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HEIGHTS OF CHARACTERS IN BLOCKS

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ABSTRACT. We give a brief survey of the role of the height of an irreducible character in a block, and describe some recent joint work with Alexander Moretó concerning the minimal non-zero height of an irreducible character in a block. In particular we present a new consequence of Dade's conjecture.

1. INTRODUCTION

Let G be a finite group and p a prime. Let B be a p-block of G with defect group D, and write $\mathrm{Irr}(B)$ for the set of irreducible characters in B. Write d for the defect of B, i.e., we have $|D| = p^d$. The *defect* $d(\chi)$ of $\chi \in \mathrm{Irr}(B)$ is the (non-negative) integer such that $p^{d(\chi)}\chi(1)_p = |G|_p$. We have $d = \max\{d(\chi) : \chi \in \mathrm{Irr}(B)\}$. Define the *height* $h(\chi)$ of $\chi \in \mathrm{Irr}(B)$ to be $d - d(\chi)$. Note that each block must possess an irreducible character of height zero.

The information implicit in the p-part of an irreducible character degree has long been recognised, even before the explicit definitions of the height and defect. For example, $\chi(1)_p = |G|_p$ if and only if $|\mathrm{Irr}(B)| = 1$ (and B is called a block of defect zero). Another early result of Brauer states that if there is $\chi \in \mathrm{Irr}(B)$ with $\chi(1)_p = |G|_p/p$, then $|D| = |G|_p/p$. A conjecture of Brauer states that D is abelian if and only if $h(\chi) = 0$ for all $\chi \in \mathrm{Irr}(B)$. Note that the height of an irreducible character of D is precisely the exponent of its degree since D possesses a unique p-block, so D is abelian if and only if $\{h(\theta) : \theta \in \mathrm{Irr}(D)\} = \{1\}$.

Let $\mathcal{O}$ be a local, complete, discrete valuation ring with field of fractions K of characteristic zero, residue field $k = \mathcal{O}/J(\mathcal{O})$ of characteristic p, and suppose that $\mathcal{O}$ contains a primitive $|G|^3$th root of unity. We consider B as a block with respect to $\mathcal{O}$, and irreducible characters are K-characters.

A reasonable amount is known about irreducible characters of height zero. For example, a result of Knörr in [5] states that $h(\chi) = 0$ if and only if D is a vertex for an $\mathcal{O}G$-module M affording χ and $p \not| \mathcal{O}-rank\ S$ for a source S of M, and the conjecture of Brauer mentioned above also sheds some light on these characters. However, relatively little is understood about character defects in general. Hence, we begin with simple numerical observations,

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which will be the subject of this overview. Much of this survey is based on recent work with Alexander Moretó in [2].

The reader unfamiliar with block theory may benefit from reading this survey with always the *principal block* $B_0(G)$ in mind. A character $\chi \in \mathrm{Irr}(G)$ lies in $B_0(G)$ if and only if for all $g \in G$,

$$\frac{\chi(g)[G : C_G(g)]}{\chi(1)} \equiv [G : C_G(g)] \mod J(\mathcal{O}).$$

In particular the trivial character lies in $B_0(G)$. The defect groups of $B_0(G)$ are the Sylow p-subgroups, and so the height of $\chi \in \mathrm{Irr}(B_0(G))$ is the exponent of p in $\chi(1)_p$. Another instance where it is simpler to consider the principal block is in the Brauer correspondence. For a subgroup $H \leq G$ and a block b of H, we write b^G for the Brauer correspondent in B when it is defined (there are several definitions of the Brauer correspondence, but these are equivalent in all cases consider here). In this survey (but *not* in general) in every instance there is a unique block of H with Brauer correspondent $B_0(G)$, and this is $B_0(H)$. In general, we write $\mathrm{Irr}(H, B)$ for the set of irreducible characters in blocks of H with Brauer correspondent B.

2. Sets of heights in blocks

We are interested in the set $Ht(B) = \{h(\chi) : \chi \in \mathrm{Irr}(B)\}$, and its relation with $Ht(D) = \{h(\theta) : \theta \in \mathrm{Irr}(D)\}$. In general, we *cannot* expect $Ht(B) = Ht(D)$.

Example 2.1. *(i) Let G be the sporadic simple group M_{12}, and $p = 2$. In this case $|D| = 2^6$ for $D \in \mathrm{Syl}_p(G)$. We have $Ht(D) = \{0, 1, 2\}$ whilst $Ht(B_0(G)) = \{0, 1, 3\}$.*

(ii) Let G be the sporadic simple group Suz, and $p = 3$. Here $|D| = 3^7$ for $D \in \mathrm{Syl}_p(G)$. We have $Ht(D) = \{0, 1, 2\}$ and $Ht(B_0(G)) = \{0, 1, 2, 3\}$.

However, the defect group often has a large influence on the block. For example if D is cyclic, then the block B is very well-understood by Dade's theory of cyclic defect groups. Much is also known when D is dihedral, semi-dihedral or generalized quaternion, in which case the block is said to be of tame type. In general the influence is much more mysterious, and we restrict our attention here to the influence of D on $Ht(B)$.

A number of conjectures already relate $Ht(B)$ with $Ht(D)$. We mentioned in the introduction:

Conjecture 2.2 (Brauer)**.** *D is abelian if and only if $Ht(B) = \{0\}$.*

For p-solvable groups, the "only if" part is well-known, and "if" was done by Gluck and Wolf in [3] using the classification of finite simple groups (CFSG). In general, there is a reduction of "only if" to quasisimple groups by Berger and Knörr in [1], but as yet no proof.

Actually, Gluck and Wolf proved more than the above: if $e = \max(Ht(B))$, then the derived length of D is at most $2e + 1$. There are a number of results and conjectures concerned with $\max(Ht(B))$, for example

Theorem 2.3 (Moretó and Navarro [7]). *Suppose G is p-solvable and B is a block of G with defect group D. Then* $\max(Ht(B)) \leq 2(\max(Ht(D)))$

Note that this does not hold in general (there are non-p-solvable counterexamples). Moretó states a number of conjectures relating to maximal heights in [6].

The following is a consequence of a conjecture of Dade, as we will see later. However, it is known for p-solvable groups (this follows from the proof of Dade's projective conjecture for p-solvable groups in [9]).

Conjecture 2.4 (Robinson). *Let B be a block of a finite group G with defect group D. Then* $\max(Ht(B)) \leq [D : Z(D)]$.

3. Heights and defects

This section concerns the number of irreducible characters with a given height or defect. The purpose here is to motivate the work in [2] and to introduce some notation necessary for stating Dade's conjecture.

Departing a little from conventional notation, write $k(B) = |\mathrm{Irr}(B)|$,

$$k_n(B) = \{\chi \in \mathrm{Irr}(B) : h(\chi) = n\}$$

and

$$k^n(B) = \{\chi \in \mathrm{Irr}(B) : d(\chi) = n\}.$$

Two important conjectures in this area are

Conjecture 3.1 (Brauer). $k(B) \leq |D|$

and

Conjecture 3.2 (Olsson). $k_0(B) \leq k_0(D) = [D : D']$

The recent solution of the $k(GV)$ problem confirms that these conjectures hold for p-solvable groups.

A problem is to find a suitable setting which takes into account all possible character defects. We introduce some more notation. For n an integer, write

$$k_{\leq n}(B) = \sum_{i=0}^{n} k_i(B).$$

We might ask whether for all n,

$$(*) \qquad k_{\leq n}(B) \leq \sum_{i=0}^{n} p^{2i} k_i(D).$$

Setting $n = 0$ gives $k_0(B) \leq k_0(D) = [D : D']$.

Setting $n = d$ (where $|D| = p^d$) gives

$$k(D) \leq \sum_{i=0}^{d} p^{2i} k_i(D) = \sum_{\chi \in \mathrm{Irr}(D)} \chi(1)^2 = |D|.$$

The inequality we are interested in arises from this and the Alperin-McKay conjecture in a special case. The Alperin-McKay conjecture arose from the observation by McKay that the non-blockwise version for $p = 2$ appeared to hold for simple groups:

Conjecture 3.3 (Alperin-McKay)**.** *Let B be a block of a finite group G with defect group D. Then $k_0(B) = k_0(N_G(D), B)$.*

The choice of height zero is important - we do *not* in general have $k_n(B) = k_n(N_G(D), B)$ (this will become clear when we look at Dade's conjecture). The Alperin-McKay conjecture is known in a large number of cases, e.g., p-solvable groups, many simple groups, etc., and is a consequence of Dade's conjecture (although not necessarily case-by-case, depending on which form of Dade's conjecture you use).

Now if $N_G(D) = D$, then the Alperin-McKay conjecture predicts that $k_0(B) = k_0(D)$.

Remark 3.4. Guralnick, Navarro and Malle in [4] showed that if $N_G(P) = P \in \mathrm{Syl}_p(G)$ and $p \geq 5$, then G is solvable. If $PSL_2(3^r)$ is not a composition factor, then this is also true for $p = 3$. Note that the Alperin-McKay is known to hold for p-solvable groups.

Suppose that $N_G(D) = D$ and that the Alperin-McKay conjecture holds for B, e.g., if G is p-solvable.

Let n be the smallest non-zero height occurring amongst $\mathrm{Irr}(B)$ (if it exists). Suppose also that

$$k_{\leq n}(B) \leq \sum_{i=0}^{n} p^{2i} k_i(D).$$

We have $k_0(B) = k_0(D)$ and $0 < k_n(B)$, so $0 < k_{\leq n}(D) - k_0(D)$. In other words, writing

$$mh(B) = min\{h(\chi) : \chi \in \mathrm{Irr}(B),\ h(\chi) \neq 0\},$$

with $mh(B) := \infty$ if $Ht(B) = \{0\}$, we expect $mh(B) \geq mh(D)$ whenever $mh(B) \neq \infty$.

Remark 3.5. Recall the conjecture of Brauer which says that $h(\chi) = 0$ for all $\chi \in \mathrm{Irr}(B)$ if D is abelian. If we assume this, then it suffices to consider D non-abelian. With a slight abuse of notation, we may take the abelian case into account by taking $mh(B) = mh(D) = \infty$ to imply that $mh(B) \geq mh(D)$.

It turns out that this consequence of $(*)$ and the Alperin-McKay conjecture does indeed hold for p-solvable groups (by a direct proof):

Theorem 3.6 (Eaton and Moretó [2]). *Suppose G is p-solvable. Then $mh(B) \geq mh(D)$.*

We are not aware of any examples (p-solvable or otherwise) where $mh(B_0(G)) \neq mh(P)$.

4. Conjectures of Dade and Robinson

We will examine the relationship between the inequalities discussed in the last section and some conjectures of Dade and Robinson. We first establish the notation necessary to present the conjectures.

A chain $\sigma : Q_0 < \cdots < Q_n$ of p-subgroups of G (with strict inequalities) is called a p-chain of length $|\sigma| = n$. Denote by $\mathcal{P}(G)$ the set of all p-chains of G. Writing $V_\sigma = Q_0$, for $H \leq G$ a p-group write $\mathcal{P}(G|H) = \{\sigma \in \mathcal{P}(G) : V_\sigma = H\}$.

Letting G act by conjugation on $\mathcal{P}(G)$, write G_σ for the stabilizer of σ in G, i.e.,

$$G_\sigma = \bigcap_{i=0}^{n} N_G(Q_i).$$

For $m \leq n$, write σ_m for the truncated chain $Q_0 < \cdots < Q_m$.

In the conjectures of Dade and Robinson, we don't need to consider all p-chains, only *radical* p-chains. Recall that a p-subgroup Q is radical if $Q = O_p(N_G(Q))$ (for example, $O_p(G)$ and defect groups are radical). A chain σ is *radical* if $Q_m = O_p(G_{\sigma_m})$ for each m, i.e., Q_0 is radical and Q_m is a radical p-subgroup of $G_{\sigma_{m-1}}$ for $m > 0$. Denote by $\mathcal{R}(G)$ the set of radical p-chains.

Let $H \lhd G$ and $\chi \in \mathrm{Irr}(G)$. Then χ is *H-projective* if $\chi(1)_p = [G : H]_p \mu(1)_p$ for $\mu \in \mathrm{Irr}(H)$ covered by χ. Write $\mathrm{Irr}(G,H)$ for the set of H-projective irreducible characters of G and $w(G,H) = |\mathrm{Irr}(G,H)|$. Note that $d(\chi) = d(\mu)$ whenever χ is H-projective.

There are several versions of Dade's conjecture, of increasing strength and complexity (designed with a reduction to finite simple groups in mind). The following conjecture, due to Robinson, is equivalent to Dade's projective conjecture:

Conjecture 4.1 (Dade and Robinson). *Let d be an integer and $\lambda \in \mathrm{Irr}(O_p(Z(G)))$. Then*

$$k^d(B,\lambda) = \sum_{\sigma \in \mathcal{R}(G)/G} (-1)^{|\sigma|} w^d(G_\sigma, B, \lambda, V_\sigma).$$

Suppose $p^d = |D|$, i.e., $k^d(B) = k_0(B)$. If $k^d(G_\sigma, B, \lambda, V_\sigma) \neq 0$, then there is $\chi \in \mathrm{Irr}(G_\sigma)$ covering $\mu \in \mathrm{Irr}(V_\sigma)$ with $1 = \chi(1)_p = [G_\sigma : V_\sigma]_p \mu(1)$. Hence V_σ is a defect group for B, and we may take $\sigma = D$. So the conjecture says

$$k^d(B,\lambda) = w^d(N_G(D), D, \lambda, P) = k^d(N_G(P), B, \lambda).$$

Summing over the λ gives the Alperin-McKay conjecture.

Let $\chi \in \mathrm{Irr}^n(B)$. If the Conjecture 4.1 holds for B, then there is a radical p-subgroup Q and $\theta \in \mathrm{Irr}^n(Q)$. Here Q is the initial term of some radical chain contributing to the right hand side. This is the motivation for the following conjecture of Robinson:

Conjecture 4.2 (Robinson)**.** *Let $\chi \in \mathrm{Irr}(B)$. Then there exists $Q \leq D$ such that $Q = O_p(N_G(Q))$ and $\theta \in \mathrm{Irr}(Q)$ such that $d(\chi) = d(\theta)$.*

It turns out that the inequality $(*)$ is also a consequence of Conjecture 4.1:

Theorem 4.3 (Eaton and Moretó [2])**.** *Suppose that the Conjecture 4.1 holds for every quotient of each subgroup of G. Then $mh(B) \geq mh(D)$.*

Remark 4.4. If $mh(D) = 1$, then it suffices to know that Conjecture 4.1 holds just for B. This is a consequence of the theory of blocks with cyclic defect groups.

Note that Conjecture 4.1 is known to hold for p-solvable groups (see [8] and [9]), so this provides an alternative (but much harder) proof of Theorem 3.6.

It is natural to conjecture the following:

Conjecture 4.5 (Eaton-Moretó)**.** *Let B be a block with defect group D. Then $mh(B) \geq mh(D)$.*

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