Perfect generalized characters inducing the Alperin-McKay conjecture

Eaton, Charles W.

2008

MIMS EPrint: **2009.68**

ISSN 1749-9097

Available online at www.sciencedirect.com
ScienceDirect
Journal of Algebra 320 (2008) 2301–2327
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Perfect generalized characters inducing the Alperin–McKay conjecture

Charles W. Eaton [1]

School of Mathematics, Alan Turing Building, The University of Manchester, Oxford Road, Manchester, M13 9PL, United Kingdom

Received 20 August 2007
Available online 23 July 2008
Communicated by Michel Broué

Abstract

It is well known that the perfect isometries predicted in Broué's conjecture do not always exist when the defect groups are non-abelian, even when the blocks have equivalent Brauer categories. We consider perfect generalized characters which induce bijections between the sets of irreducible characters of height zero of a block and of its Brauer correspondent in the normalizer of a defect group, hence providing in these cases an 'explanation' for the numerical coincidence predicted in the Alperin–McKay conjecture. In this way the perfect isometries predicted in Broué's conjecture for blocks with abelian defect groups are generalized. Whilst such generalized characters do not exist in general, we show that they do exist when the defect groups are non-abelian trivial intersection subgroups of order p^3, as well as for ${}^2B_2(q)$ for q a power of two and $PSU_3(q)$ for all q. Further, we show that these blocks satisfy a generalized version of an isotypy.

Keywords: Modular representation theory; Characters of finite groups

1. Introduction

Let G and H be finite groups. Let B be a p-block of G and b a p-block of H. Write $\mathrm{Irr}(G, B)$ (or sometimes $\mathrm{Irr}(B)$ if the meaning is clear) for the set of irreducible characters of G in B, and write $\mathbb{Z}\,\mathrm{Irr}(G, B)$ for the group of generalized characters of G all of whose irreducible con-

E-mail address: c.eaton@man.ac.uk.
[1] The author is supported by a Royal Society University Research Fellowship.

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2008.05.031

stituents are in B. Note that $\mathbb{Z}\operatorname{Irr}(G \times H^\circ, B \otimes b^\circ)$ may also be regarded as the group generated by the characters of B-b-bimodules, where b° and H° are the opposite algebra and group respectively. We define all blocks with respect to a sufficiently large complete discrete valuation ring $\mathcal{O}$ with residue field k of characteristic p and field of fractions K of characteristic zero. Characters are always defined with respect to K.

For the convenience of the reader we include some definitions found in [9]. For $\mu \in \mathbb{Z}\operatorname{Irr}(G \times H^\circ, B \otimes b^\circ)$, define $I_\mu : \mathbb{Z}\operatorname{Irr}(H, b) \to \mathbb{Z}\operatorname{Irr}(G, B)$ and $R_\mu : \mathbb{Z}\operatorname{Irr}(G, B) \to \mathbb{Z}\operatorname{Irr}(H, b)$ by

$$I_\mu(\theta)(g) = \frac{1}{|H|} \sum_{h \in H} \mu\big(g, h^{-1}\big)\theta(h)$$

and

$$R_\mu(\chi)(h) = \frac{1}{|G|} \sum_{g \in G} \mu\big(g^{-1}, h\big)\chi(g),$$

where $\theta \in \mathbb{Z}\operatorname{Irr}(H, b)$, $\chi \in \mathbb{Z}\operatorname{Irr}(G, B)$. Hence I_μ and R_μ are adjoint with respect to the usual scalar product on characters.

Given I_μ, we may recover μ as

$$\mu = \sum_{\theta \in \operatorname{Irr}(H,b)} \theta I_\mu(\theta).$$

A generalized character $\mu \in \mathbb{Z}\operatorname{Irr}(G \times H^\circ, B \otimes b^\circ)$ is called *perfect* if the following two conditions hold:

(I) for each $g \in G$ and $h \in H$, we have $\frac{\mu(g,h)}{|C_G(g)|} \in \mathcal{O}$ and $\frac{\mu(g,h)}{|C_H(h)|} \in \mathcal{O}$;
(II) if g and h are both p-singular or both p-regular, then $\mu(g, h) = 0$.

The motivation for this definition is that we think of μ as coming from a linear combination of $\mathcal{O}G$-$\mathcal{O}H$-bimodules which are projective on restriction to $\mathcal{O}G$ and to $\mathcal{O}H$. Note that induction and restriction are examples of maps I_μ and R_μ. In this case μ is the character of KG with left G-action and right H-action. We say that I_μ is a *perfect isometry* if μ is perfect and I_μ is an isometry with respect to the usual form $(-,-)_G$ on class functions.

Conjecture 1.1 *(Broué). If B has an abelian defect group D and b is the Brauer correspondent of B in $N_G(D)$, then there exists perfect $\mu \in \mathbb{Z}\operatorname{Irr}(G \times H^\circ, B \otimes b^\circ)$ such that I_μ and R_μ defined above are isometries.*

It is well known that such a perfect isometry does not necessarily exist when D is non-abelian, even when B and b satisfy various identities which occur when there is a perfect isometry (such as there being the same number of irreducible characters at each height), as is the case when D is trivial intersection. Specifically, it is expected that many of the properties which follow from the existence of a perfect isometry hold when the Brauer categories $\mathbf{Br}_B(G)$ and $\mathbf{Br}_b(H)$ are equivalent. The group ${}^2B_2(8)$ for $p = 2$ is the best-known example where a perfect isometry cannot exist (see [13] or [25]) but the Brauer categories are equivalent.

For this reason, and guided by conjectures such as those of Alperin and Dade, we take the view that if there should be a generalization of Broué's conjecture to blocks with arbitrary defect groups, then the blocks where this generalization would be most clearly apparent, would be those with trivial intersection (TI) defect groups. Recall that a block B of a group G is a *TI defect block* if a defect group D is TI, in the sense that if $g \in G \setminus N_G(D)$, then $D^g \cap D = 1$.

Some further evidence that there should be a generalization of Broué's conjecture which applies to blocks with TI defect groups comes from the following. The reader should note that if there is a perfect isometry between blocks, then their Cartan matrices have the same elementary divisors (with equal multiplicities). If q is a power of p, write $m_B(q)$ for the multiplicity of q as an elementary divisor of the Cartan matrix of B.

Proposition 1.2. *Let B be a p-block of G and suppose B has a TI defect group D. Let b be the unique block of $N_G(D)$ with Brauer correspondent B. Let q be a power of p. Then $m_B(q) = m_b(q)$.*

Proof. Since D is TI, it follows that D and 1 are the only radical p-subgroups of G contained in D. Hence by [23, 4.3] (using the complex of radical p-subgroups, and noting that only those chains whose final terms are contained in a conjugate of D contribute to the sums) we have that $m_B(1) - m_b(1) = k(B) - k(b)$. But $k(B) = k(b)$ by [4], so the result follows for $q = 1$.

Now suppose that $q \neq 1$. The result is then an easy consequence of those in [8], but we outline the proof here.

By [8, (II)1.3] and [8, (II)2.8]

$$m_B(q) = \sum_{Q \in \mathcal{P}_q(G)} m_B^{G_{p'}}(Q),$$

where $m_B^{G_{p'}}(Q) = \dim((Tr_Q^{N_G(Q)}(kC_{G_{p'}}(Q)))Br_Q(\bar{e}_B))$, and $\mathcal{P}_q(G)$ is a set of G-conjugacy class representatives of p-subgroups of order q. Only sums with Q lying in a conjugate of D contribute to the sum, and so we assume $Q \leqslant D$. Since $q \neq 1$ and D is TI, we have $N_G(Q) \leqslant N_G(D)$. Hence $m_B^{G_{p'}}(Q) = m_b^{N_G(D)_{p'}}(Q)$ for each $Q \leqslant D$. Since $N_G(D)$ controls fusion of subgroups of D, it follows that $m_B(q) = m_b(q)$ as required. $\square$

Blocks with TI defect groups were classified in [3], and it has been our strategy to use this as a tool to find and provide evidence for the shape of a generalization of Broué's conjecture. We present here an idea which has resulted from this strategy. This is at the level of characters, and we make no attempt to formulate anything more structural, in order that there should be as few restrictions as possible to the interpretation of the evidence. This being said, we see the ultimate goal of such investigations as being an explanation at the level of categories for the numerical coincidences highlighted in conjectures such as Alperin's and Dade's.

We are motivated in part by a talk given by Jonathan Alperin at Oberwolfach in 2003 where, amongst other things, he highlighted a result of Gabriel Navarro which gives a connection between the height zero characters in a case where Broué's conjecture does not necessarily hold, and suggested that this may be possible to generalize. Recall that the *defect* of an irreducible character χ of G is the (non-negative) integer $d(\chi)$ such that $p^{d(\chi)}\chi(1)_p = |G|_p$. If χ lies in a block B of G with defect group D, where $|D| = p^{d(B)}$, then $d(\chi) \leqslant d(B)$, and we call $d(B) - d(\chi)$ the *height* of χ. Write $\mathrm{Irr}_0(G, B)$ for the set of irreducible characters in B of height zero. Navarro's result says that if G is p-solvable, with $P \in \mathrm{Syl}_p(G)$ such that $N_G(P) = P$, then for each $\theta \in \mathrm{Irr}_0(P)$ and $\chi \in \mathrm{Irr}_0(G)$, we have that $\mathrm{Ind}_P^G(\theta)$ and $\mathrm{Res}_P^G(\chi)$ each have precisely one irreducible constituent of height zero.

We are interested in when the following weaker form of Navarro's result holds. For a block B of an arbitrary finite group with defect group D, let b be the unique block of $N_G(D)$ with $b^G = B$. We say that B satisfies property (P) if:

(P) *There exists perfect $\mu \in \mathbb{Z}\mathrm{Irr}(G \times N_G(D)^\circ, B \otimes b^\circ)$ such that for each $\theta \in \mathrm{Irr}_0(N_G(D), b)$ and $\chi \in \mathrm{Irr}_0(G, B)$, we have that $I_\mu(\theta)$ and $R_\mu(\chi)$ each have precisely one irreducible constituent of height zero, and this occurs with multiplicity ± 1.*

By adjointness, if B satisfies property (P), then I_μ, R_μ induce a bijection $\mathrm{Irr}_0(G, B) \leftrightarrow \mathrm{Irr}_0(N_G(D), b)$. Hence (P) implies the Alperin–McKay conjecture, which states that $|\mathrm{Irr}_0(G, B)| = |\mathrm{Irr}_0(N_G(D), b)|$.

Remark 1.3. Suppose that Brauer's conjecture that if a block B has abelian defect groups, then every irreducible character of B has height zero, holds. Then Broué's conjecture is precisely equivalent to (P) holding for all blocks with abelian defect groups.

Actually, for blocks with trivial intersection defect groups, we will often be able to show something a little stronger. Throughout write Φ_B for the character of B regarded as a left $\mathcal{O}G$-module and a right $\mathcal{O}N_G(D)$-module. Note that Φ_B is itself perfect, and I_{Φ_B} and R_{Φ_B} are 'blockwise induction and restriction,' i.e., induction or restriction, taking only components from the block or its Brauer correspondent.

(P+) *There exists $\mu \in \mathbb{Z}\mathrm{Irr}(G \times N_G(D)^\circ, B \otimes b^\circ)$ of the form $\mu = \Phi + \sum_t \epsilon_t \Gamma_{i_t} \Phi_{j_t}$, where each Γ_{i_t} is the character of a projective indecomposable module of B and each Φ_{j_t} is the character of a projective indecomposable module of b, and $\epsilon_t = \pm 1$, such that for each $\theta \in \mathrm{Irr}_0(N_G(D), b)$ and $\chi \in \mathrm{Irr}_0(G, B)$, we have that $I_\mu(\theta)$ and $R_\mu(\chi)$ each have precisely one irreducible constituent of height zero, and this occurs with multiplicity ± 1.*

Note that by [9] every generalized character μ as in (P+) is perfect, so (P+) implies (P). We see below that (P+) is important when D is TI since is implies the existence of a generalized version of an isotypy defined below. We will abbreviate 'principal indecomposable module' to PIM.

A consequence of (P+) is that the strengthening of the Alperin–McKay conjecture due to Isaacs and Navarro (see [22]) holds. This states that for each integer κ, we have $|\mathrm{Irr}_0(G, B, [c\kappa])| = |\mathrm{Irr}_0(N_G(D), b, [\kappa])|$, where $c = [G : N_G(D)]_{p'}$ and $\chi \in \mathrm{Irr}_0(G, B, [r])$ if $\chi(1)_{p'} \equiv \pm r \bmod p$. If further D is TI, then we obtain equality with congruences modulo $|D|$ when $D \in \mathrm{Syl}_p(G)$ (since in this case $[G : N_G(D)] \equiv 1 \bmod |D|$). This has already been observed to hold for blocks with TI defect groups in [4].

If G is p-solvable and $P \in \mathrm{Syl}_p(G)$ satisfies $N_G(P) = P$, then the principal block B_0 is the only p-block of G of maximal defect. Then B_0, with G acting on the left and P on the right gives μ, so that (P+) holds in this case.

The purpose of this paper is to present evidence which suggests that (P) should hold whenever $N_G(D)$ controls fusion in D, for example when D is TI.

We make it clear that although the Alperin–McKay conjecture suggests that $|\mathrm{Irr}_0(G,B)| = |\mathrm{Irr}_0(N_G(D),b)|$ for every block B, property (P) does not hold in general. For consider the principal block B of $G = PSU_3(2)$ for $p = 2$. In this case a Sylow p-subgroup P is self-normalizing and dihedral of order 8. The irreducible characters of B have degrees $1, 3, 3, 7, 6$. It is easy to see that we cannot have μ satisfying $\mu(1,h) = 0$ for every $h \in P - Z(P)$.

In some cases, for example with the unitary groups, to save notation we do not always calculate precisely which irreducible characters of positive height occur as irreducible constituents, since we are still able to compute μ. Hence for an arbitrary generalized character χ we write χ_0 for the character formed by omitting irreducible constituents of positive height, i.e., if $\chi = \sum_{\chi_i \in \mathrm{Irr}(G)} a_i \chi_i$, then $\chi_0 = \sum_{\chi_i \in \mathrm{Irr}_0(G)} a_i \chi_i$.

1.1. Generalized isotypies

Let B and b be p-blocks of finite groups G and H respectively. Suppose that $\mu \in \mathbb{Z}\mathrm{Irr}(G \times H^\circ, B \otimes b^\circ)$ is perfect.

Write $CF_{p'}(G,B,K)$ for the subspace of $CF(G,B,K)$ consisting of class functions vanishing on p-singular elements. Then R_μ and I_μ induce maps $R_{\mu,p'} : CF_{p'}(G,B,K) \to CF_{p'}(H,b,K)$ and $I_{\mu,p'} : CF_{p'}(H,b,K) \to CF_{p'}(G,B,K)$.

Let $y \in G$ be an element of p-power order, and let B_y be a block of $C_G(y)$ with $B_y^G = B$, i.e., $(\langle y \rangle, B_y)$ is a B-subpair. Define the decomposition map

$$d_G^{(y,B_y)} : CF(G,B,K) \to CF_{p'}\big(C_G(y), B_y, K\big)$$

by $d_G^{(y,B_y)}(\alpha)(x) = \alpha(xye_{B_y})$ for $x \in C_G(y)_{p'}$ (see [18]).

Suppose now that B and b share a defect group D. Fix a maximal B-subpair (D, B_D) and a maximal b-subpair (D, b_D). For each $Q \leqslant D$ this fixes a unique B-subpair $(Q, B_Q) \leqslant (D, B_D)$ and a unique b-subpair $(Q, b_Q) \leqslant (D, b_D)$. If $y \in D$, then we write, e.g., B_y for $B_{\langle y \rangle}$. We say that R_μ is *compatible with fusion* if for each $y \in D$ there exists $\mu_y \in \mathbb{Z}\mathrm{Irr}(C_G(y) \times C_H(y)^\circ, B_y \otimes (b_y)^\circ)$ such that $d_{N_G(D)}^{(y,b_y)} \circ R_\mu = R_{\mu_y,p'} \circ d_G^{(y,B_y)}$ and each μ_y gives property (P) for B_y. We say that there is a *generalized isotypy* between B and b if the Brauer categories $\mathbf{Br}_B(G)$ and $\mathbf{Br}_b(H)$ are equivalent and there is R_μ compatible with fusion (see, for example, [18] for a definition of the Brauer category).

Note that if B is a controlled block or has trivial intersection defect group D, then by the remarks in [9, 4B] $\mathbf{Br}_B(G)$ and $\mathbf{Br}_b(N_G(D))$ are equivalent, where b is the unique block of $N_G(D)$ with $b^G = B$. Whilst we are on the subject, we also note that whilst it is true that every TI block (in the sense of [1]) is a controlled block, it is not true that every block with TI defect groups is controlled, just as it is not true that every block with abelian defect groups is controlled. For example, the block given in [2, p. 127], which has abelian TI defect groups but is not TI, is not controlled.

We note that condition (P+) is particularly well suited to finding generalized isotypies:

Lemma 1.4. *Let B be a block of a finite group G with TI defect group D. Let b be the unique block of $N_G(D)$ with $b^G = B$. Suppose that condition* (P+) *holds for B. Then there is a generalized isotypy between B and b.*

Proof. We have already seen that B and b have equivalent Brauer categories. Let $y \in D$ be non-trivial. If $\chi \in \mathrm{Irr}(B)$, then $R_\mu(\chi) = \mathrm{Res}^G_{N_G(D)}(\chi) + \Lambda$, where Λ vanishes on p-singular elements, and it follows that $d_{N_G(D)}^{(y,b_y)} \circ R_\mu(\chi) = d_{N_G(D)}^{(y,b_y)} \circ \mathrm{Res}_{N_G(D)}^G(\chi)$. We have $C_G(y) \leqslant N_G(D)$. Take $\mu_y = \sum_{\chi \in \mathrm{Irr}(C_G(y))} \chi^2$, so that R_{μ_y} is the identity map and μ_y gives (P+). Hence $R_{\mu_y,p'} \circ d_G^{(y,B_y)}(\chi) = d_G^{(y,B_y)}(\chi)$, and $d_{N_G(D)}^{(y,b_y)} \circ R_\mu(\chi) = R_{\mu_y,p'} \circ d_G^{(y,B_y)}(\chi)$ as required. □

For brevity, we say that a block B of G satisfies (PI) if:

(PI) There is a generalized isotypy between B and b, where b is the unique block of $N_G(D)$ with $b^G = B$, where D is a defect group for B.

The main result of the paper, which we prove using the classification of finite simple groups, is:

Theorem 1.5. *Let B be a block of G with non-abelian TI defect group D with $|D| = p^3$. Let b be the unique block of $N_G(D)$ with $b^G = B$. Then there is a generalized isotypy between B and b.*

This, along with other evidence presented below, leads us to make the following conjecture:

Conjecture 1.6. *Let B be a block of G with TI defect group D. Let b be the unique block of $N_G(D)$ with $b^G = B$. Then there is a generalized isotypy between B and b.*

The paper is structured as follows: In Sections 2–6 we verify (P+) for various blocks of the p' central extensions of automorphisms of non-abelian simple groups having TI defect groups, including all those with non-abelian defect groups of order p^3. In Section 7 we give two examples of controlled blocks for which (P+) holds. In Section 8 we give a proof of Theorem 1.5 by means of Clifford-theoretic reductions, a classification and the calculations in the previous sections.

2. Suzuki groups in the defining characteristic

Let $G = {}^2B_2(q)$, where $q = 2^{2m+1}$ and $m \geqslant 1$. Write $r = 2^{m+1}$. Then the principal 2-block B of G is the unique 2-block of G of positive defect. Let $P \in \mathrm{Syl}_p(G)$, and let b be the Brauer correspondent of B in $N_G(P)$. The characters of G and $N_G(P)$ are determined in [27], and we use the same labelling for the characters of G (but not for the characters of $N_G(P)$, which are not all given labels in [27]).

The irreducible characters of B are: the trivial character 1; X_i for $1 \leqslant i \leqslant q/2-1$, of degree q^2+1; Y_j for $1 \leqslant j \leqslant \frac{q+r}{4}$, of degree $(q-r+1)(q-1)$; Z_k for $1 \leqslant k \leqslant \frac{q-r}{4}$, of degree $(q+r+1)(q-1)$; W_l for $i = 1, 2$, of degree $r(q-1)/2$. The characters X_i are induced from the non-trivial linear characters of $N_G(P)$.

The irreducible characters of b are: the trivial character 1; linear characters θ_i for $1 \leqslant i \leqslant q/2-1$, where $\theta_i^G = X_i$; linear characters θ_i' for $1 \leqslant i \leqslant q/2-1$, where $(\theta_i')^G = \theta_i^G$; ζ of degree $q-1$, ζ_j for $= 1, 2$, of degree $r(q-1)/2$.

The characters of the PIMs of b that we need are $\Phi_i = \theta_i + \zeta + r/2\zeta_1 + r/2\zeta_2$, for $1 \leqslant i \leqslant q/2-1$.

The irreducible characters of height zero of B decompose on restriction to $N_G(P)$ as follows:
$\mathrm{Res}_{N_G(P)}^G(1) = 1$;
$\mathrm{Res}_{N_G(P)}^G(X_i) = \theta_i + \theta_i' + \zeta + r/2\zeta_1 + r/2\zeta_2 = \theta_i' + \Phi_i$;
$\mathrm{Res}_{N_G(P)}^G(Y_j) = \zeta + (r/2-1)(\zeta_1 + \zeta_2) = -\theta_j - \zeta_1 - \zeta_2 + \Phi_j$;

$\mathrm{Res}^G_{N_G(P)}(Z_k) = \zeta + (r/2+1)(\zeta_1+\zeta_2)$. This is $-\theta_{t+k}+\zeta_1+\zeta_2+\Phi_{t+k}$ for $1 \leqslant k \leqslant \frac{q-r}{4}-1$, where $t=(q+r)/4$.

Hence $\mu = \Phi_B - \sum_{i=1}^{\frac{q+r}{4}} \Phi_i(X_i+Y_i+W_1+W_2) - \sum_{k=1}^{\frac{q-r}{4}-1} \Phi_{t+k}(X_{t+k}+Z_k-W_1-W_2)$ gives R_μ with

$R_\mu(1)=1$;
$R_\mu(X_i)=\theta_i'$;
$R_\mu(Y_j)=-\theta_j-\zeta_1-\zeta_2$;
$R_\mu(Z_k)=\theta_{t+k}+\zeta_1+\zeta_2$ for $1 \leqslant k \leqslant \frac{q-r}{4}-1$;
$R_\mu(Z_{(q-r)/4})=\zeta+(r/2+1)(\zeta_1+\zeta_2)$;
$R_\mu(W_l)=\mathrm{Res}^G_{N_G(P)}(W_l)-\sum_{i=1}^{\frac{q+r}{4}}\Phi_i+\sum_{k=1}^{\frac{q-r}{4}-1}\Phi_{t+k}$.

From the character table for G we see that $X_i+Y_i+W_1+W_2$ and $X_{t+k}+Z_k-W_1-W_2$ vanish on p-singular elements for each t and k. Hence both are $\mathbb{Z}$-linear combinations of characters of PIMs of G (by, for example [24, 2.16]), and so (P+) holds in this case.

3. $^2G_2(3)$

Let $G={}^2G_2(3)$ and $p=3$. Let B be the principal block and $P \in \mathrm{Syl}_p(G)$. Let b be the principal block of $N_G(P)$. Then $P \cong 3_-^{1+2}$ and is TI.

Here (blockwise) induction and restriction are sufficient to give (P+), i.e., $\mu=\Phi_B$. This results in the following bijection, with signs, between $\mathrm{Irr}_0(B)$ and $\mathrm{Irr}_0(b)$:

$$\begin{pmatrix}1_1\\1_2\\1_3\\7_1\\7_2\\7_3\\8_1\\8_2\\8_3\end{pmatrix} \leftrightarrow \begin{pmatrix}1_1\\1_4\\1_2\\1_6\\1_3\\1_5\\2_1\\2_2\\2_3\end{pmatrix}.$$

We note that B and b are in fact isotypic, although not via a perfect isometry satisfying (P+) (we thank Shigeo Koshitani for this observation).

4. $\mathrm{Aut}(^2B_2(32))$

Let $G={}^2B_2(32)$ and $p=5$. Let B be the principal block and $P \in \mathrm{Syl}_p(G)$. Let b be the principal block of $N_G(P)$. Note that B is the unique 5-block of positive defect of G. Write $^2B_2(32) \cong N \lhd G$, so $[G:N]=5$. Then $P \cong 5_-^{1+2}$ and is TI.

The irreducible characters of height zero of B and b are all extensions of irreducible characters of N and $N_N(P \cap N)$ respectively (actually, this happens in general by [10, Proposition 2]), and we label them accordingly: $\mathrm{Irr}_0(B)=\{1_{1,j}\colon 1 \leqslant j \leqslant 5\} \cup \{124_{1,j}\colon 1 \leqslant j \leqslant 5\} \cup \{124_{2,j}\colon 1 \leqslant j \leqslant 5\} \cup \{1024_{1,j}\colon 1 \leqslant j \leqslant 5\} \cup \{1271_{1,j}\colon 1 \leqslant j \leqslant 5\}$ and $\mathrm{Irr}_0(b)=\{1_{i,j}\colon 1 \leqslant i \leqslant 4,\ 1 \leqslant j \leqslant 5\} \cup \{4_{1,j}\colon 1 \leqslant j \leqslant 5\}$.

The characters of the PIMs of b are $\Phi_i=\sum_{1 \leqslant j \leqslant 5}(1_{i,j}+4_{1,j})+5\cdot 20_1$, for $1 \leqslant i \leqslant 4$.

The characters of the PIMs of B are $\Gamma_1=\sum_{1 \leqslant j \leqslant 5}(1_{1,j}+1024_{1,j})$; $\Gamma_2=\sum_{1 \leqslant j \leqslant 5}(124_{1,j}+$

$1271_{1,j}) + 5 \cdot 6355_1$; $\Gamma_3 = \sum_{1 \leqslant j \leqslant 5}(124_{2,j} + 1271_{1,j}) + 5 \cdot 6355_1$; $\Gamma_4 = \sum_{1 \leqslant j \leqslant 5}(1024_{1,j} + 1271_{1,j}) + 5 \cdot 6355_1$.

The irreducible characters in $\mathrm{Irr}_0(B)$ restrict to $N_G(P)$ as follows:

$\mathrm{Res}^G_{N_G(P)}(1_{1,j}) = 1_{1,j}$;

$\mathrm{Res}^G_{N_G(P)}(124_{1,j}) = \sum_{1 \leqslant k \leqslant 5}(1_{4,k} + 4_{1,k}) + 5 \cdot 20_1 - 1_{4,j} = \Phi_4 - 1_{4,j}$;

$\mathrm{Res}^G_{N_G(P)}(124_{2,j}) = \sum_{1 \leqslant k \leqslant 5}(1_{2,k} + 4_{1,k}) + 5 \cdot 20_1 - 1_{2,j} = \Phi_2 - 1_{2,j}$;

$\mathrm{Res}^G_{N_G(P)}(1024_{1,j}) = 2\Phi_1 + 2\Phi_2 + 2\Phi_3 + 2\Phi_4 + 4_{1,j} + 20_1$;

$\mathrm{Res}^G_{N_G(P)}(1271_{1,j}) = 2\Phi_1 + 3\Phi_2 + 2\Phi_3 + 3\Phi_4 + 1_{3,j} + 20_1$.

Hence we deduce that $\mu = \Phi_B - 2(\Phi_1 + \Phi_3)\Gamma_4 - \Phi_2(\Gamma_3 + 2\Gamma_4) - \Phi_4(\Gamma_2 + 2\Gamma_4)$ gives (P+) and the following bijection with signs between $\mathrm{Irr}_0(B)$ and $\mathrm{Irr}_0(b)$:

$$\begin{pmatrix} 1_{1,j} \\ 124_{1,j} \\ 124_{2,j} \\ 1024_{1,j} \\ 1271_{1,j} \end{pmatrix} \leftrightarrow \begin{pmatrix} 1_{1,j} \\ -1_{4,j} \\ -1_{2,j} \\ 4_{1,j} \\ 1_{3,j} \end{pmatrix},$$

where $1 \leqslant j \leqslant 5$.

We note that B and b are in fact isotypic, although not via a perfect isometry satisfying (P+) (we thank Shigeo Koshitani for this observation).

5. *McL*

Let $N = 3.McL$, the (perfect) triple cover of *McL*, and let $p = 5$. Let $P \in \mathrm{Syl}_5(G)$, so $P \cong 5_+^{1+2}$ and P is TI. Let B_0 be the principal 5-block, and B_1, B_2 be the two remaining blocks of positive defect. B_1 and B_2 both have maximal defect and are faithful. Further, they are Galois conjugate, so it suffices to consider just B_1. We label the characters of G as in the GAP library.

Let b_t be the Brauer correspondent of B_t in $N_G(P)$.

To simplify notation, we label the irreducible characters in the block b_t by $1_{t,1}, \dots, 1_{t,8}$, $2_{t,1}, \dots, 2_{t,4}$, $20_{t,1}, \dots, 20_{t,6}$, $24_{t,1}$, and the characters of the PIMs by

$$\Phi_{t,i} = \begin{cases} 1_{t,i} + 20_{t,1} + 20_{t,2} + 20_{t,4} + 20_{t,5} + 20_{t,6} + 24_{t,1} & \text{for } i = 1, 2, 7, 8, \\ 1_{t,i} + 20_{t,1} + \dots + 20_{t,5} + 24_{t,1} & \text{for } i = 3, 4, 5, 6, \\ 2_{t,i-8} + 2(20_{t,1} + 20_{t,2} + 20_{t,3} + 20_{t,6} + 24_{t,1}) + 20_{t,4} + 20_{t,5} & \text{for } i = 9, 10, \\ 2_{t,i-8} + 20_{t,1} + 20_{t,2} + 2(20_{t,3} + \dots + 20_{t,6} + 24_{t,1}) & \text{for } i = 11, 12, \end{cases}$$

where $n_{t,i}$ corresponds to n_{3i+t-2} in the notation of the GAP library.

We first consider the principal block B_0.

An outer automorphism of G normalizing P interchanges $1_{0,1}$ and $1_{0,2}$, $1_{0,7}$ and $1_{0,8}$, $20_{0,1}$ and $20_{0,2}$, $20_{0,4}$ and $20_{0,5}$, leaving the other irreducible characters of b_0 fixed.

We direct the reader to [7] for the characters of the PIMs of B_0, which are labelled $\Gamma_1, \dots, \Gamma_9$, $\Gamma_{11}, \Gamma_{12}, \Gamma_{13}$.

The generalized character μ_{B_0} giving (P+) in this case is rather involved, so we give it in table form. Define the generalized character μ_1 which is the linear combination of $\Phi_{0,i}\Gamma_j$ for $i = 1, \dots, 12$ and $j \in \{1, \dots, 9, 11, 12, 13\}$, with coefficients as below:

	$\Phi_{0,1}$	$\Phi_{0,2}$	$\Phi_{0,3}$	$\Phi_{0,4}$	$\Phi_{0,5}$	$\Phi_{0,6}$	$\Phi_{0,7}$	$\Phi_{0,8}$	$\Phi_{0,9}$	$\Phi_{0,10}$	$\Phi_{0,11}$	$\Phi_{0,12}$
Γ_1	0	0	0	0	0	0	0	0	0	0	0	0
Γ_2	0	0	0	0	0	0	0	0	1	0	0	−1
Γ_3	0	0	0	1	0	1	0	0	−1	0	0	1
Γ_4	0	0	1	1	0	0	0	0	−1	0	0	1
Γ_5	0	0	0	−1	0	0	0	0	0	1	1	1
Γ_6	1	1	1	0	1	1	1	1	0	0	0	0
Γ_7	1	1	1	1	1	1	1	1	0	0	0	0
Γ_8	0	1	0	1	0	0	0	1	1	0	1	1
Γ_9	1	0	0	1	0	0	1	0	1	0	1	1
Γ_{11}	−1	−1	0	0	0	0	−1	−1	4	4	3	3
Γ_{12}	1	0	0	1	0	0	0	0	2	3	3	4
Γ_{13}	0	1	0	1	0	0	0	0	2	3	3	4

Let $\mu_{B_0} = \Phi_{B_0} - \mu_1$. Then $R_{\mu_{B_0}} : \mathbb{Z}\,\mathrm{Irr}(B_0) \to \mathbb{Z}\,\mathrm{Irr}(b_0)$, giving (P+), may be given as follows (we feel it is informative in this case to include the characters of positive height):

$$\begin{pmatrix} 1_1 \\ 22_1 \\ 231_1 \\ 252_1 \\ 770_1 \\ 770_2 \\ 896_1 \\ 896_2 \\ 3520_1 \\ \\ 3520_2 \\ 4752_1 \\ 5103_1 \\ 5544_1 \\ 8019_1 \\ 8019_2 \\ 9856_1 \\ 9856_2 \\ 10395_1 \\ 10395_2 \end{pmatrix} \to \begin{pmatrix} 1_{0,3} \\ 2_{0,4} - 20_{0,1} - 20_{0,2} + 20_{0,4} + 20_{0,5} \\ 1_{0,5} - 20_{0,3} \\ 2_{0,1} + 20_{0,1} + 20_{0,2} - 20_{0,4} - 20_{0,5} \\ -1_{0,6} + 2_{0,1} - 2_{0,4} - 2.20_{0,4} - 20_{0,5} - 20_{0,3} - 24_{0,1} \\ -1_{0,6} + 2_{0,1} - 2_{0,4} - 20_{0,4} - 2.20_{0,5} - 20_{0,4} - 24_{0,1} \\ 1_{0,4} + 20_{0,3} \\ -24_{0,1} - 20_{0,4} - 20_{0,5} - 20_{0,1} - 20_{0,2} \\ 1_{0,1} + 1_{0,5} + 1_{0,7} - 1_{0,4} + 1_{0,2} - 1_{0,6} + 1_{0,8} \\ \quad -2_{0,3} - 2_{0,4} + 2(20_{0,1} + 20_{0,2}) - 3.20_{0,3} \\ 1_{0,3} + 1_{0,5} + 1_{0,4} + 1_{0,6} + 20_{0,6} \\ 2_{0,2} \\ -2_{0,3} - 20_{0,3} \\ -1_{0,6} - 20_{0,4} - 20_{0,5} - 20_{0,1} - 20_{0,2} \\ -1_{0,1} + 20_{0,3} \\ -1_{0,2} + 20_{0,3} \\ 1_{0,7} - 20_{0,3} \\ 1_{0,8} - 20_{0,3} \\ -1_{0,6} - 2_{0,1} + 2_{0,4} - 2.20_{0,1} - 20_{0,2} - 20_{0,3} - 24_{0,1} \\ -1_{0,6} - 2_{0,1} + 2_{0,4} - 20_{0,1} - 2.20_{0,2} - 20_{0,3} - 24_{0,1} \end{pmatrix}$$

An outer automorphism interchanges 770_1 and 770_2, 8019_1 and 8019_2, 9856_1 and 9856_2, 10395_1 and 10395_2, leaving every other irreducible character in B_0 fixed. An outer automorphism also interchanges Γ_{12} and Γ_{13}, leaving every other Γ_i fixed. We see that R_μ (and I_μ) is $\mathrm{Aut}(G)$-equivariant.

Now write $E = \mathrm{Aut}(G) \geqslant G$, and note that $[N_E(P) : N_G(P)] = [E : G] = 2$. Let B_E be the principal block of E and let b_E be the principal block of $N_E(P)$. Label the irreducible characters of B_E as in [7].

Then there is $\mu_{B_E} \in \mathbb{Z}\operatorname{Irr}(E \times N_E(P)^\circ, B_E \otimes b_E^\circ)$, obtained from μ_{B_0} in the natural way, giving (P+) such that $R_{\mu_{B_E}}$ gives the following bijection with signs $\operatorname{Irr}_0(B_E) \leftrightarrow \operatorname{Irr}_0(b_E)$:

$$\begin{pmatrix} 1_1 \\ 1_2 \\ 22_1 \\ 22_2 \\ 231_1 \\ 231_2 \\ 252_1 \\ 252_2 \\ 896_1 \\ 896_2 \\ 896_3 \\ 896_4 \\ 4752_1 \\ 4752_2 \\ 5103_1 \\ 5103_2 \\ 5544_1 \\ 5544_2 \\ 16038_1 \\ 19712_1 \end{pmatrix} \to \begin{pmatrix} 1_7 \\ 1_8 \\ 2_3 \\ 2_4 \\ 1_5 \\ 1_6 \\ 2_2 \\ 2_1 \\ 1_2 \\ 1_1 \\ -24_1 \\ -24_2 \\ 2_5 \\ 2_8 \\ -2_6 \\ -2_7 \\ -1_4 \\ -1_3 \\ -2_9 \\ 2_{10} \end{pmatrix}$$

Finally consider B_1. The characters of the PIMs of B_1 are labelled $\Gamma_{18}, \ldots, \Gamma_{29}$. Define the generalized character μ_2 which is the linear combination of $\Phi_{1,i}\Gamma_j$ for $i = 1, \ldots, 12$ and $j \in \{18, \ldots, 29\}$, with coefficients as below:

	$\Phi_{1,1}$	$\Phi_{1,2}$	$\Phi_{1,3}$	$\Phi_{1,4}$	$\Phi_{1,5}$	$\Phi_{1,6}$	$\Phi_{1,7}$	$\Phi_{1,8}$	$\Phi_{1,9}$	$\Phi_{1,10}$	$\Phi_{1,11}$	$\Phi_{1,12}$
Γ_{18}	0	0	0	0	0	0	0	0	0	0	0	0
Γ_{19}	0	−1	0	0	0	0	0	0	1	0	0	0
Γ_{20}	0	0	0	0	0	0	0	−1	1	0	0	0
Γ_{21}	1	0	0	0	1	1	1	0	0	−1	0	0
Γ_{22}	0	1	1	1	0	0	0	1	−1	1	0	0
Γ_{23}	0	0	0	1	1	0	1	0	2	0	0	0
Γ_{24}	−1	0	0	0	−1	0	−1	0	1	1	2	1
Γ_{25}	1	1	2	1	1	2	0	1	0	1	0	−1
Γ_{26}	0	0	−1	0	0	−1	0	0	1	2	1	2
Γ_{27}	−1	0	0	−1	−1	−1	−1	0	2	3	3	3
Γ_{28}	2	2	1	2	1	1	1	2	2	2	3	3
Γ_{29}	3	3	2	2	2	2	3	3	2	1	3	3

Let $\mu_{B_1} = \Phi_{B_1} - \mu_2$. Then $R_{\mu_{B_1}} : \mathbb{Z}\operatorname{Irr}(B_1) \to \mathbb{Z}\operatorname{Irr}(b_1)$ gives (P+) and the following bijection with signs $\operatorname{Irr}_0(B_1) \leftrightarrow \operatorname{Irr}_0(b_1)$:

$$\begin{pmatrix} 126_1 \\ 126_3 \\ 792_1 \\ 2376_1 \\ 2376_3 \\ 2772_1 \\ 4752_2 \\ 5103_2 \\ 6336_1 \\ 6336_3 \\ 8019_3 \\ 8019_5 \\ 8064_1 \end{pmatrix} \to \begin{pmatrix} -1_{1,2} \\ -1_{1,8} \\ -2_{1,1} \\ -1_{1,3} \\ -1_{1,4} \\ 2_{1,4} \\ 2_{1,2} \\ -2_{1,3} \\ 1_{1,5} \\ 1_{1,7} \\ 24_{1,1} \\ -1_{1,6} \\ -1_{1,1} \end{pmatrix}$$

6. Unitary groups

In this section we show that (P+) holds for all blocks in the defining characteristic for $SU_3(q)$, $GU_3(q)$, $SU_3(q).2$ and $GU_3(q).2$, where the extension is by a field automorphism of order two. Unfortunately general arguments such as those used in [18] to study $SL_2(q)$ may not be translated neatly into our situation, and we are forced to construct the generalized character μ in each of these cases.

Let $N = SU_3(q)$, where $q = p^m$ is odd and $m \geqslant 1$. Let $\tau \in \operatorname{Aut}(N)$ be a field automorphism of order two (coming from an automorphism of $\mathbb{F}_{q^2}$). Define $G = N\langle\tau\rangle$, and choose a Sylow p-subgroup P of N such that $\tau \in N_G(P)$. Let $d = (3, q+1)$. Then $|Z(G)| = |Z(N)| = d$, and there are precisely d blocks $C_0, \dots, C_{d-1}$ of positive defect of N, where C_0 is the principal block. These correspond to the irreducible characters of $Z(G)$, and have maximal defect. Let c_i be the unique block of $N_N(P)$ with Brauer correspondent C_i. Note that $C_1^\tau = C_2$ and $c_1^\tau = c_2$ if $d = 3$. Let B_0 be the principal block of G, and let B_1 be the unique block of G covering C_1 (and C_2). Let b_i be the unique block of $N_G(P)$ with Brauer correspondent B_i. Then each B_i and b_i also has defect group P, and b_i covers c_i.

Write $\operatorname{Irr}^\tau(H) = \{\theta \in \operatorname{Irr}(H)\colon \theta^\tau = \theta\}$.

The irreducible characters of N and $N_N(P)$ are given in [19], and we use the notation given there.

Now $N_N(P)$ has irreducible characters $\theta_1^{(u)}$ for $0 \leqslant u \leqslant q^2 - 2$; $\theta_{q(q-1)}^{(u)}$ for $0 \leqslant u \leqslant q$; $\theta_{(q^2-1)/d}^{(u,v)}$ for $0 \leqslant u, v \leqslant d-1$. If $d = 1$, then we sometimes write simply θ_{q^2-1} for $\theta_{(q^2-1)/d}^{(0,0)}$. If $v \equiv u \bmod (q^2-1)$, then by $\theta_1^{(v)}$ we mean $\theta_1^{(u)}$.

If $d = 1$, then $\operatorname{Irr}(c_0) = \operatorname{Irr}(N_N(P))$. If $d = 3$, then $\theta_1^{(u)} \in \operatorname{Irr}(c_i)$ if $u \equiv i \bmod 3$, $\theta_{q(q-1)}^{(u)} \in \operatorname{Irr}(c_i)$ if $u \equiv i \bmod 3$, and $\theta_{(q^2-1)/d}^{(u,v)} \in \operatorname{Irr}(c_i)$ if $u = i$.

Lemma 6.1.

(a) $(\theta_1^{(u)})^\tau = \theta_1^{(qu)}$, *so* $\theta_1^{(u)} \in \mathrm{Irr}^\tau(N_N(P))$ *if and only if* $q+1 \mid u$.
$(\theta_{q(q-1)}^{(u)})^\tau = \theta_{q(q-1)}^{(q+1-u)}$, *so* $\theta_{q(q-1)}^{(u)} \in \mathrm{Irr}^\tau(N_N(P))$ *if and only if* $u = 0, \frac{q+1}{2}$.
$(\theta_{(q^2-1)/d}^{(1,v)})^\tau = \theta_{(q^2-1)/d}^{(2,v)}$ *for each* v, *so* $\theta_{(q^2-1)/d}^{(u,v)} \in \mathrm{Irr}^\tau(N_N(P))$ *if and only if* $u = 0$.
Hence $|\mathrm{Irr}^\tau(N_N(P))| = q+1+d$.
(b) $\theta_1^{(u)} \in \mathrm{Irr}^\tau(N_N(P))$ *if and only if* $\theta_1^{(-qu)} \in \mathrm{Irr}^\tau(N_N(P))$.
(c) τ *cannot permute* $\theta_1^{(u)}$ *and* $\theta_1^{(-qu)}$.

Proof. (a) and (b) are immediate from the character table.

If $(\theta_1^{(u)})^\tau = \theta_1^{(-qu)}$, then $q^2-1 \mid 2qu$. But q is odd, so $q+1 \mid u$, so $\theta_1^{(u)}$ is fixed by τ, a contradiction. □

Blau and Michler, in [6], computed the number of conjugacy classes of $N_N(P)$ and N fixed by a field automorphism. By Brauer's permutation lemma the number of fixed irreducible characters is as follows:

Proposition 6.2. *(See [6].)* $|\mathrm{Irr}^\tau(N)| = q+2+d$.

The irreducible characters of N are as follows:

$\chi_1 \in \mathrm{Irr}(C_0)$;

$\chi_{q^2-q} \in \mathrm{Irr}(C_0)$;

χ_{q^3}, the Steinberg character, in a p-block of defect zero;

$\chi_{q^2-q+1}^{(u)}$, for $1 \leqslant u \leqslant q$;

$\chi_{q(q^2-q+1)}^{(u)}$, for $1 \leqslant u \leqslant q$;

$\chi_{(q-1)(q^2-q+1)}^{(u,v)}$, for $1 \leqslant u \leqslant (q+1)/3$ and $u < v < 2(q+1)/3$ (there are $\frac{1}{6}q(q-1)$ such characters if $d=1$ and $\frac{1}{6}(q+1)(q-2)$ if $d=3$);

$\chi_{q^3+1}^{(u)}$ for $1 \leqslant u \leqslant q^2-1$ such that $q-1 \nmid u$, and we identify $\chi_{q^3+1}^{(u)}$ and $\chi_{q^3+1}^{(u_1)}$ when $u_1 \equiv -qu \bmod q^2-1$ (there are $\frac{1}{2}(q+1)(q-2)$ such characters);

$\chi_{(q+1)^2(q-1)}^{(u)}$, where $q^2-q+1 \nmid u$, and we identify $\chi_{(q+1)^2(q-1)}^{(u)}$ and $\chi_{(q+1)^2(q-1)}^{(u_1)}$ when $u_1 \equiv -qu \bmod q^2-q+1$, and $\chi_{(q+1)^2(q-1)}^{(u)}$ and $\chi_{(q+1)^2(q-1)}^{(u_2)}$ when $u_2 \equiv q^2u \bmod q^2-q+1$ (there are $\frac{1}{3}q(q-1)$ such characters if $d=1$ and $\frac{1}{3}(q+1)(q-2)$ if $d=3$);

if $d=3$, then we also have $\chi_{(q-1)(q^2-q+1)/3}^{(u)} \in \mathrm{Irr}(C_0)$, for $0 \leqslant u \leqslant 2$; $\chi_{(q+1)^2(q-1)/3}^{(u,v)}$, for $0 \leqslant u \leqslant 2$ and $1 \leqslant v \leqslant 2$.

If $d=1$, then each irreducible character, except for the Steinberg character, is in C_0. If $d=3$, then $\chi_{q^2-q+1}^{(u)} \in \mathrm{Irr}(C_i)$ if $u \equiv i \bmod 3$; $\chi_{q(q^2-q+1)}^{(u)} \in \mathrm{Irr}(C_i)$ if $u \equiv i \bmod 3$; $\chi_{(q-1)(q^2-q+1)}^{(u,v)} \in \mathrm{Irr}(C_i)$ if $u+v \equiv i \bmod 3$; $\chi_{q^3+1}^{(u)} \in \mathrm{Irr}(C_i)$ if $u \equiv i \bmod 3$; $\chi_{(q+1)^2(q-1)}^{(u)} \in \mathrm{Irr}(C_i)$ if $u \equiv i \bmod 3$; $\chi_{(q+1)^2(q-1)/3}^{(u,v)} \in \mathrm{Irr}(C_v)$.

In the following table, recall that for $\chi \in \mathrm{Irr}(N)$, we denote by $\mathrm{Res}^N_{N_N(P)}(\chi)_0$ the part of the restriction whose summands have height zero.

χ	$\mathrm{Res}^N_{N_N(P)}(\chi)_0$ $(d=1)$	$\mathrm{Res}^N_{N_N(P)}(\chi)_0$ $(d=3)$
χ_1	$\theta_1^{(0)}$	$\theta_1^{(0)}$
$\chi_{q^2-q+1}^{(u)}$	$\theta_1^{((q-1)u)}$	$\theta_1^{((q-1)u)}$
$\chi_{(q-1)(q^2-q+1)}^{(u,v)}$	θ_{q^2-1}	$\theta_{(q^2-1)/3}^{(u+v,0)}+\theta_{(q^2-1)/3}^{(u+v,1)}+\theta_{(q^2-1)/3}^{(u+v,2)}$
$\chi_{q^3+1}^{(u)}$	$\theta_1^{(u)}+\theta_1^{(-qu)}+\theta_{q^2-1}$	$\theta_1^{(u)}+\theta_1^{(-qu)}+\theta_{(q^2-1)/3}^{(u+v,0)}+\theta_{(q^2-1)/3}^{(u+v,1)}+\theta_{(q^2-1)/3}^{(u+v,2)}$
$\chi_{(q+1)^2(q-1)}^{(u)}$	θ_{q^2-1}	$\theta_{(q^2-1)/3}^{(u+v,0)}+\theta_{(q^2-1)/3}^{(u+v,1)}+\theta_{(q^2-1)/3}^{(u+v,2)}$
$\chi_{(q-1)(q^2-q+1)/3}^{(u)}$	–	$\theta_{(q^2-1)/3}^{(0,u)}$
$\chi_{(q+1)^2(q-1)/3}^{(u,v)}$	–	$\theta_{(q^2-1)/3}^{(v,u)}$

We gather some facts concerning the action of τ on $\mathrm{Irr}(N)$, all of which follow from examination of the character tables, the previous lemmas and Proposition 6.2:

Lemma 6.3.

(a) $\chi_{q^3+1}^{(u)} \in \mathrm{Irr}^\tau(N)$ *if and only if* $\theta_1^{(u)} \in \mathrm{Irr}^\tau(N_N(P))$.
(b) $(\chi_{q^2-q+1}^{(u)})^\tau = \chi_{q^2-q+1}^{(q+1-u)}$. *Hence* $\chi_{q^2-q+1}^{(u)} \in \mathrm{Irr}^\tau(N)$ *if and only if* $\theta_1^{((q-1)u)} \in \mathrm{Irr}^\tau(N_N(P))$.
(c) $(\chi_{q(q^2-q+1)}^{(u)})^\tau = \chi_{q(q^2-q+1)}^{(q+1-u)}$. *Hence* $\chi_{q(q^2-q+1)}^{(u)} \in \mathrm{Irr}^\tau(N)$ *if and only if* $\theta_1^{((q-1)u)} \in \mathrm{Irr}^\tau(N_N(P))$.
(d) $(\chi_{q^3+1}^{(u)})^\tau = \chi_{q^3+1}^{(qu)}$. *Hence* $\chi_{q^3+1}^{(u)} \in \mathrm{Irr}^\tau(N)$ *if and only if* $\theta_1^{(u)} \in \mathrm{Irr}^\tau(N_N(P))$.
(e) $\chi_{(q+1)^2(q-1)}^{(u)}$ *is never fixed by* τ.
(f) *Each* $\chi_{(q-1)(q^2-q+1)/3}^{(u)}$ *is fixed by* τ.
(g) $(\chi_{(q+1)^2(q-1)/3}^{(u,1)})^\tau = \chi_{(q+1)^2(q-1)/3}^{(u,2)}$.
(h) τ *fixes precisely* $(q-3)/2$ *of the* $\chi_{q^3+1}^{(u)}$.
(i) τ *fixes precisely* $(q-1)/2$ *of the* $\chi_{(q-1)(q^2-q+1)}^{(u,v)}$ *if* $d=1$ *and* $(q-3)/2$ *if* $d=3$.
(j) $\chi_{(q-1)(q^2-q+1)}^{(u,v)} \in \mathrm{Irr}^\tau(N)$ *if and only if either* (I) $u-2v \equiv 0 \bmod (q+1)$ *or* (II) $v-2u \equiv 0 \bmod (q+1)$. *In case* (I) $q(u+v) \equiv v-2u \bmod (q+1)$, *and in case* (II) $q(u+v) \equiv u-2v \bmod (q+1)$.

The PIMs of $N_N(P)$ have characters $\Phi^{(u)}$ with $\Phi_0^{(u)} = \theta_1^{(u)} + \theta_{q^2-1}$, for $0 \leqslant u \leqslant q^2-1$ if $d=1$ and $\Phi_0^{(u)} = \theta_1^{(u)} + \theta_{(q^2-1)/3}^{(u_1,0)} + \theta_{(q^2-1)/3}^{(u_1,1)} + \theta_{(q^2-1)/3}^{(u_1,2)}$, for $0 \leqslant u \leqslant q^2-1$ if $d=3$, where $u_1 \equiv u \bmod 3$.

Just as with the Suzuki groups, it is not necessary to work with the characters of the PIMs of N themselves, but only with linear combinations of such characters. Define

$$\Gamma_1^{u,v,w} := \chi^{(u,v)}_{(q-1)(q^2-q+1)} + \chi^{(w)}_{q^3+1} - \chi^{(u+v)}_{q(q^2-q+1)} - \chi^{(v-2u)}_{q(q^2-q+1)},$$

$$\Gamma_2^{u,v,w} := \chi^{(u,v)}_{(q-1)(q^2-q+1)} + \chi^{(w)}_{q^3+1} - \chi^{(u-2v)}_{q(q^2-q+1)} - \chi^{(u+v)}_{q(q^2-q+1)},$$

$$\Gamma_3^{u,v,w} := \chi^{(u,v)}_{(q-1)(q^2-q+1)} + \chi^{(w)}_{q^3+1} - \chi^{(v-2u)}_{q(q^2-q+1)} - \chi^{(u-2v)}_{q(q^2-q+1)},$$

$$\Gamma_4^{u,v} := \chi^{(u)}_{q^3+1} + \chi^{(v)}_{(q+1)^2(q-1)} + \chi^{(u)}_{q(q^2-q+1)}.$$

If $w \equiv u - 2v \bmod (q+1)$, then $\Gamma_1^{u,v,w}$ vanishes on p-singular elements of N. If $w \equiv v - 2u \bmod (q+1)$, then $\Gamma_2^{u,v,w}$ vanishes on p-singular elements. If $w \equiv u + v \bmod (q+1)$, then $\Gamma_3^{u,v,w}$ vanishes on p-singular elements. Finally, $\Gamma_4^{u,v}$ always vanishes on p-singular elements, hence in these cases all are $\mathbb{Z}$-linear combinations of characters of PIMs.

We gather together some easy facts concerning the action of τ on the $\Gamma_i^{u,v,w}$:

Lemma 6.4.

1. *Suppose $w \equiv u - 2v \bmod (q+1)$. Then*
   (a) *$(\Gamma_1^{u,v,w})^G$ vanishes on p-singular elements of G,*
   (b) *if $\Gamma_1^{(u,v,w)}$ is τ-stable, then $\chi^{(u,v)}_{(q-1)(q^2-q+1)}$ and $\chi^{(w)}_{q^3+1}$ are τ-stable and $(\chi^{(u+v)}_{q(q^2-q+1)})^\tau = \chi^{(v-2u)}_{q(q^2-q+1)}$.*
2. *Suppose $w \equiv v - 2u \bmod (q+1)$. Then*
   (a) *$(\Gamma_2^{u,v,w})^G$ vanishes on p-singular elements of G,*
   (b) *if $\Gamma_2^{(u,v,w)}$ is τ-stable, then $\chi^{(u,v)}_{(q-1)(q^2-q+1)}$ and $\chi^{(w)}_{q^3+1}$ are τ-stable and $(\chi^{(u-2v)}_{q(q^2-q+1)})^\tau = \chi^{(u+v)}_{q(q^2-q+1)}$.*
3. *If $w \equiv u + v \bmod (q+1)$, then $(\Gamma_3^{u,v,w})^G$ vanishes on p-singular elements of G.*
4. *$(\Gamma_4^{u,v})^G$ vanishes on p-singular elements of G.*
5. *If $\Gamma_i^{(u,v,w)}$ is τ-stable, where $i = 1, 2$, then $\Gamma_i^{(u,v,w)}$ extends to generalized characters $\Gamma_i^{(u,v,w,1)}$ and $\Gamma_i^{(u,v,w,2)}$ of G which vanish on p-singular elements.*

Proof. The first four parts follow from the previous lemmas and the character table. It remains to prove the final part. Since G/N is cyclic, every G-stable irreducible character of N extends to an irreducible character of G. Suppose that Γ is the character of a PIM of N. Since p does not divide $[G:N]$, every $\mathcal{O}G$-module is N-projective, so it follows (see, for example, [5]) that there are distinct characters Γ^1 and Γ^2 of PIMs of G extending Γ. The result follows since $\Gamma_i^{(u,v,w)}$ is a $\mathbb{Z}$-linear combination of PIMs of N. □

We now have enough information to proceed to defining the perfect generalized characters. First we set up bijections between certain sets of indices for the irreducible characters. These will be used to associate irreducible characters of N to linear characters of $N_N(P)$, as well as associating characters $\chi^{(w)}_{q^3+1}$ with characters $\chi^{(u,v)}_{(q-1)(q^2-q+1)}$ as constituents of the same Γ_i.

Recall that τ fixes $(q-1)/2$ of the $\chi^{(u,v)}_{(q-1)(q^2-q+1)}$ in case $d=1$ and $(q-3)/2$ in case $d=3$. Recall also τ fixes $(q-3)/2$ of the $\chi^{(u)}_{q^3+1}$. Let J_1 be the set of pairs (u,v) with $1 \leqslant u(q+1)/3$ and $u<v<2(q+1)/3$ such that $(q+1) \mid (u-2v)$, and J_2 be the set of pairs (u,v) with $1 \leqslant u(q+1)/3$ and $u<v<2(q+1)/3$ such that $(q+1) \mid (v-2u)$ (so in either case $\chi^{(u,v)}_{(q-1)(q^2-q+1)} \in \mathrm{Irr}^{\tau}(N)$). Then $J_2 \neq \emptyset$, since $(1,2) \in J_2$. Write $J_2' = J_2 - \{(1,2)\}$ if $d=1$ and $J_2' = J_2$ if $d=3$. We have $J_1 \cap J_2' = \emptyset$ and $|J_1 \cup J_2'| = (q-3)/2$.

Write $\{1 \leqslant u < q^2-1\colon\ q-1 \nmid u,\ q+1 \mid u\} = J \cup J'$, where $J \cap J' = \emptyset$ and multiplication by $-q$ (modulo q^2-1) gives a bijection $J \leftrightarrow J'$. Precisely $|J| = (q-3)/2$ of the $\chi^{(u)}_{q^3+1}$ are fixed by τ. Choose a bijection $\alpha : J_1 \cup J_2' \to J$.

We now turn to the irreducible characters not fixed by τ. Recall that in $N_N(P)$, the linear characters not fixed by τ are the $\theta_1^{(u)}$ where $q-1 \nmid u$ and $q+1 \nmid u$.

Write $L \cup M = \{1 \leqslant u \leqslant q^2-1\colon\ q-1, q+1 \nmid u\}$ such that multiplication by q (modulo q^2-1) gives a bijection $L \to M$. Write $L = S \cup S'$, such that multiplication by $-q$ (modulo q^2-1) gives a bijection $S \to S'$, and $M = T \cup T'$ such that multiplication by $-q$ (modulo q^2-1) gives a bijection $T \to T'$ and multiplication by q gives a bijection $S \to T$.

Now choose S_1 amongst the set of those pairs (u,v) such that $\chi^{(u,v)}_{(q-1)(q^2-q+1)}$ is not fixed by τ and let S_2 be a set of integers $0 \leqslant u \leqslant q^2-q$ giving distinct $\chi^{(u)}_{(q+1)^2(q-1)}$, chosen such that:

(i) $\chi^{(u,v)}_{(q-1)(q^2-q+1)}, (\chi^{(u,v)}_{(q-1)(q^2-q+1)})^{\tau}$ for $(u,v) \in S_1$ account for all non-fixed $\chi^{(u,v)}_{(q-1)(q^2-q+1)}$;
(ii) $\chi^{(u)}_{(q+1)^2(q-1)}, (\chi^{(u)}_{(q+1)^2(q-1)})^{\tau}$ for $u \in S_2$ account for all of the $\chi^{(u)}_{(q+1)^2(q-1)}$ (recall that every irreducible character of degree $(q+1)^2(q-1)$ is moved by τ).

Note that we have $|S| = |S_1| + |S_2|$. Observe that if $x \in \{2,\dots,q\}$, then there are at most $q-2$ pairs (u,v) with $1 \leqslant u \leqslant (q+1)/3$ and $u<v<2(q+1)/3$ such that $u+v \equiv x \bmod (q+1)$. We also make the trivial observation that $1 \leqslant u+v \leqslant q$ for each $\chi^{(u,v)}_{(q-1)(q^2-q+1)}$. Hence we may choose a bijection $\beta : S_1 \cup S_2 \to S$ such that if $(u,v) \in S_1$, then $\beta(u,v) \equiv u+v \bmod (q+1)$.

Define $\mu_N \in \mathbb{Z}\,\mathrm{Irr}(N \times N_N(P)^{\circ})$ by

$$\begin{aligned}\mu_N = \Phi_N &- \sum_{(u,v)\in J_1} \Gamma_1^{u,v,\alpha(u,v)} \Phi^{(\alpha(u,v))} - \sum_{(u,v)\in J_2} \Gamma_2^{u,v,\alpha(u,v)} \Phi^{(\alpha(u,v))} \\ &- \sum_{(u,v)\in S_1} \Gamma_3^{u,v,\beta(u,v)} \Phi^{(\beta(u,v))} - \sum_{u\in S_2} \Gamma_4^{\beta(u),u} \Phi^{(\beta(u))} \\ &- \sum_{(u,v)\in S_1} \bigl(\Gamma_3^{u,v,\beta(u,v)}\bigr)^{\tau} \bigl(\Phi^{(\beta(u,v))}\bigr)^{\tau} + \sum_{u\in S_2} \bigl(\Gamma_4^{\beta(u),u}\bigr)^{\tau} \bigl(\Phi^{(\beta(u))}\bigr)^{\tau},\end{aligned}$$

where Φ_N denotes the character for induction and restriction for N, excluding the Steinberg character.

This gives the bijection (with signs) $\mathrm{Irr}_0(N) \setminus \{\chi_{q^3}\} \to \mathrm{Irr}_0(N_N(P))$ given by $\chi \to R_{\mu_N}(\chi)_0$:

$$\begin{pmatrix} \chi_1 \\ \chi_{q^2-q+1}^{(u)} \\ \chi_{q^3+1}^{(u)}\colon u \in J \\ \chi_{(q-1)(q^2-q+1)}^{(u,v)}\colon (u,v) \in J_1 \\ \chi_{(q-1)(q^2-q+1)}^{(u,v)}\colon (u,v) \in J_2' \\ \chi_{(q-1)(q^2-q+1)}^{(1,2)}\colon \text{if } d=1 \\ \chi_{q^3+1}^{(u)}\colon u \in S \\ \chi_{q^3+1}^{(qu)}\colon u \in S \\ \chi_{(q-1)(q^2-q+1)}^{(u,v)}\colon (u,v) \in S_1 \\ (\chi_{(q-1)(q^2-q+1)}^{(u,v)})^{\tau}\colon (u,v) \in S_1 \\ \chi_{(q+1)^2(q-1)}^{(u)}\colon u \in S_2 \\ (\chi_{(q+1)^2(q-1)}^{(u)})^{\tau}\colon u \in S_2 \\ \chi_{(q-1)(q^2-q+1)/3}^{(u)}\colon \text{if } d=3 \\ \chi_{(q+1)^2(q-1)/3}^{(u,v)} \end{pmatrix} \leftrightarrow \begin{pmatrix} \theta_1^{(0)} \\ \theta_1^{((q-1)u)} \\ \theta_1^{(-qu)} \\ \theta_1^{(\alpha(u,v))} \\ \theta_1^{(\alpha(u,v))} \\ \theta_{q^2-1} \\ \theta_1^{(-qu)} \\ \theta_1^{(-q^2u)} \\ \theta^{(\beta(u,v))} \\ \theta^{(q\beta(u,v))} \\ \theta^{(\beta(u))} \\ \theta^{(q\beta(u))} \\ \theta_{(q^2-1)/3}^{(0,u)} \\ \theta_{(q^2-1)/3}^{(v,u)} \end{pmatrix}$$

Note that by our definition of α and β, and the fact that restriction of characters respects the Brauer correspondence in this case, this bijection also respects the Brauer correspondence, i.e., we have bijections $\mathrm{Irr}_0(C_i) \leftrightarrow \mathrm{Irr}_0(c_i)$.

Our labelling of the terms of $\Gamma_1^{(u,v,w,1)}$ and $\Gamma_2^{(u,v,w,2)}$ will determine our choice of I_{μ_G}.

Let $\Gamma_1^{(u,v,w)}$ and $\Gamma_2^{(u,v,w)}$ be G-stable, where $w \equiv u - 2v \bmod (q+1)$ and $w \equiv v - 2u \bmod (q+1)$ respectively. Hence $q+1 \mid w$ in each case. For $i = 1, 2$, label extensions $\chi_{(q-1)(q^2-q+1)}^{(u,v,i)}$ of $\chi_{(q-1)(q^2-q+1)}^{(u,v)}$ and $\chi_{q^3+1}^{(w,i)}$ of $\chi_{q^3+1}^{(w)}$ so that, according to Lemma 6.4, for $i, j = 1, 2$,

$$\Gamma_j^{(u,v,w,i)} = \chi_{(q-1)(q^2-q+1)}^{(u,v,i)} + \chi_{q^3+1}^{(w,i)} - \big(\chi_{q(q^2-q+1)}^{(u+v)}\big)^G.$$

In this way we account for all of the τ-stable $\chi_{q^3+1}^{(u)}$, and either all or all but one of the τ-stable $\chi_{(q-1)(q^2-q+1)}^{(u,v)}$ depending on whether $d = 3$ or $d = 1$ respectively.

We must label the extensions of the remaining τ-stable irreducible characters of N. Let χ_1^i, for $i = 1, 2$ be the extensions of χ_1 to G. Let $\chi_{q^2-q+1}^{(u,i)}$ for $i = 1, 2$ be the extensions of the τ-stable $\chi_{q^2-q+1}^{(u)}$ to G. Let $\chi_{(q-1)(q^2-q+1)/3}^{(u,i)}$ for $i = 1, 2$ be the extensions of $\chi_{(q-1)(q^2-q+1)/3}^{(u)}$ to G. Let $\chi_{(q+1)^2(q-1)/3}^{(u,0,i)}$ for $i = 1, 2$ be the extensions of $\chi_{(q+1)^2(q-1)/3}^{(u,0)}$ to G.

For the remaining $\chi \in \mathrm{Irr}(N)$, we have $\chi^G \in \mathrm{Irr}(G)$.

Using the Mackey decomposition, we may compute the restrictions of the irreducible characters of G to $N_G(P)$, noting that $[N_G(P) : N_N(P)] = 2$. Every $\theta \in \mathrm{Irr}^{\tau}(N_N(P))$ extends to $N_G(P)$, and $\theta^{N_G(P)} \in \mathrm{Irr}(N_G(P))$ for every $\theta \in \mathrm{Irr}(N_N(P)) - \mathrm{Irr}^{\tau}(N_N(P))$. Choose a labelling of the extensions of the $\theta \in \mathrm{Irr}^\tau(N_N(P))$ so that the restrictions are as follows, where $i = 1, 2$, and each constituent written is irreducible:

$\chi \in \mathrm{Irr}(G)$	$\mathrm{Res}^G_{N_G(P)}(\chi)_0$ $(d=1)$	$\mathrm{Res}^G_{N_G(P)}(\chi)_0$ $(d=3)$
χ_1^i	$\theta_1^{(0,i)}$	$\theta_1^{(0,i)}$
$\chi_{q^2-q+1}^{(u,i)}$	$\theta_1^{((q-1)u,i)}$	$\theta_1^{((q-1)u,i)}$
$(\chi_{q^2-q+1}^{(u)})^G$	$(\theta_1^{((q-1)u)})^{N_G(P)}$	$(\theta_1^{((q-1)u,i)})^{N_G(P)}$
$\chi_{(q-1)(q^2-q+1)}^{(u,v,i)}$	$\theta_{q^2-1}^i$	$\theta_{(q^2-1)/3}^{(u+v,0,i)} + \theta_{(q^2-1)/3}^{(u+v,1,i)} + \theta_{(q^2-1)/3}^{(u+v,2,i)}$
$(\chi_{(q-1)(q^2-q+1)}^{(u,v)})^G$	$\theta_{q^2-1}^1 + \theta_{q^2-1}^2$	$(\theta_{(q^2-1)/3}^{(u+v,0,i)} + \theta_{(q^2-1)/3}^{(u+v,1,i)} + \theta_{(q^2-1)/3}^{(u+v,2,i)})^{N_G(P)}$
$\chi_{q^3+1}^{(u,i)}$	$\theta_1^{(u,i)} + \theta_1^{(-qu,i)} + \theta_{q^2-1}^i$	$\theta_1^{(u),i} + \theta_1^{(-qu,i)} + \theta_{(q^2-1)/3}^{(u+v,0,i)} + \theta_{(q^2-1)/3}^{(u+v,1,i)} + \theta_{(q^2-1)/3}^{(u+v,2,i)}$
$(\chi_{q^3+1}^{(u)})^G$	$(\theta_1^{(u)} + \theta_1^{(-qu,i)})^{N_G(P)} + \theta_{q^2-1}^1 + \theta_{q^2-1}^2$	$(\theta_1^{(u)} + \theta_1^{(-qu)} + \theta_{(q^2-1)/3}^{(u+v,0)} + \theta_{(q^2-1)/3}^{(u+v,1)} + \theta_{(q^2-1)/3}^{(u+v,2)})^{N_G(P)}$
$(\chi_{(q+1)^2(q-1)}^{(u)})^G$	$\theta_{q^2-1}^1 + \theta_{q^2-1}^2$	$(\theta_{(q^2-1)/3}^{(u+v,0)} + \theta_{(q^2-1)/3}^{(u+v,1)} + \theta_{(q^2-1)/3}^{(u+v,2)})^{N_G(P)}$
$\chi_{(q-1)(q^2-q+1)/3}^{(u,i)}$	–	$\theta_{(q^2-1)/3}^{(0,u,i)}$
$(\chi_{(q+1)^2(q-1)/3}^{(u,v)})^G$	–	$(\theta_{(q^2-1)/3}^{(v,u)})^{N_G(P)}$

We see that every $\theta_1^{(u)} \in \mathrm{Irr}^\tau(N_N(P))$ occurs in the restriction of precisely one τ-stable character $\chi_{q^2-q+1}^{(u)}$ (if $q-1 \mid u$) or $\chi_{q^3+1}^{(u)}$ (otherwise), hence we may indeed make a consist choice. We also see that $\theta_{q^2-1}^i$ and $\theta_{(q^2-1)/3}^{(0,v,i)}$ may be chosen consistently, and so every irreducible character of height zero of $N_G(P)$ is now accounted for.

For $i = 1, 2$ and $0 \leqslant u \leqslant q^2 - 1$ such that $q + 1 \mid u$, write $\Phi_0^{(u,i)} = \theta_1^{(u,i)} + \theta_{q^2-1}^i$ if $d = 1$ and $\Phi_0^{(u,i)} = \theta_1^{(u,i)} + \theta_{(q^2-1)/3}^{(0,0,i)} + \theta_{(q^2-1)/3}^{(,1,i)} + \theta_{(q^2-1)/3}^{(0,2,i)}$ if $d = 3$ and $u \equiv 0 \bmod 3$, and $\Phi_0^{(u,i)} = \theta_1^{(u,i)} + (\theta_{(q^2-1)/3}^{(u_1,0)} + \theta_{(q^2-1)/3}^{(u_1,1)} + \theta_{(q^2-1)/3}^{(u_1,2)})^G$, if $d = 3$ and $u_1 \equiv u \not\equiv 0 \bmod 3$.

In each case $\Phi_0^{(u,i)}$ is the 'truncation' of the character of a PIM.

Now define $\mu_G \in \mathbb{Z}\,\mathrm{Irr}(G \times N_G(P)^\circ)$ by

$$\mu_G = \Phi_G - \sum_{i=1,2} \sum_{(u,v)\in J_1} \Gamma_1^{u,v,\alpha(u,v),i} \Phi^{(\alpha(u,v,i))} - \sum_{i=1,2} \sum_{(u,v)\in J_2} \Gamma_2^{u,v,\alpha(u,v),i} \Phi^{(\alpha(u,v,i))}$$
$$- \sum_{(u,v)\in S_1} \left(\Gamma_3^{u,v,\beta(u,v)}\right)^G \left(\Phi^{(\beta(u,v))}\right)^{N_G(P)} - \sum_{u\in S_2} \Gamma_4^{\beta(u),u^G} \left(\Phi^{(\beta(u))}\right)^{N_G(P)},$$

where Φ_G denotes the character for (blockwise) induction and restriction, again excluding extensions of the Steinberg character.

This gives the bijection (with signs) $\mathrm{Irr}_0(B_0 \oplus B_1) \to \mathrm{Irr}_0(b_0 \oplus b_1)$ given by $\chi \to R_{\mu_G}(\chi)_0$:

$$\begin{pmatrix} \chi_1 \\ \chi_{q^2-q+1}^{((q+1)/2,i)} \\ (\chi_{q^2-q+1}^{(u)})^G\colon\ u \neq (q+1)/2 \\ \chi_{q^3+1}^{(u,i)}\colon\ u \in J \\ \chi_{(q-1)(q^2-q+1)}^{(u,v,i)}\colon\ (u,v) \in J_1 \\ \chi_{(q-1)(q^2-q+1)}^{(u,v,i)}\colon\ (u,v) \in J_2' \\ \chi_{(q-1)(q^2-q+1)}^{(1,2,i)}\colon\ \text{if } d=1 \\ (\chi_{q^3+1}^{(u)})^G\colon\ u \in S \\ (\chi_{(q-1)(q^2-q+1)}^{(u,v)})^G\colon\ (u,v) \in S_1 \\ (\chi_{(q+1)^2(q-1)}^{(u)})^G\colon\ u \in S_2 \\ \chi_{(q-1)(q^2-q+1)/3}^{(u,i)}\colon\ \text{if } d=3 \\ (\chi_{(q+1)^2(q-1)/3}^{(u,v)})^G \end{pmatrix} \leftrightarrow \begin{pmatrix} \theta_1^{(0),i} \\ \theta_1^{((q^2-1)/2,i)} \\ (\theta_1^{(u)})^G \\ \theta_1^{(-qu,i)} \\ \theta_1^{(\alpha(u,v),i)} \\ \theta_1^{(\alpha(u,v),i)} \\ \theta_{q^2-1}^{i} \\ (\theta_1^{(-qu)})^G \\ (\theta_1^{(\beta(u,v))})^G \\ (\theta_1^{(\beta(u))})^G \\ \theta_{(q^2-1)/3}^{(0,u,)} \\ (\theta_{(q^2-1)/3}^{(v,u)})^G \end{pmatrix}$$

It is clear that this bijection respects blocks.

We now turn our attention to $GU_3(q)$ and $PGU_3(q).2$. If $3 \nmid (q+1)$, then we are done, so let $3 \mid (q+1)$.

First consider $N = SU_3(q)$ and choose δ such that $H = GU_3(q) = N\langle\delta\rangle$ and $\delta \in N_H(P)$.

From examination of the character tables, δ fixes every irreducible character of G, except $\chi_{(q-1)(q^2-q+1)/3}^{(u)}$ (where $u \in 0,1,2$), which are permuted, and $\chi_{(q+1)^2(q-1)/3}^{(u,v)}$, where for each v, δ permutes $\chi_{(q+1)^2(q-1)/3}^{(0,v)}$, $\chi_{(q+1)^2(q-1)/3}^{(1,v)}$ and $\chi_{(q+1)^2(q-1)/3}^{(2,v)}$.

As before, we label the extensions of the irreducible characters of N according to the extensions of the projective indecomposables.

Each $\Gamma_i^{u,v,w}$ is H-stable, and there are extensions $\Gamma_i^{u,v,w,[j]}$ of $\Gamma_i^{u,v,w}$ for $j = 0,1,2$ which vanish on p-singular elements. Label extensions $\chi_{q^3+1}^{(u),[j]}$ of $\chi_{q^3+1}^{(u)}$, $\chi_{(q-1)(q^2-q+1)}^{(u,v),[j]}$ of $\chi_{(q-1)(q^2-q+1)}^{(u,v)}$, $\chi_{q(q^2-q+1)}^{(u),[j]}$ of $\chi_{q(q^2-q+1)}^{(u)}$, and $\chi_{(q+1)^2(q-1)}^{(u),[j]}$ of $\chi_{(q+1)^2(q-1)}^{(u)}$ so that

$$\Gamma_1^{u,v,w,[j]} := \chi_{(q-1)(q^2-q+1)}^{(u,v),[j]} + \chi_{q^3+1}^{(w),[j]} - \chi_{q(q^2-q+1)}^{(u+v),[j]} - \chi_{q(q^2-q+1)}^{(v-2u),[j]},$$

$$\Gamma_2^{u,v,w,[j]} := \chi_{(q-1)(q^2-q+1)}^{(u,v),[j]} + \chi_{q^3+1}^{(w),[j]} - \chi_{q(q^2-q+1)}^{(u-2v),[j]} - \chi_{q(q^2-q+1)}^{(u+v),[j]},$$

$$\Gamma_3^{u,v,w,[j]} := \chi_{(q-1)(q^2-q+1)}^{(u,v),[j]} + \chi_{q^3+1}^{(w),[j]} - \chi_{q(q^2-q+1)}^{(v-2u),[j]} - \chi_{q(q^2-q+1)}^{(u-2v),[j]},$$

$$\Gamma_4^{u,v,[j]} := \chi_{q^3+1}^{(u),[j]} + \chi_{(q+1)^2(q-1)}^{(v),[j]} + \chi_{q(q^2-q+1)}^{(u),[j]}.$$

For each of the remaining $\chi \in \mathrm{Irr}(N)$ fixed by δ, denote by $\chi^{[i]}$, for $i = 0, 1, 2$, the extensions of χ to H.

Now δ fixes every irreducible character of $N_N(P)$ of degree 1 and $q(q-1)$, and for each $u \in \{0, 1, 2\}$ permutes $\theta^{(u,0)}_{(q^2-1)/3}$, $\theta^{(u,1)}_{(q^2-1)/3}$ and $\theta^{(u,2)}_{(q^2-1)/3}$. Denote the extensions of $\theta^{(u)}_1$ to $N_H(P)$ by $\theta^{(u),[i]}_1$, and the extensions of $\theta^{(u)}_{q(q-1)}$ by $\theta^{(u),[i]}_{q(q-1)}$, for $i = 0, 1, 2$, chosen so that the restrictions of the irreducible characters of H are as follows:

$\chi \in \mathrm{Irr}_0(H)$	$\mathrm{Res}^H_{N_H(P)}(\chi)_0$
$\chi^{[i]}_1$	$\theta^{(0),[i]}_1$
$\chi^{(u),[i]}_{q^2-q+1}$	$\theta^{((q-1)u),[i]}_1$
$\chi^{(u,v),[i]}_{(q-1)(q^2-q+1)}$	$(\theta^{(u+v,0)}_{(q^2-1)/3})^{N_H(P)}$
$\chi^{(u)}_{q^3+1}$	$\theta^{(u),[i]}_1 + \theta^{(-qu),[i]}_1 + (\theta^{(u+v,0)}_{(q^2-1)/3})^{N_H(P)}$
$\chi^{(u),[i]}_{(q+1)^2(q-1)}$	$(\theta^{(u+v,0)}_{(q^2-1)/3})^{N_H(P)}$
$(\chi^{(u)}_{(q-1)(q^2-q+1)/3})^H$	$(\theta^{(0,u)}_{(q^2-1)/3})^{N_H(P)}$
$(\chi^{(u,v)}_{(q+1)^2(q-1)/3})^H$	$(\theta^{(v,u)}_{(q^2-1)/3})^{N_H(P)}$

Write $\Phi^{(u),[i]} = \theta^{(u),[i]}_1 + (\theta^{(u_1,0)}_{(q^2-1)/3})^H$, where $u_1 \equiv u \bmod 3$. Then $\Phi^{(u),[i]}$ is the character of a PIM of $N_H(P)$.

Note that the action of δ on $\mathrm{Irr}_0(N)$ and $\mathrm{Irr}_0(N_N(P))$ commutes with the action of τ (although of course this is not the case with the irreducible characters of positive height).

Define $\mu_H \in \mathbb{Z}\,\mathrm{Irr}(H \times N_H(P)^\circ)$ by

$$\mu_H = \Phi_H - \sum_{i=0,1,2}\Bigg(\sum_{(u,v)\in J_1} \Gamma_1^{u,v,\alpha(u,v),[i]}\Phi^{(\alpha(u,v)),[i]} - \sum_{(u,v)\in J_2} \Gamma_2^{u,v,\alpha(u,v),[i]}\Phi^{(\alpha(u,v)),[i]}$$
$$- \sum_{(u,v)\in S_1} \Gamma_3^{u,v,\beta(u,v),[i]}\Phi^{(\beta(u,v)),[i]} - \sum_{u\in S_2} \Gamma_4^{\beta(u),u,[i]}\Phi^{(\beta(u)),[i]}$$
$$- \sum_{(u,v)\in S_1} \big(\Gamma_3^{u,v,\beta(u,v),[i]}\big)^\tau \big(\Phi^{(\beta(u,v)),[i]}\big)^\tau + \sum_{u\in S_2} \big(\Gamma_4^{\beta(u),u,[i]}\big)^\tau \big(\Phi^{(\beta(u)),[i]}\big)^\tau\Bigg),$$

where Φ_H denotes the character for induction and restriction for N, excluding the extensions of the Steinberg character.

This gives the bijection (with signs) $\mathrm{Irr}_0(H) \setminus \{\chi^{[i]}_{q^3}\colon i = 0, 1, 2\} \to \mathrm{Irr}_0(N_H(P))$ given by $\chi \to R_{\mu_H}(\chi)_0$:

$$\begin{pmatrix} \chi_1^{[i]} \\ \chi_{q^2-q+1}^{(u),[i]} \\ \chi_{q^3+1}^{(u)}\colon u \in J \\ \chi_{(q-1)(q^2-q+1)}^{(u,v),[i]}\colon (u,v) \in J_1 \\ \chi_{(q-1)(q^2-q+1)}^{(u,v),[i]}\colon (u,v) \in J_2' \\ \chi_{q^3+1}^{(u),[i]}\colon u \in S \\ \chi_{q^3+1}^{(qu),[i]}\colon u \in S \\ \chi_{(q-1)(q^2-q+1)}^{(u,v),[i]}\colon (u,v) \in S_1 \\ (\chi_{(q-1)(q^2-q+1)}^{(u,v),[i]})^\tau\colon (u,v) \in S_1 \\ \chi_{(q+1)^2(q-1)}^{(u),[i]}\colon u \in S_2 \\ (\chi_{(q+1)^2(q-1)}^{(u),[i]})^\tau\colon u \in S_2 \\ (\chi_{(q-1)(q^2-q+1)/3}^{(0)})^H \\ (\chi_{(q+1)^2(q-1)/3}^{(0,v)})^H \end{pmatrix} \leftrightarrow \begin{pmatrix} \theta_1^{(0),[i]} \\ \theta_1^{((q-1)u),[i]} \\ \theta_1^{(-qu),[i]} \\ \theta_1^{(\alpha(u,v)),[i]} \\ \theta_1^{(\alpha(u,v)),[i]} \\ \theta_1^{(-qu),[i]} \\ \theta_1^{(-q^2u),[i]} \\ \theta_1^{(\beta(u,v)),[i]} \\ \theta_1^{(q\beta(u,v)),[i]} \\ \theta_1^{(\beta(u)),[i]} \\ \theta_1^{(q\beta(u)),[i]} \\ (\theta_{(q^2-1)/3}^{(0,0)})^{N_H(P)} \\ (\theta_{(q^2-1)/3}^{(v,0)})^{N_H(P)} \end{pmatrix}$$

Again, it is clear that R_{μ_H} respects blocks.

Now let $E = GU_3(q).2 = H\langle\tau\rangle$. Recall that $\tau \in N_E(P)$. We combine the notations for extensions to G and to H.

Define $\mu_E \in \mathbb{Z}\,\mathrm{Irr}(E \times N_E(P)^\circ)$ by

$$\begin{aligned} \mu_E = \Phi_E - \sum_{j=0,1,2} \Bigg(& \sum_{i=1,2} \sum_{(u,v)\in J_1} \Gamma_1^{u,v,\alpha(u,v),[j],i} \Phi^{(\alpha(u,v,[j],i))} \\ & - \sum_{i=1,2} \sum_{(u,v)\in J_2} \Gamma_2^{u,v,\alpha(u,v),[j],i} \Phi^{(\alpha(u,v,[j],i))} \\ & - \sum_{(u,v)\in S_1} \big(\Gamma_3^{u,v,\beta(u,v),[j]}\big)^E \big(\Phi^{(\beta(u,v)),[j]}\big)^{N_E(P)} \\ & - \sum_{u\in S_2} \Gamma_4^{\beta(u),u,[j]\,E} \big(\Phi^{(\beta(u)),[j]}\big)^{N_E(P)} \Bigg), \end{aligned}$$

where Φ_E denotes the character for (blockwise) induction and restriction, again excluding extensions of the Steinberg character.

This gives the bijection (with signs) $\mathrm{Irr}_0(E) \setminus \{\chi_{q^3}^{[j],i}\colon i, j = 0,1,2\} \to \mathrm{Irr}_0(N_E(P))$ given by $\chi \to R_{\mu_E}(\chi)_0$:

$$\begin{pmatrix} \chi_1^{[j],i} \\ \chi_{q^2-q+1}^{((q+1)/2,i),[j]} \\ (\chi_{q^2-q+1}^{(u),[j]})^E\colon\ u \neq (q+1)/2 \\ \chi_{q^3+1}^{(u,i),[j]}\colon\ u \in J \\ \chi_{(q-1)(q^2-q+1)}^{(u,v,i),[j]}\colon\ (u,v) \in J_1 \\ \chi_{(q-1)(q^2-q+1)}^{(u,v,i),[j]}\colon\ (u,v) \in J_2' \\ (\chi_{q^3+1}^{(u),[j]})^G\colon\ u \in S \\ (\chi_{(q-1)(q^2-q+1)}^{(u,v),[j]})^E\colon\ (u,v) \in S_1 \\ (\chi_{(q+1)^2(q-1)}^{(u),[j]})^E\colon\ u \in S_2 \\ (\chi_{(q-1)(q^2-q+1)/3}^{(u,i)})^E \\ (\chi_{(q+1)^2(q-1)/3}^{(0,v)})^E \end{pmatrix} \leftrightarrow \begin{pmatrix} \theta_1^{(0),[j],i} \\ \theta_1^{((q^2-1)/2,i),[j]} \\ (\theta_1^{(u)})^G \\ \theta_1^{(-qu,i)} \\ \theta_1^{(\alpha(u,v),i)} \\ \theta_1^{(\alpha(u,v),i)} \\ \theta_{q^2-1}^i \\ (\theta_1^{(-qu)})^G \\ (\theta_1^{(\beta(u,v))})^G \\ (\theta_1^{(\beta(u))})^G \\ \theta_{(q^2-1)/3}^{(0,u,)} \\ (\theta_{(q^2-1)/3}^{(v,u)})^G \end{pmatrix}$$

7. Controlled blocks

Let $p = 3$ and $G = J_2$ or J_3. Then the principal block B is not a TI block, but is however a controlled block (for principal blocks this means that whenever $Q \leqslant P \in \mathrm{Syl}_p(G)$ and $g \in G$ with $Q^g \in P$, then $g = cn$ for some $c \in C_G(Q)$ and $n \in N_G(P)$). In these case we are able to verify that property (P) holds.

Suppose first that $G = J_2$.

Now the principal block b of $N_G(P)$ has linear characters $1_1, \ldots, 1_8$, and the PIMs have characters $\Phi_1, \ldots, \Phi_8$, where 1_i is the unique linear constituent of Φ_i. There is a unique non-linear irreducible character 8_1 of b of height 0.

Using the computer algebra package GAP, the PIMs of G have characters $\Gamma_1, \ldots, \Gamma_5, \Gamma_7, \Gamma_8, \Gamma_{11}$, corresponding to irreducible Brauer characters 1_1, 13_1, 13_2, 21_1, 21_2, 57_1, 57_2, 133_1 respectively.

Then

$$\mu = \Phi - \Phi_1\Gamma_{11} - \Phi_2(\Gamma_2 + \Gamma_8) - \Phi_3(\Gamma_3 + \Gamma_4 - \Gamma_5 - \Gamma_8) - \Phi_4(-\Gamma_4 + \Gamma_8 + \Gamma_{11}) \\ - \Phi_5(\Gamma_4 + \Gamma_5) - \Phi_6(-\Gamma_5 + \Gamma_7 + \Gamma_{11}) - \Phi_7\Gamma_{11} - \Phi_8(-\Gamma_4 + \Gamma_7 + \Gamma_8 + \Gamma_{11})$$

gives (P).

This results in the following bijection, with signs, between $\mathrm{Irr}_0(B)$ and $\mathrm{Irr}_0(b)$:

$$\begin{pmatrix} 1_1 \\ 14_1 \\ 14_2 \\ 70_1 \\ 70_2 \\ 160_1 \\ 175_1 \\ 224_1 \\ 224_2 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1_1 \\ -1_3 \\ -1_2 \\ 1_6 \\ 1_4 \\ 1_5 \\ 1_7 \\ -1_8 \\ 8_1 \end{pmatrix}.$$

Now suppose $G = J_3$. Again the principal block b of $N_G(P)$ has linear characters $1_1, \dots, 1_8$, and the PIMs have characters $\Phi_1, \dots, \Phi_8$, where 1_i is the unique linear constituent of Φ_i. There is a unique non-linear irreducible character 8_1 of b of height 0.

Using the computer algebra package GAP, the PIMs of G have characters $\Gamma_1, \dots, \Gamma_7, \Gamma_9$, corresponding to irreducible Brauer characters $1_1, 18_1, 18_2, 84_1, 84_2, 153_1, 153_2, 934_1$ respectively. Then

$$\begin{aligned}\mu = \Phi - \Phi_1\Gamma_9 - \Phi_2(\Gamma_5 + 2\Gamma_9 - \Gamma_2 - \Gamma_3) - \Phi_3\Gamma_9 - \Phi_4(\Gamma_7 + 4\Gamma_9 - \Gamma_2 - 2\Gamma_3)\\ - \Phi_5(\Gamma_3 - \Gamma_9) - \Phi_6(\Gamma_2 - \Gamma_9) - \Phi_7(\Gamma_6 - \Gamma_2) - \Phi_8(\Gamma_2 + \Gamma_3 - 2\Gamma_9)\end{aligned}$$

gives (P).

This results in the following bijection, with signs, between $\mathrm{Irr}_0(B)$ and $\mathrm{Irr}_0(b)$:

$$\begin{pmatrix} 1_1 \\ 85_1 \\ 85_2 \\ 323_1 \\ 323_2 \\ 646_1 \\ 646_2 \\ 1615_1 \\ 2432_1 \end{pmatrix} \leftrightarrow \begin{pmatrix} 1_1 \\ 1_2 \\ -8_1 \\ -1_7 \\ -1_4 \\ 1_5 \\ 1_6 \\ 1_3 \\ -1_8 \end{pmatrix}.$$

We sound a note of caution here: the principal 5-block for Co_3 is also a controlled block. However, in this case the index in Co_3 of the normalizer of a Sylow 5-subgroup is congruent to 1 modulo 25 (the order of a Sylow 5-subgroup), and there is an irreducible character of degree congruent to $\pm 3 \bmod 25$, whilst no so such irreducible character exists for the normalizer of a Sylow 5-subgroup. Hence we cannot hope to verify (P+). It is not clear to us whether (P) holds in this case or not.

8. Blocks with non-abelian TI defect groups of order p^3

The blocks with TI defect groups of automorphism groups of non-abelian simple groups are determined in [3]. With a suitable reduction, this allows us to apply the classification of finite simple groups to show that (for $p \neq 11$) every block with non-abelian TI defect groups of order p^3 satisfies (PI). For $p = 11$, the group J_4 has TI defect groups, but as yet the simple kJ_4-modules are not known, it is unrealistic to hope to check this case at this stage. However, (PI) will hold for $p = 11$ provided that J_4 is not involved in G. We emphasize again that J_4 is not necessarily a counterexample to property (PI).

We first give the classification, which is a direct consequence of [3, 1.1]:

Lemma 8.1. *Let G be a central extension of an automorphism group of a non-abelian simple group and let B be a block with non-abelian TI defect group D of order p^3, such that $Z(G) \leqslant G'$. Then G and D satisfy one* (*or more*) *of the following*:

(a) $D \cong Q_8$;
(b) $D \cong 3_-^{1+2}$ *and G is* $\mathrm{Aut}({}^2G_2(3)') = {}^2G_2(3)$;

(c) $D \cong 5_+^{1+2}$ *and* G *is* $3.McL$ *or* $\mathrm{Aut}(McL)$;
(d) $D \cong 5_-^{1+2}$ *and* G *is* $\mathrm{Aut}({}^2B_2(32))$;
(e) $D \cong 11_+^{1+2}$ *and* G *is* J_4;
(f) $D \cong p_+^{1+2}$ *and* G *is* $SU_3(p)$, $PSU_3(p).2$, $GU_3(p)$ *or* $PGU_3(p).2$, *where the extension is by the unique field automorphism of order* 2 *and* p *is odd.*

Proof. We have one subtlety to address. Whereas the Schur multipliers of the simple groups are well known, it is not always so clear what the Schur multiplier of a group of automorphisms of a non-abelian simple group is. However, note that in every case considered here the outer automorphism group has cyclic Sylow l-subgroups for every prime l. Hence by [14, 3.5] the Schur multiplier of a group of automorphisms is a quotient of the Schur multiplier of the simple group in each case.

Note also that we do not need to consider $SU_3(p).2$ and $GU_3(p).2$ instead of $PSU_3(p).2$ and $PGU_3(p).2$ since in these cases, the field automorphism does not centralize non-trivial elements of the centre of $SU_3(p)$ and $GU_3(p)$. $\square$

Lemma 8.2. *Let* B *be a block of* G *with defect group* B, *and let* b *be the Brauer correspondent of* B *in* $N_G(D)$. *If there is a perfect isometry* $I_\mu : \mathbb{Z}\,\mathrm{Irr}(N_G(D), b) \to \mathbb{Z}\,\mathrm{Irr}(G, B)$, *then* (P) *holds for* B. *The analogous statement holds for generalized isotypies and isotypies.*

Proof. This is immediate, since a perfect isometry takes an irreducible character of height zero to plus/minus an irreducible character of height zero (see [9, 1.5]). $\square$

Lemma 8.3. *(See [11].) Let* B *be a* 2*-block of* G *with TI defect group* $D \cong Q_8$. *Then there is an isotypy between* B *and its Brauer correspondent* b *in* $N_G(D)$.

Proof. By [11] (and the corrected version [12]), there is an isotypy (and hence a perfect isometry) between B and its Brauer correspondent c in $C_G(Z(D))$. Since D is TI, we have $C_G(Z(D)) \leqslant N_G(D)$, and by the transitivity of the Brauer correspondence there is an isotypy between b and c. $\square$

Lemma 8.4. *Let* B *be a block of* G *with TI defect groups. Let* $N \lhd G$ *and let* b *be a block of* N *covered by* B. *Write* $I = I_G(b)$, *the stabilizer of* b *under conjugation in* G. *Then there is a unique block* B_I *of* I *covering* b *with* $(B_I)^G = B$, *and* B_I *and* B *both have defect group* D *which is TI in* I. *The blocks* B_I *and* B *are isotypic.*

Proof. The existence of B_I and its Morita equivalence with B are well known. It is clear that D is also TI in I.

Let e_{B_I} be the block idempotent in $Z(\mathcal{O}I)$ for B_I and e_B the block idempotent in $Z(\mathcal{O}G)$ for B. Then by [26, 3.1] $\mathcal{O}Ge_B \cong \mathrm{Ind}_I^G(\mathcal{O}Ie_{B_I})$, and further B and B_I are Puig equivalent. By [28, Ex. 47.3] $\mathbf{Br}_B(G)$ and $\mathbf{Br}_{B_I}(I)$ are equivalent, so by [20, 1.9] B and B_I are isotypic. $\square$

A useful result when considering groups with TI Sylow p-subgroup P is that if P is not cyclic or generalized quaternion, then $O_{p'}(G) \leqslant N_G(P)$. This generalizes considerably:

Lemma 8.5. *Let B be a block of a group G with TI defect group D, and suppose that D is not cyclic or generalized quaternion. Let $N \lhd G$ with $N \cap D = 1$. Then $N \leqslant N_G(D)$. In particular, if $Q \leqslant D$, then $N \leqslant C_G(Q)$.*

Proof. Choose a D-stable block c of N covered by B. Then c has defect zero and there is a unique block b of DN covering c, and further b has TI defect group D. The result then follows from [15, 3.3] (and [16]). □

In order to construct isotypies using correspondences similar to the Fong correspondences we need to choose extensions of stable irreducible characters of normal subgroups with some care:

Lemma 8.6. *Let G be a finite group and D a p-subgroup. Suppose $N \lhd G$ with $N \cap D = 1$ and let c be a G-stable block of defect zero of N. Suppose further that $N \leqslant C_G(D)$. Let $\mathrm{Irr}(c) = \{\zeta\}$. Then there is a central extension $\hat{G}$ of G by a cyclic p'-group $\hat{W}$ such that ζ, regarded as a character of the appropriate normal subgroup $\hat{N}$ of $\hat{G}$ identified with N, extends to some $\hat{\theta} \in \mathrm{Irr}(\hat{G})$ with $\hat{D} \leqslant \ker(\hat{\theta})$, where $\hat{D}$ is the p-subgroup of $\hat{G}$ with $D = \hat{W}\hat{D}/\hat{W}$.* □

Proof. We follow the construction of a projective representation of G extending ζ given in [21, 11.2]. Let σ be a representation of N affording ζ. Choose a transversal T of N in G such that $D \subset T$. For each $t \in T$, a non-singular matrix P_t is chosen so that $\sigma^t = P_t \sigma P_t^{-1}$. Since $D \leqslant C_G(N)$, we may take $P_d = I$ for $d \in D$. A projective representation ρ of G is defined by $\rho(nt) = \sigma(n)P_t$ for $n \in N$ and $t \in T$. Then $\rho(n) = \sigma(n)$ for $n \in N$ and $\rho(d) = \sigma(1)$ for $d \in D$.

Now it follows from the argument in [3, p. 467] that there is a projective representation ρ_1 of G extending σ with cocycle of order prime to p (the argument is phrased in terms the existence of characters of central extensions by a p'-group which extend ζ, but this implies the existence of such a projective representation). By [21, 11.2] (and the discussion preceding [21, 11.7]) it follows that ρ and ρ_1 have the same cocycle. Hence we are done by lifting ρ to a representation of a suitable central extension $\hat{G}$ and taking its character $\hat{\theta}$. □

Now let B_Y be a block of a finite group Y with defect group D and $X \lhd Y$ such that $X \cap D = 1$ and B_Y covers a Y-stable block B_X of X of defect zero. Suppose that $X \leqslant C_Y(D)$, and that the unique irreducible character ζ in B_X extends to $\theta \in \mathrm{Irr}(Y)$ with $D \leqslant \ker(\theta)$, as in Lemma 8.6. Write $\overline{Y} = Y/X$. Then there is a unique block $B_{\overline{Y}}$ such that $\mathrm{Irr}(Y, B_Y) = \{\theta\overline{\chi}\colon\ \overline{\chi} \in \mathrm{Irr}(\overline{Y}, B_{\overline{Y}})\}$, and $B_{\overline{Y}}$ has defect group $\overline{D}$ (see, for example, [17]).

Lemma 8.7. *Let $y \in D$, and write $\bar{y} = yX$. Then $C_{\overline{Y}}(\bar{y}) = \overline{C_Y(y)}$.*

Proof. Write $Q = \langle y \rangle \leqslant D$, and let $\bar{g} \in C_{\overline{Y}}(\overline{Q}) \leqslant N_{\overline{Y}}(\overline{Q})$. But $N_{\overline{Y}}(\overline{Q}) = N_Y(Q)/X$ since $X \leqslant C_Y(Q)$, so $[y, g] \in Y \cap Q = 1$ and we are done. □

Let $y \in D$, and let B_y be a block of $C_Y(y)$ with $(B_y)^Y = B_Y$. Let e_y be the corresponding primitive central idempotent (so e_1 is the central idempotent for B_Y). Then B_y covers B_X. Now ζ extends to the irreducible character $\theta_y = \mathrm{Res}^Y_{C_Y(y)}(\theta)$ of $C_Y(y)$, so as above there is a unique block $B_{\bar{y}}$ of $C_{\overline{Y}}(\bar{y})$ such that $\mathrm{Irr}(C_Y(y), B_y) = \{\theta_y\overline{\chi}\colon\ \overline{\chi} \in \mathrm{Irr}(C_{\overline{Y}}(\bar{y}), B_{\bar{y}})\}$. We have $(B_{\bar{y}})^{\overline{Y}} = B_{\overline{Y}}$ (see [17]). Define $I_{\mu_y} : CF(C_{\overline{Y}}(\bar{y}), B_{\bar{y}}) \to CF(C_Y(y), B_y)$ by $I_{\mu_y}(\overline{\alpha}) = \theta_y\overline{\alpha}$. Then each I_{μ_y} is a perfect isometry. Write $I_\mu = I_{\mu_1}$.

For each $y \in D$, write $I_{\mu_y,p'}$ for the map $CF_{p'}(C_{\overline{Y}}(\bar{y}), B_{\bar{y}}) \to CF_{p'}(C_Y(y), B_y)$ induced by I_{μ_y}.

Lemma 8.8. *With the notation above, I_μ is compatible with fusion and gives an isotypy.*

Proof. The Brauer categories $\mathbf{Br}_{B_Y}(Y)$ and $\mathbf{Br}_{B_{\overline{Y}}}(\overline{Y})$ are isomorphic in this case.

Let $\bar{\alpha} \in CF(\overline{Y}, \overline{B})$ and $h \in C_Y(y)_{p'}$. Then

$$\left(I_{\mu_y,p'} \circ d_{\overline{Y}}^{(\bar{y},B_{\bar{y}})}\right)(\bar{\alpha})(h) = \theta_y(h)\bar{\alpha}(\bar{h}\bar{y}e_{\bar{y}}).$$

The calculation of $(d_Y^{(y,B_y)} \circ I_\mu)(\bar{\alpha})(h)$ is complicated by the fact that we do not have $I_\mu(\bar{\alpha})(a) = \theta(a)\bar{\alpha}(\bar{a})$ for arbitrary $a \in \mathcal{O}Y$.

$$\begin{aligned}\left(d_Y^{(y,B_y)} \circ I_\mu\right)(\bar{\alpha})(h) &= I_\mu(\bar{\alpha})(hyB_y) = I_\mu(\bar{\alpha})\left(hy \sum_{\chi \in \mathrm{Irr}(B_y)} \frac{\chi(1)}{|C_Y(y)|} \sum_{g \in C_Y(y)} \chi\left(g^{-1}\right)g\right)\\ &= \sum_{\chi \in \mathrm{Irr}(B_y)} \left(\mathrm{Res}^Y_{C_Y(y)}\left(I_\mu(\bar{\alpha})\right), \chi\right)_{C_Y(y)} \chi(hy)\\ &= \sum_{\bar{\chi} \in \mathrm{Irr}(B_{\bar{y}})} \left(\mathrm{Res}^Y_{C_Y(y)}(\theta\bar{\alpha}), \theta_y\bar{\chi}\right)_{C_Y(y)} \theta_y(hy)\bar{\chi}(\overline{hy}).\end{aligned}$$

Since $D \leqslant \ker(\theta)$ we have $\theta_y(hy) = \theta_y(h)$. Note also that since $\bar{\chi} \leftrightarrow \theta_y\bar{\chi}$ gives a bijection $\mathrm{Irr}(B_{\bar{y}}) \leftrightarrow \mathrm{Irr}(B_y)$, we have $(\mathrm{Res}^Y_{C_Y(y)}(\theta\bar{\alpha}), \theta_y\bar{\chi})_{C_Y(y)} = (\mathrm{Res}^{\overline{Y}}_{C_{\overline{Y}}(\bar{y})}(\bar{\alpha}), \bar{\chi})_{C_{\overline{Y}}(\bar{y})}$. Hence

$$\begin{aligned}\left(d_Y^{(y,B_y)} \circ I_\mu\right)(\bar{\alpha})(h) &= \sum_{\bar{\chi} \in \mathrm{Irr}(B_{\bar{y}})} \left(\mathrm{Res}^{\overline{Y}}_{C_{\overline{Y}}(\bar{y})}(\bar{\alpha}), \bar{\chi}\right)_{C_{\overline{Y}}(\bar{y})} \theta_y(h)\bar{\chi}(\overline{hy})\\ &= \theta_y(h)\bar{\alpha}(\bar{h}\bar{y}e_{\bar{y}})\end{aligned}$$

as required. □

We give the reduction for property (PI) to the list of blocks given in Lemma 8.1.

Let G be minimized with respect to $[G : Z(G)]$, subject to the existence of a block B of G with non-abelian TI defect group D of order p^3 and not satisfying (PI). By Lemma 8.3 and [9] D is not generalized quaternion, so has p-rank at least two.

Let $N \lhd G$ with $Z(G) \leqslant N$, and let c be a block of N covered by B. Write $I = I_G(c)$. By replacing c with a conjugate block if necessary we may assume $D \leqslant I$. There is a unique block $\tilde{c}$ of DN covering c, and this has defect group D. Let $\tilde{c}_1$ be the unique block of $N_{DN}(D)$ with $(\tilde{C}_1)^{DN} = \tilde{c}$. Note that $N_{DN}(D) = DN_N(D)$, and $N_{DN}(D) \lhd N_G(D)$. We have $I_{N_G(D)}(\tilde{c}_1) \leqslant N_I(D)$, since if $g \in I_{N_G(D)}(\tilde{c}_1)$, then $\tilde{c}^g = ((\tilde{c}_1)^{DN})^g = (\tilde{c}_1^g)^{g^{-1}DNg} = (\tilde{c}_1)^{DN} = \tilde{c}$, so $g \in I_{N_G(D)}(\tilde{c}) \leqslant I_{N_G(D)}(c)$, as c is the unique block of N covered by $\tilde{c}$. Hence the Brauer correspondence gives a 1–1 correspondence between blocks of $N_I(D)$ covering $\tilde{c}_1$ and blocks of $N_G(D)$ covering $\tilde{c}_1$. Let b_I be the unique block of $N_I(D)$ with Brauer correspondent B_I in I. Since $(b_I)^G = ((b_I)^I)^G = (B_I)^G = B = b^G$, we have $(b_I)^{N_G(D)} = b$.

By Lemma 8.4 B and B_I are isotypic, as are b and b_I. Suppose that $I \neq G$. Then $[I : Z(I)] < [G : Z(G)]$, and so by minimality, and since B_I has defect group D which is TI (and possibly normal), B_I satisfies (PI). Hence, using the isotypies we have just constructed, B satisfies (PI), a contradiction. Hence $G = I$.

Suppose that $I \neq G$. Then $[I : Z(I)] < [G : Z(G)]$, and so by minimality B_I satisfies (PI), and so B satisfies (PI) (since we have established isotypies between B and B_I, B_I and b_I, b_I and b), and we have a contradiction. Hence $G = I$.

Hence every block of a normal subgroup containing $Z(G)$ covered by B is G-stable.

Let $N \lhd G$ with $D \cap N = 1$ and $Z(G) \leqslant N$, so that B covers a block c of defect zero of N (recall that $O_p(G) = 1$ since D has non-normal TI defect groups). Note that $N \leqslant C_G(D)$ by Lemma 8.5. We have that c is G-stable.

We show that $N \leqslant Z(G)$. Write $H = N_G(D)$.

By Lemma 8.6 there is a central extension $\hat{G}$ of G by a cyclic p'-group $\hat{W}$ (with $\hat{W} \leqslant [\hat{G}, \hat{G}]$) such that ζ, regarded as a character of the appropriate normal subgroup $\hat{N}$ of $\hat{G}$ identified with N, extends to $\hat{\theta} \in \mathrm{Irr}(\hat{G})$, where $\hat{D} \leqslant \ker \hat{\theta}$, with $\hat{D}$ defined as in Lemma 8.6. Let $\hat{H} \leqslant \hat{G}$ be the subgroup with $\hat{H}/\hat{N} = H$. Note that $\hat{H} = N_{\hat{G}}(\hat{D})$. Let $\hat{B}$ and $\hat{b}$ be the unique blocks of $\hat{G}$ and $\hat{H}$ respectively containing B and b respectively. Both $\hat{B}$ and $\hat{b}$ have defect group $\hat{D}$, and $\hat{D}$ is a TI subgroup of $\hat{G}$. By Lemma 8.8 B is isotypic with $\hat{B}$ and b is isotypic with $\hat{b}$.

Write $\tilde{G} = \hat{G}/\hat{N}$. As in the discussion preceding Lemma 8.5, there are blocks $\tilde{B}$ and $\tilde{b}$ of $\tilde{G}$ and $\tilde{H}$ respectively with defect group $\tilde{D} = \hat{D}\hat{N}/\hat{N}$ such that $\tilde{\chi} \leftrightarrow \hat{\theta}\tilde{\chi}$ gives a bijection $\mathrm{Irr}(\tilde{G}, \tilde{B}) \leftrightarrow \mathrm{Irr}(\hat{G}, \hat{B})$ and $\tilde{\chi} \leftrightarrow \mathrm{Res}^{\hat{G}}_{\hat{H}}(\hat{\theta})\tilde{\chi}$ gives a bijection $\mathrm{Irr}(\tilde{H}, \tilde{b}) \leftrightarrow \mathrm{Irr}(\hat{H}, \hat{b})$.

Since $\hat{D}$ is a TI subgroup of $\hat{G}$, by Lemma 8.5 $\hat{B}$ and $\hat{b}$ both satisfy the hypotheses of Lemma 8.8 with $X = \hat{N}$. Hence there are isotypies $\hat{B} \leftrightarrow \tilde{B}$ and $\hat{b} \leftrightarrow \tilde{b}$. Since $\hat{N} \leqslant C_{\hat{G}}(\hat{D})$ we have $\tilde{H} = N_{\tilde{G}}(\tilde{D})$. By [17] we have $\tilde{b}^{\tilde{G}} = \tilde{B}$.

Suppose that $N \neq Z(G)$. Then $[\tilde{G} : Z(\tilde{G}] \leqslant [G : N] < [G : Z(G)]$, so by minimality (PI) holds for $\tilde{B}$. But then we have isotypies $B \to \hat{B} \to \tilde{B} \to \tilde{b} \to \hat{b} \to b$, a contradiction. Hence there is no $N \lhd G$ with $Z(G) \leqslant N$ such that $D \cap N = 1$.

Let $F^*(G)$ be the generalized Fitting subgroup of G. We have $C_G(F^*(G)) \leqslant F^*(G)$ and $Z(G) \leqslant F^*(G)$, so $D \cap F^*(G) \neq 1$, since otherwise G is abelian, a contradiction. Write $E(G)$ for the layer of G (that is, the product of the subnormal quasisimple subgroups of G). Then $Z(G)E(G)$ is a central product $M_1 * \cdots * M_s$ of normal subgroups $M_i \lhd G$, and each M_i is the central product of $Z(G) * M_{i1} * \cdots * M_{it_i}$, where each M_{ij} is quasisimple. We have $D \cap M_i \neq 1$ for each i. Let b_i be a block of M_i covered by B with defect group $D \cap M_i$. Then b_i has non-normal TI defect groups. So $M_i/Z(G)$ has a non-trivial TI radical p-subgroup (recall that a p-subgroup Q of H is radical if $Q = O_p(N_H(Q))$). By [2] if Q is a radical p-subgroup of $H_1 \times H_2$ then $Q = Q_1 \times Q_2$, where Q_i is a radical p-subgroup of H_i, and if Q is TI, then $Q_1 = 1$ or $Q_2 = 1$. Since b_i is G-stable, this implies that $s = 1$ and $t_1 = 1$. Hence $E(G)$ is quasisimple. Write $M = E(G)Z(G)/Z(G)$, so $M = F^*(G/Z(G))$. Then $C_{G/Z(G)}(M) \leqslant M$, so $G/Z(G) \leqslant \mathrm{Aut}(M)$, where the preimage of M in G is perfect.

We are now ready to prove Theorem 1.5.

Proof of Theorem 1.5. Let B be a counterexample with $[G : Z(G)]$ minimized. Then G is a p' cyclic central extension of an automorphism group of a non-abelian simple group. Such blocks are listed in Lemma 8.1. We have seen that (P+) holds for each of the blocks on this list. Hence by Lemma 1.4 (PI) holds for each of these blocks, and we are done. □

Acknowledgments

I thank Jianbei An for communicating the example of a controlled block with non-abelian defect groups which are not trivial intersection. I have also benefitted from enlightening conversations with many people, including Joe Chuang, Meinholf Geck, Radha Kessar, Markus Linckelmann and Will Turner, as well as from Jon Alperin's talk at Oberwolfach in 2003.

References

[1] J. Alperin, M. Broué, Local methods in block theory, Ann. of Math. (2) 110 (1979) 143–157.
[2] J. An, C.W. Eaton, On TI and TI defect blocks, J. Algebra 243 (2001) 123–130.
[3] J. An, C.W. Eaton, Blocks with trivial intersection defect groups, Math. Z. 247 (2004) 461–486.
[4] J. An, C.W. Eaton, Modular representation theory of blocks with trivial intersection defect groups, Algebr. Represent. Theory 8 (2005) 427–448.
[5] D.B. Benson, Representations and Cohomology, I: Basic Representation Theory of Finite Groups and Associative Algebras, Cambridge Stud. Adv. Math., vol. 30, Cambridge Univ. Press, Cambridge, 1991.
[6] H.I. Blau, G.O. Michler, Modular representation theory of finite groups with T.I. Sylow p-subgroups, Trans. Amer. Math. Soc. 319 (1990) 417–468.
[7] T. Breuer, The GAP character table library, http://www.math.rwth-aachen.de/~Thomas.Breuer/ctbllib/.
[8] M. Broué, Brauer coefficients of p-subgroups associated with a p-block of a finite group, J. Algebra 56 (1979) 365–383.
[9] M. Broué, Isométries parfaites, types de blocs, catégories dérivées, Astérisque 181–182 (1990) 61–92.
[10] M. Cabanes, A note on extensions of p-blocks by p-groups and their characters, J. Algebra 115 (1988) 445–449.
[11] M. Cabanes, C. Picaronny, Types of blocks with dihedral or quaternion defect groups, J. Fac. Sci. Univ. Tokyo 39 (1992) 141–161.
[12] M. Cabanes, C. Picaronny, Corrected version of: 'Types of blocks with dihedral or quaternion defect groups', 1999; http://www.math.jussieu.fr/cabanes/.
[13] G. Cliff, On centers of 2-blocks of Suzuki groups, J. Algebra 226 (2000) 74–90.
[14] C.W. Eaton, On finite groups of p-local rank one and conjectures of Dade and Robinson, J. Algebra 238 (2001) 623–642.
[15] C.W. Eaton, A class of blocks behaving like blocks of p-solvable groups, J. Algebra 301 (2006) 337–343.
[16] C.W. Eaton, Corrigendum to: A class of blocks behaving like blocks of p-solvable groups, J. Algebra 319 (2008) 1823–1824.
[17] C.W. Eaton, G.R. Robinson, On a minimal counterexample to Dade's projective conjecture, J. Algebra 249 (2002) 453–462.
[18] P. Fong, M. Harris, On perfect isometries and isotypies in finite groups, Invent. Math. 114 (1993) 139–191.
[19] M. Geck, Irreducible Brauer characters of the 3-dimensional special unitary groups in non-defining characteristic, Comm. Algebra 18 (1990) 563–584.
[20] M.E. Harris, Splendid derived equivalences for blocks of finite groups, J. London Math. Soc. (2) 60 (1999) 71–82.
[21] I.M. Isaacs, Character Theory of Finite Groups, Dover Publications, Inc., 1994.
[22] I.M. Isaacs, G. Navarro, New refinements of the McKay conjecture for arbitrary finite groups, Ann. of Math. (2) 156 (2002) 333–344.
[23] R. Knörr, G.R. Robinson, Some remarks on a conjecture of Alperin, J. London Math. Soc. (2) 39 (1989) 48–60.
[24] G. Navarro, Characters and Blocks of Finite Groups, Cambridge Univ. Press, 1998.
[25] G.R. Robinson, A note on perfect isometries, J. Algebra 226 (2000) 71–73.
[26] A. Salminen, PhD thesis, Ohio State University, 2005.
[27] M. Suzuki, On a class of doubly transitive groups, Ann. of Math. (2) 75 (1962) 105–145.
[28] J. Thèvenaz, G-Algebras and Modular Representation Theory, Oxford Math. Monogr., Oxford Univ. Press, 1995.