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A continuous KN basis is a family of functions $\Phi(P,u)$
on an algebraic curve V, that is $P \in V$, numbered
by a continuous parameter u. It is assumed that $\Phi(P,u)$ is smooth in u. KN basis is characterized
by the property

$$\Phi(P,u)\Phi(P,v) = L\Phi(P,u+v)$$

where L is a linear differential operator in u, not depending on the point P.
KN basis is the basis of Fourier-Laurent transform
on the curve V.
We shall start with basic definitions. Then we focus
on the construction and the properties of differential
operator L. We demonstrate a connection of the the
multiplicative property of the KN basis of V
with the addition law on the Jacobian of V.

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P.G.Grinevich and S.P.Novikov (Topological Charge of the real
periodic finite-gap Sine-Gordon solutions: Dedicated to the
Memory of J.K.Moser, Commun. Pure Appl. Math., 56(7), 2003, 956-978) proposed an analog of the Fourier-Laurent integral transform on the Riemann surfaces. They use a continuous analog of the discrete Krichever-Novikov bases, which were introduced and studied for the needs of the quantum string theory in the late 80-s.

" Let us consider the following set of data: a nonsingular Riemann surface Γ of the genus g with marked
point $\infty \in \Gamma$ and selected local parameter near this point $z = k^{-1}$, $z(\infty) = 0$. We construct a function $\psi^0 = \psi(P, x)$ holomorphic on $\Gamma \backslash \infty$ and exponential near the infinite point:

$$\psi(z, x) = k^g \exp\{kx\}(1 + \sum_{i>0} \eta_i(x) k^{-i}). \qquad "$$

“**Problem.** Which multiplicative properties have the basic functions $\psi(P, x) = \psi_x(P)$ depending on x as parameter?”

“**Theorem**. Let $x, y \neq 0$. There exists a differential operator L in the variable x of the order g with
coefficients dependent on the both variables x, y
such that the following **Almost Graded Commutative Associative Ring Structure** is defined by the formula

$$\psi(P, x)\psi(P, y) = L\psi(P, x + y)$$

$$L = \partial_x^g + [\eta(x) + \eta(y) - \eta(x + y)]\partial_x^{g-1} + \dots \text{ ”}$$

When $g = 1$ one has: $L = \partial_x - (\zeta(x) + \zeta(y) - \zeta(x + y))$.

In [4] we construct the operator L for curves of higher genera. The algorithm is based on reduction of this problem to an effective description of the addition law on Jacobi variety.

Group of covariant shifts

Def. *Covariant shift*

$$W_{\alpha,\beta,c}(f(u)) := \exp\{\pi\imath\,(\langle 2u+\alpha,\beta\rangle + c)\} f(u+\alpha),$$

where $u, \alpha, \beta \in \mathbb{C}^g$; $c \in \mathbb{C}$, $\imath^2 = -1$ and $\langle\,\cdot\,,\,\cdot\,\rangle$ is Euclidean scalar product.

Group of covariant shifts $:= S$

$$W_{\alpha_2,\beta_2,c_2} W_{\alpha_1,\beta_1,c_1} = W_{\alpha_1+\alpha_2,\ \beta_1+\beta_2,\ c_1+c_2+\langle\beta_1,\alpha_2\rangle - \langle\alpha_1,\beta_2\rangle}$$

Representations of lattices

We use representations

$\mathbb{Z}^g \times \mathbb{Z}^g \to S$

defined by the formula

$$(n, n') \;\mapsto\; (\alpha, \beta, c) = \Big((n, n')\Omega,\; \phi(n, n')\Big),$$

Where:

(1) $\quad \Omega \in \mathrm{Sp}(2g, \mathbb{C})$:

$$\Omega^t J \Omega = J, \qquad J = \begin{pmatrix} 0 & 1_g \\ -1_g & 0 \end{pmatrix}.$$

(2) $\quad \phi : \mathbb{Z}^{2g} \to \mathbb{Z}_2$ $\quad$ is an Arf function:

for all Q_1 and Q_2 in $\mathbb{Z}^{2g}$ $\quad$ *Arf identity* holds

$$\phi(Q_1+Q_2) = \phi(Q_1)+\phi(Q_2)+Q_1 J Q_2^t \mod 2;$$

To define an Arf function ϕ fix

$(\ell, \ell') \in \mathbb{Z}^{2g}$ then

$\phi(n, n') = \langle n + \ell, n' + \ell' \rangle - \langle \ell, \ell' \rangle \mod 2.$

Explicit formula of a representation

Write Ω in block form

$$\Omega = \begin{pmatrix} \Omega_{1,1} & \Omega_{1,2} \\ \Omega_{2,1} & \Omega_{2,2} \end{pmatrix};$$

then we have a representation:

$$W_{\Omega}^{\ell,\ell'}(n,n') := W_{\alpha,\beta,c},$$

where

$$\alpha = (n\Omega_{1,1} + n'\Omega_{2,1})$$
$$\beta = (n\Omega_{1,2} + n'\Omega_{2,2})$$
$$c = \langle n+\ell, n'+\ell' \rangle - \langle \ell, \ell' \rangle$$

Construction of Sigma-function

Let $\Omega \in \mathrm{Sp}(2g, \mathbb{C})$ and $|\Omega_{1,1}| \neq 0$.
Set

$$G_\Omega(u) = \exp\left\{-\frac{\pi\imath}{2}\, u\varkappa u^t\right\}, \qquad \varkappa = \Omega_{1,1}^{-1}\Omega_{1,2}.$$

Def. $\boxed{\sigma(u,\Omega;\ell,\ell') := \sum_{(n,n')\in\mathbb{Z}^{2g}} W_\Omega^{\ell,\ell'}(n,n')\, G_\Omega(u)}$

Theorem. $\sigma(u,\Omega;\ell,\ell')$ is entire function of $u \in \mathbb{C}^g$ iff $\boxed{\mathrm{Im}\,\tau \quad \text{is positive definite.}}$

where

$$\tau = \Omega_{2,1}\Omega_{1,1}^{-1}, \quad \Rightarrow \quad \Omega_{2,2} = \tau\Omega_{1,1}\varkappa + (\Omega_{1,1}^t)^{-1}$$

Families of Sigma-functions

Theorem. Fix Ω and (ℓ, ℓ').

If $\quad W_{\Omega}^{\ell,\ell'}(k,k')\,F(u) = F(u), \quad \forall\,(k,k') \in \mathbb{Z}^{2g}$

then $\quad F(u) = \text{const} \cdot \sigma(u, \Omega; \ell, \ell').$

Fix a map

$$\boxed{\Omega : \mathbb{C}^q \to \mathrm{Sp}(2g, \mathbb{C}), \qquad |\Omega_{1,1}(\lambda)| \neq 0}$$

then we have

Family of Sigma-Functions:

$$\boxed{\sigma(u, \lambda; \ell, \ell') := \frac{\sigma(u, \Omega(\lambda); \ell, \ell')}{\sqrt{|\Omega_{1,1}(\lambda)|}}, \quad \lambda \in \mathbb{C}^q.}$$

The Heat operators

Fix an arbitrary smooth vector field

$$L=\sum_{j=1}^{q} v_j(\lambda)\frac{\partial}{\partial\lambda_j}.$$

Introduce the second order operator

$$H_\Omega=\sum_{r,s=1}^{g}(\alpha_{r,s}\partial_{r,s}+2\alpha_{g+r,s}u_r\partial_s+\alpha_{g+r,g+s}u_ru_s),$$

where $\quad (\alpha_{r,s})=L(\Omega^t)J\Omega \quad$ and $\quad \partial_{r,s}=\dfrac{\partial^2}{\partial u_r\partial u_s}$

$$\delta_\Omega(\lambda)=\frac{1}{2}\text{sk-tr}\Big(L(\Omega^t)J\Omega\Big).$$

sk-tr$M:=\sum_{i=1}^{g} m_{i,g+1-i}$ for $M=(m_{i,j})$

Lemma. For any constant $K\in \mathrm{Sp}(2g,\mathbb{C})$

$$\boxed{H_\Omega=H_{K\Omega},\qquad \delta_\Omega(\lambda)=\delta_{K\Omega}(\lambda).}$$

Lemma.

$$\Big(2L + H_\Omega + \delta_\Omega(\lambda)\Big)\frac{G_\Omega(u)}{\sqrt{|\Omega_{1,1}(\lambda)|}} = 0.$$

Lemma. For all $(k, k') \in \mathbb{Z}^{2g}$
the covariant shift $W^{\ell,\ell'}_{\Omega(\lambda)}(k, k')$
and the heat operator $2L + H_\Omega + \delta_\Omega(\lambda)$
commute as the operators on the space of smooth
functions of u and λ.

Theorem. The family $\sigma(u, \lambda; \ell, \ell')$ solves the equation

$$\boxed{\Big(2L + H_\Omega + \delta_\Omega(\lambda)\Big)\sigma(u, \lambda; \ell, \ell') = 0}$$

Example. $q = 1$, $g = 1$, $\quad L = \partial_\lambda$

$$\Omega(\lambda) = \begin{pmatrix} \omega & \omega\varkappa \\ \tau\omega & \tau\omega\varkappa + 1/\omega \end{pmatrix}$$

$\omega \neq 0$, Im$\tau > 0$, $\varkappa$ are smooth functions in λ.

$$H_\Omega = a_{1,1}\partial_u^2 + 2a_{1,2}u\partial_u + a_{2,2}u^2$$
$$\delta_\Omega(\lambda) = a_{1,2}$$

where

$$a_{1,1} = -\omega^2\partial_\lambda\tau,$$
$$a_{1,2} = \frac{\partial_\lambda\omega}{\omega} - \omega^2\,\varkappa\partial_\lambda\tau,$$
$$a_{2,2} = \partial_\lambda\varkappa - \omega^2\,\varkappa^2\partial_\lambda\tau + 2\varkappa\frac{\partial_\lambda\omega}{\omega}$$

Example. $q = 1$, g is arbitrary

$$L = \partial_\lambda$$

$$\Omega(\lambda) = \begin{pmatrix} \omega & \omega\varkappa \\ \tau\omega & \tau\omega\varkappa + (\omega^t)^{-1} \end{pmatrix}$$

where $|\omega| \neq 0$, $\tau^t = \tau$, Imτ is positive definite, $\varkappa^t = \varkappa$ are smooth $(g \times g)$-matrix functions in λ .

$$H_\Omega = (\partial_u)^t A_{1,1} \partial_u + 2u^t A_{2,1} \partial_u + u^t A_{2,2} u$$
$$\delta_\Omega(\lambda) = \mathrm{tr} A_{2,1}$$

where

$$A_{1,1} = -\omega^t (\partial_\lambda \tau) \omega,$$
$$A_{2,1} = \omega^{-1} \partial_\lambda \omega + \varkappa A_{1,1},$$
$$A_{2,2} = \partial_\lambda \varkappa + A_{2,1} \varkappa + \varkappa A^t_{2,1} - \varkappa A_{1,1} \varkappa.$$

Abelian Sigma-functions and Heat Equations in Non-holonomic frame.

Let $\quad s > n > 1, \quad \text{gcd}(n, s) = 1.$

$$f(x,y,\lambda) = y^n - x^s - \sum_{j=0}^{n-2} \sum_{i=0}^{s-2} \lambda_{ns-in-js} x^i y^j,$$

the number $m := \#\{\lambda_k \,|\, k < 0\}$ is called *modality*.

Set: $\qquad \lambda_k = 0, \quad k < 0.$

Def. The family of (n,s)-curves:

$$V = V_\lambda = \{(x,y) \in \mathbb{C}^2,\, \lambda \in \mathbb{C}^{2g-m} \,|\, f(x,y,\lambda) = 0\}$$

genus of a generic curve V_λ is $\quad g = \frac{(n-1)(s-1)}{2}$.

Example. Hyperelliptic curves are $(2, 2g+1)$-curves, $m = 0$:

$$f(x,y,\lambda) = y^2 - x^{2g+1} - \sum_{i=0}^{2g-1} \lambda_{2(2g-i+1)} x^i$$

The principal Arf function of (n,s)-curve

Def. *Weierstrass sequence* $(w_1,\dots,w_g)$ is the ordered set $\mathbb{N}\backslash\mathcal{M}$, where $\mathcal{M}=\{an+bs\}$, $a,b\in\mathbb{N}\cup 0$.

Assign $\quad w(\xi)=\sum_i \xi^{w_i} \quad$ then

$$w(\xi)=\frac{1}{1-\xi}-\frac{1-\xi^{ns}}{(1-\xi^n)(1-\xi^s)}.$$

Now $g=w(1)=\frac{(n-1)(s-1)}{2}$.

Let us define $(\pi_1,\dots,\pi_g)$ by the formula

$$\pi_k=w_{g-k+1}-(g-k), \qquad k=1,\dots,g.$$

$(w_g=2g-1$ and $\pi_1=g.)$

Def. *The principal Arf function* is defined by

$$\ell=(1,\dots,1) \quad \text{and} \quad \ell'=(\pi_1,\dots,\pi_g).$$

Let us fix this value of (ℓ,ℓ') for the rest of the talk.

Example. For hyperelliptic family $(n,s)=(2,2g+1)$

$$\ell'=(g,g-1,\dots,1).$$

Meromorphic Abelian integrals

We do the hyperelliptic case for simplicity. Set:

$$\mathrm{d}u_j(x,y) = \frac{x^{j-1}\mathrm{d}x}{2y}, \qquad j = 1,\dots,g,$$

$$\mathrm{d}r_j(x,y) = -\Big(x\partial_x\pi_+\Big(\frac{f(x,y,\lambda)}{x^{2j}}\Big)\Big)\mathrm{d}u_j(x,y),$$

where $\pi_+(\,\cdot\,)$ truncates the negative powers of x.

Def. *Universal Abelian cover of* V is the space W of pairs $\Big((x,y);[\gamma]\Big)$, where $(x,y)\in V$; $[\gamma]$ is an equivalence class of paths from $\infty\in V$ to (x,y).

We say that γ_1 and γ_2 are in $[\gamma]$ if the contour $\gamma_1\circ\gamma_2^{-1}$ is homologous to zero.

Def. Abelian maps: $A : W \to \mathbb{C}^g$, $A^* : W \to \mathbb{C}^g$,

$$\begin{aligned} A_j(x,y;[\gamma]) &= \int_\gamma \mathrm{d}u_j(x,y), \\ A_j^*(x,y;[\gamma]) &= \int_\gamma \mathrm{d}r_j(x,y). \end{aligned} \qquad j = 1,\dots,g.$$

The period map $\Omega : \mathbb{C}^{2g-m} \to \mathsf{Sp}(2g,\mathbb{C})$

Let the contours $\gamma_1, \ldots, \gamma_{2g}$ give a basis in $H_1(V,\mathbb{Z})$, such that the intersection matrix is J:
$J_{a,b} = \gamma_a \circ \gamma_b = \mathsf{sign}(b-a)\delta_{g,|b-a|}, \quad a,b = 1,\ldots,2g.$
Set for $i,j = 1,\ldots,g$,

$$\omega_{i,j} = \frac{1}{2}\oint_{\gamma_j} \mathrm{d}u_i(x,y), \quad \omega'_{i,j} = \frac{1}{2}\oint_{\gamma_{g+j}} \mathrm{d}u_i(x,y),$$
$$\eta_{i,j} = -\frac{1}{2}\oint_{\gamma_j} \mathrm{d}r_i(x,y), \quad \eta'_{i,j} = -\frac{1}{2}\oint_{\gamma_{g+j}} \mathrm{d}r_i(x,y),$$

then for $\quad \Omega(\lambda) = \begin{pmatrix} \omega & \eta \\ \omega' & \eta' \end{pmatrix}$, where $\omega = (\omega_{i,j})$, etc.,

we have *Legendre relation*

$$\Omega(\lambda) J \Omega(\lambda)^t = \frac{\pi\imath}{2} J$$

Note: For nonsingular V_λ the choice of basis contours provides $|\omega| \neq 0$ and positive definiteness of $\mathrm{Im}\,\omega'\omega^{-1}$.

Note: $K = \ell\omega + \ell'\omega'$ is the vector of *Riemann constants*.

Frame tangent to Discriminant and Heat operators

Denote by $\Delta(\lambda)$ the discriminant of $f(x,y,\lambda)$:

$$\Delta(\lambda) = 0 \Leftrightarrow \exists\,(x,y)\; f = f_x = f_y = 0$$

Consider the space $\mathcal{T}$ of polynomial vector fields tangent to $\quad \{\lambda \in \mathbb{C}^{2g-m} | \Delta(\lambda) = 0\}$.
$L \in \mathcal{T}$ implies

$$L\,\Delta(\lambda) = \varphi(\lambda)\Delta(\lambda), \quad \varphi(\lambda) \in \mathbb{C}[\lambda].$$

$\mathcal{T}$ has the basis $\{L_1, \ldots, L_{2g}\}$ over $\mathbb{C}[\lambda]$. The basis gives a non-holonomic frame which defines a nontrivial polynomial Lie algebra (the structure is described in [1]).

We use the basis vector fields $\{L_1, \ldots, L_{2g}\}$ and the period map $\Omega(\lambda)$ to construct $2g$ heat operators.

Examples of basis fields L_i

$g=1$. We have $\Delta(\lambda)=4\lambda_4^3+27\lambda_6^2$

$$L_0=4\lambda_4\partial_4+6\lambda_6\partial_6,$$
$$L_2=6\lambda_6\partial_4-\frac{4}{3}\lambda_4^2\partial_6.$$

Here $\partial_k=\dfrac{\partial}{\partial\lambda_k}$, $\deg\lambda_k=k$. Then $\deg L_j=j$.

$g=2$. The symmetric matrix T transforms the standard fields $\partial_4,\partial_6,\partial_8,\partial_{10}$ to the basis fields L_0,L_2,L_4,L_6

$$T=\begin{pmatrix} 4\lambda_4 & 6\lambda_6 & 8\lambda_8 & 10\lambda_{10} \\ * & \frac{40\lambda_8-12\lambda_4^2}{5} & \frac{50\lambda_{10}-8\lambda_4\lambda_6}{5} & -\frac{4\lambda_4\lambda_8}{5} \\ * & * & \frac{20\lambda_4\lambda_8-12\lambda_6^2}{5} & \frac{30\lambda_4\lambda_{10}-6\lambda_6\lambda_8}{5} \\ * & * & * & \frac{4\lambda_6\lambda_{10}-8\lambda_8^2}{5} \end{pmatrix}$$

Note: $\Delta(\lambda)=|T|$. The matrix T plays an important role
in Singularity Theory as the *convolution matrix*.

Examples of operators H_i

Here $D_i = \dfrac{\partial}{\partial u_i}$; $\deg u_i = -i$ $\deg H_i = i$.

$g = 1$.

$$H_0 = u_1 D_1 - 1$$
$$6H_2 = 3D_1^2 - \lambda_4 u_1^2$$

$g = 2$.

$$H_0 = u_1 D_1 + 3u_3 D_3 - 3$$

$$\begin{aligned} 10H_2 = 5D_1^2 + 10u_1 D_3 - 8\lambda_4 u_3 D_1 - \\ - 3\lambda_4 u_1^2 + (15\lambda_8 - 4\lambda_4^2) u_3^2 \end{aligned}$$

$$\begin{aligned} 5H_4 = 5D_1 D_3 + 5\lambda_4 u_3 D_3 - 6\lambda_6 u_3 D_1 - 5\lambda_4 - \\ - \lambda_6 u_1^2 + 5\lambda_8 u_1 u_3 + 3(5\lambda_{10} - \lambda_4\lambda_6) u_3^2 \end{aligned}$$

$$\begin{aligned} 10H_6 = 5D_3^2 - 6\lambda_8 u_3 D_1 - 5\lambda_6 - \\ - \lambda_8 u_1^2 + 20\lambda_{10} u_1 u_3 - 3\lambda_4\lambda_8 u_3^2 \end{aligned}$$

Abelian Sigma-function.

Our construction gives the following result.

Theorem. The heat equations

$$L_i\sigma(u,\lambda) = H_i\sigma(u,\lambda),$$

for $i \in \{nk + js\}$, $0 \leqslant j < n-1$, $0 \leqslant k < s-1$, uniquely define *the Abelian σ-function of* (n,s)-*curve*.

(1) It has the translation property

$$\sigma(u + A[\chi]) = \sigma(u)\exp\Big\{ -\Big\langle A^*[\chi], u + \frac{1}{2}A[\chi]\Big\rangle + \\ +\pi\imath\Big(\langle k+\ell, k'+\ell'\rangle - \langle \ell, \ell'\rangle\Big)\Big\},$$

where $[\chi] = \sum_{j=1}^{g}(k_j\gamma_j + k'_j\gamma_{g+j})$.

(2) It is an entire function on $\mathbb{C}^g \times \mathbb{C}^{2g-m}$.

Its power series in u and λ has rational coefficients.

(3) The grading $\deg x = n$, $\deg y = s$ and $\deg \lambda_k = k$ gives $\deg f(x,y,\lambda) = ns$ and the grading of u s.t.

$$\deg \sigma(u,\lambda) = -\sum_{j=1}^{g}\ell'_j = -\frac{(n^2-1)(s^2-1)}{24}.$$

Krichever-Novikov continuous basis

Def.

$$\boxed{\psi(x,y;[\gamma]) := \exp\left\{-\int_{[\gamma]}\Big\langle A^*((x',y'),[\gamma']),\mathrm{d}A(x',y')\Big\rangle\right\}}$$

$\psi(x,y;[\gamma])$ is the unique entire function $W \to \mathbb{C}$ with:

(1) Single essentially singular point $\infty \in V$

$$\psi \sim \xi^g(1+O(\xi)).$$

(2) No zeros and poles in $V\backslash\infty$.

$\boxed{\xi \text{ is local parameter at } \infty,\ \deg\xi = -1.}$

Def.

$$\boxed{\Psi\Big(u,(x,y)\Big) := \frac{\sigma(A(x,y;[\gamma])-u)}{\psi(x,y;[\gamma])\,\sigma(u)}\exp^{-\langle A^*(x,y;[\gamma]),u\rangle}}$$

Ψ is single-valued function $\mathbb{C}^g \times V \to \mathbb{C}$.

If $g=1$, $\quad \Psi\Big(u,(\wp(\xi),\wp'(\xi))\Big) = \dfrac{\sigma(\xi-u)}{\sigma(\xi)\sigma(u)}\exp\{u\zeta(\xi)\}.$

gives a solution of Lamè equation

$$\partial_u^2\Psi\Big(u,(x,y)\Big) - 2\wp(u)\Psi\Big(u,(x,y)\Big) = x\,\Psi\Big(u,(x,y)\Big)$$

Fix $u \in \mathbb{C}^g$. Then $\Psi\Big(u,(x,y)\Big)$ is
the **unique single-valued function** on V with:

(1) g zeros on V at $A^{-1}(u)$.

(2) Single essentially singular point $\infty \in V$

$$\Psi \sim \xi^{-g} \exp\{p(\xi^{-1}; u, \lambda)\}(1 + O(\xi)),$$

where $p(t; u, \lambda) = p_1(u, \lambda)t + \cdots + p_{2g-1}(u, \lambda)t^{2g-1}$ is fixed by the choice of $f(x, y, \lambda)$.

$p_k(u, \lambda)$ is homogeneous polynomial $\deg p_k(u, \lambda) = -k$. In general case

$$p_1(u_1, 0, \ldots, 0, \lambda) = u_1,$$
$$p_j(u_1, 0, \ldots, 0, \lambda) = 0, \qquad j > 1.$$

Note: $\Psi\big(u,(x,y)\big)$ is the *Baker-Akhiezer Function* corresponding to the degenerate set of Krichever data.

Example. For hyperelliptic curves

$$p(t;u,0)=\sum_{i=1}^{g}u_{2i-1}t^{2i-1}.$$

$$g=1,\quad p(t;u,\lambda)=u_1t,$$

$$g=2,\quad p(t;u,\lambda)=u_1t+u_3t^3,$$

$$g=3,\quad p(t;u,\lambda)=\Big(u_1+\frac{1}{2}\lambda_4u_5\Big)t+u_3t^3+u_5t^5.$$

Also, the equation

$$\partial_{u_1}^2\Phi-2\wp_{1,1}(u)\Phi=x\,\Phi,$$

where

$$\wp_{1,1}(u)=-\frac{\partial^2}{\partial u_1^2}\log\sigma(u)$$

has solutions

$\Phi_\pm=\Psi\Big(\pm u,(x,y)\Big),$

$$\begin{vmatrix}\partial_{u_1}\Phi_+ & \partial_{u_1}\Phi_-\\ \Phi_+ & \Phi_-\end{vmatrix}=2y.$$

In our notation the $\boxed{\text{“}\psi(P,t)\text{”}}$ of Grinevich and Novikov is $\boxed{\Psi(t\mathsf{e}_1,(x,y))}$, where e_1 is the 1-st ort in $\mathbb{C}^g$.

The relation defining the multiplicative structure
of the base $\{\Psi(t\mathsf{e}_1,(x,y))\}$ is a particular case of the relation

$$\Psi\Big(u,(x,y)\Big)\Psi\Big(v,(x,y)\Big) = L\Psi\Big(w,(x,y)\Big)\Big|_{w=u+v},$$

where $u,v \in \mathbb{C}^g$ and

$$L = \sum_{j=0}^{g} a_j(u,v,w)\frac{\partial^{g-i}}{\partial w_1^{g-i}}, \qquad \deg L = g.$$

We define the family of functions on V with parameter $w \in \mathbb{C}^g$

$$G_k^{(w)}(x,y) = \frac{\partial_{w_1}^k \Psi(w,(x,y))}{\Psi(w,(x,y))}, \qquad k = 0, 1, \ldots$$

Each $G_k^{(w)}(x,y)$ is rational function on V.
It has $g + k$ poles in $\{k\,\infty, A^{-1}(w)\}$.
Its coefficients are Abelian functions on the Jacobi variety of V.

$$\begin{aligned} &G_0^{(w)}(x,y) = 1, \\ &G_1^{(w)}(x,y) = -\Big(\zeta_1(A(x,y;[\gamma]) - w) + \zeta_1(w) + \\ &\qquad\qquad\qquad\qquad + \langle A^*(x,y;[\gamma]), \mathsf{e}_1\rangle\Big), \end{aligned}$$

where $\zeta_1(w) = \partial_{w_1} \log \sigma(w)$.
For $k > 1$ we have the recurrence

$$G_{k+1}^{(w)}(x,y) = \partial_{w_1} G_k^{(w)}(x,y) + G_1^{(w)}(x,y)\, G_k^{(w)}(x,y).$$

We express $G_{k+1}^{(w)}(x,y)$ as rational functions of (x,y).

Example. In the hyperelliptic case

$$G_1^{(w)}(x,y) = \frac{1}{2}\,\frac{2y + \sum\limits_{i=1}^{g} \wp_{1,1,(g-i)}(w)x^{g-i}}{x^g - \sum\limits_{i=1}^{g} \wp_{1,i}(w)x^{g-i}},$$

$$G_2^{(w)}(x,y) = x + 2\wp_{1,1}(w)$$

from the recurrence we have for $k > 2$

$$\begin{aligned} &G_k^{(w)}(x,y) = a_k + b_k G_1^{(w)}(x,y)\\ &a_{k+1} = \partial_{w_1} a_k + (x + 2\wp_{1,1}(w))b_k,\\ &b_{k+1} = \partial_{w_1} b_k + a_k, \end{aligned}$$

Clearly, a_k and b_k are polynomials in x.

$$\wp_{i,j}(w) = -\frac{\partial^2 \log \sigma(w)}{\partial w_i \partial w_j},$$

$$\wp_{i,j,k}(w) = -\frac{\partial^3 \log \sigma(w)}{\partial w_i \partial w_j \partial w_k}$$

where i, j, k are any odd integers between 0 and $2g$.

We prove that for all $u, v, w \in \mathbb{C}^g$

$$\Psi\Big(u, (x,y)\Big)\Psi\Big(v, (x,y)\Big)\Psi\Big(-u-v, (x,y)\Big) = R_{3g}^{(u,v)}(x,y),$$
$$\Psi\Big(w, (x,y)\Big)\Psi\Big(-w, (x,y)\Big) = R_{2g}^{(w)}(x,y).$$

As function on V

$R_{3g}^{(u,v)}(x,y)$ has
$3g$-tuple pole at ∞ and $3g$ zeros at
$\{A^{-1}(u), A^{-1}(v), A^{-1}(-u-v)\}$.

$R_{2g}^{(w)}(x,y)$ has
$2g$-tuple pole at ∞ and $2g$ zeros at
$\{A^{-1}(w), A^{-1}(-w)\}$.

The functions $R_{3g}^{(u,v)}(x,y)$ and $R_{2g}^{(w)}(x,y)$ define **addition and inverse operations** on $\mathrm{Sym}^g(V)$.

Theorem. The operator L is defined by the equality

$$\frac{R_{3g}^{(u,v)}(x,y)}{R_{2g}^{(u+v)}(x,y)} = \sum_{i=0}^{g} a_i(u,v,u+v)G_i^{(u+v)}(x,y).$$

This reduces the problem to comparing the coefficients at monomials, after cancelation of the common
denominator on both sides.

Example. For hyperelliptic curves we have

$$\begin{aligned} \alpha_0(u,v,w) &= 1, \\ \alpha_1(u,v,w) &= -\zeta_1(u) - \zeta_1(v) + \zeta_1(w), \\ 2\alpha_2(u,v,w) &= -\wp_{1,1}(u) - \wp_{1,1}(v) - 3\wp_{1,1}(w) + \\ &\quad + \alpha_1(u,v,w)^2\Big), \quad \text{etc.} \end{aligned}$$