*On Spherical Classes in H*QSn*

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On Spherical Classes in H_*QS^n

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Contents

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We consider the problem of determining spherical classes in H_*QS^1. We take a geometrical approach and show how existence of specific classes as a spherical class in H_*QS^1 will determine the type of homology operations that can detect the related homotopy class. Most of our results here are quite general, and can be applied to H_*QX, with X an arbitrary path connected space . We see this as an approach to attack the conjecture of Ed Curtis about spherical classes in $H_*Q_0S^0$.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

Copyright Statement

بسم الله الرحمن الرحيم

الْحَمْدُ لِلَّهِ رَبِّ الْعَالَمِينَ

الرَّحْمَنِ الرَّحِيمِ

مَالِكِ يَوْمِ الدِّينِ

إِيَّاكَ نَعْبُدُ وَإِيَّاكَ نَسْتَعِينُ

اهدِنَا الصِّرَاطَ الْمُسْتَقِيمَ

صِرَاطَ الَّذِينَ أَنعَمتَ عَلَيْهِمْ غَيْرِ الْمَغْضُوبِ عَلَيْهِمْ وَلاَ الضَّالِّينَ

Dedicated to Prophet Mohammad and his Ahl-al bait,

and to the presence of 12*th Imam, Imam Mahdi*

Dedicated to My Parents, Mohammad Taghi and Zahra, first Teachers in my life

and,

Dedicated to the spirit of my Grandfather Hadj Reza Zare, whom his memory and life has always inspired me to work for good and try to make difference in other people's life

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Chapter 1

Introduction

I believe that the theory of infinite loop spaces has been one of the most influential branches of topology, and that the machineries produced to study such spaces have found fundamental applications in other areas of mathematics, as well as theoretical physics. Nevertheless, there are still many questions in this area which are waiting to be answered, both from technical and computational point of view, as well as from theoretical point of view.

The present thesis considers a computational problem, and our aim is to tackle an outstanding conjecture in the field known as the *Curtis conjecture*. This conjecture, first stated as a theorem by Ed Curtis [C75, Theorem 7.1], predicts the type of spherical classes in the $\mathbb{Z}/2$-homology of Q_0S^0, the base point component of the infinite loop space associated with the sphere spectrum S^0. It reads as follows.

The Curtis conjecture. Let $f \in {}_2\pi_*Q_0S^0$ be a positive dimensional class with $hf \neq 0$ where $h : {}_2\pi_*Q_0S^0 \to H_*(Q_0S^0;\mathbb{Z}/2)$ is the Hurewicz homomorphism. Then f is either a Hopf invariant one element or a Kervaire invariant one element.

Let me start by explaining the terminology used in the statement of the above conjecture. But first, let us fix a notation. Assume $f^S \in {}_2\pi_*^S$ be the image of f under the isomorphism ${}_2\pi_*Q_0S^0 \to {}_2\pi_*^S$. We refer to f^S as the stable adjoint of f. Similarly, we may refer to f as the stable adjoint of f^S.

The Hopf invariant one elements. We will say $f \in {}_2\pi_*Q_0S^0$ is a Hopf invariant one element if f^S is detected by the Hopf invariant, i.e. it is detected by Sq^{2^j} in its stable mapping cone. A spherical class $\xi \in H_*(Q_0S^0;\mathbb{Z}/2)$ with $hf = \xi$ is called a Hopf invariant one element if f is a Hopf invariant one element. Shortly, we will explain what we mean by detecting.

The Kervaire invariant one elements. We say that f is a Kervaire invariant one element if f^S is detected by the Kervaire invariant, i.e. it is detected by a secondary operation arising from the Adem relation

$$Sq^{2^j}Sq^{2^j} = \sum_t Sq^{2^{j+1}-2^t}Sq^{2^t}$$

in its stable mapping cone. Given a spherical class $\xi \in H_*(Q_0S^0;\mathbb{Z}/2)$ with $hf = \xi$, we say ξ is a Kervaire invariant one element if f is a Kervaire invariant one element. Notice that the positive dimensional Hopf invariant one elements in ${}_2\pi_*^S$ are known to exist only in 1-stem, 3-stem, and 7-stem given by one of the classical Hopf invariant elements $\eta \in {}_2\pi_1^S$, $\nu \in {}_2\pi_3^S$, $\sigma \in {}_2\pi_7^S$ [A60, Theorem 1.1.1]. We also recall that there exits a Hopf invariant one element if and only if certain elements $h_i \in E_2^{1,2^i}$ are permanent cycles, where here $E_2^{*,*}$ denotes the E_2-term of the Adams spectral sequence. Regarding the Kervaire invariant one element, Browder [B69, Theorem 7.1] showed that there exits an element in ${}_2\pi_*^S$ detected by the Kervaire invariant if and only if $h_i^2 \in E_2^{2^{i+1},2}$ are permanent cycles. This means that such elements may exist only in dimensions $2^{i+1}-2$. Moreover, a very recent development [HHR09, Doomsday Theorem] is that the Kervaire invariant one elements only exist in dimensions $2^{i+1}-2$ with $i < 7$. These are known to exist when $i < 6$ and the case $i = 6$ still is open.

Henceforth, the Curtis conjecture predicts that the Hurewicz homomorphism $h : {}_2\pi_*Q_0S^0 \to H_*(Q_0S^0;\mathbb{Z}/2)$ cannot detect as many elements of ${}_2\pi_*^S$ as one may wish. In fact, it predicts that this homomorphism cannot see above the 2-line in the Adams spectral sequence for the sphere spectrum. Philosophically, this may not seem very useful as it predicts what we cannot do, where often we look for what we can do. However it is an interesting problem from computational point of view. The gaps in Curtis's argument were discovered by Wellington [W82] while he was trying to generalise Curtis's theorem to odd primes. We postpone more discussion on this to Chapter 5. Indeed, we have to say that Curtis's goal in [C75] was to describe the relation between the Dyer-Lashof algebra R and the Λ-algebra and this goal is achieved [C75, Lemma 5.4]. This result is even generalised to odd primes by Wellington [W82, Theorem 7.11, Theorem 7.12].

It has been a common belief in the community of the infinite loop space theorists that the Curtis conjecture is true, however there has been no proof of it since it was stated by Curtis in 1974. In this thesis, we are not claiming any proof of the conjecture. The approach taken seems to be a new one, the results may seem to be natural ones that one could have expected. I will suggest a road map towards the proof of this conjecture which I believe will give a resolution of this conjecture. However, the details need to be written down and completed.

Previous approaches to solve this problem have used a lot of heavy algebraic methods. Perhaps Wellington's work [W82] is the most detailed existing record of this. Other approaches include Lannes and Zarati's work, which recently has been pursued by Hung and his students, see for example [H99]. This latter approach relates the Curtis conjecture to some problems in Dickson algebra, Singer's algebraic transfer, and etc. This then may be taken as an evidence for the level of complexity of the Curtis conjecture, and perhaps justify our interest in attacking this problem. The latter approach is also very algebraic.

We will not consider use of the spectral sequences as the main tool in our approach. This has been tried previously by Curtis and later on by Wellington. Our approach seems to be a bit different compared to other works that we referred above. We will consider a more general problem on the type of spherical classes in H_*QX for any path connected space X. First, let us fix a notation. We use f^S to denote the image of $f \in {}_2\pi_*QX$ under the isomorphism ${}_2\pi_*QX \to {}_2\pi_*^SX$. We refer to f^S as the stable adjoint of f, and similarly refer to f as the stable adjoint of f^S. Sometimes we may use h^Sf to denotes hf^S. We have a conjecture on this due to Eccles which reads as following.

The Eccles conjecture. Suppose X is a path connected space. Let $f \in {}_2\pi_*QX$ be a positive dimensional class with $hf \neq 0$ where $h : {}_2\pi_*QX \to H_*(QX;\mathbb{Z}/2)$ is the Hurewicz homomorphism. Then f is either a stably spherical element or a Hopf invariant one element.

Stably spherical classes. We say $\xi \in H_*(QX;\mathbb{Z}/2)$ with $hf = \xi$ is stably spherical, if f^S is nontrivial in homology, i.e. $h^S f = hf^S \neq 0$.

The Hopf invariant one elements. Recall that $f \in {}_2\pi_* Q_0 S^0$ is a Hopf invariant one element if f^S is detected by a primary operation, namely Sq^{2^j}. We say $f \in {}_2\pi_* QX$ is a Hopf invariant one element, if f^S is detected by a primary operation in its stable mapping cone. Moreover, given a spherical class $\xi \in H_*(QX;\mathbb{Z}/2)$ with $hf = \xi$ we say ξ is a Hopf invariant one element if f is a Hopf invariant one element.

Shortly, in Chapter 2, we will show how the Curtis conjecture motivates the Eccles conjecture. We shall then postpone more discussion on the relation between the two conjectures to Chapter 5.

Although we fail to prove the Eccles conjecture as well, we do obtain some *brand new* partial results in a very general setting which could be of interest on their own. These results have been never considered before. In the case of $X = S^n$, $n > 0$, our results are more precise and we have succeeded in identifying the form of the potential spherical classes in $H_*(QS^n;\mathbb{Z}/2)$ which is the statement of Lemma 12. In the case of $X = S^0$, instead of focusing on a single space Q_0S^0, we consider the collection of spaces QS^n with $n \in \mathbb{Z}$. *Suspending up*, will help us to identify potential classes for being spherical, where *desuspending down* will help to eliminate some of the potential classes.

Our approach was suggested by Peter Eccles at the beginning of this project. It provides easy proofs of what is known before, and the author hopes that one day it may prove more than what we know now.

Through the rest of this chapter we fix our notation and recall some well known facts. First, we note that throughout this thesis, we will work with CW-complexes of finite type. Notice that the cellular approximation theorem [MT68, Theorem 1, Corollary 1, Chapter 13] allows us to choose maps, up to homotopy, to be cellular. A genuine mapping between two spaces X, Y is denoted by $X \to Y$, whereas we use $X \nrightarrow Y$ to denote a stable mapping from X to Y. Here by a stable mapping $X \nrightarrow Y$ we mean a mapping that will be realised as a genuine mapping after finitely many suspensions. A mapping $f : X \to Y$ between two path connected spaces is called a weak equivalence if it induces isomorphisms $f_* : \pi_i X \to \pi_i Y$ for $i > 0$. If X and Y are not path connected, we then require $f_* : \pi_0 X \to \pi_0 Y$ to be a 1-1 correspondence, and that f_* induces an isomorphism on higher homotopy groups between the corresponding components of X and Y. Notice that according to the Whitehead Theorem [MT68, Chapter 13, Theorem 3] a weak equivalence $f : X \to Y$ is a homotopy equivalence, i.e. there is a mapping $g : Y \to X$ such that $fg : Y \to Y$ and $gf : X \to X$ are homotopic to identity maps of Y and X respectively. By abbreviation "$X \simeq Y$" we mean X is homotopy equivalent to Y.

Infinite loop spaces. Following Adams [A78] we say X is an infinite loop space if there is a collection of spaces $\{X_i : i = 0, 1, 2, \ldots\}$ with $X = X_0$ and homotopy equivalences $X_i \to \Omega X_{i+1}$. One may replace the homotopy equivalences with homeomorphisms, since [M69, Page 472] shows that for the purpose of homotopy theory the two definitions are equivalent. Most of the time we are dealing with infinite loop spaces QX defined by

$$QX = \mathrm{colim}\,(\cdots \to \Omega^k\Sigma^k X \to \Omega^{k+1}\Sigma^{k+1} X \to \cdots)$$

where the map $\Omega^k\Sigma^k X \to \Omega^{k+1}\Sigma^{k+1} X$ is the kth loop of the suspension map $E_{\Sigma^k X} : \Sigma^k X \to \Omega\Sigma^{k+1} X$, with the suspension map $E_{\Sigma^k X}$ being the adjoint of the identity map $\Sigma^{k+1} X \to \Sigma^{k+1} X$. The space QX, with X being path connected, sometimes is referred to as the *free* infinite loop space generated by X [CLM76, Part I, Section 2, Page 39 2nd Paraghraph]. Observe that $QX = \Omega Q\Sigma X$. We can view Q as a functor from spaces to infinite loop spaces. An infinite loop space X is armed with a structure map $\theta_X : QX \to X$. This map itself is a map of infinite loop spaces, i.e. it can be delooped infinitely many number of times, and so its homotopy fibre is also an infinite loop space. Using the structure map, any mapping $f : Y \to X$ can be extended to a unique infinite loop map $QY \to X$ defined by the composite $\theta_X \circ Qf : QY \to QX \to X$.

We feel free to to use the stable splitting of loop spaces of the form $\Omega^k\Sigma^k X$ with

$k > 0$, known as the Snaith splitting [S74, Theorem 1.1], given by

$$\Omega^k\Sigma^k X \simeq_{\text{stable}} \bigvee_{r\geqslant 1} D_r(\mathbb{R}^k, X), \tag{1.1}$$

where X is assumed to be path-connected. This splitting has been obtained by use of the *stable James-Hopf invariants.* The rth stable James-Hopf invariant is a mapping

$$j_r : \Omega^k\Sigma^k X \to QD_r(\mathbb{R}^k, X).$$

Here $D_r(\mathbb{R}^k, X)$ is known as the r-adic construction on X, and is defined by

$$D_r(\mathbb{R}^k, X) = F(\mathbb{R}^k, r) \ltimes_{\Sigma_r} X^{\wedge r}$$

where $F(\mathbb{R}^k, r)$ is the configuration space of r-tuples in $\mathbb{R}^k$ with an action of the permutation group Σ_r. The space $X^{\wedge r}$ is the r-fold smash product of X with itself. In particular $D_1(\mathbb{R}^k, X) \simeq X$ where $k \geqslant 0$. We use $D_r X$ to denote $D_r(\mathbb{R}^\infty, X)$. In this case we may replace $F(\mathbb{R}^\infty, r)$ with $E\Sigma_r$, where $E\Sigma_r$ is a contractible space with a free Σ_r action such that $E\Sigma_r/\Sigma_r \simeq B\Sigma_r$. Of particular interest are $D_2(\mathbb{R}^k, S^n) \simeq \Sigma^n P_n^{n+k-1}$ where $k > 0$ [K82, Corollary 1.4]. Here P is the infinite dimensional real projective space, P^i is its i-skeleton i.e. the i-dimensional real projective space, and P_n^{n+k-1} is the truncated projective space P^{n+k-1}/P^{n-1}.

The r-adic construction can be defined for any space X, not necessary path connected. For example when $X = S^0$ we have $D_2(\mathbb{R}^k, S^0) = P_+^{k-1}$ [K82, Proposition 1.3]. However, in this case the stable equivalence (1.1) does not hold, and we have to use *group-completion* [A78, Theorem 3.2.1].

We will only use the homology of the stable James-Hopf invariants. The main result on this is due to Kuhn [K83, Proposition 2.7] and we will recall this result at Chapter 4, where we use j_r's.

We note that given any spectrum E we may use the structure maps $E_i \to \Omega E_{i+1}$ to define an infinite loop space $\Omega^\infty E = \operatorname{colim} \Omega^i E_i$. Then for a given space X, QX will be the infinite loop space associated with $\Sigma^\infty X$ where $\Sigma^\infty X$ is the suspension spectrum of the space X. We use $Q\Sigma^{-k}X$ to denote the infinite loop space associated with the k-th desuspension of $\Sigma^\infty X$. In this case we have $Q\Sigma^{-k}X = \Omega^k QX$. We will be interested in $QS^{-n} = \Omega^n QS^0$.

There are different machineries on the market to describe the geometry of infinite loop spaces such as Boardmann-Vogt-May's little n-cubes [M69], or Barratt-Eccles Γ-functor [B71],[BEa74],[BEb74],[BEc74]. Although we will not work with these models, a slight knowledge of them sometimes can help with understanding the calculations. Due to the *credit crunch* [BBC] we use the homemade Barratt-Eccles model, which has been exported very well, and nowadays in the language of operads is known as the Barratt-Eccles operad! This provides one with a simplicial model of an infinite loop space. In this setting an infinite loop space X has a Γ-structure which is given by a map $\Gamma X \to X$.

Finally we note that it is possible to have different deloopings of a given space. Such phenomenon gives rise to the notion of E_∞ *ring spaces* explored in [M77]. Of interest among such spaces is QS^0 where it has two product structures coming from the *loop sum* and *composition product* of maps of degree 1. Although some results of Madsen [M70],[M75] on the image of the homology of the J-homomorphism $O \to Q_1S^0 \to Q_0S^0$ are obtained by the interaction between the two infinite loop structures on QS^0, however we will not work with the composition product on QS^0.

Homology. Through this thesis we will only use the singular homology with $\mathbb{Z}/2$-coefficients. We denote the homology of a space by H_*X, and $\overline{H}_*X$ will denote the reduced homology of the space X. This will be the only homology theory that we will use. We use Σx to denote image of $x \in H_nX$ under the homology isomorphism $H_nX \to H_{n+1}\Sigma X$.

Since we are working with $\mathbb{Z}/2$ coefficients, the universal coefficient theorem [G75, Theorem 25.16] then allows us to have $H_nX \simeq \mathrm{Hom}_{\mathbb{Z}/2}(H^nX, \mathbb{Z}/2)$. We then may view the Kronecker pairing as $\langle -,- \rangle : H_nX \otimes H^nX \to \mathbb{Z}/2$ defined by $\langle x, y^* \rangle = x(y^*)$. Any $y^* \in H^nX$ with $\langle x, y^* \rangle = 1$ will be a cohomology dual for x. Often we use the same notation to denote both x and any cohomology class dual to it.

Steenrod operations. We will use the Steenrod operations $Sq^i : H^*X \to H^{*+i}X$. Given a mapping $f : X \to Y$ the functional operation Sq^i_f is defined on a class $y \in H^nY$ if $f^*y = 0$ and $Sq^iy = 0$. The fact that $f^*y = 0$ implies that y pulls back to a generator $y \in H^nC_f$ where C_f denotes the mapping cone of f. One then may look for the value of $Sq^i y \in H^{n+i}C_f$. This then pulls back to a class $u \in H^{n+i-1}X$. This class is not unique, and is determined up to an indeterminacy given by

$$\Upsilon = Sq^i H^{n-1}X + f^* H^{n+i-1}Y,$$

i.e. u gives rise to a class $Sq^i_f y$ in $H^{n+i-1}X/\Upsilon$. Observe that

$$Sq^i y \neq 0 \text{ in } H^*C_f \iff Sq^i_f y \neq 0 \text{ in } H^{n+i-1}X/\Upsilon.$$

This means that we may identify functional operations with the operations defined on the mapping cone. We feel free to switch between these two interpretation of these operation. Notice that if a nontrivial class $Sq^i y \in H^*C$ is given with a homology dual $u \in H_*C$, we then have

$$\langle Sq^i_* u, y\rangle = \langle u, Sq^i y\rangle = 1,$$

i.e. $Sq^i_* u$ is a homology dual to y. Here Sq^i_* is the operation dual to Sq^i. By the statement " the mapping $f : X \to Y$ is detected by Sq^i on $y \in H_nY$" we really mean that the mapping f is detected by the operation Sq^i on a dual class y in its mapping cone, i.e. by $Sq^i_f y$. We refer the reader to [MT68, Chapter 16] for the basic material on this topic.

Kudo-Araki operations. We will rely on the so-called *Kudo-Araki* operations [DL62, Definition 2.2], [CLM76, Part I, Theorem 1.1] defined on the homology of infinite loop spaces. These operations are defined on the homology of any infinite loop space A as homomorphisms

$$Q^i : H_*A \to H_{*+i}A.$$

For $a \in H_nA$, we have $Q^i a = 0$ if $i < n$, and $Q^n a = a^2$ where the square is taken with respect to the Pontryagin product in H_*A induced by the loop sum on A. These operations satisfy various forms of the Cartan formulae, Adem relations, etc. We will use very basic properties of these operations, and try to do *down to earth* calculations. We refer the reader to [CLM76, Part I, Theorem 1.1] for the full list of these properties, and their proof. During the thesis we will recall any property when needed.

The homology algebra H_*QX. The homology of spaces QX, for X path connected, is described using the operations Q^i. Let $\{x_\mu\}$ be an additive basis for $\overline{H}_*X$. Then the homology of the space QX as an algebra and as a module over the Dyer-Lashof algebra, is given by

$$H_*QX \simeq \mathbb{Z}/2[Q^I x_\mu : \text{excess}(Q^I x_\mu) > 0, I \text{ admissible}].$$

Here $I = (i_1, \ldots, i_r)$ is called admissible if $i_j \leqslant 2i_{j+1}$ for any $1 \leqslant j \leqslant r-1$. The iterated operation Q^I is defined by $Q^I x = Q^{i_1} Q^{i_2} \cdots Q^{i_r} x$. The *excess* of $Q^I x_\mu$ is defined to be $\text{excess}(Q^I x_\mu) = i_1 - (i_2 + \cdots + i_r + \dim x_\mu)$. We also allow the empty sequence ϕ to be admissible with $Q^\phi x_\mu = x_\mu$, $\text{excess}(Q^\phi x_\mu) = +\infty$, and $l(\phi) = 0$. Observe that if $\text{excess}(Q^I x) = 0$, then $i_1 = \dim Q^{i_2} \cdots Q^{i_r} x$ which means that $Q^I x$ is a square, i.e. $Q^I x = (Q^{i_2} \cdots Q^{i_r} x)^2$. Moreover, if $\text{excess}(Q^I x) < 0$ then $i_1 < \dim(Q^{i_2} \cdots Q^{i_r} x)$ which means that $Q^I x = 0$.

We may define a filtration $w : H_*QX \to \mathbb{N}$, called the weight filtration, by $w(\xi\xi') = w(\xi) + w(\xi')$ and $w(Q^I x) = 2^{l(I)}$ where $\xi, \xi' \in H_*QX$, and for $I = (i_1, \ldots, i_r)$ we have $l(I) = r$. Notice that the stable splitting of (1.1) gives a decomposition of $\overline{H}_*QX$ as $\oplus_r \overline{H}_* D_r X$. One then has that $\overline{H}_* D_r X$ is the group of elements of weight r. We have to say that the homology of QX is more complicated when X is not path connected. We will describe H_*QS^0 in Chapter 5, where we know that $\pi_0 QS^0 \simeq \mathbb{Z}$.

We have not said anything about the Dyer-Lashof algebra R, and the Λ-algebra. Appendix B contains a very brief introduction to these algebras. We will not use these algebras, although we will use some of results obtained as a relation between these two algebras. We refer the reader to [W82, Chapter 7] for a careful and clear discussion of this.

We will recall some results on any infinite loop space of finite type, obtained by Finkelstein as a generalisation of the Kahn-Priddy theorem [F77, Theorem 3.2, Proposition 6.9], which seem to be less known, and very rarely referred to in the literature. We will demonstrate some applications of these results.

Hopf algebras. We also assume that the reader is familiar with the notion of Hopf algebras, where basic material on this subject can be found in the classic paper by Milnor-Moore [MM65]. Our main tool will be the Milnor-Moore exact sequence [MM65, Proposition 4.23]

$$0 \to Pk(s_H H) \to PH \to QH \to Qk(r_H H) \to 0$$

where H is a Hopf algebra of finite type over $\mathbb{Z}/2$, P is the primitive submodule functors, Q is the indecomposable quotient module functor, $s_H : H \to H$ is the Frobenius homomorphism defined by $h \mapsto h^2$, and $r_H : H \to H$ is dual to the squaring map $s_{H^*} : H^* \to H^*$ where $H^* = \mathrm{Hom}_k(H, k)$. By $k(S)$ we mean the submodule of H generated by S, where $S \subseteq H$. We note that having fixed a basis $\{x_\mu\}$ for $\overline{H}_*X$, then H_*QX is a polynomial generated by symbols $Q^I x$ where I is admissible, and excess$(Q^I x) > 0$. In such a case, by abuse of notion, we refer to $Q^I x$ as an indecomposable. Notice that in this case any class $\xi \in H_*QX$ which has at least one term of the form $Q^I x$, with $Q^I x$ being a generator of H_*QX, will determine a nonzero class in QH_*QX, hence determining an indecomposable element.
We shall also recall some facts on the Eilenberg-Moore spectral sequence. The main material on this is to be found in [S70] together with some results borrowed from [G04]. We leave more discussion on this to Chapter 5 where we shall apply this machinery.

The organisation of this thesis is as following. Chapter 2 is a brief introduction to spherical classes, and their basic properties. We will explain our approach, and the motivations behind it in this chapter. We exhibit our main results in this chapter, and leave the proofs to the next chapters. Chapters 3 and 4 provide the reader with proof of most of our results mentioned in Chapter 2. We provide the reader with explicit calculations, which hopefully will make the material clearer. The proofs in these chapters are quite explicit, and tedious, although they are based on simple numerical facts.

Chapter 5 has two parallel purposes. Our first aim is to have a discussion on the homology of $H_*Q_0S^0$, and record some calculations that we have done in this ring, perhaps well known to the experts! On the other hand some of our results stated in Chapter 2 need some results about H_*QS^{-n} for $n = 0, 1, 2$. For this reason we start Chapter 5 by recalling some basic material about Hopf algebras and the Eilenberg-Moore spectral sequence. We then describe $H_*Q_0S^0$, and the submodule of primitives in this algebra. This will make it quite straightforward to digest the rest of the material on our calculations on the primitive classes in other algebras like H_*QP, $H_*Q\mathbb{C}P$, and H_*QS^{-1}. This enables us to prove the rest of our claims. These discussions are also fruitful as we succeed to add a bit to our current knowledge on homology algebras H_*QS^{-2}. Moreover, they make us able to do more calculations and derive perhaps some new information on homology rings $H_*Q\Sigma^{-1}P$, $H_*Q\Sigma^{-1}\mathbb{C}P$, H_*QS^{-1}, and H_*QS^{-2}. We succeed to identify some specific subalgebras of $H_*Q_0S^{-2}$ and $H_*Q_0S^{-3}$, with explicit description of its generators. These results may be of interest on their own as well. We have to say that the discussions in this chapter help us to get a general picture, and propose our road map toward the proof of the Curtis's conjecture.

Finally we like to mention that the numbering of theorems, lemmata and etc is done in two separate ways. As the Chapter 2 is about our main results, we have used single numbers in this chapter, like Theorem 1. In all of the other chapters we have have used numbering based on chapters.

Chapter 2

Statement of Results

The *Hurewicz* homomorphism

$$h : \pi_n X \to H_n X$$

is defined by sending a homotopy class $f : S^n \to X$ to $f_* g_n$ where $g_n \in H_n S^n$ is the generator. A homology class $x \in H_* X$ is called *spherical* if it is in the image of the Hurewicz homomorphism. If $x \in H_* X$ is spherical, then it has two basic properties:

- x is primitive;
- x is A-annihilated.

Here primitive is understood to be primitive with respect to the co-product induced by the diagonal map $X \to X \times X$. By $x \in H_* X$ being A-annihilated we mean that

$$Sq^i_* x = 0 \quad \text{for any } i > 0,$$

where $Sq^i_* : H_* X \to H_{*-i} X$ is the dual to the i-th Steenrod operation $Sq^i : H^* X \to H^{*+i} X$. One notes that not every class in $H_* X$ may have both properties. Hence the above properties give an upper bound on the set of all spherical classes in $H_* X$, although they do not in general characterise such classes.

The *stable Hurewicz* homomorphism

$$h^S : \pi_n^S X \to H_n X$$

is defined in a similar way. Notice that depending on the connectivity of X there exists a positive integer k such that $\pi_n^S X \simeq \pi_{n+k}\Sigma^k X$. We then define h^S to be the composite

$$\pi_n^S X \simeq \pi_{n+k}\Sigma^k X \xrightarrow{h} H_{n+k}\Sigma^k X \simeq H_n X.$$

These homomorphisms fit into a commutative diagram as following

$$\begin{array}{ccc} \pi_n^S X & \xrightarrow{h^S} & H_n X \\ \simeq \uparrow & & \uparrow \sigma_*^\infty \\ \pi_n QX & \xrightarrow{h} & H_n QX. \end{array}$$

Here σ_* is the homology suspension, and σ_*^∞ is the iterated homology suspension. We may also view σ_*^∞ as the projection onto the first factor in the Snaith splitting for QX.

We start with an example to see why calculating the image of the Hurwicz homomorphism $h : \pi_n QX \to H_n QX$ can be useful.

Example 1. Let us assume that the Curtis conjecture holds, i.e. the only spherical classes in $H_*Q_0S^0$ are the Hopf invariant one, and the Kervaire invariant one elements. It is well known that there is Hopf invariant one element in ${}_2\pi_*^S$ if and only if certain primitive elements $p'_{2^s-1} \in H_{2^s-1}Q_0S^0$ are spherical. Moreover, according to Madsen [M70, Theorem 7.3] the Kervaire invariant one elements in dimension $2^{i+1}-2$ give rise to spherical classes $(p'_{2^i-1})^2 \in H_{2^{i+1}-2}Q_0S^0$.
The Kervaire invariant one elements then die under the homology suspension $\sigma_* : H_*Q_0S^0 \to H_{*+1}QS^1$, as they are decomposable terms. On the other hand a class of the form p'_{2^s-1} suspends to $Q^{2^s-1}g_1 + Q^{2^s-2}Q^1g_1$. Notice that a spherical class in H_*QS^1 will pull back to a spherical class in $H_*Q_0S^0$. Hence, the Curtis conjecture implies that the only spherical classes in H_*QS^1 are the Hopf invariant one classes, i.e. those ones which arise from stable mapping $S^n \not\to S^1$ which are detected by a primary operation operation in their stable mapping cone. This is indeed the statement of the Eccles conjecture for $X = S^1$.
Now we want to justify why looking at homology of a mapping $S^n \to QS^1$ can be useful. Let $X = S^1$. Assume that $n > 1$ and $f \in \pi_n^S S^1$. Clearly there is no hope of

detecting this map by homology as

$$f_* : H_nS^n \to H_nS^1 \simeq 0,$$

i.e. $h^Sf = 0$. On the other hand the stable adjoint of f, say f', is an element in π_nQS^1. We know $H_*QS^1 \simeq \mathbb{Z}/2[Q^Ig_1 : \mathrm{excess}(Q^Ig_1) > 0, I \text{ admissible}]$ which is much richer than $H_*S^1 \simeq E_{\mathbb{Z}/2}(g_1)$, the exterior algebra over g_1 in dimension 1. So we may try to examine f' in homology by calculating hf', and hope that it may not fail to be trivial. In fact there are spherical classes in H_*QS^1 given by the well known classical Hopf invariant one elements. We may consider $\eta \in {}_2\pi_2QS^1 \simeq \mathbb{Z}/2$, $\nu \in {}_2\pi_4QS^1 \simeq \mathbb{Z}/8$, and $\sigma \in {}_2\pi_8QS^1 \simeq \mathbb{Z}/16$, where we have [E80, Proposition 3.4]

$$\begin{array}{rclcl} h\eta & = & Q^1g_1 & = & g_1^2, \\ h\nu & = & Q^3g_1 + Q^2Q^1g_1 & = & Q^3g_1 + g_1^4, \\ h\sigma & = & Q^7g_1 + Q^4Q^3g_1 & = & Q^7g_1 + (Q^3g_1)^2. \end{array}$$

Hence, calculating the spherical classes in H_*QS^1 can be useful. Notice that in any of the above examples any single term is primitive and A-annihilated. We will explain the reason for this. We will see in Chapter 5 that the Curtis conjecture implies that the above examples are the only possible spherical classes in H_*QS^1. Finally observe that there is no term in these homology classes belonging to H_*S^1, i.e. they have trivial image under $h^S : \pi_*^SS^1 \to H_*S^1$.

We note that a given spherical class in H_*QS^n pulls back to a spherical class in H_*QS^1. This implies that if we prove the Eccles conjecture for $X = S^1$, then it automatically holds for $X = S^n$ with $n > 0$. On the other hand notice that spheres are the building blocks of CW-complexes, hence this provides one with another motivation that the Eccles conjecture can hold for any path connected space X of finite type, if it can be proven for $X = S^1$. In order to tackle the Eccles conjecture we need to identify all A-annihilated classes in H_*QX. Our first result provides one with a *full and complete* description of indecomposable A-annihilated classes of the form Q^Ix in H_*QX where X is an arbitrary path connected space. We need the following definition to state this result. Let n be a positive integer with $n = \sum n_i2^i$

with $n_i \in \{0,1\}$. We may define a function $\rho : \mathbb{N} \to \mathbb{N}$ by $\rho(n) = \min\{i : n_i = 0\}$. We then have the following.

Theorem 2. *Suppose $Q^I x \in H_*QX$ is given, with $I = (i_1, \ldots, i_r)$ admissible, and* $\operatorname{excess}(Q^I x) > 0$. *Such a class is A-annihilated if and only if the following conditions are satisfied:*

1- $x \in \overline{H}_*X$ *is A-annihilated;*

2- $\operatorname{excess}(Q^I x) < 2^{\rho(i_1)}$;

3- $0 \leqslant 2i_{j+1} - i_j < 2^{\rho(i_{j+1})}$;

where $\operatorname{excess}(Q^I x) = i_1 - (i_2 + \cdots + i_r + \dim x)$. *If $l(I) = 1$, then the first two conditions chracterise all A-annihilated class of the form $Q^i x$ of positive excess. Notice that the fact that* $\operatorname{excess}(Q^I x) > 0$ *means that $Q^I x$ is not a square, i.e. it is an indecomposable.*

In an appendix, and based on elementary observations, we will provide the reader with a construction for sequences I which will satisfy condition 3 of Theorem 2.

Important Remark. Notice that if $Q^I x$ is an A-annihilated class with $\operatorname{excess}(Q^I x) > 0$, then I cannot have any even entry. This is easy to see, once we observe that having i_1 even implies that $\rho(i_1) = 0$. This together with condition 2 of the above theorem implies that $\operatorname{excess}(Q^I x) < 2^0 = 1$, i.e. $\operatorname{excess}(Q^I x) = 0$ which is a contradiction. Moreover, if there exists $j > 1$ with i_j even, then condition 3 implies that i_{j-1} is even. Iterated application of this will imply that i_1 is even which leads to a contradiction.

Note 3. Conditions $(2)-(3)$ were used in Curtis's work [C75, Theorem 6.3] to describe the 0-line of the E_2-term of the unstable Adams spectral sequence converging to the 2-primary component of $\pi_*\Omega^n S^{n+k}$ were his condition (1), corresponding to our condition (2), is adapted to work for any space $\Omega^n S^{n+k}$. Curtis claims that these conditions describe a basis for the 0-line of the E_2-term of the unstable Adams spectral sequence, whereas Wellington [W82, Remark 11.26] shows that this claim is not valid for the case $k = 0$. It seems that Curtis's claim can be true in odd dimensions, and fails to be true in even dimensions. Corollary 6 of ours, to be followed shortly, together with Theorem 5.23 can be thought of as a confirmation of Curtis's claim in odd dimensions for the case $k \geqslant 0$, at least in the E_1-level.

In a more careful statement by Wellington, conditions $(2)-(3)$ are used to describe A-annihilated classes in the Dyer-Lashof algebra R [W82, Theorem 11.25]. One may derive his theorem from ours by formally defining $Q^I = Q^I g_0$, where g_0 is a formal 0-dimensional A-annihilated class.

However as we observe, adding condition (1) will give a complete set of indecomposable A-annihilated classes in homology of H_*QX, where X is any path connected space. I believe that Theorem 2 is the most general possible statement which classifies A-annihilated classes in H_*QX which are of the form Q^Ix with $\mathrm{excess}(Q^Ix) > 0$, when X is path connected. Although Wellington might be aware of a such statement, but I am not aware of such a statement or anything similar to this mentioned anywhere in the literature.

All of the above results, those obtained by Curtis and Wellington as well as our Theorem 2, are based on an observation by Curtis [C75, Lemma 6.2]. We will recall this observation in an appendix as it is an important step in the proof of Theorem 2. The result of Curtis [C75, Lemma 6.2] later on was generalised to odd primes by Wellington [W82, Lemma 12.5].

Finally we note that other partial results, describing indecomopsable A-annihilated classes, may be found in a work by Snaith and Tornehave [ST82, Theorem 1.1]. But such results are restricted to a limited number of cases.

It is possible to use the conditions in Theorem 2 to do more numerical calculations in order to extract more information about indecomposable A-annihilated classes by giving an explicit description of such classes. When $X = S^1$ and $l(I)$ is not too big, it is possible to determine all indecomposable A-annihilated classes of the form $Q^I g_1$. We have the following example.

Lemma 4. *Consider $Q^I g_1$ with $\mathrm{excess}(Q^I g_1) > 0$, i.e. it is an indecomposable. If $l(I) = 1$, then this class is A-annihilated if and only if it is of the form $Q^{2^s-1}g_1$ for some $s > 1$. If $l(I) = 2$, then such a class is A-annihilated if and only if it is of the form $Q^{2^s+2^j-1}Q^{2^s-1}g_1$ with $1 \leqslant j \leqslant s-1$. If $l(I) = 3$ and $Q^{i_1}Q^{i_2}Q^{i_3}g_1$ is A-annihilated, then $Q^{i_2}Q^{i_3}g_1$ is not A-annihilated.*

Remark 5. A class of the form $Q^{2^s+2^j-1}Q^{2^s-1}g_1$ will pull back to a class of the form $Q^{2^s+2^j-1}x_{2^s-1} \in H_*Q_0S^0$ modulo decomposable terms. We will describe $H_*Q_0S^0$ in Chapter 5. One observes that $Q^{2^s+2^j-1}x_{2^s-1}$ is not A-annihilated, where this can be seen by use of $Sq_*^{2^j}$. Hence a corollary of the above lemma will be that the only A-annihilated classes of the form $Q^ix_j \in H_*Q_0S^0$ are of the form $x_{2^s-1}^2$. It is also straightforward to see that x_n is A-annihilated if and only if $n = 2^s - 1$ for some $s > 0$. These may be compared to [W82, Theorems 11.10, 11.11]

Next we mention a rather nontrivial corollary of Theorem 2 which is an outcome of its proof.

Corollary 6. *Let $x \in \overline{H}_*X$ be fixed. Suppose*

$$\xi = \sum Q^I x$$

is A-annihilated, with $\mathrm{excess}(Q^Ix) > 0$*, and I runs over certain admissible sequences. Then each term Q^Ix is A-annihilated. In particular any odd dimensional class of the above form has this property.*

Remark 7. The reader may notice that this is really a fact about the Dyer-Lashof algebra. More precisely, we may say that any two operations Q^I and Q^J which are of positive excess, and are not A-annihilated can be separated by an operation Sq_*^r. We note that this is not true in general if at least one of these iterated operations is of trivial excess. A counter example maybe found in [W82, Remark 11.26] which is discussed in Chapter 5.

Note 8. Not every spherical class can in fact be written as $\sum Q^Ix$ with x fixed. For example consider $X = P$ and let $\widetilde{\nu} \in \pi_3QP$ be the pull back of $\nu \in \pi_3Q_0S^0$ through the Kahn-Priddy map $\lambda : QP \to Q_0S^0$. This has nontrivial Hurewicz image given by

$$h\widetilde{\nu} \;\; = \;\; a_3 + a_1a_2 + a_1^3 + Q^2a_1.$$

To see this notice that the Kahn-Priddy theorem provides us with a monomorphism $H_*Q_0S^0 \to H_*QP$. In this example the image must be A-annihilated and primitive, and the above class is the only class in that dimension with these properties. We leave more discussion on this to Chapter 5.

On the other hand we note that in some still interesting cases like $X = S^n$ it is the case that any spherical class can necessarily be written as $\xi = \sum Q^I x$ where $x = g_n \in H_n S^n$ is fixed.

Corollary 6 is a weaker version of a uniqueness property that we conjecture to be true, and seems to be an argument of a purely numerical nature, like many other arguments that we make!

Conjecture 9. *Suppose $x \in H_n X$, and let $r > 0$ be given. Let I, J be two admissible sequences of length r such that $\operatorname{excess}(Q^I x) > 0$, and $\operatorname{excess}(Q^J x) > 0$. Suppose both $Q^I x$ and $Q^J x$ are A-annihilated terms. Then*

$$\dim I \neq \dim J,$$

where for $I = (i_1, \ldots, i_r)$ we have $\dim I = i_1 + \cdots + i_r$.

A rather trivial, but useful, observation is that if $\xi \in H_{*+1} Q\Sigma X$ is spherical, then it is in the image of the homology suspension

$$\sigma_* : H_* QX \to H_{*+1} Q\Sigma X.$$

This implies the following simple lemma.

Lemma 10. *Suppose $\xi \in H_{*+1} Q\Sigma X$ is spherical. Then*

$$\xi = \sum Q^I \Sigma x,$$

where the sum varies over certain terms $Q^I \Sigma x$ with I admissible, $x \in \overline{H}_* X$ not necessarily fixed, and $\operatorname{excess}(Q^I \Sigma x) \geqslant 0$.

Hence working with spaces $Q\Sigma X$ has the advantage that the spherical classes do not have *unpredicted* decomposable terms, and the only possible decomposable terms will be square ones, i.e those terms with $\operatorname{excess}(Q^I \Sigma x) = 0$. Notice that suspension kills the cup product. This then implies that any class in $H_* \Sigma X$, and consequently any class $Q^I \Sigma x \in H_* Q\Sigma X$ is primitive.

Remark 11. Similar to Note 8, not all primitive A-annihilated sums $\sum Q^I\Sigma x \in H_*Q\Sigma X$ have the property that each $Q^I\Sigma x$ is A-annihilated. We urge the reader not to confuse this with the case of Corollary 6, where there $x \in \overline{H}_*X$ was fixed. Here Σx varies as well as I. As an example, consider $Q^{67}Q^{35}Q^{19}a_{11} + Q^{67}Q^{35}Q^{23}a_7 \in H_{132}QP$. This sum is A-annihilated, and maps to $Q^{67}Q^{35}Q^{19}\Sigma a_{11} + Q^{67}Q^{35}Q^{23}\Sigma a_7 \in H_{133}Q\Sigma P$ under the homology suspension. This later class is A-annihilated, and primitive. But still neither any single term is A-annihilated.

Despite the above remark, we are able to take advantage of Corollary 6 in some special, and still important cases. Specialising to $X = S^n$ with $n > 0$, we have the following result.

Lemma 12. *Suppose $\xi = \sum Q^I g_n \in H_*QS^n$ is a spherical class. Then each $Q^I g_n$ is A-annihilated.*

The proof of this theorem when ξ is odd dimensional does not use any desuspension argument, and the claim holds for any odd dimensional A-annihilated primitive class. However, the proof for the case of even dimensional classes depends on desuspension arguments, and we need the assumption of ξ being spherical.

Notice that when $n > 1$, any spherical class $\xi_n \in H_*QS^n$ will desuspend to a spherical class $\xi_1 \in H_*QS^1$. Hence a good knowledge on spherical classes in H_*QS^1 can be very useful. According to Lemma 12, any spherical class in H_*QS^1 is of the form $\sum Q^I g_1$, such that each $Q^I g_1$ is A-annihilated with $\text{excess}(Q^I g_1) \geqslant 0$. This tells us about the possible forms of an odd dimensional spherical class $\xi_1 \in H_*QS^1$ and the terms which may contribute to a spherical class, although we will see that there will be a very few terms in any potential spherical class.

Next we have two parallel tasks. First, given $f \in {}_2\pi_n QS^1$ with $hf = \xi = \sum Q^I g_1 \in H_nQS^1$ we may try to extract more information about the stable adjoint of f, $f^S : S^n \not\to S^1$ in terms of the cohomology operations that detect f^S, where we have used $\not\to$ to indicate that the mapping is stable, i.e. it can be realised after finitely many suspensions. The second task is to try to eliminate possible terms which may contribute to a spherical class $\xi \in H_*QS^1$.

We start by dealing with the first task, and ask the question in a wider generality. Assume $\xi \in H_nQX$ is spherical, then there exists a homotopy class $f : S^n \to QX$, not necessarily unique, with $hf = \xi$. The mapping f has a stable adjoint $f^S : S^n \not\to X$ characterising a unique element in ${}_2\pi_n^S X$. If X is a suspension, then the results mentioned earlier describe a potential spherical class. One may ask, what is the relationship between the spherical class $\xi \in H_nQX$ and the stable mapping $f^S : S^n \not\to X$? More precisely, we may like to understand the order of the stable operation that detects f^S based on information about ξ.

Let me explain what is our approach to dealing with this first task. Assume that $f : S^m \to QX$ is given. Since $QX = \Omega Q\Sigma X$ then we may use adjointness to *suspend up* to obtain a mapping $f^1 : S^{m+1} \to Q\Sigma X$. Performing this operation finitely many times, depending on the connectivity of X, we end up with the stable adjoint of f, namely $f^S : S^m \not\to X$. On the other hand we may try to do the same with homology, that is assuming that $hf \neq 0$ then we may try to see what happens to hf when we suspend up under the homology suspension $\sigma_* : H_*QX \to H_*Q\Sigma X$.

Notice that if $h^S f \neq 0$, then this will imply that f^S is detected by homology. Hence we restrict our attention to the cases where $h^S f = 0$.

One of the first examples is provided by two equivalent definitions of the Hopf invariant. It is well known that $f : S^{2n+1} \to S^{n+1}$ has Hopf invariant one if and only if the adjoint mapping $S^{2n} \to \Omega\Sigma S^n$ has $g_n^2 \in H_{2n}\Omega\Sigma S^n$ as its image in homology. On the other hand f has Hopf invariant 1 if it is detected by a primary operation. This fact may be generalised as follows. Let X be a path connected space, and $f : S^{2n} \to QX$ be given with $h^S f = 0$. Then hf has $x_n^2 \in H_{2n}QX$ in its image if and only if the adjoint mapping $f^S : S^{2n} \not\to X$ is detected by the primary operation Sq^{n+1} on $x_n \in H_nX$ in its stable mapping cone [E81, Lemma 4.2], [E93, Proposition 4.4]. The following result demonstrates a full generalisation of this fact where we remove the restriction on the choice of x_n.

Lemma 13. *Suppose $f \in \pi_{2n}QS^k$, $k \geqslant 0$, with $hf = \xi_n'^2$ where $\xi_n' \in H_nQS^k$. Then the adjoint mapping $S^{2n+1} \to Q\Sigma S^k$ is detected by the primary Sq^{n+1} on $\sigma_*(\xi_n' + O)$*

in its mapping cone, where O refers to the indeterminacy which is a sum of terms of the same weight as ξ_n and of lower excess. ($O = 0$ if Conjecture 9 holds.)

Remark 14. The proof for case with $k = 0$ will be slightly different, and a bit technical. We will provide the reader with the proof of this case after a discussion of the delooped Kahn-Priddy theorem.

Note 15. Observe that in Lemma 13 the class $\xi'_n \in H_nQS^k$ will be A-annihilated and primitive. Hence we know that we may write it as $\sum Q^I g_k$ where the sum is A-annihilated. If there exists $I = \phi$, i.e. $hf = g_k^2$, then we obtain the previous generalisation in [E93, Proposition 4.4] where in this case one has trivial indeterminacy, that is $O = 0$. In this case the stable mapping f^S will be detected by a primary operation.

Remark 16. The above lemma can be generalised in the following way. Notice that in the above lemma $f \in \pi_{2n}QS^k$, i.e. $f : S^{2n} \to QS^k$. It is quite straightforward to replace S^{2n} with Y_{2n}, where Y_{2n} has the bottom cell in dimension $2n$. This part of the generalisation comes straightforward as a corollary of the above lemma. It seems that it is possible to replace Y_{2n} with a wider range of spaces. My conjecture is that still it should be true if we replace Y_{2n} with any path connected space Y, assuming that $f : Y \to QS^k$ has trivial homology in dimensions $\leqslant n$. This will ensure that the functional operation Sq^{n+1} on the mapping cone of ξ does not have any indeterminacy. However, I have not written down the proof of this last claimed generalisation. Such a generalisation would be very useful in a proof of Conjecture 26.

We have to say that in the above lemma we need to know that $\sigma_*\xi'_n \neq 0$. However we know that $\xi'_n = \sum Q^I g_k$. This means that if $\sigma_*\xi'_n = 0$, then ξ'_n it must be a square. The following result shows that this will not happen.

Theorem 17. *Let $\xi \in H_*QS^n$, with $n \geqslant 0$ be spherical. Then it is impossible to have $\xi = \zeta^{2^t}$ for any $t > 1$.*

The proof of this for the case $n = 0$ depends on the homology of QS^{-2} which will be discussed in its own place.

Note 18. One can state the above Theorem 17 in a much wider generality. One may weaken the condition of ξ being spherical and replace it by ξ being A-annihilated and primitive class which is in the image of homology suspension, i.e. $\xi = \sigma_*\xi'$ with ξ' an A-annihilated and primitive class. This still can be handled when $n = 0$ by different methods.

The method used in the proof of Theorem 17 can be employed to prove the following lemma.

Lemma 19. *Let $\xi = \zeta^2 \in H_{2d}QS^n$, $n \geqslant 0$, be spherical. Then d must be odd.*

Note 20. In the context of finding potential candidates for being spherical Theorem 17 may be compared to [W82, Theorem 11.16(i)] where it seems our theorem eliminates more classes than Wellington's. In Wellington's terminology, a tower can give rise to a spherical class, and his theorem [W82, Theorem 11.16(i)] counts the classes ζ^{2^t} with $t > 1$ among these candidates.

Let us mention the motivation behind Lemma 13 as it provides us with a kind of inductive step in our arguments. We write σ_*^k for iterations of the homology suspension, i.e. σ_*^k will be a homomorphism

$$\sigma_*^k : H_*QX \to H_{*+k}Q\Sigma^k X.$$

As we mentioned earlier, given $f \in \pi_m QX$ we use $h^S f$ to denote $f_*^S g_m$, where h and h^S fit into a commutative diagram as following

$$\begin{array}{ccc} \pi_m^S X & \xrightarrow{h^S} & H_m X \\ \simeq \uparrow & & \uparrow \sigma_*^\infty \\ \pi_m QX & \xrightarrow{h} & H_m QX. \end{array}$$

If X is a suspension, then according to Lemma 10 we know the possible form that a spherical class $\xi \in H_m QX$ is going to take. Moreover, according to Lemma 12 we have additional information when $X = S^n$ with $n > 1$. If $h^S\xi = 0$ then $\sigma_*^\infty h\xi = 0$. This shows that there exists $k \geqslant 0$, such that $\sigma_*^k h\xi$ is a decomposable. Notice that according to the Milnor-Moore exact sequence a decomposable primitive must be square of a primitive class. This implies that $\sigma_*^k h\xi$ must be a square, i.e. $\sigma_*^k h\xi = \zeta^{2^t}$

for some $t > 0$. Let $X = S^n$ with $n > 0$, then Theorem 17 implies that $t = 1$. This implies that ζ is not decomposable, i.e. $\sigma_*\zeta \neq 0$. Hence we can use the Lemma 13 which implies that if $S^{m+k} \to QS^{n+k}$ has ζ^2 in its image, then the adjoint mapping $S^{m+k+1} \to QS^{n+k+1}$ will be detected by Sq^{d+1} on $\sigma_*(\zeta + O)$ where $d = \dim \zeta$ and O is the indeterminacy, probably $O = 0$.

If $\sigma_*\zeta \notin H_*S^{n+k+1}$ then similar reasoning will show that $\sigma_*^{k_2}\zeta$ will be a decomposable class for some $k_2 > 0$. Moreover, as we explained in Note 19 it is impossible to have $\sigma_*^{k_2}\zeta = \zeta'^{2^t}$ with $t > 1$, i.e. we only can have $t = 1$.

On the other hand, observe that as excess of any single term in O is less than excess(ζ), then while we suspend, O will die under suspension before ζ.

Note 15 explained what happens if there exists $I = \phi$ in the expression for ζ. In this case $l(I) = l(\phi) = 0$. Hence the next case will be the case when there exists I in the expression for ζ with $l(I) = 1$ i.e. ζ has a term of the form $Q^i g_n$. This then implies that after finite suspension, the function will be detected by a primary operation on a square. We then have the following result.

Theorem 21. *Suppose we have a mapping $f : S^{2n+k-1} \to QS^n$ detected by Sq^k_f on g_n^2. Moreover, we require $k > 1$ to be minimum in the sense that f is not detected by $Sq^{k'}$ with $k' < k$ on an element of weight 2. Also assume that f is not detected by a primary operation on a class of weight 1. Then the adjoint of this mapping $g : S^{2n+k} \to QS^{n+1}$ is detected by a secondary operation arising from non-admissible term $Sq^k Sq^{n+1}$ on g_{n+1}.*

Notice that choosing k to be least implies that $k = 2^s$ for some $s \geqslant 0$. We observe that for $s = 0$, being detected by Sq^1_f will be the same as being detected by homology, and this will reduce to the case of Lemma 13.

Remark 22. We warn the reader that the notion of being admissible for the Steenrod operations is completely opposite to the notion of being admissible for Kudo-Araki operation. Recall that $Sq^a Sq^b$ is admissible if and only if $a \geqslant 2b$. So non-admissible means $a < 2b$. We will recall the related Adem relation in its place when we are going to use it!

Moreover, notice that in general a secondary operation determined by a relation is not unique [MT68, Page 163].

We have to say that Theorem 21 is motivated by the real life examples such as the following.

Example 23. The stable mapping $\eta : S^1 \not\to S^0$ may be realised as a genuine mapping $\eta : S^{15} \to S^{14}$ which is detected by Sq^2 on its mapping cone. Similarly the stable mapping $\sigma : S^7 \not\to S^0$ may be realised as a mapping $S^{15} \to S^8$ with adjoint $\sigma : S^{14} \to \Omega\Sigma S^7$. For the adjoint mapping σ we have $h\sigma = g_7^2$. Hence the composite $\sigma\eta : S^{15} \to \Omega\Sigma S^7$ is detected by Sq^2 on g_7^2, with trivial indeterminacy. Now Theorem 21 implies that the adjoint mapping $S^{16} \to QS^8$ is detected by the secondary operation arising from a Adem relation with the non-admissible term Sq^2Sq^8. Notice that we may consider the composite

$$S^{15} \to S^{14} \to \Omega\Sigma S^7 \to QS^7.$$

We already know that the the stable adjoint $\sigma : S^{14} \to QS^7$ satisfies $h\sigma = g_7^2$. Hence the composite $\sigma\eta : S^{15} \to QS^7$ is detected by Sq^2 on g_7^2. However, here we have a nontrivial indeterminacy coming from $Sq^2_*Q^9g_7 = g_7^2$. Now Theorem 21 implies that the adjoint mapping $S^{16} \to QS^8$ is detected by the secondary operation arising from a Adem relation with the non-admissible term Sq^2Sq^8. Notice that at this stage the above composite is in the stable range and can be seen as a mapping $S^{16} \to S^8$. Observe that this is the same $\eta_3 \in {}_2\pi_8^S$ modulo elements of higher Adams filtration, where $\eta_i \in {}_2\pi_{2^i}^S$ denotes Mahowald's family [M77, Theorem 1].

Remark 24. Observe that η_3, and in general η_i, does not give rise to a spherical class in H_*QS^1, or even in $H_*Q_0S^0$. This is quite straightforward to see based on the construction [M77, Theorem 2]. Hence the above example can be seen as an evidence that Theorem 21 admits a kind of inverse. That is given any stable mapping $S^n \not\to X$ which is detected by a secondary operation, then there exists a nonnegative integer l such that the stable adjoint mapping $S^{n+l} \to Q\Sigma^l X$ is detected by a primary operation on a square term in homology of $H_*Q\Sigma^l X$.

Notice that Theorem 21 together with our previous observations imply that if $f \in \pi_* QS^1$ such that the minimum $l(I)$ in the expression for hf is 2, or equivalently its minimum weight is 4, then the stable adjoint of f will be detected by a secondary operation. On the other hand it is easy to show that if f^S is detected by a secondary operation then it will not give rise to a spherical class in $H_* QS^1$. Summing up these two would imply the following.

Lemma 25. *It is impossible to have a spherical class in $H_* QS^1$ whose minimum weight is* 4.

This means that if there exists a spherical class in $H_* QS^1$, which is not a stably spherical, then its minimum weight will be either 2 or at least 8.

A combination of Lemma 13 and Theorem 21 provides us with a motivational result towards elements of higher weights, where we like to see them as results illustrating the inductive step in a more general picture. That is, we may hope to prove a result of the following form.

Conjecture 26. *Suppose we have a mapping $f : S^m \to QS^n$ detected by an operation of order r on ξ_n^2. Then the adjoint mapping will be detected by an operation of order $r+1$ on $\sigma_*(\xi_n + O)$, where O is a sum of terms of lower excess.*

In practice this is what one might expect and the proof of this depends on choosing the right framework towards the higher order operations, and interpreting the previous results in this term. We outline an approach that we think will lead to the proof of this claim. Of course there are some examples coming from the "real life" which provide us with some evidence for correctness of the above conjecture. Consider $\eta^3 \in {}_2\pi_3^S$ realised by the composition

$$S^3 \xrightarrow{\eta} S^2 \xrightarrow{\eta^2} Q_0 S^0$$

as an unstable mapping. We know that this mapping has Adams filtration 3. Recall that according to Example 1, the mapping η^2 is detected by homology with $h\eta^2 = x_1^2$. Moreover, the mapping η in the above composition is detected by Sq^2. Hence the

composition, i.e. η^3 is detected by Sq^2 on x_1^2. Applying Theorem 21 we see that the adjoint mapping $S^4 \to S^3 \to QS^1$ is detected by a secondary operation arising from the Adem relation $Sq^2Sq^2 = Sq^3Sq^1$ on $\sigma_* x_1 = Q^1 g_1 = g_1^2$. The above conjecture then predicts that the adjoint mapping $S^5 \to S^4 \to QS^2$ is detected by an operation of order 3 on g_2, where by other methods we know that this is the case.
We like to note that given a stable map which is detected by an operation of order r, then it implies that the Adams filtration of that stable map is at least r. Hence the above conjecture relates the minimum length of a spherical class in $H_*Q_0S^0$ to the Adams filtration of its stable adjoint. I think that this is related to the results of Lannes-Zarati, and maybe thought as a geometric approach to it. But I am not aware of all details of their work.

Now we switch to our second task, namely trying to eliminate some of the possible cases which we found during our analysis of A-annihilated primitives. We have already mentioned one of the eliminations that we meant to do, which was the statements of Theorem 17 and Note 19. These calculations are based on our analysis of primitives in $H_*Q_0S^0$, various transfer maps, and use of the Eilenberg-Moore spectral sequence. As an outcome we also calculate primitives in H_*QP and $H_*Q\mathbb{C}P$. We will give three slightly different descriptions of these primitive submodules. However, proof of all these descriptions is the same. The result for H_*QP reads as following.

Proposition 27. *Any primitive class in H_*QP belongs to the R-module generated by $p^P_{2n+1}, p^P_{i,j}$, i.e. any primitive class in H_*QP will be a linear combination of elements of the form $Q^I p^P_{2n+1}$ and $Q^K p^P_{i,j}$ with I and K are admissible. The classes p^P_{2n+1} and $p^P_{i,j}$ are defined, modulo decomposable terms, by $p^P_{2n+1} = a_{2n+1}$ and $p^P_{i,j} = Q^{2i+1}a_{2j}$. Here $a_i \in H_iP$ is a generator dual to $a^i \in H^iP$ with $a \in H^1P$ being the first universal Steifel-Whitney class.*

I am in debt to Søren Galatius for the discussions that we had during the Arolla Topology conference-2008. I learnt from him how to calculate the primitive elements using the Milnor-Moore exact sequence.

The precise description of the classes p_{2n+1} and $p^P_{i,j}$ are given in Section 5.8. A similar result for $H_*Q\mathbb{C}P$ holds.

Proposition 28. *Any primitive element in $H_*Q\mathbb{C}P$ is a linear combination of terms of the form $Q^Lp^{\mathbb{C}P}_{4n+2}$ and $Q^Kp^{\mathbb{C}P}_{i,j}$ with L and K are admissible. The classes $p^{\mathbb{C}P}_{4n+2}$ and $p^{\mathbb{C}P}_{i,j}$ are defined, modulo decomposable terms, by $p^{\mathbb{C}P}_{4n+2}=c_{4n+2}$, and $p^{\mathbb{C}P}_{i,j}=Q^{2i+1}c_{2j}$ with j even. Here $c_{2i}\in H_{2i}\mathbb{C}P$ is a dual to $c^i\in H^{2i}\mathbb{C}P$ with $c\in H^2\mathbb{C}P$ being the first universal Chern class.*

After these calculations we focus on the $H_*Q_0S^0$. We give a complete classification of potential spherical classes in $H_*Q_0S^0$ which do not correspond to the Hopf invariant one or the Kervaire invariant one elements. The result, which is the statement of Theorem 5.45 reads as following.

Theorem 29. *Let $\theta\in H_*Q_0S^0$ be a spherical class which is not a Hopf invariant one class, neither a Kervaire invariant one class. Then θ satisfies one of the the following cases.*

1- *If $\sigma_*\theta\neq 0$ and θ is an odd dimensional class, then*

$$\theta=\sum Q^Ip'_{2i+1},$$

with $l(I)>1$ such that each of terms $Q^Ip'_{2i+1}$ in the above sum is A-annihilated.

2- *If $\sigma_*\theta\neq 0$ and θ is an even dimensional class, then*

$$\theta=\sum Q^Ip'_{2i+1}+P^2,$$

with $l(I)>1$ where I has only has odd entries. In this case $(I,2i+1)$ satisfies condition 3 of Theorem 2, i.e. $0<2i_{j+1}-i_j<2^{\rho(i_{j+1})}$ for $1\leqslant j\leqslant r$ with $i_{r+1}=2i+1$. Moreover, $\mathrm{excess}(Q^Ip'_{2i+1})-1<2^{\rho(i_1)}$ for every $Q^Ip'_{2i+1}$ involved in the above sum. Here P is a primitive term. If $P\neq 0$, then it is of odd dimension. If $P=0$, then each term in the above expression for θ is A-annihilated.

3- *If $\sigma_*\theta=0$, then $\theta=\xi^2$, with ξ an odd dimensional A-annihilated primitive class, i.e.*

$$\theta=(\sum Q^Ip'_{2i+1})^2,$$

with $l(I)>0$ such that each of terms $Q^Ip'_{2i+1}$ in the above sum is A-annihilated.

We note that at the above theorem $p'_{2i+1} \in H_{2i+1}Q_0S^0$ is a certain primitive element, described in Chapter 5 as Madsen's description, such that any primitive class can be written as a sum of terms of the form $Q^Ip'_{2i+1}$.
This theorem will be clear after a long preparation on the type of primitive classes in $H_*Q_0S^0$. This preparation will be achieved after a discussion on different generating sets for $PH_*Q_0S^0$, and their relation to each other.

We then run towards studying homology of Q_0S^{-n} for $n > 0$. We like to identify subalgebras of these homology rings which will contain pull back of any class classified by Theorem 29. We succeed to give a geometric meaning for generators of specific types of these subalgebras using the classical Hopf invariant one maps, namely $\eta \in \pi_2^S$ and $\nu \in \pi_3^S$. This job is done in Remark 5.40, Note 5.41, and Remark 5.52.

Our study in subsection 5.9.1 illustrates the pattern that we look for to show to the reader. This is the pattern that appears while we desuspend a spherical class $\theta_{-k} \in H_*Q_0S^{-k}$ into a spherical class θ_{-k-1} living in $H_*Q_0S^{-k-1}$. This predicts what can happen when we try to write θ_{-k-1} in terms of primitive classes. Of course, this is based on the assumption that certain primitive classes pull back through the homology suspension and we have examined the consequences of such an assumption. This assumption may fail, but the following pattern is what I strongly suspect that will happen while we desuspend a spherical class in $H_*Q_0S^{-n}$ to a spherical class in $H_*Q_0S^{-n-1}$. We have two different types of primitive classes in $H_*Q_0S^{-k}$, namely classes $\mathsf{p}^{S^{-k}}_{(I,2i+1)}$ and $Q^Ip^{S^{-k}}_{2i+1}$. These are defined in subsection 5.9.2 in an inductive manner, and with use of homology suspension $\sigma_* : H_*Q_0S^{-k-1} \to H_*Q_0S^{-k}$. This of course does not define such classes uniquely. However, such a description is adequate to estimate the action of the Steenrod algebra on such classes, up to some indeterminacy. We have the following result which is the statement of Proposition 5.53.

Proposition 30. *Let $\theta_{-k} \in H_*Q_0S^{-k}$ be a spherical class with $\theta = \sigma_*^k\theta_{-k} \neq 0$ which is not a Hopf invariant class nor a Kervaire invariant class. Then modulo kernel of $\sigma_* : H_*Q_0S^{-k} \to H_*Q_0S^{-k+1}$ the class θ_{-k} can be written as linear combination of* *primitive terms of the following forms*

$$Q^J \mathtt{p}_K^{S-k}, \quad Q^L p_{2l+1-k}^{S-k},$$

where $(I, 2i+1) = (J, K) = (L, 2l+1)$ *with* I, J, K, L *being admissible and* J *can be the empty sequence. Here* θ *has either one of the following forms,*

$$\theta = \sum Q^I p_{2i+1} \qquad \text{modulo decomposable terms}$$
$$\theta = (\sum Q^I p_{2i+1})^2$$

satisfying one of the cases identified by Theorem 5.45*,* $(I, 2i+1)$ *admissible if* $\mathrm{excess}(I, 2i+1) > 0$.

I then claim that the Curtis conjecture is a corollary of this proposition. This is the statement of Conjecture 5.54 which reads as following.

Conjecture 31. *It is impossible to have a spherical class in* $H_*Q_0S^0$ *satisfying one of cases identified by Proposition 29.*

I give a sketch of a proof for this conjecture. I will mention where the possible gaps of such an argument can be. But I believe that this will lead us to a proof of the Curtis conjecture.

Finally in Chapter 6, which is relatively very short, I will discuss some further projects that can be done based on the work done in this thesis, or are related to the subject of this thesis.

Chapter 3

Proof of Theorem 2

Recall that having fixed an additive basis $\{x_\alpha\}$ for $\overline{H}_*X$, with X path connected, then H_*QX is a polynomial algebra with generators given by the symbols $Q^I x_\alpha$, with I admissible. Allowing the empty sequence ϕ to be an admissible sequence, with $Q^\phi x = x$, and excess$(\phi) = +\infty$, then we can see that $Q^I x$ is a decomposable if and only if excess$(Q^I x_\alpha) = 0 > 0$ where in this case $Q^I x = (Q^{i_2} \cdots Q^{i_r} x)^2$ with $I(i_1, \ldots, i_r)$. Hence, Theorem 2 determines all A-annihilated classes in H_*QX classes of the form $Q^I x$ with are not square. Notice that in general, any class in H_*QX involving at least one term $Q^I x$ of positive excess with I determines a nonzero class in QH_*QX, the module of indecomposables of H_*QX, i.e. it gives rise to an indecomposable element.

We only use the *Nishida relations*. The Nishida relation is given as following [CLM76, Part I, Theorem 1.1(9)],

$$Sq^a_* Q^b \quad = \quad \textstyle\sum_{r\geqslant 0} \binom{b-a}{a-2r} Q^{b-a+r} Sq^r_*. \tag{3.1}$$

Notice that $Sq^r_* Q^I$ with $l(I) > 1$ may be computed by iterated use of the Nishida relations. One observes that *the Nishida relations respect the length*, i.e. if

$$Sq^a_* Q^I = \sum Q^K Sq^{a^K}_*,$$

then $l(I) = l(K)$.

Let R denote the Dyer-Lashof algebra. Then according to [M75, Equation 3.2] the Nishida relations maybe used to define an action $N : A \otimes R \to R$ as following

$$N(Sq^a_*, Q^b) = \binom{b-a}{a} Q^{b-a}, \tag{3.2}$$

$$N(Sq^a_*, Q^{i_1} \cdots Q^{i_r}) = \sum \binom{i_1 - a}{a - 2t} Q^{i_1-a+t} N(Sq^t_*, Q^{i_2} \cdots Q^{i_r}). \tag{3.3}$$

In other suppose $Sq^a_* Q^I = \sum Q^K Sq^{a^K}_*$ where $a^K \in \mathbb{Z}$. Then we have

$$N(Sq^a_*, Q^I) = \sum_{a^K=0} Q^K. \tag{3.4}$$

We note that if a sequence I is admissible, then it is not clear whether or not after applying Sq^a_* we will get a sum of admissible terms, i.e we may need to use the Adem relations to rewrite terms in admissible form. This means that we may decide about vanishing or non-vanishing of a homology class $Sq^a_* Q^I x$ after rewriting it in admissible form.

Example 3.1. Consider $Q^9Q^5g_1$ which is an admissible term. One has

$$Sq^4_* Q^9 Q^5 g_1 = Q^7 Q^3 g_1,$$

where Q^7Q^3 is not admissible. Although it may look nontrivial, however the Adem relation $Q^7Q^3 = 0$ implies that $Q^7Q^3g_1 = 0$. Indeed the class $Q^9Q^5g_1$ is not A_*-annihilated, which can be seen by applying Sq^2_* as we have

$$Sq^2_* Q^9 Q^5 g_1 = Q^7 Q^5 g_1 \neq 0.$$

Notice that the right hand side of the above equation is an admissible term.

According to the above example, part of the job in distinguishing between A-annihilated and not-A-annihilated classes $Q^I x$ is to choose the right operation Sq^a_* in a way that the outcome is admissible and there is no need to use the Adem relations after the Kudo-Araki operation. The reason being that it is practically impossible to use the Adem relations when $l(I)$ is big. The following lemma tells us when it is not possible to choose the right operation! and provides us with the main tool towards the proof of Theorem 2.

Lemma 3.2. *Suppose I is an admissible sequence such that $2i_{j+1}-i_j<2^{\rho(i_{j+1})}$ for all $1\leqslant j\leqslant r-1$. Let*

$$N(Sq^a_*,Q^I)=\sum_{K\ admissible}Q^K.$$

Then

$$\mathrm{excess}(K)\leqslant\mathrm{excess}(I)-2^{\rho(i_1)}.$$

Remark 3.3. The above lemma is implied by a result of Curtis [C75, Lemma 6.2]. It also can be obtained by combining [W82, Theorem 7.11], [W82, Theorem 7.12] and [W82, Lemma 12.5]. We note that Curtis's result [C75, Lemma 6.2] is meant to tell one about the way that differentials in the Λ-algebra and the Dyer-Lashof algebra are related. We refer to Appendix B to see how the above lemma is obtained from Curtis's result. We urge the reader to take care while comparing the above statement to Wellington's and Curtis's as they let the Steenrod operation act from right, and their iterated operations Q^I, in fact λ_I in terms of the Λ-algebra, are in the reverse order which may cause a bit of confusion. Although we we will not use the result, this lemma has been generalised to odd primes by Wellington [W82, Lemma 12.15]. We will make some more comments on this in an appendix.

Now we are ready to prove Theorem 2. We break it into little lemmata.

Lemma 3.4. *Let $x\in H_*X$ be A-annihilated, and I an admissible sequence with* $\mathrm{excess}(Q^Ix)>0$ *such that*
1- $\mathrm{excess}(Q^Ix)<2^{\rho(i_1)}$*;*
2- $2i_{j+1}-i_j<2^{\rho(i_{j+1})}$ *for all* $1\leqslant j\leqslant r-1$*;*
Then Q^Ix is A-annihilated.

Proof. Let $r>0$. Then we have the following

$$Sq^a_*Q^Ix\ =\ \sum Q^KSq^{a^K}_*x\ =\ \sum_{a^K=0}Q^Kx.$$

But notice that according to Lemma 3.2

$$\mathrm{excess}(Q^Kx)\leqslant\mathrm{excess}(Q^Ix)-2^{\rho(i_1)}<0.$$

Hence the above sum is trivial, and we are done. □

This proves the Theorem 2 in one direction. Now we have to show that the reverse direction holds as well. That is we have to show if either of conditions (1)-(3) of Theorem 2 does not hold then $Q^I x$ will be not-A-annihilated.

Remark 3.5. Before proceeding, we recall a basic property of the function ρ defined before Theorem 2 which is as following. Notice that given a positive integer n, then $\rho(n)$ is the least integer t such that

$$\binom{n-2^t}{2^t} \equiv 1 \bmod 2.$$

Notice that if $n = \sum n_i 2^i$ and $m = \sum m_i 2^i$ are given with $n_i, m_i \in \{0,1\}$ then $\binom{n}{m} = 1 \bmod 2$ if and only if $n_i \geqslant m_i$ for all i. This makes it easy to verify the above property for ρ.

The next three lemmata show that if any of conditions (1), (2) or (3) doesn't hold, then $Q^I x$ will not be A-annihilated.

Lemma 3.6. *Let X be path connected. Suppose $I = (i_1, \ldots, i_r)$ is an admissible sequence, such that* $\mathrm{excess}(Q^I x) \geqslant 2^{\rho(i_1)}$. *Then such a class is not A-annihilated.*

Proof. This is quite straightforward. We may use $Sq_*^{2^\rho}$ with $\rho = \rho(i_1)$, which gives

$$Sq_*^{2^\rho} Q^I x \quad = \quad Q^{i_1-2^\rho} Q^{i_2} \cdots Q^{i_r} x + O \tag{3.5}$$

where O denotes other terms given by

$$O = \sum_{t>0} \binom{i_1 - 2^\rho}{2^\rho - 2t} Q^{i_1-2^\rho+t} Sq_*^t Q^{i_2} \cdots Q^{i_r} x.$$

Notice that $\mathrm{excess}(Q^I x) \geqslant 2^{\rho(i_1)}$ ensures that i_1 is not of the form 2^ρ. Looking at the binary expression implies that all coefficients in O are nontrivial, and O will depend on the action of Sq_*^t on terms $Q^{i_2} \cdots Q^{i_r} x$. However, all of these terms are terms of lower excess, and they will not cancel the first term in (3.5).

Notice that at the right hand side of (3.5) the term $Q^{i_1-2^\rho} Q^{i_2} \cdots Q^{i_r} x$ is obviously admissible. Moreover,

$$\mathrm{excess}(Q^{i_1-2^\rho} Q^{i_2} \cdots Q^{i_r} x) = \mathrm{excess}(Q^I x) - 2^\rho \geqslant 0.$$

This proves that $Sq^{2^\rho}_* Q^I x \neq 0$. Notice that if $\mathrm{excess}(Sq^{2^\rho}_* Q^I x) = 0$, then

$$Sq^{2^\rho}_* Q^I x = (Q^{i_2} \cdots Q^{i_r} x)^2 \neq 0.$$

This completes the proof. □

The above lemma shows that if (2) of Theorem 2 does not hold, then we will have a class which is not A-annihilated. Next, we move on to the case when condition (3) does not hold.

Lemma 3.7. *Let X be path connected. Suppose $I = (i_1, \ldots, i_r)$ is an admissible sequence, and let $Q^I x$ be given with* $\mathrm{excess}(Q^I x) > 0$ *such that $2i_{j+1} - i_j \geqslant 2^{\rho(i_{j+1})}$ for some $1 \leqslant j \leqslant r-1$. Then such a class is not A-annihilated.*

Proof. Assume that $Q^I x$ satisfies the condition above. We may write this condition as

$$i_j - 2^\rho \leqslant 2i_{j+1} - 2^{\rho+1} = 2(i_{j+1} - 2^\rho),$$

where $\rho = \rho(i_{j+1})$. This is the same as admissibility condition for the pair $(i_j - 2^\rho, i_{j+1} - 2^\rho)$. In this case we use $Sq^{2^{\rho+j}}_*$ where we get

$$Sq^{2^{\rho+j}}_* Q^I x \ = \ Q^{i_1 - 2^{\rho+j-1}} Q^{i_2 - 2^{\rho+j-2}} \cdots Q^{i_j - 2^\rho} Q^{i_{j+1} - 2^\rho} Q^{i_{j+2}} \cdots Q^{i_r} x + O \quad (3.6)$$

where O denotes other terms, and similar to previous lemma will be a sum of terms of lower excess.

The first term in right hand side of the of the above equality is admissible. Moreover,

$$\begin{aligned}
\mathrm{excess}(Sq^{2^{\rho+j}}_* Q^I x) &= (i_1 - 2^{\rho+j-1}) - (i_2 - 2^{\rho+j-2}) - (i_j - 2^\rho) - (i_{j+1} - 2^\rho) - \\
&\quad (i_{j+2} + \cdots + i_r + \dim x) \\
&= i_1 - (i_2 + \cdots + i_r + \dim x) \\
&= \mathrm{excess}(Q^I x) > 0,
\end{aligned}$$

where by abuse of notation we have written $\mathrm{excess}(Sq^{2^{\rho+j}}_* Q^I x)$ to denote the excess of the first term in (3.6). This implies that

$$Sq^{2^{\rho+j}}_* Q^I x \neq 0,$$

and hence completes the proof. □

Remark 3.8. According to the proof in this case we always have

$$\mathrm{excess}(Sq_*^{2^{\rho+j}}Q^Ix) > 0$$

which means that we always end up with an indecomposable term after applying the "right" operation, i.e. the outcome will not be a square. This little observation will be useful.

Now we show that the condition (1) is also necessary in the proof of the main theorem.

Lemma 3.9. *Let X be path connected, and $x \in \overline{H}_*X$ be not A-annihilated. Then Q^Ix is not A-annihilated.*

Proof. Let t be the least number that

$$Sq_*^{2^t}x \neq 0.$$

If $I = (i_1, \ldots, i_r)$, we apply $Sq_*^{2^{t+r}}$ to Q^Ix, where we get

$$Sq_*^{2^{r+t}}Q^Ix \;\;=\;\; Q^{i_1-2^{r+t-1}} \cdots Q^{i_r-2^t}Sq_*^{2^t}x + O,$$

where O denotes sum of other terms which are of the form Q^Jy with $\dim y > \dim Sq_*^{2^t}x$. This means that the first term in the above equality will not cancel with any of other terms.
By abuse of notation we write $\mathrm{excess}(Q^{i_1-2^{r+t-1}} \cdots Q^{i_r-2^t}Sq_*^{2^t}x)$ to denote the excess of the first term in the above equality. We have $\mathrm{excess}(Q^{i_1-2^{r+t-1}} \cdots Q^{i_r-2^t}Sq_*^{2^t}x) = \mathrm{excess}(Q^Ix) > 0$. Moreover,

$$Q^{i_1-2^{r+t-1}} \cdots Q^{i_r-2^t}$$

is admissible. Hence $Sq_*^{2^{t+r}}Q^Ix \neq 0$. Note that similar to the previous lemma we end up with an indecomposable term. □

This completes the final step in the proof of Theorem 2. Next we move on to prove Corollary 6, which is an outcome of the proof of Theorem 2. Recall that according to Corollary 6, if $\sum Q^Ix \in H_*QX$ is an A-annihilated class with $\mathrm{excess}(Q^Ix) > 0$, with $x \in \overline{H}_*X$ fixed and X path connected, then every Q^Ix in the above sum will be A-annihilated. We like to demonstrate how this will work. To make our approach more clear we start with some illustrations. Recall that a class of the form Q^Ix with I having at least one even entry is not A-annihilated. Our illustration below shows that such a class will not be a term of any A-annihilated sum of the form $\sum Q^Ix$ with $\mathrm{excess}(Q^Ix) > 0$.

Recall that Nishida relations respect the length. Hence we concentrate on terms Q^Ix with I of fixed length, and single out the not-A-annihilated terms. Although we do the proofs for $X = S^1$, but they can be adopted to any path connected space X. The proofs are based on the following Nishida relations,

$$Sq^1_*Q^{2t} = Q^{2t-1}, \tag{3.7}$$

$$Sq^1_*Q^{2t+1} = 0. \tag{3.8}$$

We also recall that the operations Sq^i_* satisfy the Cartan formula as following. If $x, y \in H_*\Omega X$, then we have [W82, Remark 1.9]

$$Sq^i_*(xy) = \sum_{j+k=i} (Sq^j_*x)(Sq^k_*y). \tag{3.9}$$

This in particular implies that

$$Sq^{2t}_*\xi^2 = (Sq^t_*\xi)^2. \tag{3.10}$$

We also have the following Cartan formulae,

$$Q^{2t}\xi^2 = (Q^t\xi)^2, \tag{3.11}$$

$$Q^{2t+1}\xi^2 = 0. \tag{3.12}$$

The following lemma illustrates how we apply these relations, and proves even more that we may expect. It shows that if Q^Ix is an indecomposable, and a term of a spherical class, then i_1 must be odd. We have the following.

Lemma 3.10. *An admissible term Q^Ig_1 with $\mathrm{excess}(Q^Ig_1) > 0$ and i_1 even is not A-annihilated. Moreover, the action of Sq^1_* separates such a term from any other class of the form Q^Jg_1 which is not A-annihilated. In particular it is impossible to have a term of this form as part of a spherical class $\xi \in H_*QS^1$.*

Notice that here being separated under an operation has its obvious meaning. We say two classes are separated under Sq^r_* if they map to different elements under this operation.

Proof. Suppose we are given distinct terms

$$Q^H g_1, \quad Q^I g_1, \quad Q^J g_1, \quad Q^K g_1,$$

of the same dimension with H, I, J, K nonempty admissible sequences, and

$$\begin{aligned} H &= (2s_1, h_2, \ldots, h_r), \\ I &= (2t_1, i_2, \ldots, i_r), \\ J &= (2t'_1 + 1, j_2, \ldots, j_r), \\ K &= (k_1, k_2, \ldots, k_r). \end{aligned}$$

Moreover, assume that $\text{excess}(Q^H g_1) > 0$, $\text{excess}(Q^I g_1) > 0$, $\text{excess}(Q^J g_1) > 0$ and $\text{excess}(Q^K g_1) = 0$. Notice that $\text{excess}(Q^K g_1) = 0$ implies that we may write

$$Q^K g_1 = (Q^L g_1)^{2^s}$$

for some $s \geqslant 1$, and L admissible. Applying Sq^1_* to these classes we obtain

$$\begin{aligned} Sq^1_* Q^H g_1 &= Q^{2s_1-1} Q^{h_2} \cdots Q^{h_r} g_1 \neq 0, \\ Sq^1_* Q^I g_1 &= Q^{2t_1-1} Q^{i_2} \cdots Q^{i_r} g_1 \neq 0, \\ Sq^1_* Q^J g_1 &= 0, \\ Sq^1_* Q^K g_1 &= 0. \end{aligned}$$

This shows that the last two, the square term and the term starting with an odd number, are annihilated by Sq^1_* where the first two terms survive. Notice that

$$\begin{aligned} \text{excess}(Sq^1_* Q^H g_1) &= \text{excess}(Q^H g_1) - 1 \geqslant 0, \\ \text{excess}(Sq^1_* Q^I g_1) &= \text{excess}(Q^I g_1) - 1 \geqslant 0. \end{aligned}$$

This shows that $Q^H g_1$ is not A-annihilated. Now we show that it cannot be term of any sphercial class.

The fact that $H \neq I$ implies that $(2s_1 - 1, h_2, \ldots, h_r) \neq (2t_1 - 1, i_2, \ldots, i_r)$, i.e. $Sq^1_* Q^H g_1 \neq Sq^1_* Q^I g_1$. This means that having a class like $Q^H g_1$ starting with an even number as a term of a spherical class $\xi \in H_*QS^1$ will contradict the fact that ξ must be A-annihilated. More precisely, according to Lemma 10 we may write $\xi = \sum Q^A g_1$ with A admissible, and excess$(Q^A g_1) \geqslant 0$. Any term $Q^A g_1$ can be in either of the forms that we have mentioed at the beginning of the proof. However, the above calculations shows that if there is a sequence A with $Q^A g_1$ of positive excess, then ξ will not be A-annihilated which is a contradiction. This completes the proof. □

According to this lemma from now on we have to concentrate on distinct terms of the form

$$Q^I g_1, \quad Q^J g_1, \quad Q^K g_1,$$

with excess$(Q^I g_1) > 0$, excess$(Q^J g_1) > 0$ and excess$(Q^K g_1) = 0$ where I, J start with odd numbers. Before proceeding more, we have a very little observation.

Lemma 3.11. *Let a be odd, and (a,b) be an admissible pair, i.e. $a \leqslant 2b$. Then $(a-1, b-1)$ is also admissible.*

We do one more example to see how other similar cases will be resolved.

Lemma 3.12. *Suppose $Q^I g_1$ is an admissible term with* excess$(Q^I g_1) > 0$*, i_1 odd and i_2 even. Then such a class is not A-annihilated. Moreover, the action of Sq^2_* separates such a class from any other $Q^J g_1$ which is not A-annihilated. In particular, this implies it is impossible to have a term of this form as part of a spherical class $\xi \in H_*QS^1$.*

Proof. Suppose we have $Q^I g_1$ with $I = (i_1, \ldots, i_r)$, i_1 odd, and excess$(Q^I g_1) > 0$. Notice that the fact that excess$(Q^I g_1) > 0$ together with the admissibility of I implies that excess$(Q^{i_2} \cdots Q^{i_r} g_1) > 0$. The Nishida relations give the following

$$\begin{aligned} Sq^2_* Q^I g_1 &= \tbinom{i_1-2}{2} Q^{i_1-2} Q^{i_2} \cdots Q^{i_r} + \tbinom{i_1-2}{0} Q^{i_1-1} Sq^1_* Q^{i_2} \cdots Q^{i_r} g_1 \\ \\ &= \begin{cases} O + Q^{i_1-1} Q^{i_2-1} Q^{i_3} \cdots Q^{i_r} g_1 & \text{if } i_2 \text{ is even,} \\ O + 0 & \text{if } i_2 \text{ is odd.} \end{cases} \end{aligned}$$

Here O refers to the other terms which in this case is given by

$$O = \binom{i_1 - 2}{2} Q^{i_1-2} Q^{i_2} \cdots Q^{i_r},$$

which can be trivial or nontrivial. Notice that if O is nontrivial it will be admissible, and wouldn't cause trouble, as it is different from the second term in the above relation. We concentrate on the second term. Notice that according to above lemma $Q^{i_1-1}Q^{i_2-1}$ is admissible, and hence $Q^{i_1-1}Q^{i_2-1}Q^{i_3}\cdots Q^{i_r}$ is admissible. Moreover

$$\begin{aligned}\text{excess}(Q^{i_1-1}Q^{i_2-1}Q^{i_3}\cdots Q^{i_r}g_1) &= (i_1-1)-((i_2-1)+i_3+\cdots+i_r+1)\\ &= i_1-(i_2+i_3+\cdots+i_r+1)\\ &= \text{excess}(Q^Ig_1)>0.\end{aligned}$$

Hence the second term is nontrivial, i.e.

$$Sq^2_*Q^Ig_1\neq 0.$$

Notice that, if we have an admissible term $Q^Hg_1\neq Q^Ig_1$ with h_1 odd and h_2 even, then it would be not A-annihilated, and separated from Q^Ig_1 under the action of Sq^2_* for obvious reasons.

If we have any term Q^Jg_1 with $\text{excess}(Q^Jg_1)=0$, then it is a square, i.e. $Q^Jg_1=(Q^Kg_1)^{2^s}$. According to (3.8) if we succeed to find an operation, say Sq^t_*, with $Sq^t_*Q^Jg_1\neq 0$, then the outcome will be again a square term. This means that Q^Ig_1 is also separated from Q^Jg_1 under the action of Sq^2_*. This completes the proof of the lemma. □

Remark 3.13. Again we observe that apart from the case when i_1 is even, in the other case applying the right operation gives us an indecomposable term. For example in the example in the above lemma we have $\text{excess}(Q^{i_1-1}Q^{i_2-1}Q^{i_3}\cdots Q^{i_r}g_1)>0$. This is true in general.

Perhaps one can see how the rest of classes Q^Ig_1 with I having at least one even entry will be excluded from being A-annihilated. The following theorem resolves the general case.

Theorem 3.14. *Suppose Q^Ig_1 is an admissible term with* $\text{excess}(Q^Ig_1)>0$*, such that $I=(i_1,i_2,\ldots,i_r)$ has at least one even entry, say i_s is even with s chosen to be the least such number. Then this class is not A-annihilated under the action of $Sq^{2^{s-1}}_*$. Moreover, the action of $Sq^{2^{s-1}}_*$ separates such a term from any other class* *of the form $Q^J g_1$ which is not A-annihilated. In particular it is impossible to have a term of this form as part of a spherical class $\xi \in H_*QS^1$.*

The proof of this theorem is analogous to the examples provided before. Indeed we have to observe that in general case O will be a sum of terms of lower excess, and order.

Remark 3.15. Notice that we may order terms of the form $Q^I g_1$, of the same dimension, by putting an order on the sequences I. For sequences of length 1 the order is just the order of natural number. Suppose we have defined the order on the sequences of length $r-1$. Given two admissible sequences I, J of length r, we assume $I > J$ if either $i_1 > j_1$ or $i_1 = j_1$ and $(i_2, \ldots, i_r) > (j_2, \ldots, j_r)$. The term order in the above paragraph refers to this order.

Notice that the above observations have a very quick corollary about possible form that a spherical can have. We have the following.

Corollary 3.16. *Let $\xi = \sum Q^I x \in H_*QX$ with X path connected, and $x \in \overline{H}_*X$ fixed. Assume that ξ is A-annihilated. Then for any term with* excess$(Q^I x) > 0$ *we have I only with odd entries. In particular, if $\xi \in H_*QS^n$ with $n > 0$ is an odd dimensional class, then*

$$\xi = \sum Q^I g_n,$$

with I admissible, having only odd entries.

Proof. The first half of the corollary is evident from the previous explanations. If is $\xi \in H_*QS^n$ then $\xi = hf$ for some $f \in \pi_*QS^n$. The class f has an isomorphic image, say $g \in \pi_{*-1}QS^{n-1}$ with $\sigma_* hg = \xi$ which means that ξ is in the image of the homology suspension $\sigma_* : H_{*-1}QS^{n-1} \to H_*QS^n$. Hence according to Lemma 6 we may write $\xi = \sum Q^I g_n$ with excess$(Q^I g_n) \geqslant 0$. As we have chosen ξ to be odd dimensional, we then don't have the case of excess$(Q^I g_n) = 0$. This proves the lemma. □

Notice that $g_n \in H_nS^n$ is primitive, and hence any single term $Q^I g_n$ is primitive. One may expect a similar result to hold for odd dimensional spherical classes in $H_*Q_0S^0$. We leave this until after some discussion on primitive classes in $H_*Q_0S^0$.

3.1 Filtering Indecomposable A-annihilated Classes using their Length

Theorem 2 provides us with a general description of indecomposable A-annihilated classes $Q^I x \in H_*QX$ for any path connected space X. As we have mentioned earlier, the case $X = S^1$ is of special importance for us. In this case Lemma 4 provides us with an explicit description of such classes when $l(I) < 3$. This section contains the calculations to verify this claim.

We start by doing the proof for when $l(I) = 1$. This is rather easier than one might guess. Clearly $g_1 \in H_1S^1$ is A-annihilated. Next in this line are classes of the form $Q^i g_1$, with i odd. The action of an operation Sq^r_* on such a class is given by

$$Sq^r_* Q^i g_1 \quad = \quad \binom{i-r}{r} Q^{i-r} g_1.$$

If $i \neq 2^\alpha - 1$, choose $m = \rho(i)$. Then $i - 2^m \equiv 2^{m+1} - 1 \bmod 2^{m+1}$ and hence the binomial coefficient above for $r = 2^m$ is odd, i.e.

$$Sq^{2^m}_* Q^i g_1 \quad = \quad Q^{i-2^m} g_1 \neq 0.$$

It is also obvious that any class of the form $Q^{2^\alpha - 1} g_1$ is A-annihilated. The triviality of the coefficients is easy to see once we look at the binary expansions.

Next in line are classes of the form $Q^a Q^b g_1$, i.e. $Q^I g_1$ with $l(I) = 2$. Let us fix our notation. For a natural number k let $\nu(k)$ be the number of 1's in its binary expansion. Then we have the following lemma.

Lemma 3.17. *Suppose k is a natural number such that the pair $(2^\alpha - 1 + k, 2^\alpha - 1)$ is admissible and the class $Q^{2^\alpha - 1 + k} Q^{2^\alpha - 1} g_1$ is of positive excess, i.e. $k > 1$. Then such a class is not A-annihilated if and only if $\nu(k) > 1$, i.e. it is A-annihilated if and only if $\nu(k) = 1$.*

Proof. First of all, notice that we like $2^\alpha - 1 + k$ to be odd, as otherwise we have the elimination by Sq^1_*. Hence we only restrict to the cases where $2^\alpha - 1 + k$ is odd, i.e. k is even. First let $\nu(k) = 1$, i.e. $k = 2^j$ for some $1 \leqslant j \leqslant \alpha - 1$. The only operation which might act nontrivially is $Sq_*^{2^j}$, but

$$Sq_*^{2^j}Q^{2^\alpha+2^j-1}Q^{2^\alpha-1}g_1 = Q^{2^\alpha-1}Q^{2^\alpha-1}g_1 = 0,$$

where it vanishes for dimensional reasons. The operations $Sq_*^{2^l}$ with $l < j$ do have trivial action for numerical reasons, i.e. the coefficients in the Nishida relation become trivial. Moreover, the operations $Sq_*^{2^l}$ with $l > j$ have trivial action for dimensional reasons.

Now assume $\nu(k) > 1$, k even. Having $k = \sum_l k_l 2^l$, let j be the minimum l such that $k_l = 1$. Then,

$$Sq_*^{2^j}Q^{2^\alpha+k-1}Q^{2^\alpha-1}g_1 = Q^{2^\alpha+k-2^j-1}Q^{2^\alpha-1}g_1 \neq 0.$$

Notice that since $\nu(k) > 1$, after taking 2^j out, still there is another l, say l' with $k_{l'} = 1$, which means that

$$\binom{k-2^j}{2^j} \equiv 1 \bmod 2.$$

This completes the proof. □

Remark 3.18. Notice that we may choose k in way to get $2^\alpha + k - 1 = 2^\beta - 1$. But this will not give an admissible term. To see this notice that

$$Q^{2^\beta-1}Q^{2^\alpha-1}g_1$$

is nontrivial if $2^\beta - 1 \geqslant 2^\alpha - 1 + 1 > 2^\alpha - 1$, which implies that $\beta > \alpha$. On the other hand the term will be admissible if

$$2^\beta - 1 \leqslant 2(2^\alpha - 1) < 2^{\alpha+1} - 1,$$

which implies that $\beta < \alpha+1$ where β is an integer, but packed between two successive integers. This is a contradiction.

The above lemma classifies all A-annihilated classes of the form $Q^iQ^{2^s-1}g_1$. This proves Lemma 2 in one direction. The following will complete the proof when $l(I) = 2$.

Lemma 3.19. *Let $Q^aQ^bg_1$ be an A-annihilated class with* excess$(Q^aQ^bg_1) > 0$*. Then $b = 2^s - 1$ for some $s > 0$.*

Proof. According to Theorem 2, this class is A-annihilated, if and only if,

$$\begin{aligned} a-(b+1) &< 2^{\rho(a)},\\ 2b-a &< 2^{\rho(b)}. \end{aligned}$$

Adding these together yields

$$0 \leqslant b-1 < 2^{\rho(b)} + 2^{\rho(a)}.$$

Notice that for any integer b, there exists $N_b \geqslant 0$ such that $b = 2^{\rho(b)+1}N_b + 2^{\rho(b)} - 1$. Also recall from the appendix A, or [C75, Proof of Lemma 6.2], that if $0 \leqslant 2b - a < 2^{\rho(b)}$, then $\rho(a) \leqslant \rho(b)$. Hence we have,

$$0 \leqslant 2^{\rho(b)+1}N_b + 2^{\rho(b)} - 1 - 1 < 2^{\rho(b)} + 2^{\rho(a)}.$$

This implies that

$$2^{\rho(b)+1}N_b - 1 - 1 < 2^{\rho(a)}.$$

Taking into account that $\rho(b) \geqslant \rho(a)$, the above inequality can hold only if $N_b = 0$, i.e. $b = 2^{\rho(b)} - 1$. This completes the proof. □

Next, we may consider the case $l(I) = 3$. A natural way to obtain an A-annihilated class of $Q^I g_1$ with $l(I) = n$ is try to find a suitable operation Q^i, and apply it to an indecomposable A-annihilated class $Q^J g_1$ with $l(J) = n - 1$. The following lemma, shows that this is not possible when $l(J) = 2$.

Lemma 3.20. *It is impossible to have a nontrivial admissible class of the form*

$$Q^{2^{\alpha+1}+2^j+k-1}Q^{2^\alpha+2^j-1}Q^{2^\alpha-1}g_1$$

which is A-annihilated.

Proof. According to previous section, we are not worried about the cases that $2^{\alpha+1} + 2^j + k - 1$ is even. So we consider the cases where it is odd. Notice that the class above is admissible only if

$$2^{\alpha+1} + 2^j + k - 1 \leqslant 2(2^\alpha + 2^j - 1) = 2^{\alpha+1} + 2^{j+1} - 2,$$

which implies that

$$1 < k < 2^j - 1.$$

If $k-1 \neq 2^b - 1$, choose $m = \rho(k)$, i.e. $k - 1 \equiv 2^m - 1 \bmod 2^{m+1}$. Then

$$\begin{aligned} Sq_*^{2^m} Q^{2^{\alpha+1}+2^j+k-1} Q^{2^\alpha+2^j-1} Q^{2^\alpha-1} g_1 &= \\ Q^{2^{\alpha+1}+2^j+k-1-2^m} Q^{2^\alpha+2^j-1} Q^{2^\alpha-1} g_1 &\neq 0. \end{aligned}$$

If $k - 1 = 2^b - 1$, then

$$\begin{aligned} Sq_*^{2^b} Q^{2^{\alpha+1}+2^j+k-1} Q^{2^\alpha+2^j-1} Q^{2^\alpha-1} g_1 &= \\ Q^{2^{\alpha+1}+2^j-1} Q^{2^\alpha+2^j-1} Q^{2^\alpha-1} g_1 &= \\ (Q^{2^\alpha+2^j-1} Q^{2^\alpha-1} g_1)^2 &\neq 0. \end{aligned}$$

This completes the proof. □

According to this lemma if $Q^{i_1}Q^{i_2}Q^{i_3}g_1$ is A-annihilated, then $Q^{i_2}Q^{i_3}g_1$ is not A-annihilated. This completes the proof for Lemma 2.

Remark 3.21. Notice that an A-annihilated class of the form $Q^{2^s+2^j-1}Q^{2^s-1}g_1$ may be identified with the class $Q^{2^s+2^j-1}\Sigma a_{2^s-1} \in H_*Q\Sigma P$ where $a_{2^s-1} \in H_{2^s-1}P$ is a dual to $a^{2^s-1} \in H^*P$. In fact these two classes have similar behavior under the avtion of the Steenrod operations. However, the class $Q^{2^s+2^j-1}a_{2^s-1} \in H_*QP$ is not A-annihilated.

3.2 Complementary Notes

Theorem 2 provides us with a complete description of A-annihilated classes in H_*QX of the form $Q^I x$ of positive excess, with X being path connected. In return, this also classifies all such classes that are not A-annihilated. For instance given a class $Q^I x \in H_*QX$, this class will not be A-annihilated, if at least one of conditions in Theorem 2 does not hold; i.e.

- x is not A-annihilated,
- $\mathrm{excess}(Q^I x) \geqslant 2^{\rho(i_1)}$;
- There exists j such that $2i_{j+1} - i_j \geqslant 2^{\rho(i_{j+1})}$;

We applied a part of such information to obtain more information on the possible form of an odd dimensional spherical class in H_*QS^n with $n > 0$ which was the statement of Corollary 3.16. Recall that according to Theorem 3.14 and Corollary 3.16 we excluded any class $Q^I g_n$ with I having at least one even entry from being a term of a spherical class. Notice that according to the remark after Theorem 2, a class $Q^I x$ of positive excess with at least one even entry will not be A-annihilated. We want to see that if more can be done, and use full power of this description. That is we like to prove Corollary 6, and Lemma 12, and show that having given an A-annihilated sum of terms $Q^I g_n$ of positive excess, then each of these terms must be A-annihilated.

Now we return to the above classification of classes $Q^I x$ which are not A-annihilated. Each of the above cases distinguishes a class of indecomposable terms $Q^I x$ that are not A-annihilated. One observes that a given class $Q^I x$ can be not-A-annihilated for multiple reasons. For instance, consider $a_2 \in H_2P$. This class is not A-annihilated as $Sq^1_* a_2 = a_1$. This then implies that $Q^5 a_2$ is not A-annihilated which comes from applying Lemma 3.9. But this also can be seen by the fact that excess$(Q^5 a_2) = 3 \not< 2^{\rho(5)} = 2$. However, for any class there is a "minimal" condition for being not-A-annihilated. To be more precise, given a class $Q^I x$ which is not A-annihilated for one of the above reasons, we can find least t such that $Sq^{2^t}_* Q^I x \neq 0$, this gives the minimal case. This makes more sense when a class $Q^I x$ is not A-annihilated, because there exists j with $2i_{j+1} - i_j \geqslant 2^{\rho(i_{j+1})}$. In this case choosing least such j, can lead us to the least operation. To see a more complex example, assume that $Q^I x$ is not A-annihilated because there exists j with $2i_{j+1} - i_j \geqslant 2^{\rho(i_{j+1})}$; and because excess$(Q^I x) \geqslant 2^{\rho(i_1)}$. In such a case we pick up least j, giving the operation $Sq^{2^t}_*$ and compare it to $Sq^{2^{\rho(i_1)}}_*$ and choose the one with least degree, i.e. let $m = \min(\rho(i_1), t)$ and then $Sq^{2^m}_*$ will be the least operation that does not kill $Q^I x$.

Now let us see how this helps to get a result sharper than Corollary 3.16. Recall that Corollary 3.16 allows us to restrict our attention to indecomposable terms $Q^I x$ with I having only odd entries. We like to show that if two indecomposable terms $Q^I x$, and $Q^J x$ are not A-annihilated for the same minimal reason, then they are *separated* from each other. We say two indecomposable classes $Q^I x$ and $Q^J x$ are separated if there exists k such that $Sq^k_* Q^I x \neq Sq^k_* Q^J x$.

First, consider the classes $Q^I x$ and $Q^J x$ of the same dimension, of positive excess with $x \in H_* X$ not A-annihilated. Then there exists the least positive integer t such that $Sq^{2^t}_* x \neq 0$. Then

$$Sq^{2^{t+r}}_* Q^I x \neq Sq^{2^{t+r}}_* Q^J x$$

where $l(I) = l(J) = r$. To see that let us assume $r = 2$, then iterated use of the Nishida relations implies that

$$\begin{aligned} Sq^{2^{t+r}}_* Q^I x &= Q^{i_1 - 2^{t+1}} Q^{i_2 - 2^t} Sq^{2^t}_* x, \\ Sq^{2^{t+r}}_* Q^J x &= Q^{j_1 - 2^{t+1}} Q^{j_2 - 2^t} Sq^{2^t}_* x. \end{aligned}$$

As $I \neq J$ the right hand side of the first and second quality are admissible terms of positive excess, in fact $\mathrm{excess}(Q^{i_1 - 2^{t+1}} Q^{i_2 - 2^t} Sq^{2^t}_* x) = \mathrm{excess}(Q^{i_1} Q^{i_2} x) > 0$. Moreover, the right hand sides of these equalities are not equal. This then verifies the above inequality. The cases with $r > 2$ can be verified in a similar way. Notice that because of the choice of t, in these cases each side of the above inequality will have only one term, namely

$$\begin{aligned} Sq^{2^{t+r}}_* Q^I x &= Q^{i_1 - 2^{t+r-1}} Q^{i_2 - 2^t} Sq^{2^t}_* x, \\ Sq^{2^{t+r}}_* Q^J x &= Q^{j_1 - 2^{t+r-1}} Q^{j_2 - 2^t} Sq^{2^t}_* x, \end{aligned}$$

which verifies that $Sq^{2^{t+r}}_* Q^I x \neq Sq^{2^{t+r}}_* Q^J x$.

Second, let $Q^I x$ and $Q^J x$ be two classes that are not A-annihilated such that $\mathrm{excess}(Q^I x) \geqslant 2^{\rho(i_1)}$, and $\mathrm{excess}(Q^J x) \geqslant 2^{\rho(j_1)}$. If we choose $\rho = \min\{\rho(i_1), \rho(j_1)\}$, then it is clear that

$$Sq^{2^\rho}_* Q^I x \neq Sq^{2^\rho}_* Q^J x.$$

Notice that we are assuming that I and J have the same length. Finally we may consider $Q^I x$, and $Q^J x$ where they are not A-annihilated because there exists least k such that $2i_{k+1} - i_k \geqslant 2^{\rho(i_{k+1})}$ and $2j_{k+1} - j_k \geqslant 2^{\rho(j_{k+1})}$. In this case we have

$$\begin{aligned} Sq^{2^{\rho(i_{k+1})+r-k}}_* Q^I x &\neq 0, \\ Sq^{2^{\rho(j_{k+1})+r-k}}_* Q^J x &\neq 0, \end{aligned}$$

where the above operations are the least operations which do not annihilate the related classes. One may choose the least among such operation, and such that

$$Sq_*^{2^{m+r-k}}Q^Ix \neq Sq_*^{2^m+r-k}Q^Jx,$$

where $m = \min\{\rho(i_{k+1}), \rho(j_{k+1})\}$.

These may be summarised by saying that if two classes Q^Ix and Q^Jx are not A-annihilated for precisely the same reason, then it is possible to find an operation which does annihilated one, and does not annihilate the other one. We like to take this further up. We are able to take this further more and obtain reasonably more general results. More precisely, we aim to show that if Q^Ix and Q^Jx are two classes of positive excess, then we can separate them from each other. We note that we like to use this result to obtain a better description of an A-annihilated primitive class in H_*QS^n with $n > 0$, i.e. we always choose $x = g_n$ which is an A-annihilated class. Hence we restrict our attention to classes of the form Q^Ig_n of positive excess with I an admissible sequence with only odd entries. Finally we comment on that how this one can be modified to obtain an analogous result in $H_*Q_0S^0$. We like to draw the reader'a attention that this result also can be seen as a result about a special class of operations in the Dyer-Lashof algebra R.

First, let Q^Ig_n and Q^Jg_n be two classes which are of positive excess, with $l(I) = l(J) = r$ of the same dimension. Suppose these classes are not A-annihilated, only for the following reasons; $2i_{k+1} - i_k \geqslant 2^{\rho(i_{k+1})}$ for some $1 \leqslant k \leqslant r$, and $2j_{k'+1} - j_{k'} \geqslant 2^{\rho(j_{k'+1})}$ for some $1 \leqslant k' \leqslant r$. Choose least k and denotes it with k_0, and similarly choose k'_0. Notice that $Sq_*^{2^{\rho(i_{k_0+1})+k_0}}$ is the least operation which does not annihilate, Q^Ig_n, and similarly $Sq_*^{2^{\rho(j_{k'_0+1})+k'_0}}$ is the least operation which does not annihilate Q^Jg_n. We have already considered the case with $k_0 = k'_0$. So, assume $k_0 < k'_0$. One can check that $Sq_*^{2^{\rho(i_{k_0+1})+k_0}}$ will not annihilate Q^Ig_n, where it annihilates Q^Jg_n. This is still quite general and we may replace g_n with any A-annihilated class in H_*X with X path connected.

The only case that we need to resolve is the following. Assume that Q^Ig_n and Q^Jg_n are given, with $l(I) = l(J) = r$. Suppose these classes are not A-annihilated for the following reason. For Q^Ig_n, there exists k such that $2i_{k+1}-i_k\geqslant 2^{\rho(i_{k+1})}$, and Q^Jg_n is not A-annihilated because excess$(Q^Jg_n)\geqslant 2^{\rho(j_1)}$. Similar to previous cases let k_0 to be the least among k's. In this case we have two different situations. $k_0>1$, $k_0=1$. If $k_0>1$, then we let $m=\min(2^{\rho(j_1)},2^{\rho(i_{k_0+1})+k_0})$. Then the operation Sq^m_* will annihilate one and will not annihilate the other one, depending what m is. We could resolve previous cases like this as well, however the way that we already did is quite adequate. Now assume $k_0=1$. Then we have

$$\begin{array}{rcl} Sq^{2^{\rho(i_2)+1}}_*Q^Ig_n &=& Q^{i_1-2^{\rho(i_1)}}Q^{i_2-2^{\rho(i_2)}}Q^{i_3}\cdots Q^{i_r}g_n,\\ Sq^{2^{\rho(j_1)}}_*Q^Jg_n &=& Q^{j_1-2^{\rho(j_1)}}Q^{i_2}\cdots Q^{i_r}g_n. \end{array}$$

We only need to consider the case $\rho(j_1)=\rho(i_2)+1$. Then

$$Sq^{2^{\rho(i_2)+1}}_*Q^Ig_n=Sq^{2^{\rho(j_1)}}_*Q^Jg_n,$$

if and only if $i_1-2^{\rho(i_2)}=j_1-2^{\rho(j_1)}$, $i_2-2^{\rho(i_2)}=j_2$ which implies that $j_1=i_1+2^{\rho(i_2)}$, $j_2=i_2-2^{\rho(i_2)}$. However, we have

$$j_1-2j_2=i_1-2^{\rho(i_2)}+2^{\rho(i_2)+1}+2^{\rho(i_2)}\geqslant 2^{\rho(i_2)},$$

which implies that (j_1,j_2) is not admissible. This contradicts the fact that J is admissible.

This completes our verification of the fact that any two distinct indecomposable classes in homology of H_*QX which are not A-annihilated can be separated by an operation. This also completes the proof of Corollary 6.

Remark 3.22. Observe that we concentrated on indecomposable terms, i.e. Q^Ix with excess$(Q^Ix)>0$. In fact shows that any two operation Q^I and Q^J of positive excess which are not A-annihilated can be separated by some operation. This also may be seen as a proof of [C75, Theorem 6.3] in odd degrees. However, this is not true if we allow operation or terms of trivial excess, i.e. square terms which eliminates claim of [C75, Theorem 6.3] to be true in even degrees. We refer the reader to see [W82, Remark 11.26] for counterexample. We have discussed this example Note 5.27.

The proof of Lemma 10 is quite clear. If $\xi\in H_{*+1}Q\Sigma X$ is a spherical class, then it is in the image of the homology suspension $\sigma_*:H_*QX\to H_{*+1}Q\Sigma X$. Notice that σ_*

kills the decomposable terms, and sends $Q^I x$ to $Q^I \Sigma x$. This implies that the terms $Q^I \Sigma x$, with excess$(Q^I \Sigma x) \geqslant 0$ are the only terms which survive after one suspension.

The last proof of this chapter is the proof of Theorem 17. That is we show that it is impossible to have a spherical class $\zeta^{2^t} \in H_*QS^n$ with $n \geqslant 0$, and $t > 1$. Here we do the proof for $n > 1$. The cases $n = 1$ and $n = 0$ will be discussed later.

Assume that $\xi_n = \zeta^4 \in H_{4d}QS^n$ is spherical. Notice that $\xi = Q^{2d}Q^d\zeta$, $\zeta = \sum Q^I g_n$ with $(2d, d, I)$ admissible, where $d = \dim\zeta$. This then pulls back to a spherical class of the form $\xi_{n-1} = Q^{2d}Q^d\zeta' + D \in H_{4d-1}QS^{n-1}$ where $\zeta' = \sum Q^I g_{n-1}$ and D denotes the decomposable part. Notice that $Q^{2d}Q^d\zeta'$ is sum of primitive classes, hence D a decomposable primitive. This implies that D must be a square which is impossible as it is an odd dimensional class. This implies that $D = 0$. Hence,

$$\xi_{n-1} = \sum Q^{2d}Q^d\zeta'.$$

Applying Sq^1_* we obtain

$$Sq^1_*\xi_{n-1} = \big(\sum Q^d\zeta'\big)^2 \neq 0.$$

This shows that ξ_{n-1} is not A-annihilated, giving us the contradiction that we were looking for, and hence proving the claim. A similar method works for higher powers of 2. This method also can be employed to prove Lemma 19 as well, when $n > 1$.

We leave the proof of Lemma 13 and Theorem 21 to the next chapter as these proofs have a different nature to the proofs in this chapter.

Chapter 4

Proof of Technical Lemmata

This chapter is dedicated to the proof of two technical lemmata. The first one is the proof of Lemma 13. The other proof is the proof of Theorem 21 which provides us with the inductive step in our approach to the Curtis conjecture.

We start by proving Lemma 13 which seems to be the most generalised case of this observation. More precisely, suppose $\xi \in \pi_{2n}QS^k$, $k > 0$, with $h\xi = \xi_n'^2$ where $\xi_n' \in H_nQS^k$. Then the adjoint mapping $S^{2n+1} \to Q\Sigma S^k$ is detected by the primary operation Sq^{n+1} on $\sigma_*(\xi_n' + O)$ in its mapping cone, where O refers to the indeterminacy which is a sum of terms of the same weight as ξ_n and of lower excess. The proof of this will use the homology of certain stable James-Hopf invariants, namely $j_{2^t} : QX \to QD_{2^t}X$. To state the result we note that for a class ξ of dimension d we have $Q_i\xi = Q^{i+d}\xi$. One then may consider iterated operations in lower indices, denoted by Q_I. The following result is due to Kuhn [K83, Proposition 2.7] which tell us how to calculate the homology of the stable James-Hopf invariants.

Proposition 4.1. *Suppose $x \in H_*X$ is primitive, with X path connected, and $I = (i_1, \ldots, i_r)$ is a nondecreasing sequence. Then, $(j_q)_*Q_Ix = 0$ unless $q = 2^t$ with $t \leqslant r$. If $t \leqslant r$, let $I = (K, J)$ with $l(J) = t$. Then*

$$(j_{2^t})_*Q_Ix = Q_K(Q_Jx)$$

modulo terms of the form $Q_{K'}(Q_{J'}x)$ such that $l(K) = l(K')$, $l(J') = l(J)$, J' is nondecreasing, and if $j_1' < j_1 = i_{r-t+1}$.

Remark 4.2. We recall that For an iterated operation $Q_I = Q_{i_1} \cdots Q_{i_r}$, the excess is i_1. In lower indices, a nondecreasing sequence I will mean admissible in our sense, where we allow $i_j \geqslant 0$. Here, by $(Q_J x)$ we mean image of $Q_J x$ in $H_*D_{2^t}X$ under the projection map given by the Snaith splitting (1.1). Notice that the above theorem does **not** imply that (K', J') is admissible. The above proposition in the case $q = 2^t$ with operations written in upper indices reads as

$$(j_{2^t})_* Q^I x = Q^K(Q^J x)$$

modulo terms of the form $Q^{K'}(Q^{J'}x)$ such that $l(K) = l(K')$, $l(J') = l(J)$, J' admissible with $0 \leqslant \text{excess}(J') < \text{excess}(J)$.

We are interested in finding the images of square A-annihilated classes under the James-Hopf invariants. Notice that square terms are of the form ζ^{2^t} with $t > 0$. The following example resolves the cases with $t > 1$. Although we will not use this at all.

Example 4.3. Let $\xi = \zeta^{2^t} \in H_*QX$ with $\xi = Q^I x$, $x \in H_*X$ primitive, $I = (i_1, \ldots, i_r)$ admissible. For simplicity we assume $t = 2$. Consider the James-Hopf map $j_{2^{r-1}} : QX \to QD_{2^{r-1}}X$. Notice that in this case $I = (i_1, J)$ with $Q^J x = \zeta^2$, a square term, i.e. $\text{excess}(Q^J x) = 0$. Hence if there is any class $Q^{J'}x$ with $\text{excess}(Q^{J'}x) < \text{excess}(Q^J x) = 0$ it must be trivial. This means that there is no indeterminacy. Hence

$$(j_2)_*\xi = (\zeta^2)^2.$$

The above example is an easy application of Kuhn's result. The situation is slightly different when $t = 1$. Notice that any spherical class $\xi \in H_*QS^n$, $n > 0$, is a square or it will become a square after finitely many suspensions, where according to Theorem 17 it will not be a higher power of 2. We give an example to show how Kuhn's result, together with being A-annihilated, enables us to do calculations with some *control* of the indeterminacy.

Example 4.4. Consider the class $\xi = (Q^{19}Q^{11}g_5)^2 \in H_{70}QS^5$. This is an A-annihilated class. We wish to calculate $(j_4)_*\xi$. Kuhn's results says that

$$(j_4)_*\xi = ((Q^{19}Q^{11}g_5))^2$$

modulo terms of the form $Q^{k'}(Q^{J'}g_5)$, where

$$\text{excess}(Q^{J'}g_5) < \text{excess}(Q^{19}Q^{11}g_5) = 3.$$

We observe that classes with $k' > \dim Q^{J'}g_5$, k' even, are excluded, and cannot take part as terms of $(j_4)_*\xi$. This can be shown just by use of Sq^1_*. We already know that $(j_4)_*\xi$ is A-annihilated. If there is an even k', then we may apply Sq^1_* to show that $Sq^1_*(j_4)_*\xi \neq 0$ which is a contradiction. Taking these to account, we may calculate that

$$\begin{array}{rcllll}(j_4)_*\xi & = & ((Q^{19}Q^{11}g_5))^2 & + & \lambda_4 Q^{37}(Q^{17}Q^{11}g_5) & + \\ & & \lambda_8 Q^{39}(Q^{16}Q^{10}g_5) & + & \lambda_{12}Q^{41}(Q^{15}Q^{9}g_5) & + \\ & & \lambda_{16}Q^{43}(Q^{14}Q^{8}g_5) & + & \lambda_{20}Q^{45}(Q^{13}Q^{7}g_5) & + \\ & & \lambda_{24}Q^{47}(Q^{12}Q^{6}g_5), & & & \end{array}$$

with $\lambda_i \in \mathbb{Z}/2$ modulo terms of the form $Q^{k'}(Q^{J'}g_5)$ with $k' = \dim Q^J g_5 = \dim Q^{J'}g_5 = 35$, i.e $Q^{k'}(Q^{J'}g_5) = ((Q^{J'}g_5))^2$. We like to evaluate the coefficients λ_i and show that they do vanish. To do so, recall that we have a commutative diagram,

$$\begin{array}{ccc} \Sigma QS^5 & \xrightarrow{j_4'} & Q\Sigma D_4 S^5 \\ \downarrow & & \downarrow \\ QS^6 & \xrightarrow{j_4} & QD_4S^6 \end{array}$$

where the map at top row, is adjoint of $j_4 : QS^5 \to QD_4S^5$. The vertical map $Q\Sigma D_4S^5 \to QD_4S^6$ is the infinite loop map obtained from the mapping induced by the evaluation mapping $\Sigma QS^5 \to QS^6$.

Applying homology to the above diagram, shows that $\lambda_i = 0$. More precisely, notice that $\xi = (Q^{19}Q^{11}g_5)^2$ dies under the homology suspension $\sigma_* : H_*QS^5 \to H_{*+1}QS^6$. Hence it maps trivially into $H_*QD_4S^6$. This shows that $(j_4)_*\xi$ must die under the evaluation map $H_*QD_4S^5 \to H_{*+1}QD_4S^6$. The first term is a square, and hence maps trivially. Other terms will map trivially if and only if $\lambda_i = 0$. Hence,

$$(j_4)_*\xi \;=\; ((Q^{19}Q^{11}g_5))^2 + \textstyle\sum_{J'} \lambda_{J'}((Q^{J'}g_5))^2$$

where $\lambda_{J'} \in \mathbb{Z}/2$. The important fact for us is that this class dies after suspending once, i.e. $\sigma_*(j_4)_*\xi = 0$. Notice that if $f : S^{70} \to QS^5$ is a mapping with $hf = \xi$, then

$$j_4 f : S^{70} \to QD_4S^5$$

is a mapping whose image is a square. This mean that the adjoint of $j_4 f$, i.e.

$$S^{71} \to Q\Sigma D_4 S^5$$

is detected by Sq^{36} on $\sigma_*((Q^{19}Q^{11}g_5)) + \sum_{J'} \lambda_{J'}((Q^{J'}g_5))$. Applying homology to the above class implies that the mapping

$$S^{71} \to QD_4 S^6$$

is also detected by Sq^{36} on $(Q^{19}Q^{11}g_6) + \sum_{J'} \lambda_{J'}\sigma_*(Q^{J'}g_5)$. Then naturality implies that the mapping

$$S^{71} \to QS^6$$

is detected by Sq^{36} on $Q^{19}Q^{11}g_6 + \lambda_{J'}\sigma_* Q^{J'}g_5$. Note that $\text{excess}(J') < \text{excess}(J')$. This means that after finitely many times suspensions, say k times, the class $\sigma_* Q^{J'}g_5$ will die before $Q^J g_6$. Hence the mapping

$$S^{71+k} \to QS^{6+k}$$

will be detected by Sq^{36} on $Q^J g_{6+k}$.

The following lemma is analogous to this example.

Lemma 4.5. *Let $Q^I g_n = (Q^J g_n)^2 \in H_*QS^n$, with $n > 0$, $\text{excess}(Q^J g_n) > 0$, and $l(J) = r$, be an A-annihilated class. Then,*

$$(j_{2^r})_* Q^I g_n = ((Q^J g_n))^2 + O^2$$

where O is the indeterminacy which is an A-annihilated sum of admissible terms of the form $Q^{J'}g_n$ with J' admissible such that $\text{excess}(Q^{J'}g_n) < \text{excess} Q^J g_n$.

Note 4.6. Notice that if O is odd dimensional, then we may assume that it is sum of A-annihilated terms.

Proof. Notice that the class $Q^I g_n = (Q^J g_n)^2 \in H_*QS^n$ dies under suspension. Consider the commutative diagram [K82, Theorem 1.2]

$$\begin{array}{ccc} \Sigma QS^n & \longrightarrow & Q\Sigma D_{2^r} S^n \\ \downarrow & & \downarrow e' \\ QS^{n+1} & \longrightarrow & QD_{2^r} S^{n+1}. \end{array}$$

This implies that in homology

$$e'_*(j_{2^r})_* Q^I g_n = 0.$$

Our claim is that this is the same as saying that

$$(j_{2^r})_* Q^I g_n = ((Q^J g_n))^2$$

modulo square terms. To see this notice that the mapping $e' : Q\Sigma D_{2^r} S^n \to QD_{2^r} S^{n+1}$ is an infinite loop map, obtained in the following manner. Recall that the Barrat-Eccles' Γ-functor provides a simplicial model for infinite loop spaces. We have the evaluation mapping

$$e : \Sigma\Gamma X \to \Gamma\Sigma X,$$

where X is an arbitrary path connected space. The space ΓX is a filtered space, i.e. there are spaces $\Gamma_k X$ such that

$$\Gamma X = \text{colim } (\cdots \subseteq \Gamma_r X \subseteq \Gamma_{r+1} X \subset \cdots).$$

This satisfies

$$D_r X \simeq \Gamma_r X / \Gamma_{r-1} X,$$

i.e. we have cofibration sequences

$$\Gamma_{r-1} X \to \Gamma_r X \to D_r X.$$

The evaluation mapping $e : \Sigma\Gamma X \to \Gamma\Sigma X$ restricts to the filtered spaces and gives mappings

$$\Sigma\Gamma_r X \to \Gamma_r \Sigma X.$$

This induces mappings

$$\Sigma D_r X \to D_r \Sigma X.$$

The mapping e' in the above diagram is infinite loop extension of this map. This means that to understand e' in homology, we only need to calculate the homology of $\Sigma D_r X \to D_r \Sigma X$, which is already induced by the evaluation mapping. Notice that these mappings fit into a commutative diagram as following

$$\begin{array}{ccc} \Sigma\Gamma X & \xrightarrow{\widetilde{j_r}} & \Gamma\Sigma D_r X \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle e'} \\ \Gamma\Sigma X & \xrightarrow{j_r} & \Gamma D_r \Sigma X \end{array}$$

where $\widetilde{j_r} : \Sigma\Gamma X \to \Gamma\Sigma D_r X$ in the top row is the adjoint of the r-th James-Hopf invariant $j_r : \Gamma X \to \Gamma D_r X$. In particular, choosing $X = S^n$ and replacing r with 2^r gives the diagram at the beginning of the proof.

As a consequence of this, we can extract enough information out of this to say more about the indeterminacy above. Notice that this description completely determines $\ker e'_*$. More precisely, a class $Q^K(Q^J g_n)$ belongs to $\ker e'_*$ if and only if excess$(Q^K(Q^J g_n)) = 0$ or excess$(Q^J g_n) = 0$. One can check that in particular if $\xi \in H_*QX$ belongs to the kernel of the evaluation $H_*QX \to H_{*+1}Q\Sigma X$, then $(j_r)_*\xi$ belongs to the kernel of e'_*. In our case, this implies that the indeterminacy must be a square. To be more precise, notice that the non-square part of the indeterminacy is a sum of terms of the form,

$$Q^{k'}(Q^{J'}g_n),$$

with $k' > \dim Q^{J'}g_n$ and J' admissible. Application of Sq^1_*, similar to the previous example, shows that k' must be odd. On the other hand $k' + \dim Q^{J'}g_n = 2\dim Q^J g_n$. Hence $\dim Q^{J'}g_n$ must be odd, which in particular implies that $Q^{J'}g_n$ is not a square, i.e. excess$(Q^J g_n) \neq 0$. Hence $Q^{J'}g_n$ is an indecomposable term. According to our observation on $\ker e'_*$, the term $Q^{k'}(Q^{J'}g_n)$ maps nontrivially to $Q^{k'}(Q^{J'}g_{n+1}) \in H_*QD_{2^r}S^n$ under e'_*. But this gives contradiction to the fact that $j_{2^r}\sigma_*\xi^2 = 0$. This is what we were looking for.

It is also quite straightforward to see that O is an A-annihilated sum of admissible terms. Noitce that $((Q^J g_n))^2$ is A-annihilated, $((Q^J g_n))^2 + O^2$ is supposed to be A-annihilated. Hence O^2, and consequently O must be A-annihilated, where

$$O = \sum Q^{J'}g_n$$

with J' admissible, and $\dim Q^J g_n = \dim Q^{J'}g_n$. This completes the proof of the Lemma. □

We have some comments in order. First, notice that if Conjecture 9, the uniqueness conjecture, holds then the indeterminacy will be trivial, i.e. $O = 0$. Second, Note that $\text{excess}(Q^J g_n) = 0$, does not imply that $Q^K(Q^J g_n)$ is a decomposable. This class is an indecomposable in $H_*QD_{2^r}S^n$ where $r = l(J)$.

Remark 4.7. The above lemma holds in a greater generality. One may replace S^n by $\Sigma^n X$ where X is any space and $n > 0$, and the proof will be analogous. Notice that the key point is that having a suspension ensures that any homology class is primitive. The full general form of this lemma has the following form. Suppose $x \in H_*X$ is primitive with $Q^I x = (Q^J x)^2$ chosen to be A-annihilated, and $\text{excess}(Q^J x) > 0$, and $l(J) = r$, where X is path connected. Then

$$(j_{2^r})_* Q^I x \quad = \quad ((Q^J x))^2 + O^2,$$

where O is an A-annihilated sum of terms $Q^{J'}x$ of lower excess, i.e. $\text{excess}(Q^{J'}x) < \text{excess}(Q^J x)$. Here $(Q^J x) \in H_*D_{2^r}X$ is the image of $Q^J x \in H_*QX$ under the James-Hopf map.

Now we are ready to carry on with the proof of Lemma 13. We recall that our object is to prove that given $f \in \pi_{2n}QS^k$, $k > 0$, with $hf = \xi = \xi_n'^2$ where $\xi_n' \in H_nQS^k$. Then the adjoint mapping $S^{2n+1} \to Q\Sigma S^k$ is detected by the primary Sq^{n+1} on $\sigma_*(\xi_n' + O)$ in its mapping cone, where O refers to the indeterminacy which is a sum of terms of the same weight as ξ_n and of lower excess. The proof is as following.

Proof. Consider the composite

$$S^{2n} \longrightarrow QS^k \xrightarrow{j_{2^r}} QD_{2^r}S^k.$$

According to Lemma 4.5 we have

$$(j_{2^r})_*\xi \quad = \quad ((Q^J g_k))^2 + O^2,$$

where O^2 denotes the indeterminacy of square terms, with O a sum of terms of excess less than $\text{excess}(Q^J g_k)$. The adjoint mapping

$$S^{2n+1} \longrightarrow \Sigma QS^k \xrightarrow{j_{2^r}} Q\Sigma D_{2^r}S^k$$

is detected by Sq^{n+1} on $\Sigma(Q^J g_k) + \sigma_* O \in H_{n+1} Q\Sigma D_{2^r} S^k$. Recall that we have a commutative diagram

$$\begin{array}{ccc} \Sigma Q S^k & \longrightarrow & Q\Sigma D_{2^r} S^k \\ \downarrow & & \downarrow \\ QS^{k+1} & \longrightarrow & QD_{2^r} S^{k+1}. \end{array}$$

In homology, this implies that the class $\Sigma(Q^J g_k) + \sigma_* O$ maps to $(Q^J g_{k+1}) + j_{2^r}\sigma_* O \in H_{n+1} QD_{2^r} S^{k+1}$, i.e. the mapping

$$S^{2n+1} \to QD_{2^r} S^{k+1},$$

is detected by Sq^{n+1} on $(Q^J g_{k+1}) + j_{2^r}\sigma_* O$. Notice that here the indeterminacy, to be detected by this operation, is trivial. Then the naturality implies that the mapping

$$S^{2n+1} \to QS^{k+1}$$

is detected by Sq^{n+1} on $(Q^J g_{k+1}) + \sigma_* O$.

Finally observe that O is sum of terms of the same dimension as $Q^J g_k$ and of lower excess, then after finitely many suspensions it will die, while $Q^J g_k$ will survive. This means that there exists $m > 0$ such that the mapping

$$S^{2n+m} \to QS^{k+m}$$

will be detected by Sq^{n+1} on $Q^J g_{k+m}$. This completes the proof. □

Remark 4.8. Notice that the proof of Lemma 13 heavily depends on the homology of the James-Hopf invariant j_{2^r}. As we pointed out in Remark 4.7 the homology of these maps on square terms could be obtained in a wider range of spaces, i.e. Lemma 4.5 can be stated in a wider generality. As a consequence Lemma 13 also can be stated in a wider generality. More precisely, in Lemma 13 we may assume $\xi \in H_{2m} Q\Sigma^n X$, $n > 0$ is spherical, and result similar to Lemma 13 holds.The proof in general case is analogous to the special case with $X = S^0$.

Now we move on to prove Theorem 21. First, we explain the motivation behind this theorem. Suppose $f \in \pi_* QS^n$ such that $hf = Q^i Q^j g_n + O$, where O denotes other terms of lower excess. Notice that here $* = i + j + n$. After finite number of suspensions, say k times, we will have an element $f_k \in \pi_{2i}QS^{n+k}$ which will map to $(Q^j g_{n+k})^2$. Now $f_k : S^{2i} \to QS^{n+k}$ is a mapping whose image in homology is a square. Lemma 13 then implies that after adjointing another times, $f_{k+1} \in \pi_{2i+1}QS^{n+k+1}$ will be detected by the primary operation Sq^{i+1} on $Q^j g_{n+k+1} + O'$ where O' is a sum of terms of lower excess. The adjoint mapping $f_{k+2} : S^{*+2} \to QS^{n+k+1}$ can be decomposed as

$$e \circ \Sigma f_{k+1} : S^{*+2} \to \Sigma QS^{n+k+1} \to QS^{n+k+2},$$

where e denotes the evaluation map. By natuarlity of the primary operations, the mapping Σf_{k+1} is detected by Sq^{i+1} on $\Sigma(Q^j g_{n+k+1} + O')$. We also have

$$e_*\Sigma(Q^j g_{n+k+1} + O') = Q^j g_{n+k+2} + \sigma_* O'.$$

This equality together with the naturality of the functional operations implies [MT68, Page 157] that f_{k+2} is detected by Sq^{i+1} on $Q^j g_{n+k+2} + \sigma_* O'$. The indeterminacies here will be trivial. We may continue to carry on with the same procedure, i.e. suspending finite number of times, say k' times, where $f_{k+k'} \in \pi_{n+k+k'}QS^{n+k+k'}$ is detected by a primary operation, on the class $g^2_{n+k+k'}$, where this class dies after another suspension. The natural guess could be that this class is detected by a secondary operation. Theorem 21 verifies this claim. Recall that according to Theorem 21, if $f : S^{2n+k-1} \to QS^n$ is detected by Sq^k on g_n^2, not detected by any operation on g_n, with k least such number, then the adjoint mapping will be detected by a secondary operation arising from the Adem relation corresponding to the pair $Sq^k Sq^{n+1}$ with $k < 2(n+1)$. Now we are ready to prove Theorem 21.

Proof of Theorem 21. First, we need to consider the factorisation of f through the Kahn-Priddy map, $\lambda_n : Q\Sigma^n P \to QS^n$, given by

$$S^{2n+k-1} \xrightarrow{f'} Q\Sigma^n P \xrightarrow{\lambda_n} QS^n.$$

The mapping λ_n satisfies the following relations

$$Sq^i_{\lambda_n} g_n = \Sigma^n a_{i-1}, \text{ for all } 2 \leqslant i \leqslant n,$$

and

$$(\lambda_n)_*(\Sigma^n a_i) = Q^i g_n, \text{ for all } i \geqslant n.$$

In particular $\Sigma^n a_n$ maps to g_n^2 under λ_n in homology. Then the naturality argument shows that f' is detected by Sq^k on $\Sigma^n g_n$ modulo indeterminacy which in this case is given by image of

$$f'^* : H^{2n+k-1}Q\Sigma^n P \to H^{2n+k-1}S^{2n+k-1}.$$

Notice that f'^* is nontrivial if and only if

$$f'_* : H_{2n+k-1}S^{2n+k-1} \to H_{2n+k-1}Q\Sigma^n P$$

is nontrivial. We do the proof of the proposition in two different cases: when the image of f'_* is trivial in homology, and when f'_* is not trivial in homology.

Case $f'_* = 0$.

This implies that f' is detected by Sq^k on $\Sigma^n a_n$. Notice that $Sq^k\Sigma^n a_n = \lambda^* Sq^k g_n^2 = 0$ in $H^*Q\Sigma^n P$, hence the functional operation $Sq^k_{f'}$ on $\Sigma^n a_n$ is defined.

The adjoint of f is given by the composite

$$g : S^{2n+k} \xrightarrow{g'} Q\Sigma^{n+1}P \xrightarrow{\lambda_{n+1}} QS^{n+1}.$$

where the mapping g' is detected by $Sq^k_{g'}$ on $\Sigma^{n+1}a_n$, and λ_{n+1}, which is delooping of λ_n, is detected by

$$Sq^i_{\lambda_{n+1}} g_{n+1} = \Sigma^{n+1} a_{i-1} \text{ for all } i \leqslant n+1,$$

and

$$(\lambda_{n+1})_*(\Sigma^{n+1}a_i) = Q^i g_{n+1} \text{ for all } i \geqslant n+1.$$

Notice that $k \leqslant 2n < 2(n+1)$. So we may consider the Adem relation

$$Sq^k Sq^{n+1} = \sum_{j=0}^{[k/2]} \binom{n-j}{k-2j} Sq^{k+n+1-j} Sq^j.$$

We want to show that the secondary relation corresponding to this relation, applied to g_{n+1} detects g. The class $Sq^j g_{n+1}$ is in dimension $n+1+j$. Since $[k/2] \geqslant j$, then

$k - j \geqslant [k/2] \geqslant j$ which implies that $k + n + 1 - j \geqslant n + 1 + j$.

Notice that $k = 2^s$ for some $s > 0$. In this case in the Adem relation above, applied to g_{n+1} all terms $Sq^{k+n+1-j}Sq^j g_{n+1}$ vanish for dimensional reasons with $j \neq [k/2] = k/2$ vanish and the Adem relation above applied to g_{n+1} reduces to

$$Sq^k Sq^{n+1} g_{n+1} = \epsilon Sq^{k/2+n+1} Sq^{k/2} g_{n+1}.$$

Regardless the coefficient ϵ, the right hand side of this term also vanishes. To see this notice that the mapping g is detected by a secondary operation related to this operation if the above Adem relation gives rise to a contradiction in the double mapping cone. More precisely, assuming that g is null homotopic we may consider a coextension of λ_{n+1} to a mapping

$$\lambda'_{n+1} : Q\Sigma^{n+1} P \cup_{g'} e^{2n+k+1} \to QS^{n+1},$$

where the double mapping cone is the cone of λ'_{n+1}. The double mapping cone is given by $QS^{n+1} \cup_{\lambda'_{n+1}} C(Q\Sigma^{n+1} P \cup_{g'} e^{2n+k+1})$. Consider the collapse mapping

$$QS^{n+1} \cup_{\lambda'_{n+1}} C(Q\Sigma^{n+1} P \cup_{g'} e^{2n+k+1}) \to \Sigma(Q\Sigma^{n+1} P \cup_{g'} e^{2n+k+1}).$$

In the right hand side of the Adem relation $Sq^k Sq^{n+1} g_{n+1} = \epsilon Sq^{k/2+n+1} Sq^{k/2} g_{n+1}$ one has $Sq^{k/2} g_{n+1} = \Sigma^{n+2} a_{k/2-1}$. Then the relation $Sq^{k/2+n+1} Sq^{k/2} g_{n+1}$ takes place in $\Sigma(Q\Sigma^{n+1} P \cup_{g'} e^{2n+k+1})$. Hence,

$$\begin{aligned} Sq^{k/2+n+1} Sq^{k/2} g_{n+1} &= Sq^{k/2+n+1} \Sigma^{n+2} a_{k/2-1} \\ &= \Sigma(Sq^{k/2+n+1} \Sigma^{n+1} a_{k/2-1}) = 0, \end{aligned}$$

where the last equation vanishes for dimensional reasons.

This then implies that Adem relation arising from the non-admissible term $Sq^k Sq^{n+1}$, applied to g_{n+1}, detects g. This completes the proof of the proposition in the case of $f'_* = 0$.

Case $f'_* \neq 0$.

First observe that f is detected by Sq^k_f on g_n^2. This implies that $k \leqslant 2n$ and

$$f_* = 0 : H_{2n+k-1} S^{2n+k-1} \to H_{2n+k-1} QS^n.$$

Suppose $f'_* : H_{2n+k-1}S^{2n+k-1} \to H_{2n+k-1}Q\Sigma^n P$ is nontrivial in homology with terms of the form $Q^I\Sigma^n a_i$ in its image, with I admissible, where $\Sigma^n a_i$ is a basis element of $H_*\Sigma^n P$. Notice that I cannot be the empty sequence. If $I = \phi$ then $i+n = 2n+k-1$, i.e. $i = n+k-1 > n$. In this case, $\Sigma^n a_i$ maps to $Q^{n+k-1}g_n \in H_{2n+k-1}QS^n$ under $(\lambda_n)_* : H_{2n+k-1}Q\Sigma^n P \to H_{2n+k-1}QS^n$. This implies that $f_* \neq 0$ which is a contradiction. It is impossible to have I of length more than 1, as in this case the dimension of $Q^I\Sigma^n a_i$ exceeds $4n$, where we know $2n+k-1 < 4n$.

Therefore, $l(I) = 1$ is the only case that f'_* can be nontrivial. In this case we may write

$$hf' = \sum Q^i\Sigma^n a_j,$$

where this is A-annihilated.

A term of the form $Q^i\Sigma^n a_j$ is nontrivial only if $i \geqslant j+n$. If $i = j+n$, then $i+j+n = 2n+k-1$ implies that $2n+k-1 = 2(n+j)$. This implies that $k = 2j+1$, where our assumption on minimality of k shows that $k = 2^s$ for some $s > 0$. So this case cannot happen, i.e. we may write

$$hf' = \sum_{i>j+n} Q^i\Sigma^n a_j,$$

Having $i+j+n = 2n+k-1$ implies that $2j < k-1 < 2n$, and hence $j < n$. Recall that a class of the form $Q^i\Sigma^n a_j$ maps to $Q^iQ^jg_n \in H_*QS^n$. The fact that $j < n$ implies that hf' maps trivially under λ_*.

The sum $\sum_{i>j+n} Q^i\Sigma^n a_j$ is supposed to be A-annihilated. Applying Sq^1_* shows that i cannot be even. Now suppose we have two terms, $Q^{i_1}\Sigma^n a_{j_1}$ and $Q^{i_2}\Sigma^n a_{j_2}$. To start, let a_{j_1} and a_{j_2} both be not-A-annihilated. Assume $j_1 < j_2$, and let t be the least integer such that $Sq^{2^t}_* a_{j_1} \neq 0$. Then we obtain immediately that

$$Sq^{2^{t+1}}_*(Q^{i_1}\Sigma^n a_{j_1} + Q^{i_2}\Sigma^n a_{j_2}) \neq 0.$$

Notice that this holds for $n \geqslant 0$. This implies that when $n > 0$, then hf' can have at most one term of the form $Q^i\Sigma^n a_j$ with a_j not-A-annihilated. As, hf' is supposed to be A-annihilated, hence in the case of existence a term of this we need a term (or terms) of the form $Q^{i_1}\Sigma^n a_{2^q-1}$ which is not A-annihilated, and

$$Q^i\Sigma^n a_j + Q^{i_1}\Sigma^n a_{2^q-1}$$

is A-annihilated. If $j < 2^q-1$, then it is straightforward to see that $Sq_*^{2^{\rho(j)+1}}$ does not kill the above sum. So assume $j > 2^q-1$. In this case, we have $i_1 > i$, and as $Q^{i_1}\Sigma^n a_{2^q-1}$ is not A-annihilated, then $i_1-(n+2^q-1) \geqslant 2^{\rho(i_1)}$. Let $\rho = \min(\rho(i_1), \rho(j)+1)$. Then the above some is not A-annihilated under $Sq_*^{2^\rho}$.
Hence hf' is a sum of A-annihilated terms, i.e. we may write

$$hf' = \sum Q^i\Sigma^n a_{2^q-1},$$

where $0 < i-(n+2^q-1) < 2^{\rho(i)}$, and i is odd. This implies that n is also odd.
We like to show that this latter possibility also cannot happen. We do this by showing that in this case it is possible to show that there is an integer $k' < k$ such that f is detected by $Sq^{k'}$ on a class of weight 2.
Consider the mapping $\lambda_n : Q\Sigma^n P \to QS^n$. Notice that in the cone of this map we have

$$Sq_*^{2^q}(\Sigma^{n+1}a_{2^q-1}) = g_n.$$

We look for an operation, say $Sq^{k'}$, to detect a connection between the cells marked with $Q^i\Sigma^{n+1}a_{2^q-1}$ and $Q^i g_n$ in the cone of $Q\Sigma^n P \to QS^n$. We examine $Sq_*^{2^{q+1}}$ on $Q^i\Sigma^{n+1}a_{2^q-1}$. Then by Nishida relation we have

$$\begin{aligned} Sq_*^{2^{q+1}} Q^i\Sigma^{n+1}a_{2^q-1} &= Q^{i-2^q}Sq_*^{2^q}\Sigma^{n+1}a_{2^q-1} + O \\ &\neq 0, \end{aligned}$$

where O denotes other terms. The only difficulty is that this is a relation in $Q(S^n \cup C\Sigma^n P)$. But we are looking for a relation in $QS^n \cup CQ\Sigma^n P$. Consider the following commutative diagram

$$\begin{array}{ccccc} Q\Sigma^n P & \longrightarrow & QS^n & \longrightarrow & Q(S^n \cup C\Sigma^n P) \\ & & \downarrow & \overset{w}{\nearrow} & \\ & & QS^n \cup CQ\Sigma^n P & & \end{array}$$

But we notice that here the mapping w respects this relation, which implies that

$$\begin{aligned} Sq_*^{2^{q+1}}Q^i\Sigma^{n+1}a_{2^q-1} &= Q^{i-2^q}Sq_*^{2^q}\Sigma^{n+1}a_{2^q-1}+O \\ &= Q^{i-2^q}g_n+O \end{aligned}$$

as a relation in $QS^n\cup CQ\Sigma^nP$, where O denotes the other terms.
Now the naturality of the primary operations implies that the mapping f is detected by $Sq^{2^{q+1}}$ on $Q^{i-2^q}g_n$. Notice that $2j<k-1$, implies that $2^{q+1}=2j\leqslant k$. We only need to resolve the case $2j=2^{q+1}=k$. Notice that in this case $i+n+j=2n+k-1=2n+2j-1$ which implies that $i=n+j-1$. But we have already assumed that $i>n+j$. Hence $2j=2^{q+1}<k$. But this contradicts minimality of k. This completes verifying indeterminacy problem, and hence the proof of the proposition. □

Notice that we already know the elements of ${}_2\pi_*^S$ which are detected by secondary operations, namely the Kervaire invariant classes, and Mahowald's η_i family. According to the construction [M77, Theorem 2], the family η_i does not give rise to a spherical class in $H_*Q_0S^0$. According to Madsen [M70, Theorem 7.3] the Kervaire invariant one elements in dimension $2^{i+1}-2$ give rise to spherical classes $(p'_{2^i-1})^2\in H_{2^{i+1}-2}Q_0S^0$ which die under the homology suspension. Hence we see that, it is not possible to have a spherical class $\xi\in H_*QS^n$, $n>0$, such that it involves a term Q^Ig_n with $l(I)=2$. This proves the Lemma 25. We will discuss this more in the next chapter. Notice that it is possible to handle this case, just based on our knowledge on the type of A-annihilated classes Q^Ig_1 with $l(I)=2$ which was provided by Lemma 4. We have the following example.

Example 4.9. Suppose $f\in\pi_*QS^1$ with nonzero Hurewicz image where its minimum weight is 4, i.e. there exists I with $l(I)=2$ such that Q^Ig_1 is a term of $hf=\sum Q^Ig_1$. According to Lemma 6 the class Q^Ig_1 must be A-annihilated. First of all notice that when $l(I)\leqslant 2$ then it is clear that such I is unique, in the following sense. Lemma 4 implies that there is at most one sequence I of length 2 in each dimension such that Q^Ig_1 A-annihilated, and if there is J of length 2 and Q^Jg_1 is A-annihilated then $\dim Q^Ig_1\neq\dim Q^Jg_1$.

There are three possible cases which we analyse separately.

(1) $Q^I g_1 = g_1^4$. This then implies that $f \in \pi_4 QS^1$. Recall that for $\nu \in \pi_4 QS^1$ we have $h\nu = Q^3 g_1 + g_1^4$. In this case we obtain $h(f+\nu) = Q^3 g_1$. The mapping $\nu + f$ pulls back to $\pi_3 Q_0 S^0$, and hence $h(\nu + f)$ pulls back to $p_3 = x_3 + x_1 x_2 + x_1^3 \in H_3 Q_0 S^0$ as a spherical class. However, this class is not A-annihilated, as we can see that $Sq^1_* p_3 = x_1^2$. This is a contradiction.

(2) $Q^I g_1 = (Q^{2^\alpha - 1} g_1)^2 = Q^{2^\alpha} Q^{2^\alpha - 1} g_1$. Hence $f \in \pi_{2^{\alpha+1}} QS^1$. This implies that the adjoint mapping $S^{2^{\alpha+1}+1} \to QS^2$ is detected by $Sq^{2^{\alpha}+1}$ on $Q^{2^\alpha - 1} g_2$. Observe that $Sq^{2^\alpha + 1} = Sq^1 Sq^{2^\alpha}$. Hence the adjoint mapping is detected by Sq^1 on $Sq^{2^\alpha}(Q^{2^\alpha - 1} g_2)$ which implicitly means that $Sq^{2^\alpha}(Q^{2^\alpha - 1} g_2) \neq 0$ in $H_* QS^2$. One can check that later claim is not true for dimensional reasons. Hence this case also will not arise.

(3) $Q^I g_1 = Q^{2^\alpha + 2^j - 1} Q^{2^\alpha - 1} g_1$ with $1 \leqslant j \leqslant \alpha - 1$. This means that

$$f : S^{2^{\alpha+1} + 2^j - 1} \to QS^1.$$

We observe that after adjointing f, $2^j - 1$ times, we obtain a mapping

$$g : S^{2^{\alpha+1} + 2^{j+1} - 2} \to QS^{2^j},$$

with $hg = (Q^{2^\alpha - 1} g_{2^j})^2$. We claim that

$$g' = j_2 \circ g : S^{2^{\alpha+1} + 2^{j+1} - 2} \to QS^{2^j} \to Q\Sigma^{2^j} P_{2^j},$$

satisfies $hg' = (\Sigma^{2^j} a_{2^\alpha - 1})^2$, i.e. there is no indeterminacy. In this case, and according to Lemma 13, the adjoint mapping $\widetilde{g'} : S^{2^{\alpha+1} + 2^{j+1} - 1} \to Q\Sigma^{2^j + 1} P_{2^j}$ will be detected by $Sq^{2^j + 2^\alpha}$ on $\Sigma^{2^j + 1} a_{2^\alpha - 1}$ in its mapping cone. We want to show that this leads to a contradiction. Notice that $j < \alpha$ which means that, similar to the case (2), we may consider the Adem relation,

$$\begin{aligned} Sq^{2^j} Sq^{2^\alpha} &= \textstyle\sum_t \binom{2^\alpha - 1 - t}{2^j - 2t} Sq^{2^j + 2^\alpha - t} Sq^t \\ &= \textstyle\sum_{t=0}^{j-1} Sq^{2^j + 2^\alpha - 2^t} Sq^{2^t} + Sq^{2^j + 2^\alpha}. \end{aligned}$$

Hence we use,

$$Sq^{2^j + 2^\alpha} = Sq^{2^j} Sq^{2^\alpha} + \textstyle\sum_{t=0}^{j-1} Sq^{2^j + 2^\alpha - 2^t} Sq^{2^t}.$$

Applying both sides of the equation to $\Sigma^{2^j+1}a_{2^\alpha-1}$ we obtain,

$$Sq^{2^j+2^\alpha}\Sigma^{2^j+1}a_{2^\alpha-1} \quad = \quad 0+\sum_{t=0}^{j-1} Sq^{2^j+2^\alpha-2^t}\Sigma^{2^j+1}a_{2^\alpha+2^t-1},$$

where the first term vanishes for dimensional reasons. Notice that for dimensional reasons we have $Sq^{2^j+2^\alpha-2^t}a_{2^\alpha+2^t-1}=0$ in H_*P. Hence the terms of the above sum have some chance to be nontrivial in the mapping cone. Notice that here $1\leqslant t\leqslant j-1$. Assume there exists a t such that $Sq^{2^j+2^\alpha-2^t}\Sigma^{2^j+1}a_{2^\alpha+2^t-1}\neq 0$ in the mapping cone. Hence the stable adjoint of g'

$$g'^S : S^{2^{\alpha+1}+2^{j+1}-1} \nrightarrow \Sigma^{2^j+1}P_{2^j},$$

is detected by $Sq^{2^j+2^\alpha-2^t}\Sigma^{2^j+1}a_{2^\alpha+2^t-1}\neq 0$ in the stable mapping cone. Hence, desuspending 2^{t+1} times, the stable mapping

$$g'^S : S^{2^{\alpha+1}+2^{j+1}-2^{t+1}-1} \nrightarrow \Sigma^{2^j-2^{t+1}+1}P_{2^j},$$

is detected by $Sq^{2^j+2^\alpha-2^t}\Sigma^{2^j-2^{t+1}+1}a_{2^\alpha+2^t-1}\neq 0$ in the stable mapping cone. This means that the stable adjoint of this mapping

$$\widetilde{g'^S} : S^{2^{\alpha+1}+2^{j+1}-2^{t+1}-2} \to Q\Sigma^{2^j-2^{t+1}}P_{2^j},$$

in homology will have $(\Sigma^{2^j-2^{t+1}}a_{2^\alpha+2^t-1})^2$ in its image, which dies after suspending once more. This contradicts the fact that $hg\neq 0$. Hence this case also cannot happen.

Now we return to our claim about the indeterminacy. Notice that according to Lemma 4.2 we have

$$hg' = (\Sigma^{2^j}a_{2^\alpha-1})^2 + O^2,$$

where in this case O^2 is a sum of terms of the form $\Sigma^{2^j}a_q$, of lower excess, and O is A-annihilated. Notice that $\Sigma^{2^j}a_q$ is in the right dimension if and only if $q=2^\alpha-1$, which is not of lower excess. This implies that $O=0$. This completes the proof for this case.

4.1 Complementary Notes

We wish to have a generalised version of Theorem 21, namely Conjecture 26. Suppose there exists a mapping $f : S^m \to QS^k$ which is detected by an operation of order r, say Φ^r, on a class $\xi_n^2 \in H_{2n}QS^k$. The mapping f has a decomposition via the Kahn-Priddy map as

$$\lambda_k f' : S^m \to Q\Sigma^k P \to QS^k.$$

Notice that the mapping $\lambda_k : Q\Sigma^k P \to QS^k$ satisfies the following relations

$$Sq^i_{\lambda_k} g_{n+1} = \Sigma^k a_{i-1}, \text{ for all } i \leqslant k,$$

and

$$(\lambda_k)_*(\Sigma^k a_i) = Q^i g_k, \text{ for all } i \geqslant k.$$

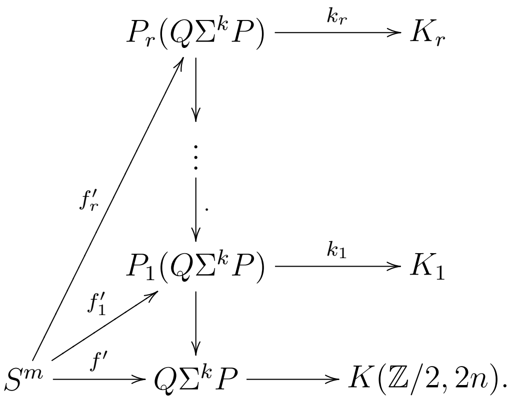
Hence if we have $k > n$, then we have ensured that λ_k is trivial in homology in dimensions $\leqslant n$. This will then satisfies the conditions mentioned in Remark 16. Hence we may deloop, or adjoint the mapping λ_k, where they both will be detected by Sq^{n+1} on $\sigma_*(\xi_n + O)$. This later fact, implies that in term of the Postnikov systems we have

$$\begin{array}{ccccc} & & P_1(QS^{k+1}) & \xrightarrow{Sq^{n+1}} & K(\mathbb{Z}/2, 2n+1) \\ & \overset{f_1}{\nearrow} & \downarrow & & \\ \Sigma Q\Sigma^k P & \xrightarrow{f_0} & QS^{k+1} & \longrightarrow & K(\mathbb{Z}/2, n+1) \end{array}$$

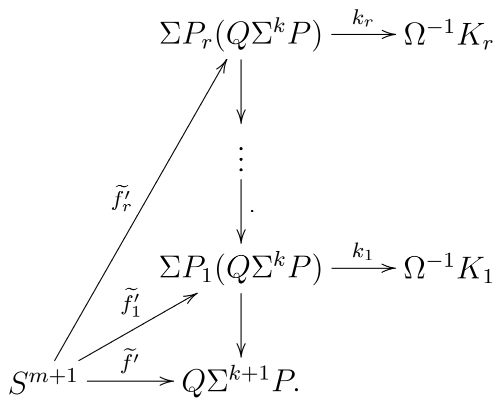
where f_0 is the adjoint of λ_k, and the composite $Sq^{n+1} f_1$ is nontrivial. Here $Sq^{n+1} : P_1(QS^{n+1}) \to K(\mathbb{Z}/2, 2n+1)$ is the class obtained from $Sq^{n+1} : K(\mathbb{Z}/2, n+1) \to K(\mathbb{Z}/2, 2n+2)$ and is the first k-invariant of the above Postnikov system.

On the other hand recall that higher order operations also satisfy naturality property. The naturality implies that the mapping f' is detected by Φ, modulo indeterminacies. Given that the mapping f' is detected by an operation of order r implies that in the Postnikov system for $Q\Sigma^k P$ the mapping f' lifts to $P_r(Q\Sigma^k P)$ and does not lift to

$P_{r+1}(Q\Sigma^k P)$. That is we have a tower of fibrations



In this tower $P_i(Q\Sigma^k P) \to P_{i-1}(Q\Sigma^k P)$ is the fibrations induced by the k-invariant $k_{i-1} : P_{i-1}(Q\Sigma^k P) \to K_{i-1}$. Here K_i is an Eilenberg-MacLane space depending on the *admissible sequence of fibrations* used to define Φ. Now adjointing any of the fibrations, we obtain a tower of maps (not necessarily fibrations)



Here $\Omega^{-1}K_i$ is the obvious delooping of an Eilenberg-MacLane space. On the other hand notice that $f_1 : Q\Sigma^{k+1}P \to P_1(QS^{k+1})$ induces a mapping of towers

$$\Sigma P_i(Q\Sigma^k P) \to P_{i+1}(QS^{k+1}).$$

This still does not complete the proof. We have to say that in the *stable range* the adjointed tower above can be thought of as a tower for $Q\Sigma^{k+1}P$ [MT68, Chapter 18, Proposition 5]. That is in a range of dimensions we may think of $P_i(Q\Sigma^{k+1}P)$ as the same as $\Sigma P_i(Q\Sigma^k P)$. Notice that the approximation in [MT68, Chapter 18, Proposition 5] is stated for an Adams resolution, but it still ought to be true in our setting. Of course we have not proved this, and this can be identified as one of the gaps in argument as well! Indeed the mapping of towers that we have constructed are genuine maps and exist, and at least will induce maps of degree 1 in terms of some unstable Adams spectral sequence.

I believe that one might be able to show that the mapping $S^{m+1} \to \Sigma P_r(Q\Sigma^k P) \to P_{r+1}(QS^{k+1})$ will not be lifted any more, and this would complete the proof that the adjoint mapping $S^{m+1} \to QS^{k+1}$ is detected by an operation of order $r+1$. We refer the reader to [H02, Section 4.3] for more details on the higher order operations.

Finally, recall that according to [L81, Theorem 1.1] the Kahn-Priddy map induces a mapping

$$E_2^{s,t} P \to E_2^{s+1,t+1} S^0,$$

where $E_2^{*,*}$ is the E_2-term of the classical Adams spectral sequence. Lin derives this by constructing a mapping from the r-th stage of the Adams resolution for P to the $(r+1)$-stage of the Adams resolution for S^0. Observe that the mapping of towers that we constructed above is an *unstable* version of the mapping constructed by Lin. This concludes this chapter!

Chapter 5

Spherical Classes in $H_*Q_0S^0$

I would like to dedicate this chapter to some discussion on the Curtis conjecture on the spherical classes in $H_*Q_0S^0$, and its relation to the Eccles conjecture on the spherical classes in H_*QX for an arbitrary path connected space X. I will also present what I believe to be a resolution to the Curtis conjecture verifying Curtis's claim on the spherical classes in $H_*Q_0S^0$. This will require us to recall some well known results on the primitive classes in $H_*Q_0S^0$ which are due to Madsen [M70, Proposition 6.7], [M75, Proposition 5.1].

My intention in this chapter is to record some of calculations that I have done. Perhaps such calculations have been well known for experts. I also will mention some partial results on the type of primitive classes in H_*QP and $H_*Q\mathbb{C}P$. All these will fit together when we consider various transfer maps among suitable spaces.

Let me first mention the two conjectures that we are going to discuss.

The Curtis Conjecture. *A positive dimensional class $\xi\in H_*Q_0S^0$ is spherical, if and only if it is a Hopf invariant one element, or a Kervaire invariant one element.*

Suppose $\xi\in H_*Q_0S^0$ is given with $hf=\xi$ where $f\in\pi_*Q_0S^0$. Moreover, let $f^S\in\pi_*^S$ be the stable adjoint of f. Recall from introduction that a class $\xi\in H_*Q_0S^0$ is a Hopf invariant one element if f^S is detected by the Hopf invariant, i.e. f^S is detected by a primary operation in its stable mapping cone. Similarly, we say ξ is a Kervaire invariant one element if f^S is detected by the Kervaire invariant.

Now suppose that X is a path connected space. Then the Eccles conjecture reads as following.

The Eccles Conjecture. *A positive dimensional class $\xi \in H_*QX$ is spherical, if and only if it is either a stably spherical class, or is a Hopf invariant one element.*

Suppose that $f \in \pi_*QX$ with $hf = \xi$, then ξ is called stably spherical if f^S is detected by homology, i.e. $h^S f \neq 0$. We say ξ is a Hopf invariant one element if $hf = \xi$, then f^S is detected by a primary operation in its mapping cone.

The two conjecture are related in a very interesting way. First we show how assuming truth of the Curtis' conjecture, one can obtain the Eccles' conjecture for $X = S^1$.

Let us assume that Curtis's conjecture holds, and let $\xi \in H_*Q_0S^0$ with $hf = \xi$, i.e. it is spherical. Then the stable mapping f^S is detected either by homology, by Hopf invariant, or by Kervaire invariant. We ignore the cases when ξ is stably spherical, as it is quite straightforward, and focus on the last two cases. Recall that only the classical cases η, ν, σ are the elements of ${}_2\pi_*^S$ which have Hopf invariant one. We already know that the stable adjoint of these classes give rise to spherical classes in $H_{*+1}QS^1$ [E80, Proposition 3.4], namely

$$\begin{aligned} h\eta &= Q^1 g_1; \\ h\nu &= Q^3 g_1 + Q^2Q^1 g_1; \\ h\sigma &= Q^7 g_1 + Q^4Q^3 g_1. \end{aligned}$$

These pull back to unique primitive elements in $H_*Q_0S^0$, where one has

$$\begin{aligned} h\eta &= p_1 \in H_1Q_0S^0; \\ h\nu &= p_3 + Q^2x_1 \in H_3Q_0S^0; \\ h\sigma &= p_7 + Q^6x_1 \in H_7Q_0S^0. \end{aligned}$$

Shortly the reader will see that the classes p_{2n+1} are generators of the submodules of primitives in $H_*Q_0S^0$. According to Madsen [M70, Theorem 7.3], later on proved in various similar and equivalent forms by Eccles [E81, Proposition 4.1] and Snaith-Tornehave [ST82, Theorem A], there is a manifold of Kervaire invariant one if and only if $(p_{2^s-1}+Q^{2^s-2}x_1)^2 \in H_{2^{s+1}-2}Q_0S^0$ is spherical. Such a class is a square, and dies under the homology suspension $H_*Q_0S^0 \to H_{*+1}QS^1$. Thus only the Hopf invariant one elements survive under the homology suspension $\sigma_*: H_*Q_0S^0 \to H_{*+1}QS^1$. Now, if there is a spherical class in $H_{*+1}QS^1$, then it pulls back to a spherical class in $H_*Q_0S^0$. The above explanation then implies that this class must be a Hopf invariant one element, and this in fact proves the Eccles conjecture for H_*QS^n with $n > 0$.

I suspect that if the Curtis conjecture holds, then it is possible to prove the Eccles conjecture for $X = P$. This will be an inductive argument. Notice that the Curtis conjecture implies the Eccles conjecture for QS^n with $n > 0$. Regarding S^1 as P^1, we may consider the (quasi-)fibration sequence

$$QS^n \to QP^n \to QP^{n+1} \to QS^{n+1}.$$

Now the inductive hypothesis tells that QP^n satisfies the Eccles conjecture. This together with our assumption on QS^n should help to prove the Eccles conjecture for QP^{n+1} completing the inductive step. This will prove the Eccles conjecture for $X = P$.

Now suppose that the Eccles conjecture holds, at least for $X = P$. Let $\xi \in H_*Q_0S^0$ be a spherical class, i.e. $\xi = hf$ for some $f \in {}_2\pi_*Q_0S^0$. According to the Kahn-Priddy theorem, there exists $f' \in \pi_*QP$ such that $f = \lambda f'$ where $\lambda : QP \to Q_0S^0$ is the Kahn-Priddy map. Assuming that $hf \neq 0$ implies that $hf' = \xi_P \in H_*QP$ is a spherical class such that $\lambda_*\xi_P = \xi$. According to the Eccles conjecture, if ξ_P is spherical, then f'^S is either detected by homology or by a primary operation in its mapping cone. On the other hand, one may observe that the stable Kahn-Priddy map $P \not\to S^0$ is an extension of $\eta : S^1 \to S^0$, and is detected by any operation Sq^{i+1}, $i > 0$, in its mapping cone [E81, Proposition 4.6]. This fact together with the second Peterson-Stein formula [PS59, Theorem 6.3] implies that $f = \lambda f'$ is either detected by the Hopf invariant, or by the Kervaire invariant. Hence we get the Curtis conjecture as a result.

5.1 Some notes on the work of Ed Curtis

The calculation of the stable homotopy groups may be based on the Adams spectral sequence [A58, Theorem 2.1]. The E_2-term of this spectral sequence is given by

$$E_2^{s,t}(X) = \mathrm{Ext}_A^{s,t}(\mathbb{Z}/2, H^*X),$$

where the Ext is calculated in the category of A-modules, and A is the mod 2 Steenrod algebra. This may be reformulated as

$$E_2^{s,t}(X) = \mathrm{Ext}_{A_*}^{s,t}(\mathbb{Z}/2, H_*X)$$

where this later Ext-term is in the category of A_*-comodules [R86, Chapter 2, Section 2]. This spectral sequence converges to ${}_2\pi_*^S X$.

There is a similar machinery for calculating the unstable homotopy groups, i.e. a spectral sequence which converges to ${}_2\pi_* X$, provided X is a "nice" space, which agrees with the Adams spectral sequence in the stable range. Such spectral sequences are known as *unstable Adams spectral sequences.* For example one may consider a spectral sequence constructed by Massey and Peterson [MP67, Theorem 23.1], or its modified version due to Bousfield and Curtis [BC70, Theorem 2.1]. By analogy the E_2-term of this spectral sequence is given by

$${}^{\text{unstable}}E_2^{s,t}(X) = \mathrm{Ext}_{\mathcal{U}}^{s,t}(\mathbb{Z}/2, H_*X),$$

where this Ext-term is in the category of unstable comodules over the Steenrod algebra, denoted by $\mathcal{U}$. However, in our case H_*QS^n is too huge and it does not seem to be practical to simply put this homology in the above formula and calculate the Ext-term and then prove some collapse! (which in fact does not collapse!) One instead needs some reductions in order to be able to do some calculations. According to [BC70, Theorem 5.2] if one can find a module over the Steenrod algebra, say M such that

$$H_*X \simeq U(M)$$

as unstable comodules over the Steenrod algebra, then one has

$${}^{\text{unstable}}E_2^{s,t}(X) = \mathrm{Ext}_A^{s,t}(\mathbb{Z}/2, M).$$

The functor U above is the free unstable A-comodule functor, and the Ext-term is in the category of A-modules. Hence if one can succeed to find M such that $H_*Q_0S^0 \simeq U(M)$, then it is possible to use the above machinery. Curtis [C75, Proposition 3.1] constructs an A-module M, and claims that it has the property that $H_*Q_0S^0 \simeq U(M)$. However, according to [W82, Proposition 2.5] this statement does not hold, i.e. the ${}^{\text{unstable}}E_2$-page is not identified correctly.
In this approach one then looks at the 0-line which filters the Hurewicz homomorphism. Those classes which survive to ${}^{\text{unstable}}E_\infty$-page will be the only candidates to be spherical. So the permanent cycles of the 0-line in this unstable spectral sequence are the only candidates to be spherical. This was the approach taken by Curtis to identify spherical classes in $H_*Q_0S^0$.

Curtis [C75, Page 235] has a more general approach to this by constructing A-modules $M(\Omega^n S^{n+k})$ and Wellington's argument only identifies mistakes for the case $k = 0$. However, it is not clear to me whether if

$$H_*QS^k \simeq U(M(\Omega^\infty S^{\infty+k})),$$

where $k \neq 0$ as A-comodules. Definition of $M(\Omega^\infty S^{\infty+k})$ [C75, Last pargraph on page 235] suggests that the free R-module generated by $M(\Omega^\infty S^{\infty+k})$ is isomorphic to H_*QS^k as A-modules. It also seems correct that the map commutes with the action of Sq^r_* and this suggests that the above isomorphism stands some chance to hold in the category of A-comodules. But it is not clear at all that if this object, the free R-module generated $M(\Omega^\infty S^{\infty+k})$, is the same as the free A-comodule generated by $M(\Omega^\infty S^{\infty+k})$. Wellington also states that the argument for the case $k \neq 0$ also is not convincing, of course without presenting a proof! I believe that a proof of this claim or disprove of it needs to be written down by showing that there is, or there is not, a map inducing such an isomorphism. We urge the reader to not confuse this with Wellington [W82, Page 163]. The homology algebra $H_*\Omega^n S^{n+k}$, when $n < \infty$ is a truncated polynomial algebra, and is seems a correct to claim that for the cases $n < \infty$ the algebra $H_*\Omega^n S^{n+k}$ does not have a free A-comodule structure, and for this reason the spectral sequence does not collapse.

We recall that earlier in Note 3, we mentioned another failure in Curtis's argument, which again seemed to fail only for the case $k = 0$ in even dimensions, and the odd dimensional cases seem to be correct, as our Lemma 6 suggests which is the statement of Theorem 5.23.

Another fact about Curtis's claim is a collapse claim of the spectral sequence, and that most potential classes identified by use of our Theorem 2 and Note 3 do not give rise to permanent cycles in the spectral sequence [C75, Proposition 6.5]. We already had a discussion on this earlier in Note 3, where our Theorem 17, and Lemma 19 seem to eliminate the cases of [C75, Proposition 6.5(3)]. We refer the reader to Note 20 for more discussion on this.

We recall that the Adams spectral sequence is the spectral sequence associated with a tower of fibrations of spectra known as the Adams resolution [R86]. In a similar fashion the unstable Adams spectral sequence is the spectral sequence associated with a tower of fibrations of spaces, expressed in terms of the Postnikov systems [MP67].

Finally, it seems that most ambiguity about this comes from the confusion about, and complexity of working with $H_*Q_0S^0$, while one does not have such a problem while working with H_*QX where X is a path connected space.

5.2 Primitives in H_*QX, and $H_*\Omega_0 QX$

We start this section by recalling some basic facts about Hopf algebras of finite type. These are to be found in Milnor-Moore's paper [MM65].

Let $H = \oplus_i H_i$ be a Hopf algebra of finite type over $k = \mathbb{Z}/2$. Here by finite type we mean H_i is finitely generated for each i. Moreover, we assume that $H_i = 0$ for $i < 0$. We say H is *connected* if the unit map $k \to H_0$ is an isomorphism. We say that H is bicommutative if it is both commutative and cocommutative. We note that it is possible to see cocommutativity of H as commutativity condition for H^* where $H^* = \mathrm{Hom}_k(H, k)$. Let PH and QH denote the submodule of primitive elements, and the quotient module of indecomposable elements in H, respectively. For commutative Hopf algebras we may relate the functors P, Q via the Frobenius homomorphism $s_H : H \to H$ given by $s_H(h) = h^2$. Notice that $s_H(hg) = s_H(h)s_H(g)$, and $s_H(h+g) = s_H(h) + s_H(g)$. Let us use $k(S)$ to denote the submodule of H generated by S where $S \subseteq H$. Notice that if H is a Hopf algebra of finite type, then H^* is also a Hopf algebra and has a Frobenius homomorphism, $s_{H^*} : H^* \to H^*$. Let $sH^* = \mathrm{im}(s_{H^*} : H^* \to H^*)$, and let $k(rH) = (k(sH^*))^*$. Then we have the following result [MM65, Propositon 4.23].

Proposition 5.1. *Suppose H is a connected bicommutative Hopf algebra of finite type over $k = \mathbb{Z}/2$. Then there is an exact sequence of the following form*

$$0 \to Pk(sH) \to PH \to QH \to Qk(rH) \to 0.$$

For Hopf algebras of finite type we may define $r_H : H \to H$ to be dual of $s_{H^*} : H^* \to H^*$. The homomorphism $r = r_H$ behaves like the *square root* homomorphism, where it maps h^2 to h and acts trivially otherwise. Bearing in mind that we are working in the graded world we have to point out that s_H maps H_n into H_{2n}. Hence $k(s_H H) = 0$ in odd degrees. Similar argument together with duality shows that $k(r_H H) = 0$ in odd degrees. Hence by the above exact sequence we have $PH \simeq QH$ in odd degrees.

This proposition is the main tool in calculating the primitive elements in certain Hopf algebras arising as homology of loop spaces. We note that in these cases the homology algebras are over $\mathbb{Z}/2$, and all have finite type. Moreover, notice that in the case of infinite loop spaces the Pontrjagin product is commutative, which means that homology algebra of such spaces are bicommutative associative Hopf algerbas. This implies that these algebras satisfy Borel's structure theorem [W78, Thm.8.11]. According to Borel's theorem such algebras are tensor product of certain algebras, which are either polynomial over one generator, or truncated polynomial over one generator.

Now assume that X is a path connected space, with an additive homogeneous basis $\{x_\alpha\}$ for $\overline{H}_*X$. Recall that the homology ring H_*QX is given as

$$H_*QX \simeq \mathbb{Z}/2[Q^I x_\alpha : \mathrm{excess}(Q^I x_\alpha) > 0, I \text{ admissible}].$$

Notice that given any space, one may define the Frobenius homomorphism $s : H^*X \to H^*X$ as before, i.e. $s(x) = x^2$. One then has the following [G04, Lemma 7.2] .

Lemma 5.2. *The cohomology algebra H^*Q_0X is a polynomial algebra if $s : H^*X \to H^*X$ is injective. Here Q_0X denotes the base point component of QX.*

Remark 5.3. We have a list of interesting spaces satisfying the above conditions, such as $X = S^0, P, P_+, \mathbb{C}P, \mathbb{C}P_+$. In general any space with polynomial cohomology will satisfy the above lemma. Such information will be very useful when one wants to calculate the homology ring $H_*\Omega_0 QX$, where $\Omega_0 QX$ denotes the base point component of ΩQX.

One of the main tools in calculating the homology of loop spaces is the Eilenberg-Moore spectral sequence. We recall the following [G04, Proposition 7.3].

Proposition 5.4. *Let X be simply connected, with H^*X polynomial. Then $H_*\Omega X$ is an exterior algebra, and the suspension*

$$\sigma_* : QH_*\Omega X \to PH_*X$$

is an isomorphism, and the Eilenberg-Moore spectral sequence

$$E^2 = \mathrm{Cotor}^{H_*X}(\mathbb{Z}/2, \mathbb{Z}/2) \Rightarrow H_*\Omega X$$

collapses. In particular,

$$H_*\Omega X \simeq E_{\mathbb{Z}/2}(\sigma_*^{-1} PH_*X),$$

where $E_{\mathbb{Z}/2}(\sigma_*^{-1}PH_*X)$ denotes the exterior algebra over $\mathbb{Z}/2$ generated by $\sigma_*^{-1}PH_*X$

This theorem provides the main tool to calculate the homology rings $H_*Q_0S^{-1}$ and $H_*\Omega_0 QP$, where one chooses $X = \overline{Q_0S^0}, \overline{QP}$. Here $\overline{Y}$ denotes the universal cover of a given space Y. We refer the reader to [G04] for the proof of the machinery provided above. We recall the calculation of $H_*Q_0S^{-1}$.

Example 5.5. First, notice that the squaring map $H^*S^0 \to H^*S^0$ is injective. This implies that $H^*Q_0S^0$ is polynomial. Recall from Appendix D that $Q_0S^0 = P \times \overline{Q_0S^0}$. Hence $H^*\overline{Q_0S^0}$ is polynomial as well. On the other hand notice that $QS^{-1} = \Omega Q_0 S^0$, which implies that $Q_0S^{-1} = \Omega\overline{Q_0S^0}$. Now putting $X = \overline{Q_0S^0}$ in Proposition 5.4 implies that $H_*Q_0S^{-1}$ is an exterior algebra, with $\sigma_* : QH_*Q_0S^{-1} \to PH_*Q_0S^0$ an isomorphism, i.e.

$$H_*Q_0S^{-1} = E_{\mathbb{Z}/2}(\sigma_*^{-1}PH_*Q_0S^0).$$

This is due to Cohen-Peterson [CP89, Theorem 1.1].

Remark 5.6. One may feel a bit uneasy about applying Lemma 5.2 to $X = S^0$, as it is not path connected. However, there is another way to see that $H^*Q_0S^0$ is polynomial. First notice that we can apply Lemma 5.2 to $X = P$. On the other hand, notice that the Kahn-Priddy map $\lambda : QP \to Q_0S^0$ induces an epimorphism in homology [KP78, Theorem 3.1]. This implies that $\lambda^* : H^*Q_0S^0 \to H^*QP$ is a monomorphism. Notice that λ^* respects the multiplication. This implies that squaring map of $H^*Q_0S^0$ is injective. Notice that $H^*Q_0S^0$ is a bicommutative Hopf algebra, and according to the Borel's structure theorem [W78, Thm.8.11] it is a tensor product of polynomial algebras, or truncations of polynomial algebras. It is not possible to have truncation in this case as it contradicts injectivity of the squaring map on $H^*Q_0S^0$. This shows that $H^*Q_0S^0$ is a polynomial algebra.

Example 5.7. Let $X = P, \mathbb{C}P$. Then the Frobenius homomorphisms $H^*X \to H^*X$ is injective. Hence one obtains the following isomorphisms,

$$\begin{aligned} H_*Q_0\Sigma^{-1}P &\simeq E_{\mathbb{Z}/2}(\sigma_*^{-1}PH_*QP) \\ H_*Q_0\Sigma^{-1}\mathbb{C}P &\simeq E_{\mathbb{Z}/2}(\sigma_*^{-1}PH_*Q\mathbb{C}P). \end{aligned}$$

We will provide the reader with three different descriptions of generating sets for the primitive submodules in $H_*Q_0S^0$, H_*QP and $H_*Q\mathbb{C}P$. Each of these descriptions will give a presentation of the above homology algebras, some easier to work with, where other ones are ideal and a bit difficult to understand very clearly.

Remark 5.8. The coproduct of the Dyer-Lashof algebra $R \to R \otimes R$ sends the operation Q^i to $\sum_j Q^{i-j} \otimes Q^j$. The coproduct of $Q^ix \in H_*QX$ is given by

$$Q^ix \mapsto \sum(Q^{i-j} \otimes Q^j)(\sum x' \otimes x'') = \sum Q^{i-j}x' \otimes Q^jx'',$$

where $(\Delta_X)_*x = \sum x' \otimes x''$, and $\Delta_X : X \to X \times X$ is the diagonal map. This implies that if x is primitive, then Q^ix is primitive, and vice versa. A very quick outcome of this is that in $H_*Q\Sigma X$ every class $Q^I\Sigma x$ is primitive. The is immediate as the suspension kills the cup product. Therefore, we restrict our attention to cases where X is not a suspension such as $X = S^0, P, \mathbb{C}P$, and try to calculate the primitive classes in H_*Q_0X.

5.3 Homology of Q_0S^0

Recall that $\pi_0QS^0 \simeq \mathbb{Z}$. Given $n : S^0 \to QS^0$ we let $[n] \in H_0Q_nS^0$ be the image of image of $1 \in \overline{H}_0S^0$, the generator of the non-base-point component in H_0QS^0 under the Hurewicz map $\pi_0QS^0 \to H_0QS^0$. Notice that in dimension 0 both π_0QS^0 and H_0QS^0 are groups under the loop sum and the Hurewicz map respects this operation. One then has that $[n] * [m] = [n+m]$. Notice that $n : S^0 \to QS^0$ extends to an infinite loop map $n : QS^0 \to QS^0$, providing necessary maps for homotopy equivalence between different path components of QS^0. The homology ring $H_*Q_0S^0$ is given by [CLM76, Part I, Lemma 4.10]

$$H_*Q_0S^0 \simeq \mathbb{Z}/2[Q^Ix_i : \mathrm{excess}(Q^Ix_i) > 0, (I,i) \text{ admissible}],$$

where $x_i = Q^i[1] * [-2]$. Here $*$ denotes the loop sum in H_*QS^0. Notice that x_i is an *indecomposable* in $H_*Q_0S^0$, where it is a *decomposable* in H_*QS^0. The class x_i maps to $Q^ig_1 \in H_{i+1}QS^1$ under the homology suspension $\sigma_* : H_*Q_0S^0 \to H_{*+1}QS^1$ [CLM76, Page 47, first line]. Hence according to the Kudo transgression theorem [CLM76, Part I, Theorem 1.1 (7)] $\sigma_*Q^Ix_i = Q^IQ^ig_1$. This implies that Q^Ix_i belongs to $\ker\sigma_*$ if and only if $\mathrm{excess}(Q^Ix_i) = 0$.

Notice that different components of QS^0 have the same homotopy type, and hence the same homology. This means that the translation by $*[j] : Q_0S^0 \to Q_{i+j}S^0$ gives the homology ring for $H_*Q_jS^0$ for $j \in \mathbb{Z}$. Notice that $[j] = [1]^{*j}$ for all $j > 0$ where $[1]^{*j}$ denotes j-fold $*$-product of $[1]$ with itself. Similarly, $[j] = [-1]^{*(-j)}$ for $j < 0$. This implies that [S09, Page 33]

$$H_*QS^0 \simeq H_*Q_0S^0[[1],[-1]],$$

where the right hand side denotes the ring of Laurent polynomials in [1] and [−1] with coefficients in $H_*Q_0S^0$.

Now assume that Q^ix_j is given such that (i,j) is not admissible. Then we have

$$\begin{aligned} Q^ix_j &= Q^i(Q^j[1]*[-2]) \\ &= Q^iQ^j[1]*[-4]+\textstyle\sum_{k>0}Q^{i-k}Q^j[1]*Q^k[-2] \end{aligned} \tag{5.1}$$

Let us first deal with the second sum. All terms in this sum will give rise to decomposable terms. To see this notice that calculating these terms reduces to computing $Q^k[-2]$, where one has

$$Q^k[-2]=\begin{cases} (Q^n[-1])^2 & \text{if } k=2n, \\ 0 & \text{if } k=2n+1. \end{cases}$$

We use a result of Priddy [P75, Lemma 2.1] computing $Q^n[-1]$.

Lemma 5.9. *For $n\geqslant 0$,*

$$Q^n[-1] \;=\; \textstyle\sum(\lambda_1,\ldots,\lambda_n)(Q^1[1])^{\lambda_1}*\cdots*(Q^n[1])^{\lambda_n}*[-2\lambda-2],$$

where the summation is taken over all sequences $(\lambda_1,\cdots,\lambda_n)$ such that $\sum i\lambda_i=n$ and $\lambda=\sum\lambda_i$. The coefficients are given by

$$(\lambda_1,\ldots,\lambda_n)=\frac{\lambda!}{\lambda_1!\cdots\lambda_n!}.$$

Writing the above sum in terms of x_i's we obtain

$$Q^n[-1] \;=\; \textstyle\sum(\lambda_1,\ldots,\lambda_n)x_1^{\lambda_1}\cdots x_n^{\lambda_n}*[-2].$$

Hence,

$$Q^{2n}[-2] \;=\; \textstyle\sum(\lambda_1,\ldots,\lambda_n)x_1^{2\lambda_1}\cdots x_n^{2\lambda_n}*[-4].$$

This means that the second sum in (5.1), any single term with $k=2n$ has the form

$$Q^{i-k}Q^j[1]*Q^k[-2] \;=\; \textstyle\sum(\lambda_1,\ldots,\lambda_n)Q^{i-k}Q^j[1]*[-4]*x_1^{2\lambda_1}\cdots x_n^{2\lambda_n},$$

where one can calculate $Q^{i-k}Q^j[1]*[-4]$ in terms of Q^sx_t for some s,t. Notice that $Q^{i-k}Q^j[1]*Q^k[-2]=0$ if k is odd. Hence all the terms in the second part are decomposable.

The first summand in (5.1), namely $Q^iQ^j[1]*[-4]$ may or may not give an indecomposable term, which will depend on the Adem relation for Q^iQ^j. More precisely, if the Adem relation for $Q^iQ^j = \sum Q^aQ^b \neq 0$, then modulo decomposable terms we have the following equality

$$\begin{aligned} Q^iQ^j[1]*[-4] &= \sum Q^aQ^b[1]*[-4] \\ &= \sum Q^a x_b. \end{aligned}$$

And if $Q^iQ^j = 0$, then clearly we are left just with the second summand in (5.1). This completes the proof of the following observation.

Lemma 5.10. *Let $Q^ix_j \in H_*Q_0S^0$ be given with $i > 2j$ together with the Adem relation $Q^iQ^j = \sum Q^aQ^b$. Then modulo decomposable terms, we have*

$$Q^ix_j = \sum Q^a x_b.$$

Moreover, if $Q^iQ^j = 0$, then Q^ix_j will be a sum of decomposable terms.

For instance, one may calculate that modulo decomposable terms

$$Q^6x_2 = Q^5x_3.$$

We like to conclude this section by an observation on the form of indecomposable A-annihilated primitive classes $\xi \in H_*Q_0S^0$. But first we recall some facts about the square root map $r : H_*Q_0S^0 \to H_*Q_0S^0$. Using the Nishida relations, one can verify that

$$rQ^{2n} = Q^n r, \tag{5.2}$$

$$rQ^{2n+1} = 0. \tag{5.3}$$

Recall that in homology the Kahn-Priddy $\lambda : QP \to Q_0S^0$ is given by $\lambda_*a_i = x_i$ [KP78, Theorem 3.1]. Combining this with the action of square root map $r_P : H_*P \to H_*P$ we obtain

$$\begin{aligned} rx_{2i} &= x_i, \\ rx_{2i+1} &= 0. \end{aligned}$$

These observations completely determine the action of the square root map $r : H_*Q_0S^0 \to H_*Q_0S^0$. We will use this in the proof of the next lemma.

Lemma 5.11. *Suppose $\xi_0 \in H_*Q_0S^0$ is an A-annihilated primitive class with $\sigma_*\xi_0 \neq 0$. Then*

$$\xi_0 = \sum Q^I x_{2i+1}$$

modulo decomposable terms, where $(I, 2i+1)$ runs over certain admissible sequences of positive excess.

Proof. The fact that $\sigma_*\xi_0 \neq 0$ implies that modulo decomposable terms

$$\xi_0 = \sum Q^I x_n$$

where (I, n) is admissible with $\text{excess}(Q^I x_n) > 0$. The fact that ξ_0 is an indecomposable primitive implies that indecomposable part of ξ_0 belongs to the kernel of the square root map $r : H_*Q_0S^0 \to H_*Q_0S^0$. Notice that if we have two distinct admissible sequences (J, j) and (K, k) with only even entries, then $rQ^J x_j \neq rQ^K x_k$. Hence the decomposable part of ξ_0 belongs to the kernel of r if and only if every $Q^I x_n$ belong to the kernel. We show that assuming $n \neq 2i+1$ leads to a contradiction.

Assume that n is even. Since $Q^I x_n$ belong to $\ker r$, then I must have at least one odd entry. Let $s_0 = \max(s : 1 \leqslant s \leqslant r, i_s \text{ is odd})$. Then i_{s_0+1} is even. Notice that if $s_0 = r$, then we have x_n with n even. In this case one applies $Sq_*^{2^{s_0}}$ to ξ_0. According to our explanations in Remark 3.22 all terms of the form $Q^I x_n$ with $\text{excess}(Q^I x_n) > 0$ are separated under the action of this operation from each other. Moreover, notice that

$$\text{excess}(Sq_*^{2^{s_0}} Q^{I_0} x_n) = \text{excess}(Q^{I_0} x_n) > 0,$$

which implies that the outcome is not a decomposable, and hence is separated from any other decomposable term. This implies that $Sq_*^{2^{s_0}}\xi_0 \neq 0$ which contradicts the fact that ξ_0 must be A-annihilated. Hence n must be odd. This implies that modulo decomposable terms

$$\xi = \sum Q^I x_{2i+1},$$

with $(I, 2i+1)$ admissible. □

5.4 Primitive Classes in $H_*Q_0S^0$ and applications

We are interested in primitive classes of $H_*Q_0S^0$ for three different reasons. First, a simple description allows us to prove Theorem 17 for the case $n = 1$. Second, recall from Proposition 5.4 and Example 5.7 that our presentation of $H_*Q_0S^{-1}$ depends on our description of $PH_*Q_0S^0$. This means that there is a matter of choice in presenting $H_*Q_0S^{-1}$. There is a tension between choosing the nicest description, and choosing a workable description. We discuss three slightly different generating sets for $PH_*Q_0S^0$ which yields three different, but related, presentations of $H_*Q_0S^{-1}$. Finally, reviewing the description of $PH_*Q_0S^0$ will make it easy to understand the analogous computations of $PH_*Q\mathbb{C}P$ and PH_*QP. Notice that these later results effectively calculate the homology rings $H_*Q\Sigma^{-1}\mathbb{C}P$ and $H_*Q\Sigma^{-1}P$.
The useful and workable description for us is due to Madsen [M70, Proposition 6.7]. This will be the second set of generators that we will describe.

First Description. Let $a_i \in H_iP$ be the generator, dual to a^i where $H^*P \simeq \mathbb{Z}/2[a]$. The class a_i maps to x_i under $\lambda_* : H_iQP \to H_iQ_0S^0$, where λ is the Kahn-Priddy map [KP78, Theorem 3.1]. This makes it easy to see that x_i almost behaves like $a_i \in H_iP$. One has that

$$\Delta_* x_i = \sum_j x_{i-j} \otimes x_j,$$

where Δ_* is the coproduct homomorphism. The action of the Steenrod algebra on x_i is also similar to its action on a_i and is given by

$$Sq^k_* x_i = \binom{i-k}{k} x_{i-k}.$$

The fact that x_i is indecomposable in $H_*Q_0S^0$ implies that in odd degrees there exists a unique primitive in $H_*Q_0S^0$ corresponding to this indecomposable term, i.e.

$$p_{2n+1} = x_{2n+1} + D_{2n+1}$$

where D_{2n+1} denotes the decomposable terms. The submodule of primitives in $H_*Q_0S^0$ is spanned by terms of the form $Q^I p_{2n+1}$. The proof of this, explained below, is by induction on the length of I. Notice that here we may take I to be admissible but we don't require $(I, 2n+1)$ to be admissible.

This is straightforward once we use the Milnor-Moore exact sequence. Notice that any primitive class $p \in H_*Q_0S^0$ can be either a square of a primitive, or has an indecomposable term $Q^I x_i$ which belongs to the kernel of the square root map $r : H_*Q_0S^0 \to H_*Q_0S^0$. According to relations (5.2) and (5.3) an indecomposable class of the form $Q^I x_i$ belongs to $\ker r$ only if I has at least one odd entry, or i is odd.

Recall that Q^aQ^b is admissible if $a \leqslant 2b$. If $a > 2b$, then it can be written a sum of admissible sequences using the Adem relations given by

$$Q^aQ^b = \sum_{a+b \leqslant 3t} \binom{t-b-1}{2t-a} Q^{a+b-t}Q^t.$$

If we have an admissible pair $Q^{i_1}Q^{i_2}$ with i_1 odd and i_2 even, then we have the Adem relation

$$Q^{2i_2}Q^{i_1-i_2} = Q^{i_1}Q^{i_2} + O', \tag{5.4}$$

where O' is a sum of admissible terms of lower excess.

Example 5.12. We may calculate that

$$Q^{32}Q^5 \quad = \quad Q^{21}Q^{16} + Q^{20}Q^{17} + Q^{19}Q^{18},$$

where the sum of terms of lower excess is given by $Q^{20}Q^{17} + Q^{19}Q^{18}$. We then obtain

$$Q^{21}Q^{16} \quad = \quad Q^{32}Q^5 + Q^{20}Q^{17} + Q^{19}Q^{18}.$$

Notice that on the right hand side we have $Q^{19}Q^{18}$. In this case we may apply the above Adem relation to obtain $Q^{19}Q^{18} = Q^{36}Q^1$. Hence we obtain

$$Q^{21}Q^{16} \quad = \quad Q^{32}Q^5 + Q^{20}Q^{17} + Q^{36}Q^1.$$

According to Lemma 5.10 this implies that

$$Q^{21}x_{16} \quad = \quad Q^{32}x_5 + Q^{20}x_{17} + Q^{36}x_1.$$

modulo decomposable terms. The fact that we obtain admissible terms O' of lower excess implies that the above process ends after a finite number of steps. This is explained below.

Remark 5.13. Let $Q^{i_1}Q^{i_2}Q^{i_3}$ be admissible with i_1 the only odd entry of (i_1, i_2, i_3). We may apply the above Adem relation to obtain

$$\begin{aligned} Q^{i_1}Q^{i_2}Q^{i_3} &= (Q^{2i_2}Q^{i_1-i_2} + O)Q^{i_3} \\ &= Q^{2i_2}Q^{i_1-i_2}Q^{i_3} + O', \end{aligned}$$

where O and O' are admissible terms of lower excess. The term $Q^{i_1-i_2}Q^{i_3}$ is admissible with $i_1 - i_2$ odd, and i_3 even. Hence we may apply the above Adem relation again to obtain

$$Q^{i_1}Q^{i_2}Q^{i_3} = Q^{2i_2}Q^{2i_3}Q^{i_1-i_2-i_3} + O'',$$

Notice that $Q^{2i_2}Q^{2i_3}$ is admissible. This implies that we may make iterated use of the above Adem relation. We observe that if $I_r = (i_1, i_2, \ldots, i_r)$ is an admissible sequence with i_1 the only odd entry, then modulo admissible terms of lower excess we have

$$Q^{I_r} = Q^{2i_2}Q^{2i_3}\cdots Q^{2i_r}Q^d,$$

where $d = i_1 - (i_2 + \cdots + i_r)$. Notice that $Q^{2i_2}Q^{2i_3}\cdots Q^{2i_r}$ is admissible. Finally observe that because the difference includes only terms of lower excess, this means this process ends after a finite number of steps.

The above explanations imply if Q^IQ^i is an admissible term with i even and I with one odd entry, then we may write it as a sum of terms Q^JQ^j with j odd. Hence a term which belongs to the kernel of the square map, can always be written in terms of Q^Ix_{2n+1} or more precisely in terms of Q^Ip_{2n+1}. This proves our claim that any primitive class in $H_*Q_0S^0$ is a sum of certain terms of the form Q^Ip_{2n+1} with I admissible.

Note 5.14. It is possible to give an explicit description of the decomposable part of p_{2n+1}, namely D_{2n+1}. Notice that given p_{2n+1} we obtain another class by squaring, namely p_{2n+1}^2. Hence, one may put $p_{2n} = p_n^2$ [W82, Lemma 5.2]. Then D_{2n+1} is determined using the Newton polynomials [W82, Definition 5.1] as

$$D_{2n+1} = \sum_{i=1}^{2n} x_i p_{2n+1-i}.$$

The above description will be useful, when we describe the primitives in other homology rings such as $H_*Q\mathbb{C}P$, and $H_*Q_0S^{-1}$.

Notice that it could happen to have a term $Q^I p_{2n+1}$ with I admissible and $(I, 2n+1)$ not admissible. However, if $(I, 2n+1)$ is an admissible sequence, then the fact that $p_{2n+1} = x_{2n+1}$ modulo decomposable terms, allows one to have the following.

Lemma 5.15. *If $(I, 2n+1)$ is admissible, then modulo decomposable terms*

$$Q^I p_{2n+1} = Q^I x_{2n+1}.$$

Due to this, we may write excess$(Q^I p_{2i+1})$ to denote excess$(Q^I x_{2i+1})$. Hence if excess$(Q^I p_{2i+1}) > 0$ it refers to a primitive class with an indecomposable term, where excess$(Q^I p_{2i+1}) = 0$ means that $Q^I p_{2i+1}$ is a decomposable primitive, and so it is a square. We then the following result.

Corollary 5.16. *Let $\xi_0 \in H_*Q_0S^0$ be A-annihilated primitive class with $\sigma_*\xi_0 \neq 0$. Then*

$$\xi_0 = \sum Q^I p_{2i+1}$$

with $(I, 2i+1)$ admissible modulo decomposable terms. If ξ_0 is odd dimensional, then the decomposable part is trivial. If ξ_0 is even dimensional, then the decomposable part is either trivial or square of a primitive.

Proof. Notice that $\xi_0 = \sum Q^I x_{2i+1}$ modulo decomposable terms. Previous lemma allows us to replace $Q^I x_{2i+1}$ with $Q^I p_{2i+1}$ modulo decomposable terms. Therefore $\xi_0 = \sum Q^I p_{2i+1}$ modulo decomposable terms. However this decomposable part is primitive, hence it must be square. If ξ_0 is an odd dimensional class, then the decomposable part is trivial. If ξ_0 is even dimensional then it is either square or trivial. This completes the proof. □

This later observation gives more information on the type of admissible sequences I in the expression for an A-annihilated primitive class. Moreover, recall from Corollary 6 that we really need to focus on sequences I with $l(I) > 1$. We have the following lemma.

Lemma 5.17. *Suppose $\xi_0 = \sum Q^I p_{2i+1}$ is an odd dimensional A-annihilated primitive class, then each entry of I must be odd.*

Proof. The proof of this is quite analogous to the proof of Corollary 3.16. For example applying Sq^1_* shows that i_1 cannot be even. Notice that in odd dimensional we don't have any decomposable, so terms $Q^I p_{2i+1}$ with $\mathrm{excess}(Q^I p_{2i+1}) = 1$ will not be annihilated by any other term under the action of Sq^1_*. Similar technique applies to the other terms. Moreover, during this process we don't get to the stage of applying an operation on p_{2i+1}'s, i.e. we are not worried about these classes. This completes the proof. □

Notice that we know what happens when there exists a term $Q^I p_{2i+1}$ with $l(I) = 0$ as it corresponds to a Hopf invariant one element. On the other hand this proof will not work, as in this case we directly deal with terms x_{2i+1}, $Q^{2i}x_1$ and the decomposable terms, where when $l(I) > 0$ we do not touch these terms we only play with the operations Q^I.

Remark 5.18. It is still possible to use Sq^1_* in even dimensions to show that in any term $Q^I p_{2i+1}$ with $\mathrm{excess}(Q^I p_{2i+1})$ has to start with i_1 chosen to be odd, however we cannot go further, for reasons stated in 3.2 and the counter example given by Note 5.27.

Our description of primitive classes also allows us to complete the proof of Theorem 17 for the case $n = 1$. The result reads as following.

Lemma 5.19. *Let $\xi \in H_*QS^1$ be a spherixal class. Then it is not possible to have $\xi = \zeta^{2^t}$ for $t > 1$.*

Proof. We do the proof for $t = 2$, and the other cases are similar. Assume $\xi_1 = \zeta^4$ with $\zeta = \sum Q^I Q^i g_1$ and (I, i) admissible. Hence $\xi_1 = Q^{2d}Q^d\zeta$ where $d = \dim \zeta$. The class ξ_1 pulls back to a spherical class $\xi_0 \in H_{4d-1}Q_0S^0$. Therefore we may apply Lemma 5.16 to write

$$\xi = \sum Q^J p_{2j+1},$$

with $(J, 2j+1)$ admissible. The class ξ_1 can have a term of the form $Q^{2d}Q^dQ^IQ^ig_1$ if and only if ξ_0 has a term Q^Jp_{2j+1} such that $\sigma_*Q^Jp_{2j+1} = Q^{2d}Q^dQ^IQ^ig_1$ which implies that $(J, 2j+1) = (2d, d, I, i)$, i.e. $J = (2d, d, I)$. However, according to Lemma 5.17, J cannot have any even entry. This gives the contradiction that we were looking for, and completes the proof. □

Using a similar technique, one may complete the proof of Lemma 19 for the case $n = 1$. We have the following.

Lemma 5.20. *Let $\xi = \zeta^2 \in H_{2d}QS^1$ be spherical. Then d must be odd.*

Proof. Using the method of proof in previous lemma, one can show that if d is even, then pull back of $\xi_0 \in H_{2d-1}QS^1$ will not be A-annihilated. □

Madsen's Description. This is the second description of the submodules of primitives in $H_*Q_0S^0$, and it slightly differs from the first one. However, its behavior under the action of the Steenrod operations is more in our favor.

Notice that $p_1 = x_1$ is a primitive class, hence $p'_{2n+1} = p_{2n+1} + Q^{2n}x_1$ will be a primitive class. Letting $p'_1 = p_1 = x_1$ we obtain another set of primitives $\{p'_{2n+1} : n \geqslant 0\}$ which is in one to one correspondence with $\{p_{2n+1} : n \geqslant 0\}$. One observes that $\{Q^Ip'_{2n+1} : I \text{ admissible }, n \geqslant 0\}$ also generates the submodules of primitives in $H_*Q_0S^0$. This is the set of generators identified by Madsen [M70, Proposition 6.7]. The primitive class p'_{2n+1} belongs to the image of

$$SO \xrightarrow{J} Q_1S^0 \xrightarrow{*[-1]} Q_0S^0,$$

where $J : SO \to Q_1S^0$ is the J-homomorphism. More precisely, assuming that $p^{SO}_{2n+1} \in H_{2n+1}SO$ is the unique primitive, then its image in $H_*Q_0S^0$ under the above composite is given by $p'_{2n+1} = p_{2n+1} + Q^{2n}x_1$ [E80, Lemma 3.6]. Notice that $Q^{2n}x_1 = Q^{n+1}x_n$ modulo decomposable terms. This later equality comes from the Adem relation $Q^{2n}Q^1 = Q^{n+1}Q^n$ together with Lemma 5.10.

The action of the Steenrod algebra on H_*SO is given by

$$Sq^k_*p^{SO}_{2n+1} = \binom{2n+1-k}{k} p^{SO}_{2n+1-k}.$$

This makes it easy to see that the primitive classes p'_{2n+1} behave similar to x_{2n+1} under the action of the Steenrod algebra, i.e.

$$Sq^k_* p'_{2n+1} = \binom{2n+1-k}{k} p'_{2n+1-k}.$$

Note that the above action is trivial when k is odd.

Note 5.21. One may use the action of the Steenrod algebra on p'_{2n+1} to obtain the action of the Steenrod algebra on p_{2n+1}. For instance $Sq^1_* p'_{2n+1} = 0$, hence we obtain

$$Sq^1_* p_{2n+1} \quad = \quad Sq^1_* Q^{2n} x_1 \quad = \quad Q^{2n-1} x_1.$$

Consider the Adem relation

$$Q^{2n-1}Q^1 \quad = \quad \begin{cases} Q^n Q^n & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

This, together Lemma 5.10, implies that modulo decomposable terms,

$$Q^{2n-1}x_1 \quad = \quad \begin{cases} x_n^2 & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

According to the mixed Cartan formula for the operations Sq^i_*, the action of the Steenrod operation maps primitive classes to primitive classes. Therefore, we obtain

$$Sq^1_* p_{2n+1} \quad = \quad \begin{cases} p_n^2 & \text{if } n \text{ is odd,} \\ p^2 & \text{if } n \text{ is even,} \end{cases}$$

where p is a primitive class arising from decomposable terms. This maybe summarised as,

$$Sq^1_* p_{2n+1} = p_{2n}.$$

We refer the reader to [W82, Lemma 5.2] to see that

$$Sq^k_* p_n = \binom{n-k-1}{k} p_{n-k}.$$

Notice that mod 2, we have

$$\binom{2i-2k}{2k} = \binom{2i-2k+1}{2k}.$$

This implies that p_{2n+1}, and p'_{2n+1} behave in the same way under the operations Sq^{2k}_*. However, the primitive classes p'_{2n+1} are annihilated under the operations Sq^{2k+1}_* whereas the primitive classes p_{2n+1} have chance to survive under these operation, e.g. $Sq^1_* p_3 = x_1^2$.

Remark 5.22. We like to draw the reader's attention to the behavior of $Q^I p_{2n+1}$, and $Q^I p'_{2n+1}$ under the homology suspension. First let $I = \phi$. Recall that modulo decomposable terms,

$$\begin{aligned} p_{2n+1} &= x_{2n+1}, \\ p'_{2n+1} &= x_{2n+1} + Q^{2n} x_1. \end{aligned}$$

We then obtain,

$$\begin{aligned} \sigma_* p_{2n+1} &= Q^{2n+1} g_1, \\ \sigma_* p'_{2n+1} &= Q^{2n+1} g_1 + (Q^n g_1)^2. \end{aligned}$$

Now suppose $I = (i_1, \ldots, i_r)$ is a sequence with i_r odd such that $(I, 2n+1)$ is admissible. The equations (3.11) and (3.12) imply that

$$\sigma_* Q^I Q^{2n} x_1 = Q^I Q^{2n} g_1^2 = Q^I (Q^n g_1)^2 = 0.$$

In fact we don't need to restrict to i_r, similar statement holds if we assume only I has at least one odd entry.

Notice that $Q^I Q^{2n} x_1$ is a primitive class, which can be written in terms of $Q^J x_j$ modulo decomposable terms, where (J, j) is admissible. Any class $Q^J x_j$ with (J, j) dies under suspension, if and only if $\text{excess}(Q^J x_j) = 0$, i.e. $Q^J x_j$ is decomposable. Hence $Q^I Q^{2n} x_1$ is a decomposable primitive, and hence a square term. We then observe that if we choose I to be even dimensional, then $Q^I Q^{2n} x_1$ is odd dimensional which makes it impossible to be a square, hence $Q^I Q^{2n} x_1 = 0$. In this case we have

$$\sigma_* Q^I p'_{2n+1} \;=\; Q^I Q^{2n+1} g_1 \;=\; \sigma_* Q^I p_{2n+1},$$

as well as

$$Q^I p_{2n+1} = Q^I p_{2n+1}.$$

The above remark tell us how the primitive classes behave under the homology suspension. We like to combine this with Lemma 5.17. We restrict our attention to the classes $Q^I p_{2i+1}$ with $l(I) > 0$. The result reads as following.

Lemma 5.23. *Suppose $\xi_0 \in H_*Q_0S^0$ is an odd dimensional A-annihilated primitive class. Then*

$$\xi_0 = \sum Q^I p'_{2i+1},$$

with $(I, 2i+1)$ admissible.

Proof. According to Lemma 5.8, we have $\xi_0 = \sum Q^I p_{2i+1}$ with $(I, 2i+1)$ admissible, and I having only odd entries. Notice that modulo decomposable terms $p_{2i+1} = p'_{2i+1} + Q^{2i}x_1$. Hence we may write

$$\xi_0 \;=\; \sum Q^I p_{2i+1} \;=\; \sum Q^I(p'_{2i+1} + Q^{2i}x_1) \;=\; \sum Q^I p'_{2i+1},$$

where according to previous remark $Q^I Q^{2i}x_1$ is trivial. This completes the proof. □

The advantage of this description is that the primitive classes p'_{2i+1} have the same behavior as x_{2i+1} under the Steenrod operations, where they also take care of the decomposable parts. In the light of this observation we may see $Q^I p'_{2i+1}$ like $Q^I Q^i$ and apply our observation at Remark 3.22 to this situation. The result reads as following.

Theorem 5.24. *Suppose $\xi_0 \in H_*Q_0S^0$ is an odd dimensional spherical class. Then*

$$\xi_0 = \sum Q^I p'_{2i+1},$$

with $(I, 2i+1)$ admissible, and any $Q^I p'_{2i+1}$ is A-annihilated.

Notice that this is not true with $Q^I p'_{2i+1}$ replaced with $Q^I x_{2i+1}$. We note that this claim does hold for any A-annihilated primitive classes. This does not imply that each $Q^I p_{2i+1}$ is A-annihilated, however all of these terms magically must add to an A-annihilated class. This has the immediate corollary which completes proof of Lemma 12.

Corollary 5.25. *Suppose $\xi_1 \in H_*QS^1$ is an even dimensional spherical class . Then*

$$\xi_1 = \sum Q^I g_1,$$

where I is admissible, $\mathrm{excess}(Q^I g_1) \geqslant 0$*, and each term is A-annihilated.*

Next we turn our attention to even dimensions where we are given an A-annihilated primitive class $\xi \in H_*Q_0S^0$ with $\sigma_*\xi \neq 0$. According to Corollary 5.16 modulo decomposable terms we have

$$\xi = \sum Q^I p_{2i+1},$$

with $(I, 2i+1)$ admissible. Recall from Remark 5.18 that in the above expression i_1 is ought to be odd which implies that $Q^I Q^{2i} x_1$ is either trivial, or square of a primitive class. A result similar to Theorem 5.12 holds.

Theorem 5.26. *Suppose $\xi \in H_*Q_0S^0$ is an even dimensional A-annihilated primitive class with $\sigma_*\xi \neq 0$. Then*

$$\xi = \sum Q^I p'_{2i+1} + P^2,$$

where $(I, 2i+1)$ is admissible, with $\text{excess}(I, 2i+1) > 0$. *Here P^2 refers to the decomposable primitive part. Moreover, I has only odd entries and $(I, 2i+1)$ satisfies condition 3 of Theorem 2, i.e.*

$$0 < 2i_{j+1} - i_j < 2^{\rho(i_{j+1})},$$

where $I = (i_1, \ldots, i_r)$ and $i_{r+1} = 2i + 1$.

Proof. The first part of the claim is quite clear. We only note that $\sigma_*\xi \neq 0$ is an A-annihilated primitive class of odd dimension living in H_*QS^1. Lemma 12, then implies that every $Q^I p'_{2i+1}$ involved in the above expression for ξ must satisfy condition 3 of Theorem 2. □

Note 5.27. For reasons explained in Section 3.2 we cannot claim that every term $Q^I p'_{2i+1}$ of an even dimensional A-annihilated primitive class $\xi \in H_*Q_0S^0$ will be A-annihilated. We provide the reader with an example. Recall from [W82, Remark 11.26] that the class

$$(Q^{2062}Q^{1031}Q^{519}Q^{263}Q^{135}Q^{71}Q^{39})^2 + Q^{4120}Q^{2062}Q^{1031}Q^{519}Q^{263}Q^{135}Q^{71}Q^{39}$$

is an A-annihilated term in the Dyer-Lashof algebra, R. This implies that

$$(Q^{2062}Q^{1031}Q^{519}Q^{263}Q^{135}Q^{71}p'_{39})^2 + Q^{4121}Q^{2061}Q^{1031}Q^{519}Q^{263}Q^{135}Q^{71}p'_{39}$$

is an A-annihilated primitive in $H_*Q_0S^0$. However, each term is not A-annihilated under the action of Sq^2_*.

Next, we move on to our third description of primitive classes in $H_*Q_0S^0$. Although the above results and remarks shows that the first two descriptions are adequate for most of our purposes, i.e. eliminating potential spherical classes in $H_*Q_0S^0$; however the following description will be useful in describing homology of $H_*Q_0S^{-1}$ in a relation free fashion.

An alternative description of $PH_*Q_0S^0$. There is an alternative way to describe primitive classes in $PH_*Q_0S^0$. Notice that any indecomposable class Q^Ix_i with i odd or I with at least one odd entry corresponds to a unique primitive class modulo decomposable terms. Let I_r denote an admissible sequence $(i_1,\ldots,i_r)$ such that i_1 is the only is odd entry of I. Let $p_{I_r,i}$ be the primitive class corresponding to the indecomposable term $Q^{I_r}x_i$ where i is even. We retain p_{2n+1} to denote the primitive class corresponding to x_{2n+1}. Then any of primitive class in $H_*Q_0S^0$ will be linear combination of terms of the form $Q^Jp_{I_r,i}$ and Q^Kp_{2k+1} with (J,I_r,i), and $(K,2k+1)$ admissible. Such a generating set seems to be more convenient when we consider the problem of finding a generating set for $H_*Q_0S^{-1}$ with no relations among its generators.
It is possible to rewrite primitives expressed in previous generators in this setting. For instance, consider to $Q^4x_1=Q^4p_1$. One then has $Q^4x_1=Q^3x_2$ modulo decomposable terms, i.e.

$$Q^4x_1=p_{(3),2}.$$

5.5 Primitive Classes in H_*QP

This section verifies Proposition 27. The calculations here are in the same line as for $H_*Q_0S^0$. We first concentrate on odd degrees, and then try to generalise the situation to even degrees.

Notice that in odd degrees we have indecomposable terms $a_{2n+1}\in H_{2n+1}QP$.

Therefore, we obtain unique primitive class

$$p^P_{2n+1} = a_{2n+1} + D^P_{2n+1}$$

where D^P_{2n+1} denotes the decomposable terms. Similar to Q_0S^0 we may define $p^P_{2n} = (p^P_n)^2$, and

$$D^P_{2n+1} = \sum_{i=1}^{2n} a_i p^P_{2n+1-i}.$$

Clearly, the elements $Q^I p^P_{2n+1}$ generate a submodule of primitive elements in H_*QP. Notice that

$$\lambda_* Q^I p^P_{2n+1} = Q^I p_{2n+1},$$

where $\lambda : QP \to Q_0S^0$ is the Kahn-Priddy map. Moreover, and similar to Q_0S^0, we may consider to primitive classes $p^P_{2n+1} + Q^{2n}a_1$ which behave like a_{2n+1} under the action of the Steenrod algebra and map to $p'_{2n+1} = p_{2n+1} + Q^{2n}x_1$. Hence λ_* is an epimorphism $PH_*QP \to PH_*Q_0S^0$.

Apart from the above elements, we may consider indecomposable terms of the form $Q^{2i+1}a_{2j}$ which belong to the kernel of the square root map $r_P : H_*QP \to H_*QP$. Hence we obtain another set of primitive classes

$$p^P_{i,j} = Q^{2i+1}a_{2j} + D^P_{i,j},$$

where as before $D^P_{i,j}$ denotes the decomposable part. Now we show that the R-module generated by p^P_{2n+1} and $p^P_{i,j}$ contains all primitive classes of H_*QP which is the statement of Proposition 27.

Proof of Proposition 27. This an obvious analogue to the the proof of the similar statement about $PH_*Q_0S^0$ and is based on induction on the length. First notice that given a decomposable primitive, we can write it as a square of primitives of lower length, where one can use the inductive hypothesis. So we need to prove this for the indecomposable primitives. Such a primitive class must belong to the kernel of the square root map $r_P : H_*QP \to H_*QP$. The fact that

$$\begin{aligned} rQ^{2n} &= Q^n r, \\ rQ^{2n+1} &= 0, \end{aligned}$$

together with the action of r_P on H_*P mentioned above implies that a sum of the form $Q^Ia_i+Q^Ja_j$ belongs to the kernel of r_P if and only if the both terms belong to ker r_P. However, our calculations above has determined all terms Q^Ia_i which belong to ker r_p. This completes the proof. □

Notice that the generators provided by the above proposition provide a splitting of the PH_*QP into the primitives submodules generated by $Q^Ip^P_{2n+1}$ and $Q^Kp^P_{i,j}$. Also notice that we may replace p^P_{2n+1} with $p^P_{2n+1}+Q^{2n}a_1$ and obtain a similar statement as following.

Corollary 5.28. *Any primitive class in H_*QP belongs to the R-module generated by $p^P_{2n+1}+Q^{2n}a_1, p^P_{i,j}$, i.e. any primitive class in H_*QP will be a linear combination of elements of the form $Q^I(p^P_{2n+1}+Q^{2n}a_1)$ and $Q^Kp^P_{i,j}$.*

Remark 5.29. Consider the Kahn-Priddy map $\lambda : QP \to Q_0S^0$. The class $p^P_{i,j}$ maps to a primitive in $H_*Q_0S^0$ under λ_*. Let $(2i+1,2j)$ be admissible. Then

$$\lambda_*p^P_{i,j}=\lambda_*Q^{2i+1}a_{2j}=Q^{2i+1}x_{2j}\neq 0.$$

Recall that given (i_1,i_2) admissible, we have the Adem relation $Q^{i_1}Q^{i_2}=Q^{2i_2}Q^{i_1-i_2}$ modulo admissible terms of lower excess. This allows us to have

$$\begin{aligned}\lambda_*p^P_{i,j} &= \lambda_*Q^{2i+1}a_{2j} = Q^{2i+1}x_{2j}\\ &= Q^{4j}x_{2i+1-2j},\end{aligned}$$

modulo decomposable terms, and indecomposable terms of lower excess. One may replace $x_{2i+1-2j}$ by $p_{2i+1-2j}$ modulo decomposable terms. Hence we may write

$$\lambda_*p^P_{i,j}=Q^{4j}p_{2i+1-2j}$$

modulo decomposable terms and admissible terms of lower excess. That is we may write

$$\lambda_*p^P_{i,j}=Q^{4j}p_{2i+1-2j}+O+D,$$

where O represents the sum of admissible terms of lower excess and D denotes the decomposable part. In a given specific example, depending on the coefficients in the

Adem relation for the non admissible pair $(4j, 2i+1-2j)$, one may work out $O+D$. However, a term of the form $\lambda_* p^P_{i,j}$ will not be of interest for us as it is not A-annihilated which can be seen by applying Sq^1_*. We also note that if $(2i+1, 2j)$ is not admissible, then $\lambda_* p^P_{i,j}$ may or may not be trivial. This again will depend on the Adem relation for the pair $(2i+1, 2j)$.

Note 5.30. Recall that we established an alternative generating set for $PH_*Q_0S^0$. By analogy we may consider the primitive elements p^P_{2k+1}, $p^P_{I_r,i}$ defined in a similar way. Then any primitive class in H_*QP can be written as a linear combination of terms of the form $Q^K p^P_{2k+1}$, $Q^J p_{I_r,i}$ with K, (J, I_r) admissible. Notice that if we choose $(K, 2k+1)$ and (J, I_r, i) admissible then we have,

$$\begin{aligned}\lambda_* Q^K p^P_{2k+1} &= Q^K p_{2k+1},\\ \lambda_* Q^J p^P_{I_r,i} &= Q^J p_{I_r,i},\end{aligned}$$

where as usual λ denotes the Kahn-Priddy map.

5.6 Primitive Classes in $H_*Q\mathbb{C}P$

Similar to H_*QP, we are able to have a similar performance here, and give a complete description of the primitive classes in $H_*Q\mathbb{C}P$. This will give the proof of Proposition 28.

First, we fix our notation. Notice that $H^*\mathbb{C}P \simeq \mathbb{Z}/2[c : \deg c = 2]$, where $c \in H^2\mathbb{C}P$ is the $\mathbb{Z}/2$-reduction of the first universal Chern class. Hence, we obtain an additive basis for $\overline{H}_*\mathbb{C}P$ with generators $c_{2i} \in \overline{H}_{2i}\mathbb{C}P$ such that $\langle c_{2i}, c^j\rangle = \delta_{ij}$. This allows us to represent $H_*Q\mathbb{C}P$ as

$$\mathbb{Z}/2[Q^I c_{2i} : \text{excess}(Q^I c_{2i}) > 0, I \text{ admissible}].$$

The situation here is different from H_*QP as the indecomposable classes c_{2i} are in even dimensions. However, notice that the square root map $r_{\mathbb{C}P} : H_*Q\mathbb{C}P \to H_*Q\mathbb{C}P$ on generators c_{2i} is given as following,

$$r_{\mathbb{C}P} c_{2^k i} = \begin{cases} c_{2^{k-1}i} & \text{if } k > 1 \\ 0 & \text{otherwise} \end{cases}$$

where k in the is the highest power of 2 in $2^k i$, i.e. i is odd. This implies that if n is odd, then c_{2n} belongs to the kernel of the square root map. Therefore we obtain unique primitives

$$p^{\mathbb{C}P}_{4n+2} \quad = \quad c_{4n+2} + D^{\mathbb{C}P}_{4n+2} \in H_{4n+2}Q\mathbb{C}P.$$

Observe that $\mathbb{C}P$ really looks like P, but in a larger *scale*, i.e. we may do similar stuff as we did with QP. Let $p^{\mathbb{C}P}_{4n} = (p^{\mathbb{C}P}_{2n})^2$. Then we have,

$$D^{\mathbb{C}P}_{4n+2} = \sum_{i=1}^{2n} c_{2i} p^{\mathbb{C}P}_{4n+2-2i}.$$

Remark 5.31. Notice that one may define a homomorphism $H_iP \to H_{2i}\mathbb{C}P$ by sending a_i to c_{2i}. This induces a mapping $H_*P \to H_*\mathbb{C}P$ which does not respect grading. However, this map is an isomorphism of coalgebras, formalising what we did above, and what follows. Notice that we are **not** claiming that this is an isomorphism $H_*QP \to H_*Q\mathbb{C}P$.

Now consider $Q^I c_{2j}$ with j even. Such a class belongs to the kernel of the square root map if I has at least one odd entry. Hence, when j is even, we obtain another set of primitives

$$p^{\mathbb{C}P}_{i,j} \quad = \quad Q^{2i+1}c_{2j} + D^{\mathbb{C}P}_{i,j} \in H_{2i+2j+1}Q\mathbb{C}P.$$

Proposition 28 claims that the R-module spanned by the two sets of primitives above captures all of the primitive classes in $H_*Q\mathbb{C}P$. The rest of the proof is analogous to the proof of a similar claim about the primitive classes in H_*QP, and is based on induction on the length. We leave the details to the reader.

Note 5.32. Similar to the cases of Q_0S^0 and QP we may define an alternative generating set for primitive in $H_*Q\mathbb{C}P$. In this case, modulo decomposable terms, we have

$$p^{\mathbb{C}P}_{4k+2} \quad = \quad c_{4k+2} \tag{5.5}$$

$$p^{\mathbb{C}P}_{I_r,i} \quad = \quad Q^{I_r}c_{2i}, \tag{5.6}$$

where i is chosen to be even.

5.7 The Complex Transfer and $H_*Q_0S^{-1}$

The space QS^{-n} is defined to be the n-th loop space of QS^0. It also can be seen as the infinite loop space associated with the n-th desuspension of the sphere spectrum. The homology of these spaces, and their geometry still is not completely understood. In fact we only know about the ring structure of $H_*Q_0S^{-1}$ [CP89, Theorem 1.1], and partially know about the homology of Q_0S^{-2} [CP89, Theorem 1.2]. However, it is possible to identify particular subrings of H_*QS^{-2} by giving a reasonably good description of its "generators". We guess that this is a general pattern, i.e. every $H_*Q_0S^{-n}$ will have a subring whose generators are obtained in similar way that one obtains generators for $H_*Q_0S^{-1}$. Moreover, this subring is going to contain pull back of spherical classes of $H_*Q_0S^0$. We will inform the reader about our thoughts on this.

Previously, some knowledge of $H_*Q_0S^0$ allowed us to prove partial results on the type of possible spherical classes in $H_{*+1}QS^1$, namely Theorem 17 in the case $n = 1$. Similarly, we shall use our description of specific subrings of H_*QS^{-2} to prove Theorem 17 for the case $n = 0$.

As we mentioned earlier in Example 5.7, the homology ring $H_*Q_0S^{-1}$ is known to be an exterior algebra [CP89, Theorem 1.1]. We combine this with homology of the complex transfer to give a description of generators of $H_*Q_0S^{-1}$, and determine the structure of $H_*Q_0S^{-1}$ as an R-module, as well as its A-module structure. However, as we have seen earlier there are different choices of generating sets for the submodule of primitives $PH_*Q_0S^0$. This means that there are different ways of representing $H_*Q_0S^{-1}$. We will give all of these three descriptions. The workable description, will enable us to do some calculations. Although in such a description there are some relations in the algebra, which we shall explain. We have succeeded in identifying a subring of $H_*Q_0S^{-1}$ which contains a pull back of any spherical class from $H_*Q_0S^0$. Applying the machinery of the Eilenberg-Moore spectral sequence allows one to calculate $H_*Q\Sigma^{-1}P$, and $H_*Q\Sigma^{-1}\mathbb{C}P$, and to consider the homology of the looped transfer maps. These results may be well known for experts, but we are not aware of them stated anywhere in the literature.

The S^1-transfer is a map $\lambda_{\mathbb{C}} : Q\Sigma\mathbb{C}P_+ \to QS^0$. The homology of this map is known based on the work of Mann-Miller-Miller [MMM86, Lemma 7.4]. However, we will make some observations, which shortens calculation of the homology of $\lambda_{\mathbb{C}}$.

The complex transfer factors through the complex J-homomorphism

$$J_{\mathbb{C}} : U \to Q_1 S^0.$$

Using the translation map $*[-1]$ we then will land in Q_0S^0. The reader will need to take this to account to avoid confusion about our homological calculations while comparing to the result of Mann-Miller-Miller[MMM86, Lemma 7.4]. We urge the reader to notice that the map $\lambda_{\mathbb{C}}$ is an infinite loop map, obtained as the infinite loop extension of the following composite,

$$\Sigma\mathbb{C}P_+ \longrightarrow U \longrightarrow Q_1S^0 \longrightarrow Q_0S^0.$$

The mapping $\lambda_{\mathbb{C}}$ may be viewed as an extension of $\nu : S^3 \not\to S^0$ where S^3 sits as the bottom cell of $\Sigma\mathbb{C}P$, and the inclusion maps g_3 to Σc_2 in homology. Hence Σc_2 maps to $x_3 + O_3$. Using this, and the action of Steenrod algebra on $\Sigma\mathbb{C}P$, one may observe that Σc_{2i} maps to $x_{2i+1}+O_{2i+1}$. On the other hand, notice that Σc_i is primitive. Also, the image must have the same behavior under the action of the Steenrod algebra as Σc_{2i}. Hence we obtain,

$$(\lambda_{\mathbb{C}})_*\Sigma c_{2i} = p_{2i+1} + Q^{2i}x_1 = p'_{2i+1}.$$

Moreover, notice that Σc_0 maps to $x_1 = p_1 = p'_1$ where c_0 is the generator coming from the disjoint base point. This then allows one to calculate $(\lambda_{\mathbb{C}})_* : H_*Q\Sigma\mathbb{C}P_+ \to H_*Q_0S^0$. Notice that this in particular implies that $(\lambda_{\mathbb{C}})_* : PH_*Q\Sigma\mathbb{C}P_+ \to PH_*Q_0S^0$ is an epimorphism.

Remark 5.33. The above equations allows us to calculate that

$$(\lambda_{\mathbb{C}})_*(\Sigma c_{2i} + Q^{2i}\Sigma c_0) = p_{2i+1}.$$

Recall from Example 5.7 that $\sigma_* : QH_*Q_0S^{-1} \to PH_*Q_0S^0$ is an isomorphism. Also notice that $\sigma_* : QH_*Q\mathbb{C}P_+ \to PH_*Q\Sigma\mathbb{C}P_+$ is an isomorphism. This makes it easy to see that $(\Omega\lambda_{\mathbb{C}})_*: QH_*Q\mathbb{C}_+P \to QH_*Q_0S^{-1}$ is an epimorphism. These fit into a commutative diagram as following

$$\begin{array}{ccc} PH_*Q\Sigma\mathbb{C}P_+ & \xrightarrow{(\lambda_{\mathbb{C}})_*} & PH_*Q_0S^0 \\ \uparrow{\scriptstyle\sigma_*} & & \uparrow{\scriptstyle\sigma_*} \\ QH_{*-1}Q\mathbb{C}P_+ & \xrightarrow{(\Omega\lambda_{\mathbb{C}})_*} & QH_{*-1}Q_0S^{-1} \end{array}$$

which is the observation of [G04, Diagram 1.3].

Before proceeding further, we establish our notation for $H_*Q_0S^{-1}$. Recall from Proposition 5.4 that $\sigma_*: QH_*Q_0S^{-1} \to PH_*Q_0S^0$ is an isomorphism, and $H_*Q_0S^{-1} \simeq E_{\mathbb{Z}/2}(\sigma^{-1}PH_*Q_0S^0)$. We note that there will be a matter of choice here, and depending on the choice of a basis for $PH_*Q_0S^0$ we get a presentation for $H_*Q_0S^{-1}$. However, in this case we observe that the complex transfer allows us to have a natural choice for the generators of $H_*Q_0S^{-1}$ providing us with a type of "geometric" meaning for these generators. We shall investigate cases for each choice of a basis that we make. First we consider the primitive classes $p'_{2i+1} \in H_*Q_0S^0$ and the R-module spanned by these elements which captures $PH_*Q_0S^0$. We may consider unique elements $w'_{2i} \in QH_{2i}Q_0S^{-1}$ with $\sigma_*w'_{2i} = p'_{2i+1}$. However, the fact that $(\lambda_{\mathbb{C}})_*\Sigma c_{2i} = p'_{2i+1}$ allows one to define

$$(\Omega\lambda_{\mathbb{C}})_*c_{2i} = w'_{2i}.$$

Notice that the space Q_0S^{-1} is an infinite loop space, and hence we may consider terms of the form $Q^Iw'_{2i} \in H_*Q_0S^{-1}$ which satisfy $\sigma_*Q^Iw'_{2i} = Q^Ip'_{2i+1}$. The fact that elements of the form $Q^Ip'_{2i+1}$ generate all primitives in $H_*Q_0S^0$ implies that elements of the form $Q^Iw'_{2i}$ generate $QH_*Q_0S^{-1}$, and therefore $H_*Q_0S^{-1}$ is the exterior algebra generated by $Q^Iw'_{2i}$ with I admissible. This also determines the action of the Dyer-Lashof algebra on the homology ring $H_*Q_0S^{-1}$. Moreover, our definition of the generators w'_{2i} allows us to derive the action of the Steenrod operation on these classes, namely we have

$$Sq^{2k}_*w'_{2i} = \binom{i-k}{k}w'_{2i-2k}.$$

This together with the Nishida relations describes the action of the Steenrod algebra on the generators $Q^Iw'_{2i}$, and hence completely determines the action of the Steenrod algebra on the homology ring $H_*Q_0S^{-1}$.

Although we have identified generators of $H_*Q_0S^{-1}$, however there are some relations among these generators. For example consider $Q^3x_1 = x_1^4 = Q^2Q^1x_1 \in H_4Q_0S^0$. Hence in $H_*Q_0S^{-1}$ we have

$$Q^3w_0 = Q^2Q^1w_0.$$

Notice that in this case the above equality strictly holds as it happens in an odd dimension. The ambiguity about this comes from the description of primitives in $H_*Q_0S^0$. Notice that $PH_*Q_0S^0$ is generated by $Q^Ip'_{2n+1}$, however we don't assume any admissibility condition on the pair $(I, 2n+1)$. For instance $Q^4x_1 = Q^3x_2$ modulo decomposable terms. In this case, the left side pulls back to Q^4w_0, however we don't have good description of the pull back of the right hand side of the equation. We sum up the above discussion as following.

Theorem 5.34. *The homology algebra $H_*Q_0S^{-1}$ as an R-module is given by*

$$E_{\mathbb{Z}/2}(Q^Iw'_{2i} : I \text{ admissible}, \dim I > 2i),$$

with $w'_{2i} = (\Omega\lambda_{\mathbb{C}})_*c_{2i}$ which satisfies $\sigma_*w'_{2i} = p_{2i+1}$. Two generators $Q^Iw'_{2i}$ and $Q^Jw'_{2j}$ may be identified if and only if they map to the same element in $H_*Q_0S^0$ under the homology suspension $\sigma_* : H_{*-1}Q_0S^{-1} \to H_*Q_0S^0$. The behavior of generators w'_{2i} under the Steenrod operation is very much like $c_{2i} \in H_{2i}\mathbb{C}P$, i.e.

$$Sq_*^{2k}w'_{2i} = \binom{i-k}{k}w'_{2i-2k}.$$

This together with the Nishida relations completely determines the A-module structure of $H_*Q_0S^{-1}$.

Remark 5.35. One may choose to work with p_{2i+1}. In a similar fashion we obtain indecomposable terms $w_{2i} \in H_{2i}Q_0S^{-1}$. Moreover, since $(\lambda_{\mathbb{C}})_*(\Sigma c_{2i}+Q^{2i}\Sigma c_0) = p_{2i+1}$, then we may define w_{2i} by the following equation

$$(\Omega\lambda_{\mathbb{C}})_*(c_{2i} + Q^{2i}c_0) = w_{2i}.$$

We then observe that $H_*Q_0S^{-1}$ will be an exterior algebra over the generators Q^Iw_{2i} where $\sigma_*Q^Iw_{2i} = Q^Ip_{2i+1}$. In some cases, it is more convenient to use this description.

Observe that $w_0 = w'_0$, and $w_{2i} = w'_{2i} + Q^{2i}w_0$ which give a one to one correspondence between the two set of "generators". Finally notice that $\overline{Q}^I w_{2i}$ can be identified with $Q^I w_{2i}$.

Note 5.36. We can determine the action of the Steenrod algebra on the generators $Q^K w_{2k}$. This can be done by using the above definition, i.e. the equation

$$(\Omega\lambda_{\mathbb{C}})_*(c_{2k} + Q^{2k}c_0) = w_{2k}.$$

This will allow us to calculate the action of the Steenrod algebra on w_{2k} as,

$$\begin{aligned} Sq^{2r}_* w_{2k} &= (\Omega\lambda_{\mathbb{C}})_*(Sq^{2r}_* c_{2k} + Sq^{2r}_* Q^{2k}c_0) \\ &= (\Omega\lambda_{\mathbb{C}})_* \tbinom{k-r}{r} c_{2k-2r} + \tbinom{2k-2r}{2r} Q^{2k-2r}c_0) \\ &= \tbinom{k-r}{r} w_{2k-2r}. \end{aligned}$$

Next we like to calculate $Sq^{2r+1}_* w_{2k}$. But we deal with it in a slightly different way, and we note that the previous calculation also can be done in this way. We perform as following. Observe that w_{2k} maps to p_{2k+1} under the homology suspension. Recall that

$$\begin{aligned} Sq^{2r+1}_* p_{2k+1} &= \tbinom{2k-2r-1}{2r+1} p_{2k-2r} = \tbinom{2k-2r-1}{2r+1} p^{2^{\phi(2k-2r)}}_{2m+1} \\ &= \tbinom{2k-2r-1}{2r+1} p_{2k-2r} = \tbinom{2k-2r-1}{2r+1} Q^{2^{\phi(2k-2r)-1}(2m+1)} \cdots Q^{2m+1} p_{2m+1}, \end{aligned}$$

where $2^{\phi(n)}$ is the largest power of 2 in n. The above equation implies that

$$Sq^{2r+1}_* w_{2k} = \binom{2k-2r-1}{2r+1} Q^{2^{\phi(2k-2r)-1}(2m+1)} \cdots Q^{2m+1} w_{2m}.$$

This really motivates one to work with the generators w'_{2k} since they have much better behavior under the Steenrod operations.

So far we have used two set of generators for $PH_*Q_0S^0$ together with the S^1-transfer to identify two sets of generators for $H_*Q_0S^{-1}$, namely $\{Q^I w'_{2i}\}$ and $\{Q^I w_{2i}\}$. However, there are some relations among the generators in both of these sets. It is possible to give an alternative description of generators of $H_*Q_0S^{-1}$ which does not seem to have the above problem, i.e. there are no relations among its generators. To do this, we may use our alternative description of the primitive classes in $H_*Q_0S^0$, namely primitives $p_{I_r,i}$ and p_{2k+1}. These give rise to unique classes generators $w_{2k} \in QH_*Q_0S^{-1}$, and $w_{I_r,i-1} \in QH_*Q_0S^{-1}$ such that

$$\begin{aligned} \sigma_* w_{2k} &= p_{2k+1}, \\ \sigma_* w_{I_r,i-1} &= p_{I_r,i}. \end{aligned}$$

Equations of type (5.2) allows one to see the action of the Kudo-Araki operations in these generators. Moreover, as we have assumed admissibility in the construction of $p_{I_r,i}, p_{2k+1}$ it completely makes sense to define *excess* as following,

$$\begin{aligned} \text{excess}(Q^J w_{I_r,i-1}) &= j_1 - (j_2 + \cdots + j_t + \dim w_{I_r,i-1}), \\ \text{excess}(Q^K w_{2k}) &= \text{excess}(K) - 2k, \end{aligned}$$

where $J = (j_1, \ldots, j_t)$. The fact that $\sigma_* : QH_*Q_0S^{-1} \to PH_*Q_0S^0$ is an isomorphism implies the following.

Proposition 5.37. *As an R-module $H_*Q_0S^{-1}$ is given by the exterior algebra generated by elements $Q^K w_{2k}$, $Q^J w_{I_r,i-1}$ of positive excess where $(K, 2k+1)$, (J, I_r, i) are admissible.*

This description gives complete information about the R-module structure of $H_*Q_0S^{-1}$. We note that it is possible to identify the generators $w_{I_r,i}$ in a "geometric" way and provide the reader with a natural choice for these generators. This is of course can be done if we know which element of $H_*Q\Sigma\mathbb{C}P_+$, say $\sigma_* c_{I_r,i}$, maps to $p_{I_r,i}$. Notice that our basic knowledge on $\lambda_{\mathbb{C}}$ is that $(\lambda_{\mathbb{C}})_*\Sigma c_{2i} = p'_{2i+1}$. Hence to calculate $c_{I_r,i}$ we first need to write $p_{I_r,i}$ in terms of $Q^J p'_{2j+1}$ which may involve iterated application of Adem relations of the form (5.4). This suggest that $\sigma_* c_{I_r,i}$ will not be a single term. For the time being short and the fact that we will not use this description we leave the rest of this to the reader to investigate.

The action of the Steenrod operations on the generators $w_{I_r,i-1}$ does not seems very clear. Although one may try to suspend this class to get an primitive class in $H_*Q_0S^0$, namely $p_{I_r,i}$. According to the diagonal Cartan formula, performing any Steenrod operation on a primitive class will result in a primitive class. One then may pull back the resulting class to $H_{*-1}Q_0S^{-1}$. However, this requires a tedious application of the Nishida relations which I have not yet done!

The fact that there are no relations among the generators provided by Proposition 5.37 makes such a description much clearer, rather than considering an exterior algebra over generators $Q^I w_{2i}$ with $(I, 2i+1)$ not necessarily admissible.

Note 5.38. Our previous results on the odd dimensional spherical classes ensures that any odd dimensional spherical class pulls back to a spherical class whose indecomposable part falls into the subalgebra

$$E_{\mathbb{Z}/2}(Q^K w_{2k} : (K, 2k+1) \text{ admissible}, \text{excess}(Q^K w_{2k} > 0).$$

Hence, for the sake of calculating pull back of spherical classes from $H_* Q_0 S^0$ into $H_{*-1} Q_0 S^{-1}$ it is enough to work with the description provided by Theorem 5.34 whereas the fact that $(K, 2k+1)$ ensures that this falls into a part of algebra that there is not relation among its generators.

Note 5.39. Notice that usually the homology of a space becomes *larger* after looping. However, based on this description the homology of the $Q_0 S^{-1}$ is *smaller* than the homology of $Q_0 S^0$. This is evident from the fact that

$$QH_* Q_0 S^{-1} \simeq PH_* Q_0 S^0 \rightarrowtail H_* Q_0 S^0.$$

One may conclude that there exists a monomorphism of Abelian groups $H_n Q_0 S^{-1} \to H_{n+1} Q_0 S^0$. Notice that the fact that $H_* QS^{-1}$ is an exterior algebra implies that $PH_* Q_0 S^{-1}$ injects in $QH_* Q_0 S^{-1} \simeq PH_{*-1} Q_0 S^0$. This implies that $H_* Q_0 S^0$ and $H_{*-1} Q_0 S^{-1}$ *have the same spherical classes.*

Remark 5.40. It is possible to identify specific subalgebras of $H_* QS^{-1}$. To do so, we use $\eta \in \pi_0^S S^{-1} \simeq \pi_0 QS^{-1} \simeq \mathbb{Z}/2$. The classes $0, \eta \in \pi_0 QS^{-1}$ map to $[0], [\eta] \in H_0 QS^{-1}$ under the Hurewicz homomorphism $\pi_0 QS^{-1} \to H_0 QS^{-1}$. This makes it straightforward to see that $[0] * [\eta] = [0 + \eta] = [\eta]$ and

$$Q^0[\eta] = [\eta] * [\eta] = [2\eta] = [0] = [0] * [0] = Q^0[0].$$

Here $*$ denotes the Pontryagin product arising from the loop sum on QS^{-1}. Applying the Adem relation to $Q^i([0] * [0])$ implies that $Q^i[0] = 0$ for all $i > 0$. This observation together with the Adem relations applied to $Q^i([\eta] * [0])$ shows that $Q^i[\eta] * [0] = Q^i[\eta]$.

Indeed this is we should have expected that $[0]$ will play the role of a unit in H_*QS^{-1}. On the other hand, the mapping $\eta : S^0 \to QS^{-1}$ extends to an infinite loop map $\eta : QS^0 \to QS^{-1}$. On the level of π_0, or equivalently on the level of H_0, this induces the projection $\eta_* : \mathbb{Z} \to \mathbb{Z}/2$, i.e.

$$\eta_*[n] = \begin{cases} [\eta] & \text{if } n \text{ is odd,} \\ [0] & \text{if } n \text{ is even.} \end{cases}$$

Hence one may work out the image of $\eta_* : H_*QS^0 \to H_*QS^{-1}$ as following

$$\begin{aligned} \eta_* x_i &= \eta_*(Q^i[1] * [-2]) \\ &= \eta_* Q^i[1] * \eta_*[-2] \\ &= Q^i[\eta] * [0] \\ &= Q^i[\eta]. \end{aligned}$$

Applying the homology suspension we obtain

$$\sigma_* Q^i[\eta] = \eta_*\sigma_* x_i = \eta_* Q^i g_1 = Q^i x_1 \neq 0,$$

which shows that $Q^i[\eta] \neq 0$. Notice that $Q^i x_1$ depends on the Adem relation for Q^iQ^1, and will be trivial or maybe nontrivial modulo decomposable terms. However, the decomposable part always has nontrivial terms, for example $Q^3x_1 = x_1^4$. In fact, as x_1 is primitive, $Q^i x_1$ is primitive. If $Q^iQ^1 = 0$, then the decomposable part of $Q^i x_1$ is the square of a primitive and one may work out what this primitive is.

The above calculation allows us to calculate the image of $\eta_* : H_*QS^0 \to H_*QS^{-1}$ completely. Moreover, observe that

$$(\eta_* x_i)^2 = (Q^i[\eta])^2 = Q^{2i}([\eta] * [\eta]) = Q^{2i}[0] = 0,$$

and this calculations is consistent with the fact that H_*QS^{-1} is an exterior algebra. This together with induction on length, and the fact that $(Q^i\xi)^2 = Q^{2i}\xi^2$ implies that $(Q^I[\eta])^2 = 0$ for any I. Hence the subalgebra obtained in this way will be the exterior algebra over the generators $Q^I[\eta]$ with I admissible. This also verifies that $\ker \eta_*$ is not very small.

In particular, putting $i = 1$ we obtain $\eta_* x_1 = Q^1[\eta]$. Notice that

$$\sigma_*(Q^1[\eta]) = \eta_*\sigma_* x_1 = \eta_* g_1^2 = x_1^2 = p_1'^2,$$

where this class is related to the Kervaire invariant one element $\eta^2 \in \pi_2^S$. Hence the Hurewicz image of $\eta^2 \in \pi_1 QS^{-1}$ is given by $Q^1[\eta] \in H_1 QS^{-1}$.

Indeed this is picture of a general pattern. Recall that $Q^{2k}Q^1 = Q^{k+1}Q^k$, and

$$Q^{2k-1}Q^1 = \begin{cases} Q^kQ^k & \text{if } k \text{ is odd,} \\ 0 & \text{if } k \text{ is even.} \end{cases}$$

The class x_i suspends to $Q^i g_1$ which is a primitive class. Hence $\eta_* Q^i g_1$ will be a primitive class. If $i = 2k$, then

$$\eta_* Q^i g_1 = Q^{2k} x_1 = x_k^2 + D.$$

If $i = 2k-1$, then modulo decomposable terms one has

$$\begin{aligned} \eta_* Q^{2k-1} g_1 &= Q^{2k-1} x_1 \\ &= \begin{cases} Q^k x_k & \text{if } k \text{ is odd,} \\ 0 & \text{if } k \text{ is even.} \end{cases} \end{aligned}$$

Hence $Q^{2k-1}[\eta]$ maps to x_k^2 under σ_* modulo decomposable terms. Recall that x_k^2, with k odd, gives a primitive $(p_k')^2$. Hence, a spherical class involving x_k^2 will pull back to a term in $H_* QS^{-1}$ which will involve $Q^{2k-1}[\eta]$. Of interest will be classes with $k = 2^s - 1$ which may give rise to the possible Kervaire invariant one classes.

This also gives another set of relations among the generators of $H_* Q_0 S^{-1}$, namely

$$Q^{2k-1}[\eta] = Q^{2i+1} w'_{2i},$$

where $k = 2i+1$. We note that here $[\eta]$ plays the role of $w_0 = w'_0$.

Notice that $Q^i[\eta]$ lives in the component $Q_{2\eta} S^{-1} = Q_0 S^{-1}$. This means that the subalgebra generated by $Q^I[\eta]$ is a subalgebra of $H_* Q_0 S^{-1}$. The fact that QS^{-1} is an infinite loop space implies that $Q_\eta S^{-1}$ and $Q_0 S^{-1}$ have the same homology induced by the homotopy equivalence $*[\eta] : Q_0 S^{-1} \to Q_\eta S^{-1}$. This allows us to consider the subalgebra generated by $Q^I[\eta] * [\eta]$ of $H_* Q_\eta S^{-1}$. We then obtain a subalgebra of $H_* QS^{-1}$ given by polynomials in $[\eta]$ and $[0]$ with coefficients of the form $Q^I[\eta]$.

Finally, we note that this subalgebras does not seem to capture all of the homology rings $H_* Q_0 S^{-1}$ and $H_* QS^{-1}$ respectively.

Note 5.41. The above remark provides us with the motivating example to identify important subalgebras of H_*QS^{-n}. The start point is to consider $\pi_0QS^{-n}\simeq\pi_n^S$ together with the Hurewicz homomorphism

$$h:\pi_0QS^{-n}\to H_0QS^{-n}.$$

For any $\alpha\in\pi_n^S$, we may define $[\alpha]=h\alpha\in H_0QS^{-n}$. Then it is clear that $[\alpha]*[\beta]=[\alpha+\beta]$ for $\alpha,\beta\in\pi_0QS^{-n}$, where $*$ denotes the product induces by the loop sum. One then may try to work out the action of the Dyer-Lashof algebra on the subalgebra generated by symbols $Q^I[\alpha]$ as we did before. In particular, applying the Cartan formula implies that $Q^i[0]=0$ for all $i\geqslant 0$. In practice this does not seem possible for all n, as we are still far from having a complete description of π_n^S. But, on the other hand any calculation of H_*QS^{-n} using any other method will be quite interesting, if we can couple it with this observation.

More interesting cases will appear when π_0QS^{-n} has more than one component. Consider QS^{-9} with $\pi_0QS^{-9}\simeq\mathbb{Z}/2\oplus\mathbb{Z}/2\oplus\mathbb{Z}/2$. Having different components in π_0QS^{-9} shows the sort of the geometric complications that may appear in study of these spaces. Suppose γ_i, $i=1,2,3$, denote the generators for the first, second and third copies of $\mathbb{Z}/2$ in π_0QS^{-9} respectively. We let $[\gamma_i]=h\gamma_i$. Notice that in this case Q_0S^{-9} has 8 components of the same homotopy type. Notice that $\gamma_i^2=0$. We know that terms of the form $Q^i[\gamma_i]$ live in the component $Q_{2\gamma_i}S^{-9}=Q_0S^{-9}$. Hence, we may consider the subalgebras of $H_*Q_0S^{-9}$ generated by the symbols of the following forms,

$$Q^{I_1}[\gamma_1],\quad I_1\text{ admissible}$$
$$Q^{I_2}[\gamma_2],\quad I_2\text{ admissible}$$
$$Q^{I_3}[\gamma_3],\quad I_3\text{ admissible.}$$

The fact that there are no relations among the generators γ_i implies that there are no multiplicative relations among the three set of generators provided above. These are the only pieces that we get. This can be seen as an evidence for some stable splitting of the space Q_0S^{-9}. We observe that $Q^J[\gamma_i+\gamma_j]=Q^J([\gamma_i]*[\gamma_j])$ with $i\neq j$ can be calculated using the Cartan formula.

We can apply similar techniques as we did before and use the translation maps $*[\gamma_i]: Q_0S^{-9} \to Q_{\gamma_i}S^{-9}$ and $*[\gamma_i+\gamma_j]: Q_0S^{-9} \to Q_{\gamma_i+\gamma_j}S^{-9}$ to obtain subalgebras of the other components, and hence a subalgebra of H_*QS^{-9}.

Finally we note that we used the word “subalgebra” as we still don’t know the type of the algebraic structure of $H_*Q_0S^{-9}$. However, one still will be able to calculate the R-module structure as well as the A-module structures as we did for H_*QS^{-1}. I postpone pursuing this way for future work (if I secured a job)! We note that the spaces that we have considered in this note are infinite loop spaces of the form QS^{-n} with $n > 0$ which makes their homology algebras, the homology of the base point components, to be bi-associative and and bi-commutative Hopf algebras. This means that in these cases we may apply Borel’s structure theorem which says that these algebras are tensor product of polynomial algebras and their truncations.

5.8 Homology of $Q\Sigma^{-1}\mathbb{C}P$ and Q_0S^{-2}

Through this section we will work with the generators $w'_{2i} \in H_{2i}Q_0S^{-1}$. Recall that $(\Omega\lambda_{\mathbb{C}})_*c_{2i} = w'_{2i}$. This allowed us to calculate

$$Sq^{2j}_*w'_{2i} = \binom{i-j}{j}w'_{2i-2j}.$$

Similarly, the co-product in $H_*Q_0S^{-1}$ is given by

$$\Delta_*w'_{2i} = \sum_k w'_{2i-2k}\otimes w'_{2k},$$

Next, we need to determine the primitive classes in $H_*Q_0S^{-1}$. We use the homology of $\Omega\lambda_{\mathbb{C}}$ together with naturality to calculate the square root map,

$$r_{S^{-1}}: H_*Q_0S^{-1} \to H_*Q_0S^{-1},$$

on the *generators* w'_{2i}. Notice that all odd dimensional classes belong to the kernel of this. Moreover, given an even dimensional class ξ of dimensional $2n$, then $r\xi = Sq^n_*\xi$. Hence, to calculate $r_{S^{-1}}w'_{2i}$ we need to calculate $Sq^i_*w_{2i}$, and then see if this yields a trivial or nontrivial class in $QH_*Q_0S^{-1}$. This implies that the class w'_{2i} belongs to the kernel of the square root map $QH_*Q_0S^{-1} \to QH_*Q_0S^{-1}$, if i is odd. Hence, analogous to $H_*Q\mathbb{C}P$ we obtain a set of primitives,

$$p_{4n+2}^{S^{-1}} = w'_{4n+2} + D_{4n+2}^{S^{-1}} \in H_{4n+2}QS^{-1}.$$

Here, and similar to previous cases, we have

$$D_{4n+2}^{S^{-1}} = \sum_{i=1}^{2n} w'_{2i} p_{4n+2-2i}^{S^{-1}}.$$

On the other hand, having $Q^I w'_{2i}$ with i even, such a class belongs to the kernel of the square root map if I has at least one odd entry. According to this, we obtain another set of primitives,

$$p_{i,j}^{S^{-1}} = Q^{2i+1} w'_{2j} + D_{i,j}^{S^{-1}} \in H_{2i+2j+1}QS^{-1},$$

where j is an even number. Then one may check that any primitive element in $H_*Q_0S^{-1}$ is a linear combination of terms of the form $Q^L p_{4n+2}^{S^{-1}}$ and $Q^K p_{i,j}^{S^{-1}}$. The proof of this claim is similar to the proof of similar claims on primitive classes in $H_*Q\mathbb{C}P$ and H_*QP. Notice that, in homology, we have the following obvious relations

$$\begin{aligned}(\Omega\lambda_{\mathbb{C}})_* p_{4n+2}^{\mathbb{C}P} &= p_{4n+2}^{S^{-1}},\\ (\Omega\lambda_{\mathbb{C}})_* p_{i,j}^{\mathbb{C}P} &= p_{i,j}^{S^{-1}}.\end{aligned}$$

Our description of primitive classes in $H_*Q\mathbb{C}P$, provided by Proposition 28 and proved in section 5.6, allows us to apply the Eilenberg-Moore spectral sequence machinery, described by Proposition 5.4, to describe $H_*Q\Sigma^{-1}\mathbb{C}P$. Notice that here, similar to $H_*Q_0S^{-1}$, there is a matter of choice ranging from the workable description to the ideal ones. We choose $p_{4n+2}^{\mathbb{C}P}$ and $p_{i,j}^{\mathbb{C}P}$ to work with and obtain unique indecomposable classes $v_{4n+1}^{\mathbb{C}P} \in QH_{4n+1}Q\Sigma^{-1}\mathbb{C}P$ and $v_{i,j-1}^{\mathbb{C}P} \in QH_{2i+2j}Q\Sigma^{-1}\mathbb{C}P$ such that

$$\begin{aligned}\sigma_* v_{4n+1}^{\mathbb{C}P} &= p_{4n+2}^{\mathbb{C}P},\\ \sigma_* v_{i,j-1}^{\mathbb{C}P} &= p_{i,j}^{\mathbb{C}P}.\end{aligned}$$

This yields the following presentation.

Proposition 5.42. *As an R-module $H_*Q\Sigma^{-1}\mathbb{C}P$ is given by the exterior algebra over the generators $Q^I v_{4n+1}^{\mathbb{C}P}$ and $Q^L v_{i,j-1}^{\mathbb{C}P}$ with I and L admissible, and* $\dim I > 4n+1$

and $\dim L > 2i+2j$. *The generators* $Q^I v^{\mathbb{C}P}_{4n+1}$ *are independent from each other for different choices of admissible* I. *Two generators of the form* $Q^L v^{\mathbb{C}P}_{i,j-1}$ *are identified if they map to the same class in* $H_*Q\mathbb{C}P$ *under the homology suspension.*

It is possible to work out the action of the Steenrod algebra and give the A-module structure of $H_*Q\Sigma^{-1}\mathbb{C}P$. Although such a description and its consequences can be of its own interest, we like to use this to obtain some information on some subrings of H_*QS^{-2} as R-modules. In order to do this, we define $v_{4n+1} \in H_{4n+1}QS^{-2}$ and $v_{i,j-1} \in H_{2i+2j}QS^{-2}$ by the following equations

$$\begin{aligned} v_{4n+1} &= (\Omega^2\lambda_{\mathbb{C}})_* v^{\mathbb{C}P}_{4n+1}, \\ v_{i,j-1} &= (\Omega^2\lambda_{\mathbb{C}})_* v^{\mathbb{C}P}_{i,j-1}. \end{aligned}$$

The fact that $H_*Q\Sigma^{-1}\mathbb{C}P$ is an exterior algebra, implies that image of $(\Omega^2\lambda_{\mathbb{C}})_*$ is also an exterior algebra. So we may consider the following subalgebra of H_*QS^{-2} given by

$$E_{\mathbb{Z}/2}(Q^I v_{4n+1}, Q^L v_{i,j-1} : I, L \text{ admissible}).$$

Consider the following commutative diagram

$$\begin{array}{ccc} H_*Q\mathbb{C}P & \xrightarrow{(\Omega\lambda_{\mathbb{C}})_*} & H_*Q_0S^{-1} \\ \uparrow{\scriptstyle \sigma_*} & & \uparrow{\scriptstyle \sigma_*} \\ H_{*-1}Q_0\Sigma^{-1}\mathbb{C}P & \xrightarrow{(\Omega^2\lambda_{\mathbb{C}})_*} & H_*Q_0S^{-2}. \end{array}$$

Notice that we demonstrated that $(\Omega\lambda_{\mathbb{C}})_*$ is an epimorphism when restricted to the submodule of primitives, which implies that primitive classes H_*QS^{-1} pull back to H_*QS^{-2}. Notice that $\sigma_* : QH_{*-1}Q\Sigma^{-1}\mathbb{C}P \to PH_*Q\mathbb{C}P$ is an isomorphism. The way that we defined $v_{4n+1}, v_{i,j-1} \in H_*QS^{-2}$ implies that

$$\begin{aligned} \sigma_* v_{4n+1} &= p^{S^{-1}}_{4n+2}, \\ \sigma_* v_{i,j-1} &= p^{S^{-1}}_{i,j}. \end{aligned}$$

One then can apply Kudo's transgression theorem to obtain

$$\begin{aligned} \sigma_* Q^I v_{4n+1} &= Q^I p^{S^{-1}}_{4n+2}, \\ \sigma_* Q^L v_{i,j-1} &= Q^L p^{S^{-1}}_{i,j}. \end{aligned}$$

Important Note. We like to draw the reader's attention to the fact that according to the above equations any given primitive class in $H_*Q_0S^{-1}$ definitely pulls back to $H_*Q_0S^{-2}$.

Now we are in a position to complete the process of elimination of spherical classes. We have the following.

Theorem 5.43. *Let $\xi = \zeta^{2^t} \in H_*Q_0S^0$ be a spherical class with $\sigma_*\zeta \neq 0$. Then it is impossible to have $t > 1$.*

Proof. We do the proof for $t = 2$, i.e. $\xi = \zeta^4 = Q^{2d}Q^d\zeta$ where $d = \dim\zeta$. The approach is similar to our proof of the analogous theorem for H_*QS^n with $n > 0$, but with a little bit more complication. The fact that ξ is an A-annihilated primitive, implies that ζ is also an A-annihilated primitive class. Hence we may write $\zeta = \sum Q^I p'_{2i+1} + P^2$ with $(I, 2i+1)$ admissible, and P a primitive. According to Theorem 5.24 if ζ is odd dimensional, then $P^2 = 0$ and all terms $Q^I p'_{2i+1}$ in the expression for ζ are A-annihilated. On the other hand, if ζ is even dimensional then according to Theorem 5.26 we know that I has only odd entries with $(I, 2i+1)$ satisfying condition 3 of Theorem 2. Notice that we don't know very much about P^2. But still we know that we may write ζ as a sum, $Q^J p'_{2j+1}$ where J is admissible but $(J, 2j+1)$ is not necessarily admissible. Hence we may write

$$\xi = \sum Q^{2d}Q^dQ^L p'_{2l+1},$$

with L admissible, taking all of the above terms into one big sum, where some $(L, 2l+1)$ are admissible and some are not. Such a class pulls back to a $4d-1$ dimensional class $\xi_{-1} \in H_{4d-1}Q_0S^{-1}$ given by

$$\xi_{-1} = \sum Q^{2d}Q^dQ^L w'_{2l} + D_{-1},$$

where D_{-1} denotes the decomposable part. This is an odd dimensional primitive class, i.e. its indecomposable part must belong to the kernel of the square root map. Hence either d is odd, L has at least one odd entry, or l is odd. Hence we may rewrite the above class as

$$\xi_{-1} = \sum_{l \text{ odd}} Q^{2d}Q^dQ^L w'_{2l} + \sum_{l \text{ even}} Q^{2d}Q^dQ^L w'_{2l} + D_{-1}.$$

We have already calculated the set of primitive classes in H_*QS^{-1}, hence we write

$$\xi_{-1} = \sum_{l \text{ odd}} Q^{2d}Q^dQ^Lp_{2l}^{S^{-1}} + \sum_{l \text{ even}} Q^{2d}Q^Kp_{k,l}^{S^{-1}} + D_{-1},$$

where D_{-1} is an odd dimensional decomposable primitive class. Hence $D_{-1} = 0$, i.e.

$$\xi_{-1} = \sum_{l \text{ odd}} Q^{2d}Q^dQ^Lp_{2l}^{S^{-1}} + \sum_{l \text{ even}} Q^{2d}Q^Kp_{k,l}^{S^{-1}}.$$

Notice that $Q^dQ^Lp_{2l}^{S^{-1}}$ and $Q^Kp_{k,l}^{S^{-1}}$ are of dimension $2d-1$. We plan make use of Sq^1_*, but not here as in this exterior algebra $Sq^1_*\xi_{-1}$ will be a square which is trivial in the exterior algebra. Instead we desuspend once more. In this case the spherical class ξ_{-1} pulls back to a spherical class $\xi_{-2} \in H_{4d-2}Q_0S^{-2}$. Notice that this class is unique, while we fix the homotopy class that maps to ζ^4 under the Hurewicz homomorphism. On the other hand, recall that $(\Omega\lambda_{\mathbb{C}})_* : H_*Q\mathbb{C}P \to H_*QS^{-1}$ is an epimorphism, when restricted to the primitive submodules. Applying the Eilenberg-Moore spectral sequence machinery shows that this gives rise to a unique class in $\xi_{-2}^{\mathbb{C}P} \in QH_*Q\Sigma^{-1}\mathbb{C}P$. We then obtain a set of classes $\xi_{-2}^{\mathbb{C}P} + D_{-2}^{\mathbb{C}P} \in H_*Q\Sigma^{-1}\mathbb{C}P$ where $D_{-2}^{\mathbb{C}P}$ runs over decomposable terms, such that

$$(\Omega\lambda_{\mathbb{C}})_*\sigma_*(\xi_{-2}^{\mathbb{C}P} + D_{-2}^{\mathbb{C}P}) = \xi_{-1}.$$

This implies that $\xi_{-2} = (\Omega^2\lambda_{\mathbb{C}})_*(\xi_{-2}^{\mathbb{C}P} + D_{-2}^{\mathbb{C}P})$ for any choice of $D_{-2}^{\mathbb{C}P}$. Hence ξ_{-2} is in the exterior subalgebra generated by $\mathrm{im}(\Omega^2\lambda_{\mathbb{C}})_*$. Hence, we may write

$$\xi_{-2} = \sum_{l \text{ odd}} Q^{2d}Q^dQ^Lv_{2l-1} + \sum_{l \text{ even}} Q^{2d}Q^Kv_{k,l-1} + D_{-2}.$$

where D_{-2} denotes the decomposable part. The classes $Q^dQ^Lv_{2l-1}$ and $Q^Kv_{k,l-1}$ are of dimension $2d-2$. Although one may decide to rewrite this sum in terms of primitives, however this form is enough for our claim. Observe that

$$Sq^1_*\xi_{-2} = \sum_{l \text{ odd}} Q^{2d-1}Q^dQ^Lv_{2l-1} + \sum_{l \text{ even}} Q^{2d-1}Q^Kv_{k,l-1} + Sq^1_*D_{-2}.$$

Notice that $Sq^1_*D_{-2}$ is a decomposable. On the other hand terms $Q^{2d-1}Q^dQ^Lv_{2l-1}$ and $Q^{2d-1}Q^Kv_{k,l-1}$ are separated under this action as the map to distinct terms under the homology suspension, which also shows that these classes do not belong to

$\ker \sigma_*$. This shows that $Sq^1_*\xi_{-2} \neq 0$. But this is a contradiction. This completes the proof. □

Remark 5.44. The above calculation is based on explicit knowledge of the type of primitive classes in $H_*Q_0S^{-1}$. However, we may have a less detailed proof. Notice that in the above lemma $\xi = Q^{2d}Q^dp$ for some A-annihilated primitive class $p \in H_dQ_0S^0$. According to Example 5.7, the class p has a unique pull back in $q_{-1} \in QH_{d-1}Q_0S^{-1}$. This implies that if $f \in \pi_{4d}Q_0S^0$ with $hf = \xi$, then its isomorphic pull back satisfies $f_{-1} \in \pi_{d-1}Q_0S^{-1}$ with $\xi_{-1} = hf_{-1} = Q^{2d}Q^dq_{-1}$ modulo decomposable classes. The fact that ξ_{-1} is primitive implies that either q_{-1} is in $\ker r_{S^{-1}}$ or d is odd.

If $q_{-1} \in \ker r_{S^{-1}}$, then $q_{-1} = p_{-1}$ modulo decomposable terms, for some $p_{-1} \in PH_*Q_0S^{-1}$. This then implies that $\xi_{-1} = Q^{2d}Q^dp_{-1}$ modulo decomposable primitives. However, ξ_{-1} lives in odd dimensions which implies that the decomposable part has to be trivial. Hence

$$\xi_{-1} = Q^{2d}Q^dp_{-1}.$$

Such a class pulls back to $H_{4d-2}Q_0S^{-2}$ to a spherical class ξ_{-2}. We may write $\xi_{-2} = Q^{2d}Q^dq_{-2}$ modulo decomposable terms. Notice that ξ_{-2} is a spherical class and hence A-annihilated. Similar to the previous proof, applying Sq^1_* shows that $Sq^1_*\xi_{-2} \neq 0$ which is a contradiction for ξ being A-annihilated.

If it happens that $q_{-1} \notin \ker r_{S^{-1}}$, then d has to be odd. Hence $Q^dq_{-1} = p_{-1}$ modulo decomposable terms, for some $p_{-1} \in PH_*Q_0S^{-1}$. We then have $\xi_{-1} = Q^{2d}p_{-1}$ modulo decomposable terms. Similarly, for dimensional reasons the decomposable part is trivial. In this case, ξ_{-1} pulls back to a spherical class $\xi_{-2} \in H_{4d-2}Q_0S^{-2}$ with $\xi_{-2} = Q^{2d}q_{-2}$ modulo decomposable terms. Applying Sq^1_* will give the contradiction that we were looking for. This completes the proof.

5.9 The Curtis Conjecture

This section contains the ideas that I believe proves the Curtis conjecture, or at least provides a reasonable, and fruitful, approach towards identifying the spherical classes in $H_*Q_0S^0$. I will sketch a hand wavy argument which I believe can lead to a complete proof of the Curtis conjecture. The main idea is to do the same as we did in the proof of Theorem 5.43. This section tries to justify that these ideas do work and this is the right attitude to tackle the conjecture. This also proves to be fruitful, and one obtains other results that are interesting on their own.

Throughout this section we will distinguish between a homology class and its cohomology dual. We also distinguish between π_* and ${}_2\pi_*$, but we keep H_* to denote the homology with $\mathbb{Z}/2$-coefficients.

Observe that the main idea in the proof of Theorem 5.43 was to write a spherical class in terms of primitive classes, and pull it back twice and use some Steenrod operation to prove the theorem. We would like to follow the same line, write classes in terms of primitives and pull them back as much as we need and then apply some Steenrod operation. The calculations may seem more hand wavy, but the right manipulation shows that these are justifiable arguments.

One of the tools that we use is the Milnor-Moore exact sequence to get some idea about primitive classes in our homology algebras. Notice that QS^n is an infinite loop space for $n \in \mathbb{Z}$. This implies that the Pontrjagin product in H_*QS^n is commutative. Moreover, the co-product in H_*QS^n comes from the cup product in cohomology which is already commutative. Hence, H_*QS^n is a bicommutative Hopf algebra and we can apply Milnor-Moore exact sequence to study primitive classes in these algebras.

The calculation of the homology algebra H_*QS^n is based on applying a suitable spectral sequence to the path-loop fibration

$$QS^n \to PQS^n \to QS^{n+1}$$

and using our knowledge of the homology algebra H_*QS^{n+1}. Having said this, our objective is not to calculate H_*QS^n for $n < 0$, nor do we want to calculate spherical classes in H_*QS^n for $n < 0$. We only wish to obtain some information to use in our approach to eliminate the *rest* of the unwanted potential classes in $H_*Q_0S^0$ from being spherical in $H_*Q_0S^0$. More precisely, we would like to use this technique to show that only the Hopf invariant, and the Kervaire invariant elements give rise to spherical classes in $H_*Q_0S^0$.

Let me explain what we mean by the unwanted potential classes. First recall that according to Lemma 5.11 any A-annihilated primitive class $\xi \in H_*Q_0S^0$ with $\sigma_*\xi \neq 0$ can be written as

$$\xi = \sum Q^I x_{2i+1}$$

modulo decomposable terms, where $(I, 2i+1)$ runs over certain admissible sequences. Now, let $\theta \in H_*Q_0S^0$ be a spherical class. Then we have two separate cases: $\sigma_*\theta = 0$, and $\sigma_*\theta \neq 0$.

If $\sigma_*\theta \neq 0$, then we may assume that $\theta = \sum Q^I x_{2i+1}$ modulo decomposable terms. From Lemma 4 and Remark 5 we know that if there exists $I = \phi$ in the above expression for θ, then θ corresponds to a Hopf invariant one element. Hence, if $\theta = \sum Q^I x_{2i+1} \in H_*Q_0S^0$ is a spherical class with $\sigma_*\theta \neq 0$, which is not a Hopf invariant one element, then $l(I) > 0$ for any I involved in the expression for θ, i.e. $\min(l(I)) > 0$.

If $\sigma_*\theta = 0$, then $\theta = \xi^{2^t}$ for some $t > 0$, and $\sigma_*\xi \neq 0$. According to Theorem 17 we have $t = 1$, i.e. $\theta = \xi^2$ with $\xi = \sum Q^I x_{2i+1}$ modulo decomposable terms. Moreover, Lemma 19 tells us that ξ must be an odd dimensional class. Similar to the previous case, if there exists $I = \phi$, then combining Remark 5 and Lemma 13 will imply that θ corresponds to a Kervaire invariant one element. Hence in this case, and similar to the previous case, the classes which are not Kervaire invariant one elements are those A-annihilated primitive classes θ will be those ones with $\min l(I) > 0$. Notice that if $l(I) > 0$, we then may apply Theorem 5.24 to the class ξ. This implies that we may write $\xi = \sum Q^I p'_{2i+1}$ with every $Q^I p'_{2i+1}$ in the sum being an A-annihilated class.

The following theorem summarises the above discussion, which classifies the type of classes that we want to be eliminate from being spherical.

Theorem 5.45. *Let $\theta \in H_*Q_0S^0$ be a spherical class which is not a Hopf invariant* *one class, neither a Kervaire invariant one class. Then θ satisfies one of the the following cases.*

1- *If $\sigma_*\theta\neq 0$ and θ is an odd dimensional class, then*

$$\theta=\sum Q^I p'_{2i+1},$$

with $l(I)>1$ such that each of terms $Q^I p'_{2i+1}$ in the above sum is A-annihilated.

2- *If $\sigma_*\theta\neq 0$ and θ is an even dimensional class, then*

$$\theta=\sum Q^I p'_{2i+1}+P^2,$$

with $l(I)>1$ where I has only has odd entries. In this case $(I,2i+1)$ satisfies condition 3 of Theorem 2, i.e. $0<2i_{j+1}-i_j<2^{\rho(i_{j+1})}$ for $1\leqslant j\leqslant r$ with $i_{r+1}=2i+1$. Moreover, $\mathrm{excess}(Q^I p'_{2i+1})-1<2^{\rho(i_1)}$ for every $Q^I p'_{2i+1}$ involved in the above sum. Here P is a primitive term. If $P\neq 0$, then it is of odd dimension. If $P=0$, then each term in the above expression for θ is A-annihilated.

3- *If $\sigma_*\theta=0$, then $\theta=\xi^2$, with ξ an odd dimensional A-annihilated primitive class, i.e.*

$$\theta=(\sum Q^I p'_{2i+1})^2,$$

with $l(I)>0$ such that each of terms $Q^I p'_{2i+1}$ in the above sum is A-annihilated.

In all of the above cases $(I,2i+1)$ is supposed to be admissible.

Proof. Part 1 is the statement of Theorem 5.24. Part 3 follows from discussions above together with Theorem 5.24.

We focus on part 2. The first half is the statement of Theorem 5.26. Notice that in this case θ suspends to the odd dimensional spherical class $\sigma_*\theta=\sum Q^I Q^{2i+1}g_1\in H_*QS^1$, and hence according to Lemma 12 each of terms $Q^I Q^{2i+1}g_1$ must be A-annihilated. In particular this implies that $\mathrm{excess}(Q^I Q^{2i+1}g_1)<2^{\rho(i_1)}$. The fact that $\mathrm{excess}(Q^I Q^{2i+1}g_1)=\mathrm{excess}(Q^I p'_{2i+1})-1$ shows that $\mathrm{excess}(Q^I p'_{2i+1})-1<2^{\rho(i_1)}$. Now, we only need to show that if $P\neq 0$, then it is an odd dimensional class, and if $P=0$ then each term in the above sum is A-annihilated.

First, assume that $P \neq 0$. Then we may write

$$\theta = \sum Q^I p'_{2i+1} + Q^d P,$$

where $d = \dim P$. Similar to the proof of Theorem 5.43, we can show that this class pulls back to a spherical class $\xi_{-1} \in H_{2d-1}Q_0S^{-1}$, which may be written as

$$\xi_{-1} = \sum Q^I w'_{2i} + Q^d q_{-1},$$

modulo decomposable terms, where $\sigma_* q_{-1} = P$. The fact that ξ_{-1} is an odd dimensional class implies that it belongs to the kernel of the square root map. We note that our assumption that $l(I) > 0$ with I only having odd entries implies that $\sum Q^I w'_{2i}$ belongs to this kernel. Hence we need to have $Q^d q_{-1}$ to belong to this kernel. This means that either q belongs to this kernel, or d is odd. Assuming that d is even, implies that q_{-1} has to belong to this kernel. Hence we get a unique primitive class p_{-1} corresponding to q_{-1} modulo decomposable terms. A trick similar to the one in the proof of Theorem 5.43 implies that ξ_{-1} pulls back to a spherical class ξ_{-2} in $H_*Q_0S^{-2}$. Now applying Sq^1_* leads to a contradiction. This shows that d cannot be even.

Next, assume that $P = 0$. In this case we have an A-annihilated sum of terms $Q^I p'_{2i+1}$ of positive excess. Remark 3.22 now implies that each of the terms $Q^I p'_{2i+1}$ must be A-annihilated. □

The above theorem sets the target for us and identifies the classes that we were looking for to eliminate from being spherical in $H_*Q_0S^0$. In particular eliminating all cases mentioned by the above theorem will verify Curtis's conjecture. As we mentioned earlier, at the beginning of this section, our approach to do this elimination will be similar to the proof of Theorem 5.43. Notice that there is a narrow distinction between Hopf invariant and Kervaire invariant classes and other potential classes. The Hopf invariant one and the Kervaire invariant one elements do exist, if p'_{2^s-1} and $(p'_{2^s-1})^2$ are spherical in $H_*Q_0S^0$. We know that every spherical class ought to be A-annihilated. In the case of Hopf invariant and Kervaire invariant classes we end up with A-annihilated classes only for numerical reasons. More precisely, in these cases the action of the Steenrod algebra on p'_{2^s-1} and $(p'_{2^s-1})^2$ depends on the coefficients of the form $\binom{2^s-1-t}{t} \equiv 0 \bmod 2$.

However, in the cases other than the Hopf invariant and Kervaire invariant one elements there is kind of a dimensional reason. According to the above theorem in other cases we deal with classes of the form $Q^I p'_{2i+1}$ with $\mathrm{excess}(Q^I p_{2i+1}) < 2^{\rho(i_1)}$, square of such terms, or terms $Q^I p'_{2i+1}$ with $\mathrm{excess}(Q^I p'_{2i+1}) - 1 < 2^{\rho(i_1)}$. Our approach is to show that the assumption of such a class being spherical will imply that it can be desuspended enough times to a class which is not A-annihilated, hence giving a contradiction. We may gain this goal if we succeed to show that a given primitive class $Q^I p'_{2i+1} \in H_*Q_0S^0$ desuspends to a class of the form $Q^{i_1} q_{-(2^{\rho(i_1)}+1)}$ where $q_{-(2^{\rho(i_1)}+1)} \in H_{*-(2^{\rho(i_1)}+1)}Q_0S^{-(2^{\rho(i_1)}+1)}$. Then it is quite straightforward to see that

$$Sq_*^{2^{\rho(i_1)}} Q^{i_1} q_{-(2^{\rho(i_1)}+1)} = Q^{i_1-2^{\rho(i_1)}} q_{-(2^{\rho(i_1)}+1)} \neq 0$$

which will give us the contradiction that we were looking for. Of course we know that in general a spherical class can be a sum of primitive terms $Q^I p'_{2i+1}$, and hence we have to choose the right term to work with and pursue the above approach.

Finally, notice that having a spherical classes $\theta \in H_nQ_0S^0$, we obtain $f \in {}_2\pi_nQ_0S^0$ with $hf = \theta$. It is possible to have different elements in ${}_2\pi_nQ_0S^0$ mapping to θ under the Hurewicz homomorphism. However, we fix f at the beginning. We use f_{-k} to denote the unique desuspension of f into ${}_2\pi_{n-k}Q_0S^{-k}$. Notice that f desuspends as far as $f_{-n} \in {}_2\pi_0Q_0S^{-n}$ with $\sigma_*^n hf_{-n} = \theta$. This implies that $\sigma_*^k hf_{-n} \neq 0$ for $1 \leqslant k \leqslant n$, which in return defines unique desuspensions of θ into spherical classes $\theta_{-k} \in H_{n-k}Q_0S^{-k}$. Notice that this can be thought of an obstruction for being spherical. More precisely, if θ_{-i} for some $i > 0$ does not desuspend to $H_{n-i-1}Q_0S^{-i-1}$, then θ cannot be spherical in $H_nQ_0S^0$.

In order to follow this line, we need to identify subalgebras of $H_*Q_0S^{-k}$ which will contain pull back of a spherical class $\theta \in H_*Q_0S^0$.

Remark 5.46. Recall that a given primitive class in $H_*Q_0S^0$ does pull back to a class in $H_*Q_0S^{-1}$. Similarly, according to the Important Note before Theorem 5.43, a given primitive class in $H_*Q_0S^{-1}$ does pull back to a class in $H_*Q_0S^{-2}$. Such pull back classes give rise to nontrivial classes in the quotient module of indecomposables. So we may talk about the primitive classes they may give rise to, depending on their action under the square root map. However, it is not clear that at each stage a primitive class in $H_*Q_0S^{-k}$ will pull back to a class in $H_{*-1}Q_0S^{-k-1}$.

We proceed based on the assumptions that primitives pull back, and examine the implications. We also give clear description, and prove the truth of a such claim in some special cases, verified by Remark 5.52. Notice that claiming that a give primitive class in H_*X pulls back to a class in $H_{*-1}\Omega X$ is the same as claiming that there is a surjection

$$QH_{*-1}\Omega X \twoheadrightarrow PH_*X.$$

Remark 5.47. Previously, the use of the complex transfer $\lambda_{\mathbb{C}}: Q\Sigma\mathbb{C}P_+ \to Q_0S^0$ allowed us to give a geometric meaning to generators $w_{2i} \in H_{2i}QS^{-1}$ and define them uniquely. However, we lack such a tool in general case. Although it seems possible to use the quaternionic transfer $\lambda_{\mathbb{H}}: Q\mathbb{H}P^\tau \to Q_0S^0$ to obtain geometric description for generators of H_*QS^{-3}. Here $\mathbb{H}P^\tau$ is James's quasi-projective space [A61, Proposition 5.3].

Usually, the spaces QS^{-n} for $n > 0$ are not path connected. The reason is that $\pi_0QS^{-n} \simeq \pi_n^S$ which is not always trivial. This implies that $QS^{-n} \simeq \pi_n^S \times Q_0S^{-n}$. Notice that here we only work with loop structure arising from the loop sum on these spaces. Now consider the path connected space Q_0S^{-n} with $\pi_1Q_0S^{-n} \simeq \pi_{n+1}^S$. Notice that the groups π_{n+1}^S are Abelian. Moreover, the space Q_0S^{-n} is an infinite loop space, in particular it is an associative H-space. Hence according to [CLM76, Part I, Lemma 4.7] there is a map $K(\pi_{n+1}^S, 1) \to Q_0S^{-n}$ inducing the weak homotopy equivalence

$$\overline{Q_0S^{-n}} \times K(\pi_{n+1}^S, 1) \to Q_0S^{-n},$$

We may loop this equivalence to get the following weak homotopy equivalence

$$\Omega\overline{Q_0S^{-n}} \times \pi_{n+1}^S \to \Omega Q_0S^{-n} = QS^{-n-1},$$

where $\overline{QS^{-n}}$ denotes a suitable model of the universal cover for a space QS^{-n}. Notice that $QS^{-n-1} \simeq Q_0S^{-n-1} \times \pi_{n+1}^S$. Restricting to the base point components then gives the weak homotopy equivalence

$$\Omega\overline{Q_0S^{-n}} \to Q_0S^{-n-1}.$$

Notice that according to [CLM76, Part I, Lemma 4.8] $\overline{Q_0S^{-n}}$ is an infinite loop space and the inclusion map $\overline{Q_0S^{-n}} \to Q_0S^{-n}$ is a map of infinite loop spaces. This then induces an infinite loop structure on Q_0S^{-n-1}. Regarding the application of spectral sequences to calculate Q_0S^{-n-1} the equivalence $\Omega\overline{Q_0S^{-n}} \to Q_0S^{-n-1}$ then tells us to apply a suitable spectral sequence to the following fibration

$$Q_0S^{-n-1} \to P\overline{Q_0S^{-n}} \to Q_0S^{-n},$$

where the base space is simply connected, which eases the calculations. Notice that if a primitive class in $H_*Q_0S^{-n}$ pulls back to $H_{*-1}Q_0S^{-n-1}$, then we may look at the suspension homomorphism in the Eilenberg-Moore spectral sequence associated with the above fibration. Notice that we may continue doing the above argument and inductively conclude that $Q_0S^{-n} = \Omega^nQ_0S^0\langle n\rangle$ where $Q_0S^0\langle n\rangle$ denotes the n-connected cover of Q_0S^0.

Remark 5.48. Notice that it is easy to derive the equivalence $\Omega Q_0S^{-n} \to Q_0S^{-n-1}$. This can be obtained once we observe that $\overline{Q_0S^{-n}}$ by the fibration sequence

$$\overline{Q_0S^{-n}} \to Q_0S^{-n} \to K(\pi^S_{n+1}, 1).$$

We then may loop the above fibration, and see that $\Omega\overline{Q_0S^{-n}}$ and Q_0S^{-n-1} have the same homotopy groups. We also like to note that we have note claimed that $\overline{Q_0S^{-n}} \times K(\pi^S_{n+1}, 1) \to Q_0S^{-n}$, is an equivalence of H-spaces, nor general infinite loop spaces.

Now we return to our main problem. If we assume that the primitive classes of $H_*Q_0S^{-n}$ at any stage pulls back to some classes in $H_*Q_0S^{-n-1}$, we then are following an inductive argument. We have done the base case of this induction in Section 5.7 where we fixed a basis for $PH_*Q_0S^0$. Let us quickly review that work, without using the complex transfer.

5.9.1 Homology of Q_0S^{-1} without the complex transfer!

We fix p'_{2i+1} as a basis for primitives in $H_*Q_0S^0$. According to our discussion in Remark 5.46, assuming that p'_{2i+1} to $H_*Q_0S^{-1}$ (which in this case we know it will definitely do), we obtain an indecomposable class $w'_{2i} \in QH_{2i}Q_0S^{-1}$ with $\sigma_*w'_{2i} = p'_{2i+1}$. The fact that Q_0S^{-1} is an infinite loop space allows us to consider Q^Iw_{2i} with $\sigma_*Q^Iw'_{2i} = Q^Ip_{2i+1}$. Notice that Q^Ip_{2i+1} is a primitive class, and hence there is a class $q^{S^{-1}}_{I,2i} \in QH_*Q_0S^{-1}$ with $\sigma_*q^{S^{-1}}_{I,2i} = Q^Ip_{2i+1}$. This implies that $Q^Iw'_{2i} = q^{S^{-1}}_{I,2i}$ mod $\ker\sigma_*$. Although in this case the fact that $\sigma_* : QH_*Q_0S^{-1} \to PH_*Q_0S^0$ is an isomorphism implies that $Q^Iw'_{2i} = q^{S^{-1}}_{I,2i}$. But we don't know whether this holds in general. However, we choose to work with $Q^Iw'_{2i}$ while we do the rest of the work.

Now, a given class $\sum Q^Ip_{2i+1} \in H_*Q_0S^0$ will pull back to a class of the form $\sum Q^Iw'_{2i}$ modulo decomposable terms. Such a class ought to be primitive. This implies that $\sum Q^Iw'_{2i}$ belong to the kernel of the square root map $r_{S^{-1}} : QH_*Q_0S^{-1} \to QH_*Q_S^{-1}$. Recall from relations (5.2) and (5.3) that $rQ^{2i} = Q^ir$ and $rQ^{2i+1} = 0$, where r denotes the square root map. Hence, we only need to calculate $r_{S^{-1}}w'_{2i}$. On the other hand recall that for a given $2n$ dimensional class ξ we have $r\xi = Sq^n_*\xi$ [W82, Definition 2.1]. This observation together with the stability of the Steenrod operations allows us to redo the calculation of $r_{S^{-1}}w'_{2i}$ as following,

$$\begin{aligned}\sigma_*r_{S^{-1}}w'_{2i} &= \sigma_*Sq^i_*w'_{2i}\\ &= Sq^i_*p'_{2i+1}\\ &= \binom{i+1}{i}p_{i+1} = \begin{cases}0 & \text{if } i \text{ is odd,}\\ p_{i+1} & \text{if } i \text{ is even.}\end{cases}\end{aligned}$$

This implies that modulo $\ker\sigma_*$ we have,

$$r_{S^{-1}}w'_{2i} = \begin{cases}0 & \text{if } i = 2k+1,\\ w'_{2k} & \text{if } i = 2k.\end{cases}$$

Recall from Proposition 5.4 and Example 5.5 that $\sigma_* : QH_*Q_0S^{-1} \to PH_*Q_0S^0$ is an isomorphism. This implies that w'_{2i} belongs to the kernel of $r_{S^{-1}} : QH_*Q_0S^{-1} \to QH_*Q_0S^{-1}$ only if $i = 2k+1$ for some k. Of course, we validated that this point in section 5.7 using the complex transfer.

Now assume that $\theta = \sum Q^I p'_{2i+1}$ with $(I, 2i+1)$ admissible, and I only formed of odd entries. If we suppose that that θ is a spherical class, it then must pull back to a spherical class whose image in $QH_*Q_0S^{-1}$ is given by $\sum Q^I w'_{2i}$ modulo $\ker\sigma_*$. The fact that θ_{-1} is a primitive class implies that $\sum Q^I w'_{2i} \in \ker r_{S^{-1}}$. Notice that $w'_{2i} \notin \ker r_{S^{-1}}$ for i even. However, having $I = (i_1, \ldots, i_r)$ formed only of odd entries, we know that $Q^{i_r} w'_{2i} \in \ker r_{S^{-1}}$. Similarly, if i is odd and $w'_{2i} \notin \ker r_{S^{-1}}$, we may then consider to $Q^{i_r} w'_{2i} \in \ker r_{S^{-1}}$. In these cases, modulo decomposable terms, we obtain primitive elements,

$$\begin{aligned} p_{2i}^{S^{-1}} &= w'_{2i} && \text{if } i = 2k+1; \\ p_{i,j}^{S^{-1}} &= Q^{2i+1} w'_{2j} && \text{if } j = 2k. \end{aligned}$$

Hence, modulo decomposable classes, the image of θ_{-1} in $H_*Q_0S^{-1}$ belongs to the R-module spanned by $p_{2i}^{S^{-1}}$ and $p_{i,j}^{S^{-1}}$.

We now apply Lemma 5.46 to the primitive classes $p_{2i}^{S^{-1}}$ and $p_{i,j}^{S^{-1}}$, and work out their pull backs into classes in $QH_*Q_0S^{-2}$. Similarly, the spherical class θ_{-2} must be primitive. On then can apply naturlaity of the Steenrod operations to work out the action of the square root map. The fact that θ_{-2} is primitive forces its image in $QH_*Q_0S^{-2}$ to belong to the kernel of the square root map

$$r_{S^{-2}} : QH_*Q_0S^{-2} \to QH_*Q_0S^{-2}.$$

We have the following observation on the primitive classes in $H_*Q_0S^{-2}$.

Lemma 5.49. *Let $\theta \in H_*Q_0S^0$ be a spherical class. Then, modulo the kernel of $\sigma_* : H_*Q_0S^{-2} \to H_*Q_0S^{-1}$ this class pulls back to a primitive class belonging to the R-submodule of $H_*Q_0S^{-2}$ generated by the primitive classes $p_{2i-1}^{S^{-2}}$, $p_{i,j-1}^{S^{-2}}$ and $p_{f,i,j-1}^{S^{-2}}$ such that*

$$\begin{aligned} \sigma_* p_{2i-1}^{S^{-2}} &= p_{2i}^{S^{-1}}, \\ \sigma_* p_{i,j-1}^{S^{-2}} &= p_{i,j}^{S^{-1}}, \\ \sigma_* p_{f,i,j-1}^{S^{-2}} &= Q^{2f+1} p_{i,j}^{S^{-1}}. \end{aligned}$$

In particular, modulo decomposable terms, we have

$$\sigma_* p_{f,i,j-1}^{S^{-2}} = Q^{2f+1} Q^{2i+1} w'_{2j}.$$

Proof. Let $v_{2i-1} \in QH_*Q_0S^{-2}$ be a class with $\sigma_*v_{2i-1} = p_{2i}^{S^{-1}}$. This is an odd dimensional class, and hence belongs to the kernel of $r_{S^{-2}}$. Hence we obtain

$$p_{2i-1}^{S^{-2}} = v_{2i-1} \text{ mod decomposable terms.}$$

On the other hand, let $v_{i,j-1} \in H_{2i+2j}Q_0S^{-2}$ be a class such that $\sigma_*v_{i,j-1} = p_{i,j}^{S^{-1}}$. Notice that this class is defined if $j = 2k$. We need to apply Sq_*^{i+j} to this class. This yields the following modulo decomposable terms,

$$\begin{aligned}
\sigma_*r_{S^{-2}}v_{i,j-1} &= \sigma_*Sq_*^{i+j}v_{i,j-1}\\
&= Sq_*^{i+j}p_{i,j}^{S^{-1}}\\
&= Sq_*^{i+j}Q^{2i+1}w'_{2j}\\
&= \binom{i-j+1}{i-j}Q^{i+1}Sq_*^{j}w'_{2j}\\
&= \begin{cases} Q^{i+1}w'_{2k} & \text{if } i = 2l;\\ 0 & \text{if } i = 2l+1; \end{cases}
\end{aligned}$$

where $j = 2k$. Notice that according to the above calculation, if $j = 2k$ then

$$Sq_*^{i+j}p_{i,j}^{S^{-1}} = \begin{cases} p_{l,k}^{S^{-1}} & \text{if } i = 2l;\\ 0 & \text{if } i = 2l+1; \end{cases}$$

The above calculations also show that $v_{i,j-1}^{S^{-2}}$ can fall into $\ker r_{S^{-2}}$, if $i = 2l+1$. Hence, modulo decomposable terms, we obtain the following set of primitive classes,

$$\begin{aligned}
p_{i,j-1}^{S^{-2}} &= v_{i,j-1} && \text{if } i = 2l+1 \text{ and } v_{i,j-1} \in \ker r_{S^{-2}};\\
p_{f,i,j-1}^{S^{-2}} &= Q^{2f+1}v_{i,j-1} && \text{if } i = 2l+1 \text{ and } v_{i,j-1} \notin \ker r_{S^{-2}};\\
p_{f,i,j-1}^{S^{-2}} &= Q^{2f+1}v_{i,j-1} && \text{if } i = 2l.
\end{aligned}$$

Now having a spherical class $\theta = \sum Q^Ip_{2i+1}$, it pulls back to a class which belongs to the R module spanned by the primitives which suspends to p_{2i+1}'s. Applying the Lemma 5.46 then implies the lemma. □

Remark 5.50. Let $2i+1$ be an odd number which is not of the form $2^\alpha - 1$. Then $2i+1 = 2^{\rho+1}N + 2^\rho - 1$ for some integer $N > 0$ with $\rho = \rho(2i+1)$. The fact $2i+1 = 2^\rho(2N+1) - 1$ implies that $i = 2^{\rho-1}(2N+1) - 1$. Assuming that $\rho > 1$, implies that i is of the form $2k+1$. Notice that an A-annihilated term, $Q^Ip'_{2i+1}$ will have most entries which are not of the form $2^\alpha - 1$. This comes from the constructions that we have provided in Appendix C. In fact most of the sequences I satisfying condition 3 of Theorem 2 have this property. This means that in practice we will not have so many primitive classes of the second and third form above involved in the expression for a spherical class.

We hope that the above lemma illustrates the pattern that we claim to happen while we desuspend a spherical class $\theta \in H_*Q_0S^0$. However, the calculation become more delicate and tedious. The reason being that during these calculations we need to find out about the action of the square root map. However, as we observed to do this we need to use the homology suspension. We lack this part of information, and calculation after this involves indeterminacies up to $\ker \sigma_*$.
Of course for our purpose this is not that bad. More precisely, if the kernel becomes very huge then it may imply that we cannot pull back further which as we noted, before Lemma 5.46, is an obstruction for being spherical. On the other hand if $\ker \sigma_*$ does behave in our favorite manner, then we can continue with our calculations and finish the proof. We like to present one more example to illustrate the heavy calculations that has to be carried out.

Example 5.51. Let $\theta \in H_*Q_0S^0$ be a spherical class. Then we like to calculate, modulo the kernel of $\sigma_* : H_*Q_0S^{-3} \to H_*Q_0S^{-2}$, the subalgebra of $H_*Q_0S^{-3}$ which will contain the pull back of a spherical class $\theta \in H_*Q_0S^0$.
Lemma 5.49 implies that a spherical $\theta \in H_*Q_0S^0$ pulls back to a class which, modulo the kernel of $\sigma_* : H_*Q_0S^{-2} \to H_*Q_0S^{-1}$, belongs to the R-module generated by the primitive classes $p^{S^{-2}}_{2i-1} \in H_{2i-1}Q_0S^{-2}$, $p^{S^{-2}}_{i,j-1} \in H_{2i+2j}Q_0S^{-2}$, and $p^{S^{-2}}_{f,i,j-1} \in H_{2f+2i+2j+1}Q_0S^{-2}$ such that

$$\begin{aligned}
\sigma_* p^{S^{-2}}_{2i-1} &= p^{S^{-1}}_{2i}, \\
\sigma_* p^{S^{-2}}_{i,j-1} &= p^{S^{-1}}_{i,j}, \\
\sigma_* p^{S^{-2}}_{f,i,j-1} &= Q^{2f+1} p^{S^{-1}}_{i,j}.
\end{aligned}$$

Based on the assumption that such classes do pull back, we obtain indecomposable classes $q^{S^{-3}}_{2i-2} \in QH_{2i-2}Q_0S^{-3}$, $q^{S^{-3}}_{i,j-2} \in QH_{2i+2j-1}Q_0S^{-3}$, and $q^{S^{-3}}_{f,i,j-2} \in QH_{2f+2i+2j}Q_0S^{-3}$

such that

$$\begin{aligned}\sigma_* q_{2i-2}^{S^{-3}} &= p_{2i-1}^{S^{-2}},\\ \sigma_* q_{i,j-2}^{S^{-3}} &= p_{i,j-1}^{S^{-2}},\\ \sigma_* q_{f,i,j-2}^{S^{-3}} &= p_{f,i,j-1}^{S^{-2}}.\end{aligned}$$

The class $q_{i,j-2}^{S^{-3}}$ is odd dimensional, and so gives rise to a primitive class $p_{i,j-2}^{S^{-3}}$. These other two classes are even dimensional. Hence, we need to look at the action of the square root map $r_{S^{-3}} : H_*Q_0S^{-3} \to H_*Q_0S^{-3}$ on these classes. First, we deal with $q_{2i-2}^{S^{-3}}$ which is of dimension $2i-2$, and we consider the action of Sq_*^{i-1} on this coupled with the action of homology suspension $H_*Q_0S^{-3} \to H_*Q_0S^{-2}$. Then, modulo decomposable terms, we have

$$\begin{aligned}\sigma_* Sq_*^{i-1} q_{2i-2}^{S^{-3}} &= Sq_*^{i-1}\sigma_* q_{2i-2}^{S^{-3}}\\ &= Sq_*^{i-1} p_{2i-1}^{S^{-2}}\\ &= Sq_*^{i-1} v_{2i-1}.\end{aligned}$$

At this point we need to re-use the homology suspension to calculate $Sq_*^{i-1}v_{2i-1}$. Notice that in this case v_{2i-1} suspends to $p_{2i}^{S^{-1}}$ where is defined for $i = 2k+1$. This yields, modulo decomposable terms, the following

$$\begin{aligned}\sigma_* Sq_*^{i-1} v_{2i-1} &= Sq_*^{i-1} p_{2i}^{S^{-1}}\\ &= Sq_*^{i-1} w'_{2i}\\ &= \tbinom{i+1}{i-1} w'_{i+1}\\ &= \tfrac{(i+1)i}{2} w'_{i+1}\\ &= \tfrac{(2k+2)(2k+1)}{2} w'_{i+1}\\ &= \begin{cases} 0 & \text{if } k \text{ odd;}\\ w'_{i+1} & \text{if } k \text{ even.}\end{cases}\end{aligned}$$

This implies that $r_{S^{-3}} q_{2i-2}^{S^{-3}} \neq 0$ if $i = 2k+1$ with k is even. Hence, $q_{2i-2}^{S^{-3}}$ may belong to $\ker r_{S^{-3}}$ only if $i = 2k+1$ and k is odd. Hence, we obtain the following corresponding primitive classes which modulo decomposable terms are given by

$$\begin{array}{ll} q_{2i-2}^{S^{-3}} & \text{if } i = 2k+1 \text{ and } k \text{ is odd, with } q_{2i-2}^{S^{-3}} \in \ker r_{S^{-3}};\\ q_{2i-2}^{S^{-3}} & \text{if } i = 2k+1 \text{ and } k \text{ is odd, with } q_{2i-2}^{S^{-3}} \notin \ker r_{S^{-3}};\\ Q^{2f+1} q_{2i-2}^{S^{-3}} & \text{if } i = 2k+1 \text{ and } k \text{ is even.}\end{array}$$

We note that this will be the largest possible set of primitives that one can get out of $q^{S^{-3}}_{2i-2}$, and perhaps knowing more about the kernel of $\sigma_*: H_*Q_0S^{-3} \to H_*Q_0S^{-2}$ will eliminate some of these cases. Previously the fact that $\sigma_*: QH_*Q_0S^{-1} \to PH_*Q_0S^0$ being an isomorphism helped us to get lower number of indecomposables. This fact also helped us to calculate the action of the Steenrod operations Sq^i_* on indecomposable terms w'_{2i} up to a smaller indeterminacy.

Finally we note that one may employ similar methods to calculate the action of $r_{S^{-3}}$ on indecomposable classes $q^{S^{-3}}_{f,i,j-2} \in QH_{2f+2i+2j}Q_0S^{-3}$. This class suspends to $p^{S^{-2}}_{f,i,j-1}$ which suspends to $Q^{2f+1}Q^{2i+1}w'_{2j}$ modulo decomposable terms. Recall that such a class is defined when j is even. We need to calculate $Sq^{i+j+f}_* q^{S^{-3}}_{f,i,j-2}$ and see when it possibly vanishes and when it does not vanish. Similar to what we did before, we look at $Sq^{i+j+f}_*Q^{2f+1}Q^{2i+1}w'_{2j}$. Recall that here j is even, and i can be either odd or even. We have the following, up to decomposable terms,

$$\begin{aligned} Sq^{i+j+f}_*Q^{2f+1}Q^{2i+1}w'_{2j} &= \binom{f-i-j+1}{f-i-j} Q^{f+1}Sq^{i+j}_*Q^{2i+1}w'_{2j} \\ &= \begin{cases} Q^{f+1}Sq^{i+j}_*Q^{2i+1}w'_{2j} & \text{if } f,i \text{ both even, or both odd;} \\ 0 & \text{if } f,i \text{ have different parity.} \end{cases} \end{aligned}$$

Moreover, recall that

$$\begin{aligned} Sq^{i+j}_*Q^{2i+1}w'_{2j} &= \binom{i-j+1}{i-j} Q^{i+1}Sq^{j}_*w'_{2j} \\ &= \begin{cases} Q^{i+1}w'_{2k} & \text{if } i = 2l; \\ 0 & \text{if } i = 2l+1; \end{cases} \end{aligned}$$

This implies that

$$Sq^{i+j+f}_*Q^{2f+1}Q^{2i+1}w'_{2j} = \begin{cases} Q^{f+1}Q^{i+1}w'_{2k} & \text{if } f,i \text{ both even;} \\ 0 & \text{otherwise.} \end{cases}$$

This means that the class $q^{S^{-3}}_{f,i,j-2}$ can belong to $\ker r_{S^{-3}}$ only if $i = 2l+1$ or f is odd and i is even. Analysing different combinations will give at most five possible sets of primitives. We really like to leave the rest to the reader to verify for himself/herself. We have to say that, perhaps applying some of the number theory used in describing

A-annihilated classes will help to eliminate some of these cases. It is also possible to do more calculations, and identify some of these classes and show that they can be defined uniquely. We investigate this in the next remark. This concludes this long example!

Remark 5.52. It is still possible to show uniqueness of some primitive classes in $H_*Q_0S^{-2}$ and $H_*Q_0S^{-3}$ and the indecomposable classes that they project to. We follow a pattern similar to Remark 5.40. In this case, we consider to $\nu \in \pi_3^S \simeq \pi_1^S S^{-2} \simeq \pi_0^S S^{-3}$. This will give rise to infinite loop maps, denoted with ν,

$$\begin{array}{rcl} \nu : QS^1 & \to & Q_0S^{-2}, \\ \nu : Q_0S^0 & \to & Q_0S^{-3}. \end{array}$$

One may use this to identify specific subalgebras, as we did in Remark 5.40 and discussed in Note 5.41. But we like to show how these can be used to show that specific generators are uniquely determined, i.e. the suspension homomorphism on these generators is monomorphism.

First, notice that $\nu : QS^3 \to Q_0S^0$ sends $g_3 \in H_3QS^3$ to $p_3' \in H_3Q_0S^0$. This is an infinite loop map, so it sends Q^ig_3 to Q^ip_3'. If we choose $i > 6$, then Q^ip_3' will not be admissible. Applying Lemma 5.10 implies that we need to use the Adem relation for the pair Q^iQ^3.

If we choose $i \equiv 1 \bmod 4$, i.e. $i = 4k+1$ for some k, then we have

$$Q^iQ^3 = Q^{2k+3}Q^{2k+1}.$$

This implies that, modulo terms of lower excess, we may write

$$\nu_*(Q^{4k+1}g_3) = Q^{4k+1}p_3' = Q^{2k+3}p_{2k+1}'$$

where now the right hand side is admissible. Now we desuspend once, and consider $\nu : QS^2 \to Q_0S^{-1}$. We know that $\sigma_*\nu_*g_2 = \nu_*g_3 = p_3'$. We also know that g_2 is primitive. This implies that $\nu_*g_2 = p_2^{S^{-1}}$, and that $\sigma_*\nu_*(Q^{4k+1}g_2) = Q^{2k+3}p_{2k+1}'$. Again, notice that $Q^{4k+1}g_2$ is primitive. Taking these to account, we have

$$\nu_*(Q^{4k+1}g_2) = \begin{cases} Q^{2k+3}p_{2k}^{S^{-1}} & \text{if } k = 2l+1; \\ p_{k+1,k}^{S^{-1}} & \text{if } k = 2l. \end{cases}$$

We now can play the same game and desuspend once more to work with $\nu : QS^1 \to Q_0S^{-2}$. This bit gives new information. In particular, we know that $\sigma_*\nu_*g_1 = p_2^{S^{-1}}$. The fact that g_1 is primitive, implies that $\nu_*g_1 = v_1 = p_1^{S^{-2}}$, and that

$$\sigma_*\nu_*(Q^{4k+1}g_1) = \begin{cases} Q^{2k+3}p_{2k}^{S^{-1}} & \text{if } k = 2l+1; \\ p_{k+1,k}^{S^{-1}} & \text{if } k = 2l. \end{cases}$$

Combining this with the primitivity of $Q^{4k+1}g_1$ implies that

$$\nu_*(Q^{4k+1}g_1) = \begin{cases} Q^{2k+3}p_{2k}^{S^{-2}} & \text{if } k = 2l+1; \\ p_{k+1,k-1}^{S^{-2}} & \text{if } k = 2l. \end{cases}$$

which uniquely defines $p_{2k-1}^{S^{-2}}$, as well as $p_{k+1,k-1}^{S^{-2}}$.

In other cases, when we can do similar job, provided we know the "right" Adem relation. For instance, one can check that for $i = 4k+3$ we have the following Adem relation

$$Q^{4k+3}Q^3 = \begin{cases} Q^{2k+3}Q^{2k+3} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

This implies that modulo terms of lower excess

$$\nu_*(Q^{4k+3}g_3) = Q^{4k+3}p'_3 = \begin{cases} Q^{2k+3}p'_{2k+3} & \text{if } k \text{ is even,} \\ 0 \text{ or } p^2 & \text{if } k \text{ is odd,} \end{cases}$$

where p is a primitive! We may desuspend once to see that if k is even,

$$\nu_*(Q^{4k+3}g_2) = Q^{2k+3}p_{2k+2}^{S^{-1}}.$$

In this case we obtain,

$$\nu_*(Q^{4k+3}g_1) = Q^{2k+3}p_{2k+1}^{S^{-2}},$$

which in return uniquely defines $v_{2k+1} \in QH_{2k+1}Q_0S^{-2}$.

We note that if $i = 4k+2$, then in the Adem relation for Q^iQ^3 we will have terms of the form $Q^{2a}Q^b$ whereas we know this will not appear in the expression for a spherical class. So, we will not consider them here.

We may play a similar game, and use the map $\nu : Q_0S^0 \to Q_0S^{-3}$ to identify some of primitive classes in $H_*Q_0S^{-3}$ in a unique way. Notice that the primitive classes in

$H_*Q_0S^0$ which map to $Q^{4k+1}g_1$ and $Q^{4k+3}g_1$ are p_{4k+1} and p_{4k+3} provided by our first description of primitive classes in $H_*Q_0S^0$ (see section 5.4 for this description). This implies that,

$$\nu_*(p_{4k+1}) = \begin{cases} Q^{2k+3}p^{S^{-3}}_{2k-1} & \text{if } k = 2l+1, \\ p^{S^{-3}}_{k+1,k-2} & \text{if } k = 2l. \end{cases}$$

Similarly, if k is even we have

$$\nu_*(p_{4k+3}) = Q^{2k+3}p^{S^{-3}}_{2k}.$$

This later one, uniquely defines $q^{S^{-3}}_{2k}$ with k even. The first two also give unique identification of primitive and indecomposable classes considered in Example 5.51. Finally, we like to note similar job can be done using $\sigma \in \pi^S_7$. We leave this as an exercise to the very diligent reader! We note that this will give unique identification of some important primitive classes in $H_*Q_0S^{-n}$ for $n < 8$. We also recommend that the relation between these and the way that the homology suspension acts would be of interest, especially when we consider calculating the suspension homomorphism arisen from the Eilenberg-Moore spectral sequence.

5.9.2 The Curtis conjecture continued

The discussion in previous section shows the pattern that happens while we desuspend a primitive class to a primitive class. We may summarise it as following.

There are two basic types of primitive classes in $H_*Q_0S^{-k}$. The first type of these classes are those ones which pull back to an indecomposable $q^{S^{-k-1}} \in QH_*Q_0S^{-k-1}$ with trivial image under the square root map $QH_*Q_0S^{-k-1} \to QH_*Q_0S^{-k-1}$ which straight away gives rise to a unique primitive class in $p^{S^{-k-1}} \in H_*Q_0S^{-k-1}$. In this case,

$$\sigma_*p^{S^{-k-1}} = p^{S^{-k}}.$$

The second type of primitive classes are those primitive classes $\mathsf{p}^{S^{-k}} \in H_*Q_0S^{-k}$ which pull back to an indecomposable class $\mathsf{q}^{S^{-k-1}} \in QH_*Q_0S^{-k-1}$ with a nontrivial action under the square root map $QH_*Q_0S^{-k-1} \to QH_*Q_0S^{-k-1}$. In this case, applying an operation Q^{2i+1} to $\mathsf{q}^{S^{-k-1}}$ will give rise to a class $Q^{2i+1}\mathsf{q}^{S^{-k-1}}$ with the trivial action under the square root map. This gives rise to a primitive class $\mathtt{p}_{2i+1}^{S^{-k-1}} \in H_*Q_0S^{-k-1}$ which modulo decomposable terms is defined by

$$\mathtt{p}_{2i+1}^{S^{-k-1}} = Q^{2i+1}\mathtt{q}^{S^{-k}}.$$

In this case

$$\sigma_*\mathtt{p}_{2i+1}^{S^{-k-1}} = Q^{2i+1}\mathtt{p}^{S^{-k}}.$$

On the other side, having a primitive class $\mathtt{p}_{2i+1}^{S^{-k}} \in H_*Q_0S^{-k}$ which pulls back to an indecomposable class $\mathtt{q}_{2i+1}^{S^{-k-1}} \in QH_*Q_0S^{-k-1}$ with the trivial action under the square root map $QH_*Q_0S^{-k-1} \to QH_*Q_0S^{-k-1}$, gives rise to a primitive class $\mathtt{p}_{2i+1}^{S^{-k-1}}$ with

$$\sigma_*\mathtt{p}_{2i+1}^{S^{-k-1}} = \mathtt{p}_{2i+1}^{S^{-k}}.$$

Now let $l(I) = r$ with $I = (2i_1+1, J)$ an admissible sequence of odd numbers. By analogy we may consider primitive classes which modulo decomposable terms are given by one of the following equations

$$\begin{aligned} \mathtt{p}_I^{S^{-k-1}} &= \mathtt{q}_I^{S^{-k-1}}, \\ \mathtt{p}_I^{S^{-k-1}} &= Q^{2i_1+1}\mathtt{q}_J^{S^{-k-1}}, \end{aligned}$$

respectively satisfying one of the following equations,

$$\begin{aligned} \sigma_*\mathtt{p}_I^{S^{-k-1}} &= \mathtt{p}_I^{S^{-k}}, \\ \sigma_*\mathtt{p}_I^{S^{-k-1}} &= Q^{2i_1+1}\mathtt{p}_J^{S^{-k}}. \end{aligned}$$

In either of the cases the following identity holds

$$\sigma_*^{k+1}\mathtt{p}_I^{S^{-k-1}} = Q^I p.$$

The above discussion, combined with Theorem 5.45, is summarised in the following proposition.

Proposition 5.53. *Let $\theta_{-k} \in H_*Q_0S^{-k}$ be a spherical class with $\theta = \sigma_*^k\theta_{-k} \neq 0$ which is not a Hopf invariant class nor a Kervaire invariant class. Then modulo the kernel of $\sigma_* : H_*Q_0S^{-k} \to H_*Q_0S^{-k+1}$ the class θ_{-k} can be written as a linear combination of primitive terms of the following forms*

$$Q^J\mathtt{p}_K^{S^{-k}}, \quad Q^L p_{2l+1-k}^{S^{-k}},$$

where $(I, 2i+1) = (J, K) = (L, 2l+1)$ with I, J, K, L being admissible and J can be the empty sequence. Here θ has either one of the following forms,

$$\theta = \sum Q^I p_{2i+1} \qquad \text{modulo decomposable terms}$$
$$\theta = (\sum Q^I p_{2i+1})^2$$

satisfying one of the cases identified by Theorem 5.45*, $(I, 2i+1)$ admissible if* $\mathrm{excess}(I, 2i+1) > 0$.

Now we consider to the action of the Steenrod algebra on the primitive classes discussed above. If $\mathsf{p}_I^{S^{-k-1}} = Q^{2i_1+1}\mathsf{q}_J^{S^{-k-1}}$ is given modulo decomposable terms, then we may calculate action of the Steenrod algebra using the Nishida relations, provided that we understand the action of the Steenrod operations on $\mathsf{q}_J^{S^{-k-1}}$. Notice that the action of the Steenrod operations on $\mathsf{q}_J^{S^{-k-1}}$ can be calculated by using the suspension argument as we did before, of course modulo the kernel of $\sigma_* : H_*Q_0S^{-k-1} \to H_*Q_0S^{-k}$. Although in our approach, explained after the proof of Theorem 5.45, after we desuspended enough times we will need not to know about the action of the Steenrod algebra on $\mathsf{q}_J^{S^{-k-1}}$ and only the Nishida relations together with the right choice of the operation will finish off the proof of the Curtis conjecture. Moreover, having given the primitive classes $\mathsf{p}_I^{S^{-k-1}} = Q^{2i_1+1}\mathsf{q}_I^{S^{-k-1}}$ modulo decomposable terms, the fact that $\mathsf{q}_I^{S^{-k-1}}$ will suspend (probably iterated suspensions) to $Q^{i_1}\xi$ for some ξ allows us to treat $Sq^a_*\mathsf{q}_I^{S^{-k-1}}$ in the same way as we treat $Sq^a_*\xi_{-k-1}$ for some ξ_{-k-1} modulo the kernel of the (iterated) suspension.
Finally, notice that having $Q^I p_{2i+1} \in H_nQ_0S^0$ with $\mathrm{excess}(Q^I p_{2i+1}) > 0$ implies that $l(I, 2i+1) < n$. Recall that if $\theta \in H_nQ_0S^0$ is a spherical class satisfying Theorem 5.45, then either $\theta = \sum Q^I p_{2i+1}$ mod decomposable terms with $(I, 2i+1)$ admissible for terms with $\mathrm{excess}(I, 2i+1) > 0$, or it will be the square of terms of the form $\sum Q^I p_{2i+1}$ modulo decomposable terms with $(I, 2i+1)$ admissible. The fact that $l(I, 2i+1) < n$ implies that for some entries of I or $2i+1$ we need to have $\rho(i_j)$ to be large enough. Recall from Remark 5.50 and Example 5.51 that having ρ small could prevent a class from pulling back *enough* times.

I like to conclude this section with the following conjecture which I believe is a corollary of Proposition 5.53.

Conjecture 5.54. *It is impossible to have a spherical class satisfying one of the cases provided by Theorem* 5.45*.*

Notice that this claim proves the Curtis conjecture. I believe that the following sketchy proof will lead to the complete resolution of the above conjecture. The only gap here will be a reasonable description of the kernel of the suspension map $H_*Q_0S^{-n-1} \to H_*Q_0S^{-n}$. In fact, it is enough to know which indecomposable classes survive under this homomorphism. Such an information, will then complete the proof.

Sketch of Proof. Let $\theta \in H_nQ_0S^0$ be a spherical class which satisfies one of the cases identified by Theorem 5.45. First assume that θ is odd dimensional. Theorem 5.45 implies that

$$\theta = \sum Q^I p'_{2i+1},$$

with all terms A-annihilated. We know that θ must pull back to a spherical class $\theta_{-n} \in H_0Q_0S^{-n}$, i.e. θ admits n pull backs to A-annihilated primitive classes. Having sequences $I = (i_1, \ldots, i_r)$ with $l(I) = r > 1$ given by the above sum, let $I^0 = (i_1^0, \ldots, i_{r_0}^0)$ be the sequence whose i_1 has the least $\rho(i_1)$, and let $Q^{I^0}p_{2i^0+1}$ be the term related to this sequence in the above sum. Notice that for any sequences $(I, 2i+1)$ in the above sum we have $2^{\rho(i_1)} < i_1 < n$. This means that we can consider $\theta_{-2^{\rho(i_1^0)}-1} \in H_{n-2^{\rho(i_1)}-1}Q_0S^{-2^{\rho(i_1)}-1}$ with i_1^0 chosen as above. We know that this class will suspend to θ. Hence, we can apply Proposition 5.53 to this class, and predict its form. In particular, we like to see how the class $Q^{I^0}p_{2i^0+1}$ pulls back. Proposition 5.53 implies that we may write $\theta_{-2^{\rho(i_1^0)}-1}$ as a linear combination of terms of the form

$$Q^J \mathrm{p}_K^{S^{-k}}, \quad Q^L p_{2l+1-k}^{S^{-k}}.$$

In particular we need to have a term in one of the following forms

$$Q^{J^0} \mathrm{p}_{K^0}^{S^{-k}}, \quad Q^{L^0} p_{2l^0+1-k}^{S^{-k}},$$

with $(J^0, K^0) = (L^0, 2l^0+1) = (I^0, 2i^0+1)$. If any of the above primitive classes appear in the sum for $\theta_{-2^{\rho(i_1^0)}-1}$, then one of the following equalities will hold respectively,

$$\begin{array}{rcl}\sigma_*^{2^{\rho(i_1^0)}+1}Q^{J^0}\mathtt{p}_{K^0}^{S-k} & = & Q^{I^0}p_{2i^0+1}\\ \sigma_*^{2^{\rho(i_1^0)}+1}Q^{L^0}p_{2l^0+1-k}^{S-k} & = & Q^{I^0}p_{2i^0+1}.\end{array}$$

Let us denote such a class in $H_{n-2^{\rho(i_1^0)}-1}Q_0S^{-2^{\rho(i_1^0)}-1}$ with $P(I_0)$. Notice that in the above sum for θ all sequences $(I,2i+1)$ are admissible. This allows us to define the notion of excess for classes of the form $Q^{J^0}\mathtt{p}_{K^0}^{S-k}$ with

$$\text{excess}(Q^J\mathtt{p}_K^{S-k}) = \text{excess}(J,K) - k.$$

Notice that in this case such a term will be trivial if it is of negative excess. Notice that we have chosen I^0 with i_1^0 having the least $\rho(i_1)$. This implies that applying $Sq_*^{2^{\rho(i_1^0)}}$ to the class $\theta_{-2^{\rho(i_1^0)}-1}$ all terms apart from the term suspending to $Q^{I^0}p_{2i^0+1}$ will vanish. Moreover, $\text{excess}(Sq_*^{2^{\rho(i_1^0)}}P(I_0)) = 1$ which shows that this class is nontrivial, and is not a decomposable. My claim is that this proves that

$$Sq_*^{2^{\rho(i_1^0)}}\theta_{-2^{\rho(i_1^0)}-1} \neq 0.$$

This contradicts the fact this $\theta_{-2^{\rho(i_1^0)}-1}$ is spherical, and completes the proof in this case.

The proof for the other cases mentioned in Theorem 5.45 is similar. I believe this concludes the proof. □

5.10 Homology of $Q_0\Sigma^{-1}P$

In this section we turn our attention to the homology ring of $Q_0\Sigma^{-1}P := \Omega_0QP$. This will be done in a similar way as we did for Q_0S^{-1}. The Frobenius homomorphism $H^*P \to H^*P$ is given by the squaring map, and is a monomorphism. Therefore H^*QP is a polynomial algebra. This allows one to apply Proposition 5.4 to the case with $X = \overline{QP}$, the universal cover of QP. Notice that $\Omega_0QP = \Omega\overline{QP}$. Hence, one obtains that

$$H_*\Omega_0QP \simeq E_{\mathbb{Z}/2}(\sigma_*^{-1}PH_*\overline{QP}).$$

We have determined the submodule of primitive classes in H_*QP. Hence we only need to fill in the gaps by determining the action of the Dyer-Lashof algebra on $H_*Q_0\Sigma^{-1}P$. Recall that we had two sets of generators for PH_*QP, namely the submodules spanned by terms of the form $Q^I p^P_{2i+1}$ and $Q^J p^P_{j,k}$. This uniquely determines elements $w^{\Sigma^{-1}P}_{2i} \in QH_{2i}Q_0\Sigma^{-1}P$, and $w^{\Sigma^{-1}P}_{j,k} \in QH_{2i+2j}Q_0\Sigma^{-1}P$. Notice that the homology of $Q_0\Sigma^{-1}P$ is an exterior algebra, so terms are either indecomposable, or product of indecomposable terms. The action of the Dyer-Lashof algebra R also can be determined in a similar way as we did for $H_*Q_0S^{-1}$. We set $\mathrm{excess}(Q^I w^{\Sigma^{-1}P}_{2i}) = \mathrm{excess}(I) - 2i$, and $\mathrm{excess}(Q^J p^{\Sigma^{-1}P}_{j,k}) = \mathrm{excess}(J,j) - 2k$. Then we have the following.

Theorem 5.55. *The R-module structure of $H_*Q_0\Sigma^{-1}P$ is given by*

$$E_{\mathbb{Z}/2}(Q^I w^{\Sigma^{-1}P}_{2i}, Q^J p^{\Sigma^{-1}P}_{j,k} : \mathrm{excess}(Q^I w^{\Sigma^{-1}P}_{2i}) > 0, \mathrm{excess}(Q^J p^{\Sigma^{-1}P}_{j,k}) > 0),$$

where I and J run over admissible sequences. Two generators $Q^I w^{\Sigma^{-1}P}_{2i}$ and $Q^J p^{\Sigma^{-1}P}_{j,k}$ maybe identified, if they may to the same element under the homology suspension $\sigma_* : H_*Q_0\Sigma^{-1}P \to H_*QP$.

Our next objective is to determine the action of the Steenrod algebra on this ring. Previously, we used the homology of $\Omega\lambda_{\mathbb{C}}$ to ease the calculation of the action of the Steenrod operations on $H_*Q_0S^{-1}$. We may use a similar trick here. Recall that we have the transfer map $t_{S^1} : Q\Sigma\mathbb{C}P_+ \to QP_+$ associated with the S^1-fibration $S^1 \to P \to \mathbb{C}P$. In homology we have,

$$(t_{S^1})_*\Sigma c_{2i} = p^P_{2i+1}.$$

Hence, we conclude that

$$(\Omega t)_* c_{2i} = w^{\Sigma^{-1}P}_{2i} + D.$$

This determines the action of the Steenrod algebra on the *generators* given by $w^{\Sigma^{-1}P}_{2i}$, where together with the Nishida relations one obtains the action of the Steenrod algebra on terms of the form $Q^I w^{\Sigma^{-1}P}_{2i}$. Regarding the generators $w^{\Sigma^{-1}P}_{j,k}$, we are not able to perform such a calculation. The reason being that the class $w^{\Sigma^{-1}P}_{j,k}$ suspends to $Q^{2j+1}a_{2k}$ modulo decomposable terms. But $Q^{2j+1}a_{2j}$ is not in the image of t_* which prevents $w_{j,k}^{\Sigma^{-1}P}$ from being in the image of $(\Omega t_{S^1})_*: H_*Q\mathbb{C}P_+ \to H_*Q_0\Sigma^{-1}P$. However, we are able to follow a more direct path, namely we combine our knowledge of the A-module structure of H_*QP, with the fact $\sigma_*: QH_{*-1}Q_0\Sigma^{-1}P \to H_*QP$ is an isomorphism, to understand the A-module structure of $H_*Q_0\Sigma^{-1}P$. More precisely, we need to study the action of Sq^r_* on the *generators* $w_{j,k}^{\Sigma^{-1}P}$. Such information together with the Nishida relations determines the action of the Steenrod algebra on the classes of the form $Q^I w_{j,k}^{\Sigma^{-1}P}$. We have the following result on about the action of the Steenrod algebra on terms of the form $w_{j,k}^{\Sigma^{-1}P}$.

Lemma 5.56. *Let $r = 2r'$ be a positive integer. Then the action Sq^r_* on $w_{j,k}^{\Sigma^{-1}P}$ is given as following*

$$\begin{aligned} Sq^r_* w_{j,k}^{\Sigma^{-1}P} &= \textstyle\sum_{t'} \binom{2j+1-r}{r-2(2t')}\binom{2k-2t'}{2t'} w^{\Sigma^{-1}P}_{j+t'-r',k-t'} \\ &\quad + \textstyle\sum_{t''} \binom{2j+1-r}{r-2(2t''+1)}\binom{2k-(2t''+1)}{2t'+1} Q^{2j+1-r+2t'+1} w^{\Sigma^{-1}P}_{2k-2t''-2}. \end{aligned}$$

Proof. To see the action of Sq^r_* on $w_{j,k}^{\Sigma^{-1}P}$ notice that the class $w_{j,k}^{\Sigma^{-1}P}$ maps to the primitive class in H_*QP involving $Q^{2j+1}a_{2k}$. The Nishida relations imply

$$\begin{aligned} Sq^r_* Q^{2j+1}a_{2k} &= \textstyle\sum_t \binom{2j+1-r}{r-2t} Q^{2j+1-r+t} Sq^t a_{2k} \\ &= \textstyle\sum_t \binom{2j+1-r}{r-2t}\binom{2k-t}{t} Q^{2j+1-r+t} a_{2k-t}. \end{aligned}$$

The later sum splits into to sums, depending on parity of t, resulting in the following relation,

$$\begin{aligned} Sq^r_* Q^{2j+1}a_{2k} = &\textstyle\sum_{t'} \binom{2j+1-r}{r-2(2t')}\binom{2k-2t'}{2t'} Q^{2j+1-r+2t'} a_{2k-2t'} + \\ &\textstyle\sum_{t''} \binom{2j+1-r}{r-2(2t''+1)}\binom{2k-(2t''+1)}{2t'+1} Q^{2j+1-r+2t'+1} a_{2k-(2t''+1)}. \end{aligned}$$

Such a class pulls back into $H_{*-1}Q_0\Sigma^{-1}P$. This proves the lemma. □

Remark 5.57. Observe that our main interest is in the classes of the form $w_{2i}^{\Sigma^{-1}P}$ as spherical classes in $H_*Q_0S^{-1}$ will pull back to homology classes involving $w_{2i}^{\Sigma^{-1}P}$. However, the action of the Steenrod algebra on the classes of the form $Q^I w_{j,k}^{\Sigma^{-1}P}$ is interesting, and perhaps tells us that the space $Q_0\Sigma^{-1}P$ will not (stably) *split*.

5.11 Finkelstein-Kahn-Priddy Theorem

The Barratt-Eccles Γ^+ functor provides one with a simplicial model for infinite loop spaces. That is having a connected simplicial complex X, the space $|\Gamma^+ X|$ has the same homotopy type as $Q|X|$, where $|X|$ denotes the topological realisation of X. Using this model, Finkelstein has generalised the Kahn-Priddy theorem. The following is the topological version of his result.

Let A be an infinite loop space, path connected and of finite type. Then there is a composite of maps given by

$$QA \xrightarrow{(\theta_A, j_2)} A \times QD_2 A \longrightarrow QA.$$

This composite induces an isomorphism in homology [F77, Theorem 3.2], and an equivalence on ${}_2\pi_*$-homotopy [F77, Corollary 6.8]. Here $\theta_A : QA \to A$ is the structure map of A, and j_2 is the 2nd stable James-Hopf map. Kuhn has generalised this to odd primes [K84, Theorem 1.1].

An example of this is obtained by taking $A = S^1$. Recall form the introduction that $D_2 S^1 \simeq \Sigma P$. Hence one has the ${}_2\pi_*$-equivalence

$$QS^1 \to S^1 \times Q\Sigma P \to QS^1.$$

Notice that $QS^1 \simeq S^1 \times \overline{QS^1}$. Hence looping the above composite we obtain,

$$\mathbb{Z} \times \Omega\overline{QS^1} \to \mathbb{Z} \times QP \to \mathbb{Z} \times \Omega\overline{QS^1}.$$

Notice that $\Omega\overline{QS^1} \simeq Q_0 S^0$. This recovers the Kahn-Priddy theorem, by restricting the above composite to the base point components, as

$$Q_0 S^0 \to QP \to Q_0 S^0.$$

Now we are ready to complete the proof of Lemma 13, and carry on with its proof for the case $k = 0$. We prove the following.

Lemma 5.58. *Suppose we have a mapping $f : S^{2n} \to Q_0 S^0$ with $hf = \xi_n^2$. Then the adjoint mapping $S^{2n+1} \to QS^1$ is detected by the primary operation Sq^{n+1} on $\sigma_*(\xi_n + O)$ where O denotes a sum of terms of lower excess.*

Proof. Our explanations above, imply that the ${}_2\pi_*$-equivalence $Q_0S^0 \to QP \to Q_0S^0$ given by the Kahn-Priddy theorem can be delooped once, i.e. the mapping $t : Q_0S^0 \to QP$ is a loop map. This means that t_* respects products, in particular $t_*\xi_n^2 = (t_*\xi_n)^2$. Notice that according to the Kahn-Priddy theorem t_* is an injection which means $t_*\xi_n^2 = (t_*\xi_n)^2 \neq 0$. On the other hand, recall that according to Lemma 19, $\sigma_*\xi_n \neq 0$. According to the above calculations, we have a commutative diagrams in homology

$$\begin{array}{ccc} H_{*+1}QS^1 & \xrightarrow{(j_2)_*} & H_{*+1}Q\Sigma P \\ {\scriptstyle\sigma_*}\uparrow & & \uparrow{\scriptstyle\sigma_*} \\ H_*Q_0S^0 & \xrightarrow{t_*} & H_*QP. \end{array}$$

We can use this diagram to calculate $t_*\xi_n$, up to some indeterminacy given by decomposable terms. Notice that $\xi_n = \sum Q^I x_i$ modulo decomposable terms, with (I, i) admissible. Hence we have

$$\begin{aligned} \sigma_*t_*\xi_n &= (j_2)_*\sigma_*\xi_n \\ &= (j_2)_*\sum Q^I Q^i g_1 \\ &= \sum Q^I\Sigma a_i + O, \end{aligned}$$

where O denotes a sum of terms of lower excess, given by homology of j_2. Observe that the suspension homomorphism $\sigma_* : H_*QP \to H_*Q\Sigma P$ kills decomposable terms, and is injective on the other terms. Hence we can deduce that

$$t_*\xi_n = \sum Q^I a_i,$$

modulo other term of lower excess and possible decomposable terms. Now one may apply a technique similar to Lemma 13, in cases $k > 0$, to verify the claim in this case. We do this by showing that the adjoint mapping $S^{2n+1} \to Q\Sigma P$ is detected by a primary operation. The naturality of primary operations then would imply that the mapping $S^{2n+1} \to QS^1$ is detected by a primary operation.

Notice that $t_*hf = (\sum Q^I a_i + O)^2$, where O denotes a sum of decomposable terms and terms of lower excess. Choose I with maximum length, say $l(I) = r$. Consider $j_{2^r} : QP \to QD_{2^r}P$. We claim that

$$(j_{2^r})_*t_*hf = ((\sum Q^I a_i))^2,$$

modulo terms of lower excess, where $(\sum Q^I a_i)$ denotes the image of $\sum Q^I a_i \in H_*QP$ under

$$(j_{2^r})_* : H_*QP \to H_*QD_{2^r}P.$$

First observe that according to Proposition 4.1 all terms $Q^J a_j$ with $l(J) < l(I)$ die under $(j_{2^r})_*$. We have to be careful as here a_i's, $i \neq 1$, are not primitive. However, we may use a trick. To obtain the other terms, notice that we have diagrams similar to those ones in the proof of Lemma 4.5. More precisely, we have

$$\begin{array}{ccc} H_{*+1}Q\Sigma P & \longrightarrow & H_{*+1}QD_{2^r}\Sigma P \\ \sigma_* \uparrow & & \uparrow e'_* \\ H_*QP & \longrightarrow & H_*QD_{2^r}P \end{array}$$

which shows that $\sum Q^I a_i + O$ maps to $(\sum_{l(I)=r} Q^I a_i + O)$ modulo $\ker e'_*$. Recall that the proof of Lemma 4.5 tells us what can be known about $\ker e'_*$. This implies that

$$\begin{aligned} (j_{2^r})_* t_* hf &= (j_{2^r})_* (\sum Q^I a_i + O)^2 \\ &= ((\sum\nolimits_{l(I)=r} Q^I a_i + O))^2 + O', \end{aligned}$$

where by $(\sum_{l(I)=r} Q^I a_i + O)$ we mean the image of $\sum Q^I a_i + O$ under the James-Hopf invariant $j_{2^r} : QP \to QD_{2^r}P$. The term O' is also another sum of terms of lower excess. Analysis similar to what we did in in the proof of Lemma 4.5 shows that this must be a square as well. This means that Hurewicz image of $j_{2^r} f : S^{2n} \to QP \to QD_{2^r}P$ will be a square, say α^2 where $\alpha \in H_{2n}D_{2^r}P$. Applying Lemma 13, then implies that the adjoint mapping $S^{2n+1} \to Q\Sigma D_{2^r}P$ is detected by a primary operation on $\sigma_*\alpha$. One then can verify that this implies that the adjoint mapping $S^{2n+1} \to Q\Sigma P$ is detected by a primary operation on $\sigma_* t_* \xi_n$. The Kahn-Priddy map send the class class $t_*\xi_n \in H_*QP$ to $\xi_n \in H_*Q_0S^0$ modulo terms of lower excess. Finally, the naturality of the primary operations implies that the adjoint mapping $S^{2n+1} \to QS^1$ is detected by a primary operation on $\sigma_*(\xi_n + O)$ where O denotes sum of terms of lower excess. This completes the proof. □

We can obtain further examples of application of Finkelstein's result. Notice that

$\pi_1 Q_0 S^0 \simeq \pi_1 QP \simeq \mathbb{Z}/2$. Hence we have homotopy equivalences as,

$$\begin{array}{lcl} Q_0S^0 & \simeq & P \times \overline{Q_0S^0}, \\ QP & \simeq & P \times \overline{QP}. \end{array}$$

Notice that $Q_0S^{-1} \simeq \Omega\overline{Q_0S^0}$, and $Q_0\Sigma^{-1}P := \Omega_0 QP \simeq \Omega\overline{QP}$. This implies that looping the above equivalence, we obtain

$$\mathbb{Z}/2 \times Q_0S^{-1} \to \mathbb{Z}/2 \times Q_0\Sigma^{-1}P \to \mathbb{Z}/2 \times Q_0S^{-1}.$$

Similarly, restricting to the base point component we obtain,

$$Q_0S^{-1} \to Q_0\Sigma^{-1}P \to Q_0S^{-1},$$

inducing a ${}_2\pi_*$-isomorphism. This means that the mapping $Q_0S^{-1} \to Q\Sigma^{-1}P$ is a double loop map.

Remark 5.59. Notice that P itself is an infinite loop space of finite type, hence we have ${}_2\pi_*$-equivalence

$$QP \to P \times QD_2P \to QP,$$

where after looping, we have the ${}_2\pi_*$-equivalence

$$\mathbb{Z}/2 \times Q_0\Sigma^{-1}P \to \mathbb{Z}/2 \times Q\Sigma^{-1}D_2P \to \mathbb{Z}/2 \times Q_0\Sigma^{-1}P.$$

We restrict to the base point which gives the ${}_2\pi_*$-equivalence

$$Q_0\Sigma^{-1}P \to Q\Sigma^{-1}D_2P \to Q_0\Sigma^{-1}P.$$

Hence we end up with the following ${}_2\pi_*$-equivalence

$$Q_0S^{-1} \to Q_0\Sigma^{-1}P \to Q\Sigma^{-1}D_2P \to Q_0\Sigma^{-1}P \to Q_0S^{-1}.$$

However, we will not use this.

Chapter 6

Further projects

The followings are some problems that, I think, are related to the subject of this thesis, one may help the other one to be solved.

The Problem in ju- and jo-theory. Let ku denote the connective K-theory spectrum. Let ju be the fibre of $\psi^3 - 1 : ku \to ku$ where ψ^3 is the Adams operation. It is known [S02, Theorem 4.2] that if $f \in {}_2\pi^S_{2^{n+1}-2}P$ maps nontrivially under

$$h_{ju} : {}_2\pi^S_{2^{n+1}-2} \to ju_{2^{n+1}-2}P$$

then under the Kahn-Priddy map $\lambda : \pi^S_* P \to {}_2\pi^S_*$ the class f will map to an element in ${}_2\pi^S_{2^{n+1}-2}$ which is detected by the Kervaire invariant.

In a similar fashion one may define jo as the fibre of $\psi^3 - 1 : ko \to ko$ where ko denotes the connective real K-theory. Let JO denote the base point component of $\Omega^\infty jo$. In fact JO can be thought of fibre of $\psi^3 - 1 : BO \to BO$ regarded as a map of infinite loop spaces, with respect to the tensor product of bundles, where $BO = \Omega^\infty ko$. Then it is well known [S79, Diagram 7.3] that a solution to the Adams conjecture [Q71, Theorem 1.1] gives a mapping $JO \to SG$ making the following diagram commutative

$$\begin{array}{ccc} SO & \xrightarrow{J} & SG \\ \downarrow & \nearrow & \\ JO & & \end{array}$$

where SG denotes $Q_1 S^0$ with infinite loop space structure arising from the composition of maps of degree 1. This diagram is almost equivalent to the Adams conjecture.

This later claim can be derived from [A78, Theorem 5.1.1]. It is quite striking to me, to see in what extent we can relate the problem of finding spherical classes in $H_*Q_0S^0$ to finding spherical classes in jo_*. The above result on $ju_{2^{n+1}-2}P$ provides some evidence that jo, or more precisely ju is doing the same work as the infinite loop structure is doing, and the question is *why* this happens? If we can find a map between $H_*Q_0S^0$ and jo_* then this might be helpful in giving a shorter proof of the Curtis conjecture.

Geometry and homology of Q_0S^{-n}. Our work on identifying primitive classes in $H_*Q_0S^{-n}$ for $n > 0$ was quite fun, as we fed in a good amount of the information coming from the classical homotopy theory to derive more insight into these spaces. This was done while we did Remark 5.40, Note 5.41 and Remark 5.52. I would like to do more calculation in homology to get a better picture about these spaces. I believe that this homology will reveal more information about the geometry of these spaces.

Appendix A

Nishida relations, Adem relations

We include this short chapter to verify that applying the Nishida relations and Adem relations results in terms of lower excess.

Let $\xi \in H_*QX$ be an n dimensional class, and consider $Q^a\xi$ with $a > n$. The Nishida relations yields the following

$$Sq_*^r Q^a \xi = \sum_{t \geqslant 0} \binom{a-r}{r-2t} Q^{a-r+t} Sq_*^t \xi.$$

We have previously noted that it is not guaranteed that every term $Q^{a-r+t}Sq_*^t\xi$ is admissible. In any case we have

$$\text{excess}(Q^{a-r+t}Sq_*^t\xi) = a - n - (r - 2t).$$

Notice that to have nontrivial coefficients in the Nishida relations we need $r - 2t > 0$. Hence we have

$$\text{excess}(Q^{a-r+t}Sq_*^t\xi) \leqslant \text{excess}(Q^a\xi).$$

If we assume that $\xi = Q^J x$ is such that $Q^{a-r+t}Sq_*^t\xi$ is not admissible, then we have to use the Adem relations to rewrite this term in admissible form. Recall that if Q^aQ^b is a non admissible, i.e. $a > 2b$, then we have

$$Q^aQ^b = \sum_{a+b \leqslant 3t} \binom{t-b-1}{2t-a} Q^{a+b-t}Q^t.$$

To have a nontrivial binomial coefficient in the Adem relation we need $t > b$ which implies that

$$\text{excess}(Q^{a+b-t}Q^t) = a + b - 2t < a - b = \text{excess}(Q^aQ^b).$$

This implies that applying the Adem relations reduces the excess. This together with the previous observation on the Nishida relations implies that applying the Nishida relations, and then the Adem relations, we will end up with terms of lower excess. We have used this fact during the text. Thought, Lemma 3.2 is a much sharper result.

Appendix B

Comments on Lemma 3.2

We said that Lemma 3.2 is obtained by [C75, Lemma 6.2], and also can be obtained by combining [W82, Weillington, Theorem 7.11], [W82, Theorem 7.12] and [W82, Lemma 12.5]. We like to show how one can see this.

We the definition of the Λ-algebra. Let F be the free graded associated algebra over $\mathbb{Z}/2$ generated with generators λ_i in grading i. We then have that the Λ-algebra is the quotient of this algebra by the ideal generated by the Adem relations. We keep $\lambda_i \in \Lambda_i$ to denote the generators of this quotient algebra. We then have that the Dyer-Lashof algebra is the quotient of the Λ-algebra by the ideal generated by all terms λ_I with $\text{excess}(I) < 0$. The action of the Steenrod algebra on both of these algebras is given by the Nishida relations. Chapter 3 demonstrated how we can use these relations. We refer the reader to [W82, Chapter 7] for more discussion on this material.

The first result [W82, Weillington, Theorem 7.11] is about the differential in the Λ algebra. We mentioned the part related to prime $p = 2$. For the moment we respect the notation of Wellington as he lets the Steenrod operations act from right, and the admissibility is defined in the reverse order.

Theorem B.1. *The differential ∂ of the Λ-algebra is related to the Steenrod operations when* $\text{excess}(I) \geqslant 0$ *and I is admissible by*

$$\partial\lambda_I = \sum_{j\geqslant 1}(\lambda_I Sq_*^j)\lambda_{j-1}.$$

The second result [W82, Theorem 7.12] shows how the differential of the Λ-algebra gives information about A-module structure of the Dyer-Lashof algebra R.

Theorem B.2. *Let I be admissible,* $\text{excess}(I) \geqslant 0$*, and suppose that*

$$\partial\lambda_I = \sum_{K \text{ admissible}} \alpha_K \lambda_K$$

where $\alpha_K \in \mathbb{Z}/2$. Then the following relation holds in R,

$$\lambda_I Sq_*^j = \sum \alpha_K \lambda_{K'}$$

where $K = (K', j-1)$ and $\text{excess}(K') \geqslant 0$*.*

This is in fact a natural consequence of the theorem above. We note that here we have used the same symbol λ_I for elements of the Λ-algebra and the Dyer-Lashof algebra R. The final result [C75, Lemma 6.2], [W82, Lemma 12.5] that we need reads as following.

Lemma B.3. *Let λ_I be given with* $\text{excess}(I) \geqslant 0$ *such that $I = (s_1, \ldots, s_r)$ satisfies $2s_j - 2^{\rho(s_j)} < s_{j+1}$ for $1 \leqslant j \leqslant m-1$. Assume*

$$\partial\lambda_I = \sum_{K \text{ admissible}} \alpha_K \lambda_K,$$

then for those $K = (K', k)$ with $\text{excess}(K') \geqslant 0$ *we have that*

$$\text{excess}(K') \leqslant \text{excess}(I) - 2^{\rho(s_r)}.$$

We note that there is misprint in [W82, Lemma 12.5(i)] in direction of sign at the inequality above.

Now for $I = (s_1, \ldots, s_r)$ we let $I^{\text{reverse}} = (i_1, \ldots, i_r)$ where $i_j = s_{r-(j-1)}$. Moreover, let $Q^{I^{\text{reverse}}} = \lambda_I$ and let $Sq_*^r Q^{I^{\text{reverse}}} = \lambda_I Sq_*^r$. Notice that I^{reverse} is admissible in our sense if λ_I is admissible in Wellington's definition. Lemma 3.2 of ours now follows.

Appendix C

Constructing A-annihilated sequences

Theorem 2 on A-annihilated classes of the form $Q^I x \in H_*QX$ with $\mathrm{excess}(Q^I x) > 0$ has two fundamental parts; namely our understanding of A-annihilated classes in H_*X, and the existence of sequences of positive integers I satisfying conditions 2-3 of Theorem 2. The aim of this section is to give a construction of sequences which satisfy only condition 3 of Theorem 2. This construction, at least in theory, will determine all such sequences in a unique way. I would like to see this as a proof for the uniqueness conjecture that I have mentioned, however I am not confident that this construction gives a complete proof, for reasons to be explained.

Let $I = (i_1, \ldots, i_r)$ be a sequence satisfying condition 3, i.e.

$$0 \leqslant 2i_{j+1} - i_j \leqslant 2^{\rho(i_{j+1})}.$$

Notice that given an integer n, then we may write

$$n = 2^{\rho(n)+1} N_n + 2^{\rho(n)} - 1,$$

for some $N_n \geqslant 0$. Suppose we are given a pair of integers (m, n), $m > n$, such that

$$0 \leqslant 2n - m < 2^{\rho(n)}.$$

This is the same as assuming

$$2n - 2^{\rho(n)} < m \leqslant 2n.$$

This implies that

$$\rho(m) \leqslant \rho(n).$$

To construct a sequence I of length r, consider an r-tuple of nondecreasing positive integers,

$$\rho_1 \leqslant \rho_2 \leqslant \cdots \leqslant \rho_r.$$

Choose a nonnegative integer N_r, and let $i_r = 2^{\rho_r+1}N_r + 2^{\rho_r} - 1$. We want to find $i_{r-1} = 2^{\rho_{r-1}+1}N_{r-1} + 2^{\rho_{r-1}} - 1$ such that

$$2i_r - 2^{\rho_r} < i_{r-1} \leqslant 2i_r.$$

Plugging in the value of i_{r-1}, i_r, gives the boundary conditions on N_{r-1},

$$2^{\rho_r+2}N_r + 2^{\rho_r} - 1 < 2^{\rho_{r-1}+1}N_{r-1} + 2^{\rho_{r-1}} \leqslant 2^{\rho_r+2}N_r + 2^{\rho_r+1} - 1.$$

This can be refined as

$$2^{\rho_r+2}N_r + 2^{\rho_r} \leqslant 2^{\rho_{r-1}+1}N_{r-1} + 2^{\rho_{r-1}} < 2^{\rho_r+2}N_r + 2^{\rho_r+1}.$$

Hence we have,

$$2^{\rho_r-\rho_{r-1}+1}N_r + 2^{\rho_r-\rho_{r-1}-1} \leqslant N_{r-1} + \frac{1}{2} < 2^{\rho_r-\rho_{r-1}+1}N_r + 2^{\rho_r-\rho_{r-1}}.$$

As N_{r-1} is an integer, hence one has

$$2^{\rho_r-\rho_{r-1}+1}N_r + 2^{\rho_r-\rho_{r-1}-1} \leqslant N_{r-1} < 2^{\rho_r-\rho_{r-1}+1}N_r + 2^{\rho_r-\rho_{r-1}}.$$

This means that there are $2^{\rho_r-\rho_{r-1}-1}$ choices for N_{r-1}.
Continuing in this way we can construct such sequences which only will satisfy condition 3 of Theorem 2. Notice that

$$2^{\rho_i-\rho_{i-1}+1}N_i + 2^{\rho_i-\rho_{i-1}-1} \leqslant N_{i-1}$$

for any $1 \leqslant i < r$. This then implies that having fixed a nondecreasing r-tuple of positive integers,

$$\rho : \rho_1 \leqslant \rho_2 \leqslant \cdots \leqslant \rho_r,$$

then different choices for N_i will give different sequences in different dimensions. However, it is possible to have two different sequences, say ρ, ρ', but giving two r-tuples in the same dimensions. As an example, let $r = 2$. Then $(17, 15)$ and $(21, 11)$ both are sequences with satisfy conditions 3, and both are in dimension 32. Notice that

$$\rho(17) = 1 \quad < \quad \rho(15) = 4$$
$$\rho(21) = 1 \quad \leqslant \quad \rho(11) = 3.$$

Now we give some specific examples of constructing such sequences which seem to be more applicable.

Example C.1. This is the simplest possible case when we choose

$$\rho: \quad \rho_1 = \rho_2 = \cdots = \rho_r.$$

Let choose an specific fixed value for ρ_i, say $\rho_i = 2$. However in this case we don't restrict ourselves to some specific length. We have $i_r = 2^3 N_r + (2^3 - 1)$. Let us choose $N_r = 1$, then $i_r = 11$. Now set $i_{r-1} = 2i_r - (2^2 - 1)$, and inductively set $i_{r-j} = 2i_{i-j+1} - (2^2 - 1)$. Then it is easy to see that $i_j \equiv 2^2 - 1 \bmod 2^3$. For example continuing in this way for 3 times we obtain the sequence

$$(67, 35, 19, 11).$$

This automatically satisfies conditions 2-3 of Theorem 2, i.e. $Q^{67}Q^{35}Q^{19}Q^{11}$ is an A-annihilated class in the Dyer-Lashof algebra R. This also implies that

$$Q^{67}Q^{35}Q^{19}p'_{11}, \quad Q^{67}Q^{35}Q^{19}p'_{11}$$

are A-annihilated classes in $H_*Q_0S^0$. Notice that $Q^{67}Q^{35}Q^{19}p'_{11}$ is a primitive A-annihilated class.

As an other example let choose $\rho_i = \rho = 3$, then $i_j = 2^4 N_j + (2^3 - 1)$. Let choose $N_r = 2$, then $i_r = 39$. Now let $i_j = 2i_{j+1} - (2^3 - 1)$. If we look for a sequence I such that excess$(I) < 2^\rho = 8$ we then obtain the sequence

$$(1031, 519, 263, 135, 71, 39)$$

which means $Q^{1031}Q^{519}Q^{263}Q^{135}Q^{71}Q^{39}$ is an A-annihilated class in the Dyer-Lashof algebra. This is the sequence used in [W82, Remark 11.26] to construct a sum of even degree which is A-annihilated, but its terms are not. We analysed this example in Note 5.27.

Finally notice that in general we have an observation similar to the last part of Lemma 4 as following.

Lemma C.2. *Suppose* $Q^{i_1}Q^{i_2}\cdots Q^{i_r}x \in H_*QX$ *is* A*-annihilated, then* $Q^{i_2}\cdots Q^{i_r}x \in H_*QX$ *is not* A*-annihilated.*

The above lemma implies that if we want to obtain an A-annihilated class by applying operation Q^i to another A-annihilated class ξ, then we need at least two operations to achieve this.

One can see that our construction here is the most general one, obtained by properties of sequences I satisfying condition 3 of Theorem 2. One observes that condition 2, i.e. $\mathrm{excess}(Q^I x) < 2^{\rho(i_1)}$ tells us when the construction has to terminate.

Appendix D

Geometric decompositions

In most of the Chapter 5, we used various decompositions of spaces into product of two spaces. Here we like to give a brief background on what we did.

Suppose X is given such that $\pi_0 X \simeq G$ is an Abelian group, and X an H-space. Let X_g denote the component corresponding to $g \in G$. Then we have a decomposition of X as $X_0 \times G$, where X_0 denotes the base point component of X.
Now assume X is path connected. In general, an unstable decomposition $X \simeq Y \times Z$ may be obtained by a pair of mappings $Y \leftarrow X \rightarrow Z$ with $X \rightarrow Y \times Z$ inducing an isomorphism on homotopy groups. If X is an H-space, we may think of pair of maps $Y \rightarrow X \leftarrow Z$ where $Y \times Z \rightarrow X$ induces a π_*-isomorphism.

Example D.1. Consider QS^0, where $\pi_0 QS^0 \simeq \mathbb{Z}$ gives the decomposition $QS^0 \simeq \mathbb{Z} \times Q_0 S^0$. One observes that $\pi_1 Q_0 S^0 \simeq \pi_1^S \simeq \mathbb{Z}/2$, giving rise to a mapping $Q_0 S^0 \rightarrow K(\mathbb{Z}/2, 1) \simeq P$. Notice that the fibre of this map is $\overline{Q_0 S^0}$ the universal cover of $Q_0 S^0$. On the other side, the Kahn Priddy map $P \rightarrow Q_0 S^0$ provides us with a splitting

$$P \times \overline{Q_0 S^0} \rightarrow Q_0 S^0.$$

Recall that if X is an infinite loop space, then $\overline{X}$ is an infinite loop space and the inclusion $\overline{X} \rightarrow X$ is a map of infinite loop spaces [CLM76, Lemma 4.8], where $\overline{X}$ is a suitable model for the universal cover of X. This may be generalised to higher n-connected covers of an infinite loop space.
A wide range of examples is provided by connected infinite loop spaces. Let X be

such a space with structure map $\theta_X : QX \to X$ and let X' denote its fibre. As we mentioned previously this is also an infinite loop space. The inclusion, also known as the suspension map, $E : X \to QX$ has the property [BEa74, Definition 3.7], [BEb74, Proposition 3.1] that $\theta_X \circ E = 1_X$. This then gives the splitting

$$X' \times X \to QX.$$

Example D.2. Let $\theta : QK(\mathbb{Z}/2, n) \to K(\mathbb{Z}/2, n)$ be the structure map which induces an isomorphism on π_n, and its fibre can be viewed as $QK(\mathbb{Z}/2, n)\langle n\rangle$, the n-connected cover of $QK(\mathbb{Z}/2, n)$ with first nontrivial homotopy group in dimension $n+1$. The inclusion $K(\mathbb{Z}/2, n) \to QK(\mathbb{Z}/2, n)$ then gives the splitting

$$K(\mathbb{Z}/2, n) \times QK(\mathbb{Z}/2, n)\langle n\rangle \to QK(\mathbb{Z}/2, n).$$

There are interesting and familiar example of this decomposition. If we choose $n = 1$, we obtain

$$P \times \overline{QP} \to QP.$$

Of course one may replace $\mathbb{Z}/2$ with any Abelian group. For instance we may choose to work with $\mathbb{Z}$. In this case for $n = 1$ we obtain,

$$S^1 \times \overline{QS^1} \to QS^1$$

where looping this equivalence gives $QS^0 \simeq \mathbb{Z} \times \Omega\overline{QS^1}$ which implies that $Q_0S^0 \simeq \Omega\overline{QS^1}$. Another related example for us is given by the case of $K(\mathbb{Z}, 2)$ where we obtain the splitting

$$\mathbb{C}P \times Q\mathbb{C}P\langle 2\rangle \to Q\mathbb{C}P.$$

If we loop the above splitting we get the following splitting

$$S^1 \times \Omega Q\mathbb{C}P\langle 2\rangle \to Q\Sigma^{-1}\mathbb{C}P.$$

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