E. Then E must have one of the four forms given by Proposition 4.1. Clearly if E is the zero matrix then $M = N = \emptyset$ and (ii) holds. If E has the form $\begin{pmatrix} 0 & x \\ y & x+y \end{pmatrix}$ for $x, y \in \bar{\mathbb{R}}$ with $x + y \leq 0$, then it is readily verified that $PC(E) = \{y\}$ and $PR(E) = \{x\}$ and hence (i) holds. An entirely similar argument holds if E has the form $\begin{pmatrix} x+y & x \\ y & 0 \end{pmatrix}$, where this time $PC(E) = \{-x\}$ and $PR(E) = \{-y\}$. Finally, if E has the form $\begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix}$ with $x + y \leq 0$ then a simple computation shows that $PC(E) = [y, -x]$ and $PR(E) = [x, -y]$ so that once again (ii) holds.

Conversely, suppose (i) holds, say $M = \{x\}$ and $N = \{y\}$, where $\{x, y\} \neq \{-\infty, \infty\}$ so that $x + y$ is well-defined. If $x + y \leq 0$ then $x, y \neq \infty$ and the matrix $\begin{pmatrix} 0 & y \\ x & x+y \end{pmatrix}$ is an idempotent by Proposition 4.1, and is easily seen (by computing the projective row and column spaces) to lie in the claimed $\mathcal{H}$-class. On the other hand, if $x + y \geq 0$ then $x, y \neq -\infty$, so we have $-x, -y \in \bar{\mathbb{R}}$ and $(-x) + (-y) \leq 0$. It follows by Proposition 4.1 that the matrix $\begin{pmatrix} -x-y & -y \\ -x & 0 \end{pmatrix}$ is idempotent and once again it is easily verified that it lies in $R_M \cap L_N$.

Finally, suppose (ii) holds. If M is empty then so is N, and the zero matrix is an idempotent in $R_M \cap L_N$. Suppose, then, that M is a closed interval $[x, y]$ with $x, y \in \hat{\mathbb{R}}$ and $x < y$. Then $y \neq -\infty$ so $-y$ is well-defined, and $x + (-y) < 0$. Hence, by Proposition 4.1, the matrix $\begin{pmatrix} 0 & -y \\ x & 0 \end{pmatrix}$ is idempotent. Once more, it is straightforward to verify that this matrix lies in $R_M \cap L_N$. □

Having ascertained which $\mathcal{H}$-classes are maximal subgroups, it remains to identify the algebraic structure of each. It is a basic fact of semigroup theory (see for example [6]) that maximal subgroups (if any) within the same $\mathcal{D}$-class are all isomorphic, so it suffices to study one maximal subgroup in each $\mathcal{D}$-class.

Theorem 4.4. *The maximal subgroups in the $\mathcal{D}$-class of elements with row and column space isometric to a closed convex subset $M \subseteq \hat{\mathbb{R}}$ are isomorphic to:*

(i) the trivial group, if $M = \emptyset$;

(ii) the additive group $\mathbb{R}$ of real numbers, if M is a point, or an interval with precisely one real endpoint;

(iii) the direct product $\mathbb{R} \times S_2$, if M is an interval with two real endpoints;

(iv) the wreath product $\mathbb{R} \wr S_2$, if $M = \hat{\mathbb{R}}$.

Proof. If $M = \emptyset$ then the only matrix in $R_M \cap L_M$ is the zero matrix, so this $\mathcal{H}$-class is isomorphic to the trivial group.

Now suppose $M = \{x\}$ is a singleton. Since maximal subgroups in a $\mathcal{D}$-class are always isomorphic, by Theorem 3.7 it suffices to consider the case that $M = \{-\infty\}$. By Theorem 4.3, $R_M \cap L_M$ contains an idempotent. Reference to Figure 4 shows that

$$R_M \cap L_M = \{W_a \mid a \in \mathbb{R}\}$$

where

$$W_a = \begin{pmatrix} a & -\infty \\ -\infty & -\infty \end{pmatrix}.$$

Direct calculation shows that $W_a W_b = W_{a+b}$ for all $a, b \in \mathbb{R}$ so that $R_M \cap L_M$ is isomorphic to the additive group $\mathbb{R}$ as required.

Next suppose $M = [x, y]$ is an interval with distinct real endpoints, so that $x - y < 0$. Then by Theorem 4.3, setting $N = -M = [-y, -x]$ we have that the $\mathcal{H}$-class $R_M \cap L_N$ contains an idempotent. A direct computation using Figure 4 shows that

$$R_M \cap L_N = \{X_a, Y_a \mid a \in \mathbb{R}\}$$

where

$$X_a = \begin{pmatrix} a & a - y \\ a + x & a \end{pmatrix} \text{ and } Y_a = \begin{pmatrix} a & a - x \\ a + y & a \end{pmatrix}.$$

Simple calculation, recalling the fact that $x-y<0$, shows that $X_aX_b = X_{a+b}$, $X_aY_b = Y_bX_a = Y_{a+b}$ and $Y_aY_b = X_{a+b+(y-x)}$ for all $a, b \in \mathbb{R}$. We deduce that X_0 is idempotent and hence is the identity of $R_M \cap L_N$ and that the X_a's form a central subgroup isomorphic to the real numbers. Moreover, choosing $z = (x-y)/2$ we see that $(Y_z)^2 = X_0$ and every element Y_b can be written in the form Y_zX_a for some $a \in \mathbb{R}$. We have shown that $R_M \cap L_N$ is the product of commuting subgroups with trivial intersection, one of them isomorphic to $\mathbb{R}$ and the other to S_2. It follows that the subgroup is isomorphic to $\mathbb{R} \times S_2$, as claimed.

Now suppose M is an interval with one real and one infinite endpoint. By Theorem 3.7 we may assume that $M = [x, \infty]$. Set $N = -M = [-\infty, -x]$. Then by Theorem 4.3 we have that $R_M \cap L_N$ contains an idempotent. Another reference to Figure 4 reveals that

$$R_M \cap L_N = \{Z_a \mid a \in \mathbb{R}\}$$

where

$$Z_a = \begin{pmatrix} a & -\infty \\ a+x & a \end{pmatrix}.$$

Once again, we find that $Z_aZ_b = Z_{a+b}$ so that $R_M \cap L_N$ is isomorphic to the additive group $\mathbb{R}$

Finally, if $M = \hat{\mathbb{R}}$ then we also have $N = \hat{\mathbb{R}}$, and $R_M \cap L_N$ is the group of units. We remarked in Section 2 that it is known that the group of units of $M_n(\bar{\mathbb{R}})$ is isomorphic to the permutation group wreath product $\mathbb{R} \wr (S_n, \{1, \dots, n\})$. In the case $n = 2$, since the right translation action of S_2 on itself is isomorphic to its standard action on $\{1, 2\}$, the group of units is also isomorphic to the wreath product $\mathbb{R} \wr S_2$ of abstract groups. $\square$

We remarked in Section 2 that it is known that every group admitting a faithful representation by finite dimensional tropical matrices has an abelian subgroup of finite index [10]. In the case of groups admitting faithful 2×2 tropical matrix representations, we can now be rather more precise.

Corollary 4.5. *Every group admitting a faithful representation by 2×2 tropical matrices is either torsion-free abelian or has a torsion-free abelian subgroup of index 2.*

In general, we conjecture that a group admitting a faithful representation by $n \times n$ tropical matrices must have a torsion free abelian subgroup of index at most $n!$. Note that the conjectured bound is sharp, since $M_n(\bar{\mathbb{R}})$ has group of units $\mathbb{R} \wr (S_n, \{1, \dots, n\})$, in which the least index of a torsion free abelian subgroup is exactly $n!$.

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