Model reduction of switched dynamical systems

Chahlaoui, Younes

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Younes Chahlaoui

Centre for Interdisciplinary Computational and Dynamical Analysis (CICADA)
School of Mathematics
University of Manchester
Manchester
UK
Younes.Chahlaoui@manchester.ac.uk

ABSTRACT. A hybrid dynamical system is a system described by both differential equations (continuous flows) and difference equations (discrete transitions). It has the benefit of allowing more flexible modeling of dynamic phenomena, including physical systems with impact such as the bouncing ball, switched systems such as the thermostat, and even the internet congestion as examples. Hybrid dynamical systems pose a challenge since almost all reduction methods cannot be directly applied. Here we show some recent developments in the area of model reduction of switched dynamical systems.

RÉSUMÉ. Un système dynamique hybride est un système décrit par des équations différentielles (flux continus) et des équations de différences (transitions discrètes). L'utilisation de ce genre de systèmes permet une modélisation plus souple des phénomènes dynamiques, y compris les systèmes physiques avec impact comme la balle bondissante, les systèmes commutés comme le thermostat, et même la congestion de l'internet comme simples exemples. Les systèmes dynamiques hybrides constituent un défi, car presque toutes les méthodes de réduction ne peuvent pas être appliquées directement. Nous montrons ici quelques idées récentes pour la réduction des systèmes dynamiques hybrides commutés.

KEYWORDS : Model order reduction, switched dynamical systems

MOTS-CLÉS : Réduction d'ordre, systèmes dynamiques commutés

1. Introduction [1]: Switched dynamical systems

A switched linear system is a hybrid system which consists of several linear subsystems and a rule that orchestrates the switching among them. It is mathematically described by

$$\begin{array}{rcll} \delta x(t) & = & A_\sigma x(t) + B_\sigma u(t) + E_\sigma d(t), & x(t_0) = x_0, \\ y(t) & = & C_\sigma x(t) + G_\sigma w(t), & \end{array} \tag{1}$$

where $x(t)$ is the state, $u(t)$ is the controlled input, $y(t)$ is the measured output, $d(t)$ and $w(t)$ stand for external signals such as perturbations, σ is the piecewise constant signal taking values from an index set $M = \{1, \ldots, m\}$, A_k, B_k, C_k, E_k and G_k $k \in M$ are matrices of appropriate dimensions. The nominal system is the system free of disturbances, that is

$$\begin{array}{rcl} \delta x(t) & = & A_\sigma x(t) + B_\sigma u(t), \\ y(t) & = & C_\sigma x(t). \end{array} \tag{2}$$

In this paper, we will denote system (2) by $\Sigma(A_k, B_k, C_k)_M$.

Switched systems can be used to model systems that are subject to known or unknown abrupt parameter variations such as synchronously switched linear systems [9], networks with periodically varying switchings [2], and sudden change of system structures due to various reasons [14]. For example, the failure of a component or subsystem may have taken place in so short a time interval as to be considered an instantaneous event by comparison with the nominal time constants of the plant model. Hence, switching among different system structures is an essential feature of many engineering and practical real world systems. Figure 1 shows a schematic of a switched system.

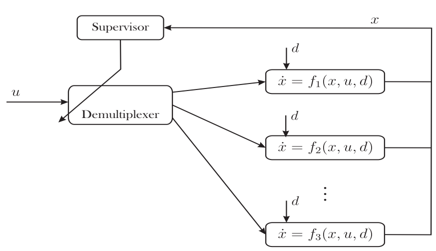

Figure 1.

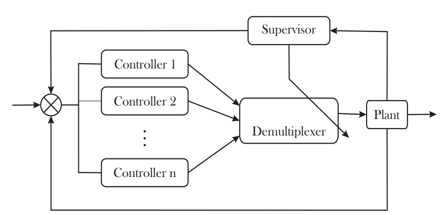

Figure 2.

Moreover, when we try to control a single process by means of multi controller switching (Figure 2), the overall system can be also described by a switched system. In the literature, this multi controller switching scheme is also known as the hybrid control architecture. It provides an effective and powerful mechanism to cope with highly complex

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systems and/or systems with large uncertainties [10, 11, 16]. For these systems, multi controller switching among smooth controllers provides a good conceptual framework to solve the problem. For example, as a common practice in stabilizing an LTI system, we use a hybrid controller that involves switching between a time-optimal or near time-optimal controller when the state is far from the equilibrium, and a linear controller near the equilibrium. This strategy can steer the system to the equilibrium quickly without exciting high-frequency dynamics or exceeding realistic actuator bandwidth.

Practically, many real world systems have or will benefit from the study of switched systems including power electronics, automotive control, aircraft or air traffic control, network and congestion control, to cite a few.

Switched linear systems are relatively easy to handle as many powerful tools from linear and multilinear analysis are applicable to cope with them. These systems seem to be accurate enough to represent many practical engineering systems with complex dynamics, and provide additional insight to some sophisticated problems [13]. Unfortunately, very often some of the subsystems, and consequently the whole system, are large complex mathematical models. However, in control design or simulation it is common practice to work with as simple models as possible, because they are easier to analyze and evaluate. There is a strong need for methods and tools that can take a complex switched model and deduce simple switched models for various purposes such as control design. One way to do this is by the use of model order reduction.

2. Model order reduction

A simple but good model captures much knowledge. It points out the basic properties and can give good insight about the process. For simple linear time-invariant models there is a well-established theory and commercially available tools for design of controllers with given specifications. Real experiments or simulations using more complex models are then used to verify that the designed controller works well. For nonlinear models the methods are much less developed. It is simple to derive a linearization in symbolic form from a nonlinear model. It is much more difficult to give explicit expressions for stationary operating points since these calculations involve nonlinear equation systems. The hybridity of the switched system adds another difficulty as the switching is ruled by some physically or security limitations. And for instance no explicit numerical method exists for model reduction of hybrid systems or switched dynamical systems.

The main idea in model reduction is that a high-dimensional state vector actually belongs to a low-dimensional subspace. Provided that the low-dimensional subspace is known, the original model can be projected on it to obtain a required low-dimensional approximation. The goal of every model reduction method is to find such a low-dimensional subspace [3].

Unfortunately existing model reduction methods do not respect the hybrid structure of the system $\Sigma(A_k, B_k, C_k)_M$. The reduced model will not necessarily have subsystems

$\Sigma(\hat{A}_k, \hat{B}_k, \hat{C}_k)$, and even if so, the subsystems $\Sigma(\hat{A}_k, \hat{B}_k, \hat{C}_k)$ will not necessarily correspond one by one to the subsystems $\Sigma(A_k, B_k, C_k)$ of the original system $\Sigma(A_k, B_k, C_k)_M$. One needs to go back to the original definition of model reduction of dynamical systems in order to see what needs to be changed to fit to the hybrid properties of the system.

The typical problem of traditional model reduction consists in approximating a system $\mathcal{S}$ by a system $\hat{\mathcal{S}}$ while minimizing the L_2 norm:

$$\left(\int_0^\infty |y(t) - \hat{y}(t)|^2 dt \right)^{\frac{1}{2}} \tag{3}$$

where y is the output of $\mathcal{S}$ and $\hat{y}$ is the output of $\hat{\mathcal{S}}$. This kind of reduction is often not adequate for hybrid systems, especially for the problems of formal verification and switching. Formal verification is a typical safety problem and it consists in determining if any trajectory of $\mathcal{S}$ starting in a given set of initial conditions enters a given set of unsafe states $\mathcal{U}$. If the reduced system $\hat{\mathcal{S}}$ is used to solve this problem, we will not be able to guaranty that the trajectories of $\mathcal{S}$ do not enter $\mathcal{U}$. This is due to the fact that minimizing (3) will give us only an upper bound on the general error for the trajectories. The same problem occurs for the switching. To explain this, let us consider that our switched system is composed by two subsystems Σ_1 and Σ_2, and a guard which consists in a value that when the output of the subsystem Σ_1 reaches it switches to the subsystem Σ_2, in this case we say that we hit the guard (Figure 3).

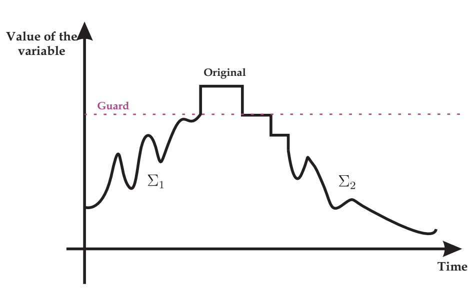

Figure 3. *A simple switched system.*

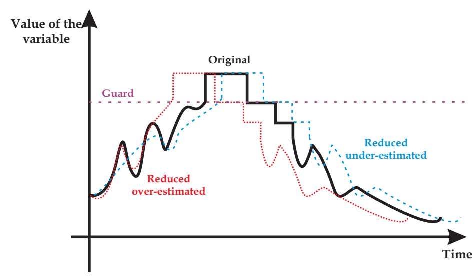

Figure 4. *Behavior of different reduced switched models.*

Any reduced model based on an upper bound, as for (3), will result in either an over-estimated reduced model or an under-estimated reduced model. Both reduced models could hit the guard far away from when the original system is actually hitting the guard, which will result in two different behaviors completely different from the behavior of the original system (Figure 4).

In order to overcome all these kind of problems, we can adopt the $\mathcal{L}_\infty$ norm

$$\sup_{t \in [0,\infty)} |y(t) - \hat{y}(t)|.$$

Other recent techniques for hybrid systems exist that are based on the notion of bisimulation [5, 6, 15]. The fundamental difference between model reduction and bisimulation [6] is that the latter tries to find a system simulating the original system (almost the same output) but not necessarily for the same input, while the reduced model should have an output very close to the output of the original system but for the same input.

3. Model order reduction for switched dynamical systems

As a special but very important technique to study dynamical systems, the input/output approach provides an attractive framework that was widely developed in the last century. Almost all model reduction techniques are made for that approach. The first idea to do model reduction on hybrid systems is to use this well developed input/output framework. The principle is very simple: we will reduce independently each subsystem $\Sigma(A_k, B_k, C_k)$ alone, then collect all the reduced subsystems $\Sigma(\hat{A}_k, \hat{B}_k, \hat{C}_k)$ to construct the hybrid reduced system $\Sigma(\hat{A}_k, \hat{B}_k, \hat{C}_k)_M$. The switching rule (the guard) is just copied into the reduced model. With this operation we will have the following features. On the one hand, each subsystem will be well approximated by the corresponding reduced subsystem in the sense of input/output behavior. For the same input, both subsystem and corresponding reduced subsystem will have very close outputs for a defined norm. As a result, if the switching rule occurs for the subsystem $\Sigma(A_k, B_k, C_k)$ based on some constraints on its output, it will also occur at almost the same time for the reduced subsystem $\Sigma(\hat{A}_k, \hat{B}_k, \hat{C}_k)$ as a result of the input/output model reduction operation. On the other hand, the switching rule will be almost the same for both systems $\Sigma(\hat{A}_k, \hat{B}_k, \hat{C}_k)_M$ and $\Sigma(A_k, B_k, C_k)_M$. Unfortunately, this approach is time consuming as one needs to consider each subsystem independently. But it will try to match the input/ouput behavior. In general, we could use either balanced truncation or Hankel norm approximation to do the model reduction of the subsystems. These are the best methods if we do not have any special knowledge about the switching time and any other property of the subsystem at hand. If this is the case, one could consider a moment matching method that will match the input/output behavior around the switching moment. This will preserve the switching at the same moment for both systems, the original and the reduced (Figure 5).

Another approach is to consider the hybrid system $\Sigma(A_k, B_k, C_k)_M$ as a whole. For this we propose to use the time-varying model reduction. First, let us consider the switching sequence of σ over $[t_0, t_1)$ $(t_0, \sigma(t_0+)), (s_1, \sigma(s_1+)), \ldots, (s_l, \sigma(s_l+)) = (s_i, \sigma(s_i+))_{i=0}^{l}$ [13]. The system $\Sigma(A_k, B_k, C_k)_M$ is equivalent to the time-varying system $\{A_k, B_k, C_k\}$ where $k = 0, \ldots, l$. On each interval $[s_i, s_{i+1})$ the dynamics of the system are given by the LTI system $\{A_i, B_i, C_i\}$. It is a special case of time-varying systems, an interval linear time-invariant system. But the dynamic of the system is following the time evolution. Many techniques are already available for this kind of systems [3, 4]. And almost all background for LTI theory can be generalized for that kind of systems.

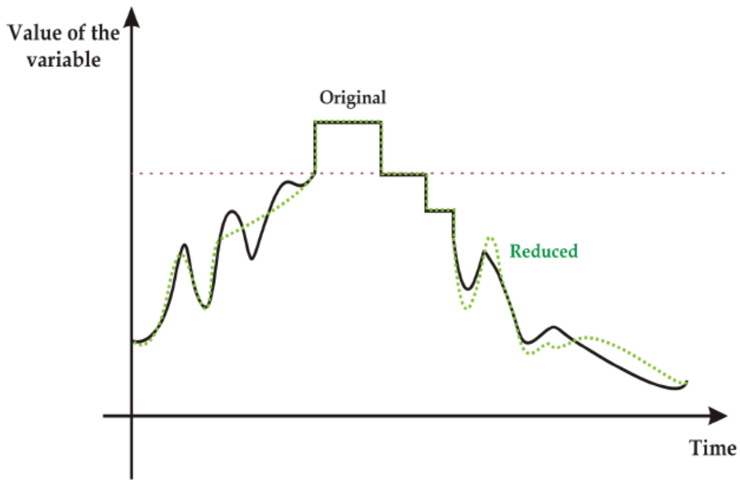

Figure 5. *Preservation of the switching moment.*

The key idea is to use the sub-gramians, which are the gramians (energy matrix functions) of each subsystem. The stability of the hybrid system is equivalent to the existence of solutions to the following systems of Linear Matrix Inequalities (LMIs)

$$\mathcal{P} = \mathcal{P}^T \in \mathbb{R}^{N\times N}, \quad \mathcal{Q} = \mathcal{Q}^T \in \mathbb{R}^{N\times N}, \quad \epsilon_{1,2} \in \mathbb{R}, \epsilon_{1,2} > 0$$

$$\mathcal{P} \succ 0, \quad A_\sigma^T \mathcal{P} + \mathcal{P} A_\sigma \leq -\epsilon_1 I < 0,$$

$$\mathcal{Q} \succ 0, \quad A_\sigma \mathcal{Q} + \mathcal{Q} A_\sigma^T \leq -\epsilon_2 I < 0.$$

We can use the idea of balanced truncation with these two gramians to come up with a balanced truncated like reduced model. With this approach we will preserve the stability for the reduced model. Here one should notice that we supposed implicitly that each subsystem is stable. This is not always the case. In fact, either two stable or unstable subsystems could generate a stable switched system [7, 8, 12].

In all previous approaches we copied the guard of the original system. Another class of model reduction for switched systems can be generated if we accept to change the guard also. The idea is that the guard will be relaxed following the quality of the reduced model. One has to define an error margin from the error between the output of the original system and the output of the reduced model. Different scenarios can be adapted following the purpose of the switched system considered. For example for robustness, one can relax the guard to two new guards, the first guard as an lower-guard and the second one as an upper-guard (Figure 6). When the system will enter the region between the two guard, it will be in a transition state, but it will switch only if the upper-guard is hit. With this approach

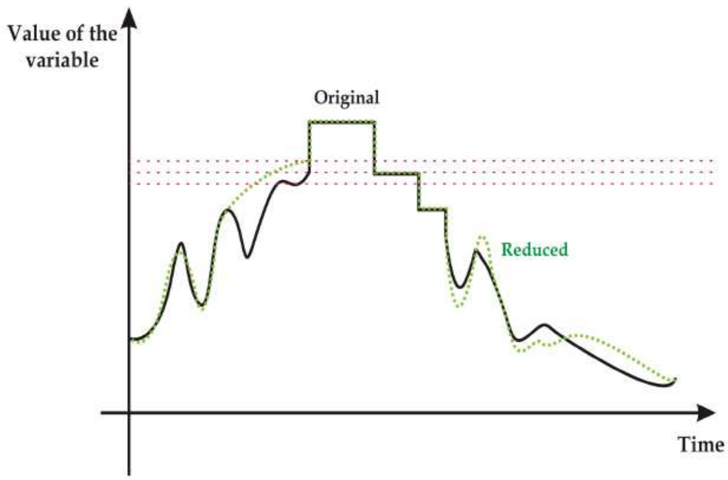

Figure 6. *Relaxation of the guard.*

we can accept to have less accurate reduced subsystems, but the switched reduced model will match much better the general behavior of the original switched system.

4. Conclusions

In this paper we considered model reduction of switched dynamical systems. Many ideas can be considered following what aspects and properties of the original system we would like to keep in the reduced model. A more analytic study should be done in order to compare between all these techniques.

5. References

[1] A.C. ANTOULAS, "Approximation of Large-Scale Dynamical Systems", *Society for Industrial and Applied Mathematic*, 2005.

[2] R.W. BROCKETT, JR. WOOD "Electrical networks containing controlled switches", *Applications of Lie Groups Theory to Nonlinear Networks Problems, Supplement to IEEE Inter Symp Circuit*, 1974, pp 1-11.

[3] Y. CHAHLAOUI, "Low-rank approximation and model reduction", *PhD thesis number 14/2003, Université catholique de Louvain*, 2003.

[4] Y. CHAHLAOUI, P. VAN DOOREN, "Model reduction of time-varying systems", *In: Dimension Reduction of Large-Scale Systems, Springer Berlin Heidelberg*, vol. 45, pp 131-148, 2005.

[5] A. GIRARD, G.J. PAPPAS, "Approximate bisimulation relations constrained linear systems", *Automatica*, vol. 43, num. 8, 2007.

[6] E. HAGHVERDI, P. TABUADA, G.J. PAPPAS, "Bisimulation relations for dynamical, control and hybrid systems", *Theoretical Computer Science*, vol. 342, num. 2-3, 2005.

[7] J. HESPANHA, "Uniform Stability of Switched Linear Systems: Extensions of LaSalle's Invariance Principle", *IEEE Transactions on Automatic Control*, vol. 49, num. 4, 2004.

[8] J. HESPANHA, A.S. MORSE, "Stability of Switched Systems with Average Dwell-time", *In Proc. of the 38th Conf. on Decision and Contr.*, 1999.

[9] T.L. JOHNSON, "Synchronous switching linear systems", *Proc of 24th IEEE CDC*, 1985, pp 1699-1700.

[10] I. KOLMANOVSKY, N.H. MCCLAMROCH, "Hybrid feedback laws for a class of cascade nonlinear control systems", *IEEE Trans. Autom. Control*, vol. 41, num. 9, 1996.

[11] A. LEONESSA, W.M. HADDAD, V.S. CHELLABOINA, "Nonlinear system stabilization via hierarchical switching control", *IEEE Trans. Autom. Control*, vol. 46, num. 1, 2001.

[12] D. LIBERZON, J. HESPANHA, A.S. MORSE, "Stability of switched linear systems: a Lie-algebraic condition", *Syst. & Contr. Lett.*, vol. 37, num. 3, 1999.

[13] Z. SUN, S.S. GE, "Switched linear systems. Control and design.", *Communications and Control Engineering. London: Springer.*, 2005.

[14] D.D. SWORDER, "Control of systems subject to sudden changes", *Proc IEEE*, vol. 64, num. 8, 1976.

[15] A. VAN DER SCHAFT, "Equivalence of dynamical systems by bisimulation", *IEEE Transactions on Automatic Control*, vol. 49, num. 12, 2004.

[16] J. ZHAO, M.W. SPONG, "Hybrid control for global stabilization of the cart-pendulum system", *Automatica*, vol. 37, num. 12, 2001.