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INFINITE DIMENSIONAL SECOND ORDER DIFFERENTIAL EQUATIONS VIA T^2M

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Abstract. The vector bundle structure obtained on the second order (acceleration) tangent bundle T^2M of a smooth manifold M by means of a linear connection on the base provides an alternative way for the study of second order differential equations on manifolds of finite and infinite dimension. Second order vector fields and their integral curves provide a new way of solving a wide class of second order differential equations on Fréchet manifolds and may be used also to describe geodesic curves on a Riemannian manifold. The new technique proposed is illustrated by concrete examples within the framework of Banach and Fréchet spaces as well as on Lie groups.

Introduction

Second order differential equations on manifolds have received renewed geometric attention in recent years from interactions with jet fields, linear and non-linear connections, Lagrangians and Finsler structures (cf., for instance, [2], [3], [22], [24]). On the other hand, the potential applications of this subject reach beyond classical Differential Geometry, having, for example, a central role in the theory of time-dependent Lagrangian particle systems (see [21], [23]). Sufficient methods for the study of equations of such type have so far been developed only for those known as sprays, which correspond to linear connections.

In the present work we propose an alternative way of studying second order differential equations on a smooth manifold M. We work mainly with the second order tangent bundle T^2M of M, consisting of all equivalence classes of curves in M that agree up to their acceleration. T^2M can be endowed with a vector bundle structure in the presence of a linear connection on M (see [5], [6]). Although this bundle structure is strongly dependent on the choice of the linear connection on the base manifold, the corresponding local sections can be used to describe in detail second order differential equations on M.

The proposed methodology is suitable for Banach modelled manifolds, and serves also as a basis for the study of second order differential equations on a wide class of Fréchet manifolds. The Fréchet problem is complicated by lack of a general solvability theory for ordinary differential equations on the models; that inhibits the establishment of existence and uniqueness of solutions from initial conditions, analogous to the cases of finite dimensional and Banach spaces. However, if one restricts to the category of Fréchet manifolds that can be viewed as projective limits of Banach manifolds, then

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the difficulty is eased. This approach proves to be compatible with the taking of projective limits, so leading to a new way of studying and solving second order differential equations on Fréchet manifolds.

Our approach is clarified by some concrete applications-examples in the last section of the paper: Second order vector fields may be used for the description of a class of geodesic curves on infinite dimensional Riemannian manifolds. On the other hand, the case of a Banach or Fréchet space endowed with the canonical flat connection and that of a smooth Lie group with the direct connection are separately studied.

1. Second Order Vector Fields

In this section we define and study the basic notion for the description of second order differential equations on a smooth manifold M of finite or infinite dimension: That of *second order vector fields*.

The second order tangent bundle of M, T^2M, is the set of all classes $[(c,x)]_2$ of smooth curves $c : (-\varepsilon, \varepsilon) \to M$, $\varepsilon > 0$, with respect to the equivalence:

$$c_1 \approx_x c_2 \Leftrightarrow c_1(0) = c_2(0),\ \ c_1'(0) = c_2'(0) \text{ and } c_1''(0) = c_2''(0).$$

In general T^2M fails to be vector bundle over M in contrast to the classical case of (first-order) tangent bundles, as a result of the incompatibilities between the nonlinearity of acceleration and the structure of a vector bundle. However, the presence of a linear connection

$$\nabla : T(TM) \longrightarrow TM$$

on the base manifold M, gives the opportunity to overcome these difficulties endowing T^2M with a natural vector bundle structure.

To be more precise, let $\mathbb{E}$ be the (finite dimensional or Banach) space model of M, $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in I}$ a corresponding atlas, $\{(\pi_M^{-1}(U_\alpha), \tau_\alpha)\}_{\alpha\in I}$ the arising local vector coordinate system of the tangent bundle TM of M and $\{(\pi_{TM}^{-1}(\pi_M^{-1}(U_\alpha)), \widetilde{\tau}_\alpha)\}_{\alpha\in I}$ the analogous trivialization of $T(TM)$. Then, adopting the formalism of [27], the local expressions

$$\nabla_\alpha := \tau_\alpha \circ \nabla \circ (\widetilde{\tau}_\alpha)^{-1} : \phi_\alpha(U_\alpha) \times \mathbb{E} \times \mathbb{E} \times \mathbb{E} \longrightarrow \phi_\alpha(U_\alpha) \times \mathbb{E},$$

of ∇ have the form

$$\nabla_\alpha(y,u,v,w) = (y, w + \Gamma_\alpha(y)(u,v)), \quad a \in I,$$

where $\{\Gamma_\alpha\}_{\alpha\in I}$ is the family of Christoffel symbols of ∇:

$$\Gamma_\alpha : \phi_\alpha(U_\alpha) \longrightarrow \mathcal{L}_s^2(\mathbb{E}\times\mathbb{E}, \mathbb{E}); \quad \alpha \in I,$$

$\mathcal{L}_s^2(\mathbb{E}\times\mathbb{E}, \mathbb{E})$ denoting the space of bilinear symmetric mappings from $\mathbb{E}\times\mathbb{E}$ to $\mathbb{E}$. Based on the above we have defined in [5] a vector bundle structure on T^2M over M with fiber type $\mathbb{E}\times\mathbb{E}$. The corresponding local trivializations have the form:

$$\begin{aligned}\Phi_\alpha \ &:\ (\pi_M^2)^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{E} \times \mathbb{E} \\ &:\ [(c,x)]_2 \mapsto \big(x, (\phi_\alpha \circ c)'(0), (\phi_\alpha \circ c)''(0) \\ &\qquad\qquad + \Gamma_\alpha(\phi_\alpha(x))((\phi_\alpha \circ c)'(0), (\phi_\alpha \circ c)'(0))\big)\end{aligned}$$

if π_M^2 stands for the projection $\pi_M^2([(c,x)]_2) = x$.

It is obvious that there is a strong dependence of the vector bundle structure defined on T^2M on the choice of the linear connection ∇ of M. However, these structures are classified if one enables the notion of conjugate connections. More precisely, the vector bundle structures induced on T^2M by two linear connections ∇, ∇' of M are isomorphic if the connections are conjugate by means of a diffeomorphism of M (i.e. the connections commute with the first and second differential of the diffeomorphism, see [6] for details).

Taking into account this characterization, we may proceed with the definition of the notion of second order vector fields.

Definition 1.1. A section $\xi : M \to T^2M$ of the second order vector bundle T^2M , i.e. a smooth map satisfying

$$\pi^2_M \circ \xi = id_M,$$

where id_M denotes the identity map of M, is called a *second order vector field* on the base manifold M.

Of course, this property is sensitive also to the choice of the initial connection ∇, and a change of choice causes corresponding changes in the set of second order vector fields.

The second order vector fields may be viewed also as derivations in the following way: We consider the set of real numbers $\mathbb{R}$ as a 1-dimensional smooth manifold endowed with the identity total chart and the canonical flat connection with Christoffel symbols:

$$\Gamma : \mathbb{R} \longrightarrow \mathcal{L}(\mathbb{R}, \mathcal{L}(\mathbb{R}, \mathbb{R}))$$

which are vanishing everywhere $\Gamma(y) = 0$, $y \in \mathbb{R}$. Then, the corresponding second order tangent bundle $T^2\mathbb{R}$ becomes a vector bundle with total vector chart

$$\Psi : T^2\mathbb{R} \longrightarrow \mathbb{R}^3 : [(c, x)]_2 \mapsto \big(x, c'(0), c''(0)\big).$$

Based on this construction we may let each second order vector field on M act as a derivation on the set of smooth functions $C^\infty(M, \mathbb{R})$ as follows:

$$\xi : C^\infty(M, \mathbb{R}) \longrightarrow C^\infty(M, \mathbb{R}^2) : f \mapsto \xi(f),$$

where

$$\xi(f)(x) = T^2_x f(\xi(x)).$$

Here $T^2_x f$ denotes the second order differential of f on the fiber over x:

$$\begin{aligned} T^2_x f \quad &: \quad T^2_x M \longrightarrow T^2_{f(x)}\mathbb{R} \equiv \mathbb{R}^2 : \\ &: \quad [(c, x)]_2 \mapsto [(f \circ c, f(x))]_2 \equiv \big((f \circ c)'(0), (f \circ c)''(0)\big). \end{aligned}$$

The above functor is well defined and independent of the choice of the curve c as one may easily check. However, although the previous definition is a natural extension of the classical (first-order) case, the existence of a corresponding Lie bracket operator seems to be unreachable due to the fact that the result of this derivation does not remain in the same space.

2. Differential Equations of Second Order on a Banach manifold

Having established in the previous section all the necessary background notions/mechanisms, we proceed here to the study of second order differential equations on a smooth manifold M modelled on a Banach space $\mathbb{E}$.

Let ξ be a second order vector field on M.

Definition 2.1. An *integral curve* of ξ is a smooth map $\theta : J \to M$, defined on an open interval J of $\mathbb{R}$, if it satisfies the condition

$$T_t^2\theta(\partial_t) = \xi(\theta(t)), \tag{1}$$

where ∂_t is the second order tangent vector of $T_t^2\mathbb{R}$ induced by a curve $c : \mathbb{R} \to \mathbb{R}$ with $c'(0) = 1, c''(0) = 1$.

Note that if we restrict ourselves to the case where the base manifold M is a Banach space $\mathbb{E}$ with differential structure induced by the total chart $(\mathbb{E}, id_{\mathbb{E}})$, then the first part of the above condition reduces to the second derivative of θ:

$$T_t^2\theta(\partial_t) = \theta''(t) = D^2\theta(t)(1,1).$$

In other words, the previous definition gives a natural generalization of the notion of second derivative on a manifold M. On the other hand, it offers the opportunity to approach ordinary differential equations of order two on M. Namely, the next result holds.

Theorem 2.2. *Let ξ be a second order vector field on M. Then, the existence of an integral curve θ of ξ is equivalent to the solution of a system of second order differential equations on $\mathbb{E}$.*

Proof. Keeping the formalism of Section 1, we consider $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in I}$ a smooth atlas of M and $\{(U_\alpha, \Phi_\alpha)\}_{\alpha\in I}$ the corresponding local trivialization of T^2M. Then, the local expression of the second derivative $T_t^2\theta(\partial_t)$ takes the form

$$\begin{aligned}
\Phi_\alpha(T_t^2\theta(\partial_t)) &= \Phi_\alpha([(\theta\circ c, \theta(t))]_2) \\
&= \big(\theta(t), (\phi_\alpha\circ\theta\circ c)'(0), (\phi_\alpha\circ\theta\circ c)'(0)''(0) + \\
&\quad \Gamma_a((\phi_\alpha\circ\theta)(t))((\phi_\alpha\circ\theta\circ c)'(0), (\phi_\alpha\circ\theta\circ c)'(0))\big)
\end{aligned}$$

However,

$$\begin{aligned}
(\phi_\alpha\circ\theta\circ c)'(0) &= D(\phi_\alpha\circ\theta)(c(0))(c'(0)) = T_t(\phi_\alpha\circ\theta)(1) = (\phi_\alpha\circ\theta)'(t), \\
(\phi_\alpha\circ\theta\circ c)''(0) &= D^2(\phi_\alpha\circ\theta)(c(0))(c'(0), c'(0)) + D(\phi_\alpha\circ\theta)(c(0))(c''(0)) \\
&= D^2(\phi_\alpha\circ\theta)(t)(1,1) + D(\phi_\alpha\circ\theta)(t)(1) \\
&= (\phi_\alpha\circ\theta)''(t) + (\phi_\alpha\circ\theta)'(t)
\end{aligned}$$

As a result,

$$\begin{aligned}
\Phi_\alpha(T_t^2\theta(\partial_t)) &= \big(\theta(t), (\phi_\alpha\circ\theta)'(t), (\phi_\alpha\circ\theta)''(t) + (\phi_\alpha\circ\theta)'(t) + \\
&\quad \Gamma_a((\phi_\alpha\circ\theta)(t))((\phi_\alpha\circ\theta)'(t), (\phi_\alpha\circ\theta)'(t))\big)
\end{aligned}$$

and the local expression of (1) takes the form

$$\begin{aligned} (\phi_\alpha \circ \theta)'(t) &= \Phi_\alpha^{(2)}(\xi(\theta(t))), \\ (\phi_\alpha \circ \theta)''(t) + (\phi_\alpha \circ \theta)'(t) &+ \\ \Gamma_a((\phi_\alpha \circ \theta)(t))((\phi_\alpha \circ \theta)'(t), (\phi_\alpha \circ \theta)'(t)) &= \Phi_\alpha^{(3)}(\xi(\theta(t))), \quad a \in I, \end{aligned} \tag{2}$$

where $\Phi_\alpha^{(2)}, \Phi_\alpha^{(3)}$ stand for the projection of Φ_α to the second and third factor respectively. □

We have proved in this way that integral curves of second order vector fields generalize the notion of second order differential equations on manifolds. Taking also into account that the differentiability of all the involved functions guarantees the satisfaction of the necessary Lipschitz conditions we may state that

Theorem 2.3. *Every second order vector field ξ on M admits locally a unique integral curve satisfying an initial condition of the form $\theta(0) = x$, $T_t\theta(\partial_t^1) = y$, for arbitrary choice of $x \in M$, $y \in T_{\theta(t)}M$, where ∂_t^1 stands for the basic vector field of $T_t\mathbb{R}$.*

Remark 2.4. It is clear from the proof of Theorem 2.2 that the second order differential equations described by our approach depend not only on the choice of the second order vector field but also on the host geometric background of the manifold, as expressed by the chosen linear connection.

3. The Fréchet case

In this section we expand the methodology proposed for the study of second order differential equations to the framework of Fréchet modelled manifolds. The general case of a second (or even first) order differential equation on this type of manifolds cannot be confronted successfully using the classical pattern of Banach modeled manifolds. For, on the model spaces an ordinary differential equation may admit no, one or multiple solutions for the same initial condition.

These analytical problems with several applications in theoretical physics (see, e.g. [1], [19, 20], [25]) led a number of authors to propose different methods for the study of certain types of differential equations in Fréchet spaces (see [4], [7], [14], [16]).

In a series of previous papers of the third author ([10], [11], [12]) a new way leading to the solution of a wide class of such types of equations is proposed. This stems from the fact that every Fréchet space is isomorphic to a projective limit of Banach spaces, and the taking of projective limits is compatible with differentiation. These techniques can also be combined with the new approach of second order differential equations proposed in the previous sections, to provide a way out of the difficulties described.

More precisely, let M be a smooth manifold modeled on the Frèchet space $\mathbb{F}$. Since always $\mathbb{F}$ can be realized as a projective limit of Banach spaces $\mathbb{F} \simeq \varprojlim\{\mathbb{E}^i; \rho^{ji}\}_{i,j\in\mathbb{N}}$, we assume that the manifold itself is obtained as the limit of a projective system of Banach modeled manifolds $\{M^i; \varphi^{ji}\}_{i,j\in\mathbb{N}}$ and that is covered by a system of "projective limit" charts:

For each $x = (x^i) \in M$ there exists a projective system of local charts $\{(U^i_\alpha, \phi^i_\alpha)\}_{i\in\mathbb{N}}$ such that $x^i \in U^i_\alpha$ and the corresponding limit $\varprojlim U^i_\alpha$ is open in M.

Let ∇ be a linear connection on M realized also as a projective limit of connections on the factors M^i. This is equivalent to the fact that the corresponding Christoffel symbols commute with the connecting morphisms of the tangent bundles of M which have the form:

$$g^{ji}_k : T^kM^j \to T^kM^i : \; [f,x]^j_k \longmapsto [\phi^{ji} \circ f, \phi^{ji}(x)]^i_k,$$

where $k = 1, 2$ denotes the order of the tangent bundle.

Under these conditions, M can be endowed with a Fréchet manifold structure modeled on $\mathbb{F}$ via the charts $\{(\varprojlim U^i_\alpha, \varprojlim \phi^i_\alpha)\}_{a\in I}$. For the differentiability of mappings in this framework we adopt the definition of J. A. Leslie ([17, 18]).

On the other hand, the tangent bundles TM and T^2M of M are endowed also with Fréchet manifold structures of the same type modeled on $\mathbb{F}^2$, $\mathbb{F}^4$ respectively. The corresponding local structures are defined by the differentials $\{\varprojlim(T\phi^i_\alpha)\}_{a\in I}$ for the first order tangent bundle and by the projective limits of the trivializations

$$\begin{aligned}\Phi^i_\alpha \;&:\; (\pi^2_{M^i})^{-1}(U^i_\alpha) \longrightarrow U^i_\alpha \times \mathbb{E}^i \times \mathbb{E}^i \\ &:\; [(c,x)]_2 \longmapsto \big(x, (\phi^i_\alpha \circ c)'(0), (\phi^i_\alpha \circ c)''(0) \\ &\qquad\qquad + \Gamma^i_\alpha(\phi^i_\alpha(x))((\phi^i_\alpha \circ c)'(0), (\phi^i_\alpha \circ c)'(0))\big),\end{aligned}$$

for T^2M.

Based on the above constructions, we may prove the following main result.

Theorem 3.1. *Every second order vector field ξ on M obtained as projective limit of second order vector fields $\{\xi^i$ on $M^i\}_{i\in\mathbb{N}}$ admits locally a unique integral curve θ satisfying an initial condition of the form $\theta(0) = x$ and $T_t\theta(\partial_t) = y$, for every choice of $x \in M$, $y \in T_{\theta(t)}M$.*

Proof. Since each ξ^i is a second order vector field on the Banach modeled manifold M^i, Theorem 2.3 ensures the existence of an integral curve θ^i satisfying:

$$\begin{aligned}(\varphi^i_\alpha o\theta^i)'(t) &= \Phi^{2,i}_\alpha(\xi^i(\theta^i(t))), \\ &\quad (\varphi^i_\alpha o\theta^i)''(t) + (\varphi^i_\alpha o\theta^i)'(t) + \\ \Gamma^i_\alpha(\varphi^i_\alpha o\theta^i)(t)[(\varphi^i_\alpha o\theta^i)'(t), (\varphi^i_\alpha o\theta^i)'(t)] &= \Phi^{3,i}_\alpha(\xi^i(\theta^i(t))),\end{aligned}$$

under the initial conditions $\theta^i(0) = x^i := \varphi^i(x)$, and $T_t\theta^i(\partial_t) = y^i := T_{\theta(t)}\varphi^i(y)$, if $\varphi^i : M = \varprojlim M^i \to M^i$, $i \in \mathbb{N}$, are the canonical projections of the projective limit.

We claim that $\theta := \varprojlim \theta^i$ exists and fulfils the conditions of the theorem. Indeed, we initially observe that for each pair of indices $j \geq i$, $\varphi^{ji} \circ \theta^j$ is also an integral curve of ξ^i since:

$$\begin{aligned}(\varphi^i_\alpha \circ (\varphi^{ji} \circ \theta^j))'(t) &= (\rho^{ji} \circ \varphi^j_\alpha \circ \theta^j)'(t) = \rho^{ji}((\varphi^j_\alpha \circ \theta^j)'(t)) = \\ &= \rho^{ji}(\Phi^{2,j}_\alpha(\xi^j(\theta^j(t)))) = \Phi^{2,i}_\alpha(g^{ji}_2(\xi^j(\theta^j(t)))) \\ &= \Phi^{2,i}_\alpha(\xi^i(\varphi^{ji} o\theta^j(t))) \;;\end{aligned}$$

and

$$\begin{aligned}
&(\varphi_\alpha^i \circ (\varphi^{ji} \circ \theta^j))''(t) + (\varphi_\alpha^i \circ (\varphi^{ji} \circ \theta^j))'(t) \\
&+\Gamma_\alpha^i(\varphi_\alpha^i \circ (\varphi^{ji} o \theta^j)(t))[(\varphi_\alpha^i \circ (\varphi^{ji} o\theta^j))'(t), (\varphi_\alpha^i \circ (\varphi^{ji} o\theta^j))'(t))] \\
&= (\rho^{ji} \circ \varphi_\alpha^j \circ \theta^j)''(t) + (\rho^{ji} \circ \varphi_\alpha^j \circ \theta^j)'(t) + \\
&+\Gamma_\alpha^i((\rho^{ji} \circ \varphi_\alpha^j \circ \theta^j)(t))[(\rho^{ji} \circ \varphi_\alpha^j \circ \theta^j)'(t), (\rho^{ji} \circ \varphi_\alpha^j \circ \theta^j)'(t))] \\
&= \rho^{ji}((\varphi_\alpha^j \circ \theta^j)''(t) + (\varphi_\alpha^j \circ \theta^j)'(t) + \\
&+\Gamma_\alpha^j((\varphi_\alpha^j \circ \theta^j)(t))[(\varphi_\alpha^j \circ \theta^j)'(t), (\varphi_\alpha^j \circ \theta^j)'(t)]) \\
&= \rho^{ji}(\Phi_\alpha^{3,j}(\xi^j(\theta^j(t)))) = \Phi_\alpha^{3,i}(g_2^{ji}(\xi^j(\theta^j(t)))) \\
&= \Phi_\alpha^{3,i}(\xi^i(\varphi^{ji} o \theta^j(t))) \qquad .
\end{aligned}$$

On the other hand,
$\theta^i(0) = x^i$ and $T_t\theta^i(\partial_t) = y^i$, give $(\varphi^{ji} \circ \theta^j)(0) = x^i$ and $T_{\theta^j(t)}\varphi^{ji} o\theta^j(\partial_t) = y^i$. As a result, $\varphi^{ji} \circ \theta^j$ and θ^i will coincide as integral curves of the same second order vector fields over the same initial conditions. Therefore, $\theta = \varprojlim \theta^i$ exists and is smooth as a projective limit of smooth functions (see [13]).

On the other hand,

$$(\varphi_\alpha \circ \theta)'(t) = ((\varphi_\alpha^i \circ \theta^i)'(t))_{i\in N} = (\Phi_\alpha^{2,i}(\xi^i(\theta^i(t)))_{i\in N} = \Phi_\alpha^2(\xi(\theta(t)))$$

and

$$\begin{aligned}
&(\varphi_\alpha \circ \theta)''(t) + (\varphi_\alpha \circ \theta)'(t) + \Gamma_\alpha((\varphi_\alpha \circ \theta)(t))[(\varphi_\alpha \circ \theta)'(t), (\varphi_\alpha \circ \theta)'(t)] \\
&= ((\varphi_\alpha^i \circ \theta^i)''(t) + (\varphi_\alpha^i \circ \theta^i)'(t) + \Gamma_\alpha^i((\varphi_\alpha^i \circ \theta^i)(t))[(\varphi_\alpha^i \circ \theta^i)'(t), (\varphi_\alpha^i \circ \theta^i)'(t)])_{i\in N} \\
&= (\Phi_\alpha^{3,i}(\xi^i(\theta^i(t))))_{i\in N} = \Phi_\alpha^3(\xi(\theta(t)))
\end{aligned}$$

We have proved in this way that θ is the desired integral curve of the second order vector field ξ. The uniqueness of it under the given initial conditions is obtained following similar reasoning and by checking that each projection of θ via the canonical mappings $\varphi^i : M \to M^i$ is the unique integral curve of ξ^i satisfying $\theta^i(0) = x^i$ *and* $T_t\theta^i(\partial_t^1) = y^i$. □

4. Applications - Examples.

Geodesic curves of Riemannian manifolds.

The new approach to second order differential equations on manifolds proposed in the previous sections, gives also a very simple way to describe a class of geodesic curves in a Riemannian manifold. To be more precise let (M, g) be an infinite dimensional Riemannian manifold endowed with a smooth atlas $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in I}$ and ∇ a Riemannian connection characterized by the Christoffel symbols

$$\{\Gamma_\alpha : \phi_\alpha(U_\alpha) \longrightarrow \mathcal{L}_s^2(\mathbb{E}\times\mathbb{E}; \mathbb{E})\}_{\alpha\in I}.$$

Keeping the formalism of Sections 1 and 2, let ξ be a second order vector field on M induced by a constant curve:

$$\xi(x) = [(c_x, x)]_2,$$

where $c_x(t) = x$, $t \in [0, 1]$. Then, the local expression of ξ takes the form:

$$\begin{aligned}\Phi_\alpha([(c_x,x)]_2) &= \big(x,(\phi_\alpha\circ c_x)'(0),(\phi_\alpha\circ c_x)''(0)\\ &\qquad +\Gamma_\alpha(\phi_\alpha(x))[(\phi_\alpha\circ c_x)'(0),(\phi_\alpha\circ c_x)'(0)]\big)\\ &= (x,0,0).\end{aligned}$$

As a result, equation (2) that provides the corresponding integral curves θ of ξ through x (see Theorem 2.3) will reduce to

$$(\phi_\alpha\circ\theta)''(t)=0,\quad a\in I,$$

which ensures that θ is a geodesic curve of M. We have proven in this way that *the integral curves of second order vector fields induced by constant functions are geodesics.*

This result holds also for every second order vector field ξ that locally fulfils

$$\Phi^2_\alpha\circ\xi=\Phi^3_\alpha\circ\xi,\quad a\in I.$$

In this case, equation (2) reduces to

$$(\phi_\alpha\circ\theta)''(t)+\Gamma_a((\phi_\alpha\circ\theta)(t))((\phi_\alpha\circ\theta)'(t),(\phi_\alpha\circ\theta)'(t))=0,\quad a\in I,$$

which is exactly the local condition that a geodesic of M has to satisfy.

In the sequel we give two more examples of applications that clarify further our method.

Example 1. Second Order Differential Equations on the model space.

If we consider the manifold $M=\mathbb{E}$ endowed with the differential structure induced by the total chart $(\mathbb{E},id_{\mathbb{E}})$, and consider the canonical flat connection with trivial Christoffel symbols $\Gamma(x)(u)=0$, for each $(x,u)\in\mathbb{E}\times\mathbb{E}$, then the second order tangent bundle $T^2\mathbb{E}$ becomes a vector bundle with (total) vector chart

$$\Phi:T^2\mathbb{E}\to\mathbb{E}\times\mathbb{E}\times\mathbb{E}:[(c,x)]_2\longmapsto\big(x,c'(0),c''(0)\big).$$

This is the case either for a Banach or a Fréchet model space. In this way, if ξ is a second order vector field on $\mathbb{E}$ and $\theta:\mathbb{R}\to\mathbb{E}$ a corresponding integral curve, equations (2) reduce to

$$\begin{aligned}\theta'(t) &= \Phi^2_\alpha(\xi(\theta(t))),\\ \theta''(t)+\theta'(t) &= \Phi^3_\alpha(\xi(\theta(t))),\end{aligned}$$

which is an ordinary differential equation of second order on $\mathbb{E}$. Always this can be solved uniquely under given initial conditions in the case of Banach spaces as well as in the Fréchet framework if we assume that the vector field ξ is a projective limit (cf. [12]).

Moreover, the integral curves of a second order vector field induced by a constant function will satisfy the equation:

$$\theta''(t)=0,$$

therefore, it will be a line $\theta(t)=at+b$, $a,b\in\mathbb{R}$.

Example 2. Second Order Differential Equations on Lie groups.
Let G be a Lie group modelled on $\mathbb{E}$ endowed with the so-called *direct connection* ∇^G, that is the unique connection which is (μ, id_G)-conjugate with the canonical flat connection of the trivial bundle $(G\times\mathcal{G}, pr_1, G)$, where

$$\mu : G \times \mathcal{G} \xrightarrow{\simeq} TG : (g, h) \mapsto T_e L_g(h)$$

denotes the left parallelization of G and $\mathcal{G}$ the Lie algebra of G. If $b_a(x)$ gives the local expression of the isomorphism $T_e L_x : T_e G \to T_x G$ with respect to the chart (U_a, ϕ_a), then the Christoffel symbols of ∇^G take the form

$$\Gamma_a^G(x)(k, h) = -Db_a(x)(k, b_a^{-1}(x)(h)); \quad x \in \phi_a(U_a), \ k, h \in \mathbb{E},$$

(for a complete presentation of the notion of direct connection and the relevant proofs we refer to [26]). As a result, equations (2) take in this case the form

$$\begin{aligned} (\phi_\alpha \circ \theta)'(t) &= \Phi_\alpha^2(\xi(\theta(t))), \\ (\phi_\alpha \circ \theta)''(t) + (\phi_\alpha \circ \theta)'(t) &- \\ Db(x)((\phi_\alpha \circ \theta)'(t), b^{-1}(x)((\phi_\alpha \circ \theta)'(t))) &= \Phi_\alpha^3(\xi(\theta(t))), \end{aligned}$$

which is the local form of a second order differential equation on G.

As in the first example, always the above equations admit solutions in the Banach case. For Fréchet Lie groups obtained as projective limits, the differential equation at hand also can be solved uniquely with respect to given initial conditions if the second order vector field ξ can be realized as a projective limit. In this case the problem is equivalent to a countable system of differential equations on the Banach factors (cf. [9]).

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