X_M-*Harmonic Cohomology and Equivariant Cohomology on Riemannian Manifolds With Boundary*

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X_M-Harmonic Cohomology and Equivariant Cohomology on Riemannian Manifolds With Boundary

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Abstract

Given a Riemannian manifold M with boundary and a torus G which acts by isometries on M and let X be in the Lie algebra of G and corresponding vector field X_M on M, we consider Witten's coboundary operator $\mathsf{d}_{X_M} = \mathsf{d} + \iota_{X_M}$ on invariant forms on M. In [1] we introduce the absolute X_M-cohomology $H^*_{X_M}(M) = H^*(\Omega^*_G, \mathsf{d}_{X_M})$ and the relative X_M-cohomology $H^*_{X_M}(M, \partial M) = H^*(\Omega^*_{G,D}, \mathsf{d}_{X_M})$ where the D is for Dirichlet boundary condition and Ω^*_G is the invariant forms on M. Let δ_{X_M} be the adjoint of d_{X_M} and the resulting *Witten-Hodge-Laplacian* is $\Delta_{X_M} = \mathsf{d}_{X_M}\delta_{X_M} + \delta_{X_M}\mathsf{d}_{X_M}$ where the space $\ker\Delta_{X_M}$ is called the X_M-harmonic forms. In this paper, we prove that the (even/odd) X_M-harmonic cohomology which is the X_M-cohomology of the subcomplex $(\ker\Delta_{X_M}, \mathsf{d}_{X_M})$ of the complex $(\Omega^*_G, \mathsf{d}_{X_M})$ is enough to determine the total absolute and relative X_M-cohomology. As conclusion, we infer that the free part of the absolute and relative equivariant cohomology groups are determined by the (even/odd) X_M-harmonic cohomology when the set of zeros of the corresponding vector field X_M is equal to the fixed point set F for the G-action.

Keywords: Algebraic topology, equivariant topology, manifolds with boundary, cochain complex, group actions, equivariant cohomology.
MSC 2010: 57R19, 55N91, 57R91

1 Introduction

In [3], S.Cappell, D. DeTurck et al. present the following main theorem,

Theorem 1.1 *[3]. Let M be a compact, connected, oriented smooth Riemannian manifold of dimension n with boundary. Then the cohomology of the complex* $(\mathrm{Harm}^*(M), \mathsf{d})$ *of harmonic forms on M is given by the direct sum of the de Rham cohomology:*

$$H^k(\mathrm{Harm}^*(M), \mathsf{d}) \cong H^k(M, \mathbb{R}) + H^{k-1}(M, \mathbb{R})$$

for $k = 0, 1, \ldots, n$ and $\mathrm{Harm}^*(M) = \ker\Delta$ *where Δ is the Laplacian operator.*

The principle idea of this paper is to adapt theorem 1.1 in terms of our operators $\mathsf{d}_{X_M}, \delta_{X_M}$ and Δ_{X_M} in order to study the X_M-*harmonic cohomology* when the manifold in question has a boundary and then we relate the X_M-harmonic cohomology with the free part of the relative and absolute equivariant cohomology.

More precisely, in this paper, we consider a compact, connected, oriented, smooth Riemannian manifold M (with or without boundary) and we suppose G is a torus acting by isometries on M and denote by Ω^k_G the k-forms invariant under the action of G. Given X in the Lie algebra of G and corresponding vector field X_M on M, in [1], we consider Witten's coboundary operator $\mathsf{d}_{X_M} = \mathsf{d} + \iota_{X_M}$. This operator is no longer homogeneous in the degree of the invariant form: if $\omega \in \Omega^k_G$ then $\mathsf{d}_{X_M}\omega \in \Omega^{k+1}_G \oplus \Omega^{k-1}_G$. Note then that $\mathsf{d}_{X_M} : \Omega^{\pm}_G \to \Omega^{\mp}_G$, where $\Omega^{\pm}_G$ is the space of invariant forms of even $(+)$ or odd $(-)$ degree. A Riemannian metric on M leads to an L^2-inner product

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge (\star\beta)$$

which is defined on $\Omega^*_G(M)$, where $\star : \Omega^*_G \to \Omega^{n-*}_G$ is the Hodge star operator and then it leads us to the formal adjoint $\delta_{X_M} = -(\mp 1)^n \star \mathsf{d}_{X_M} \star : \Omega^{\pm}_G \to \Omega^{\mp}_G$ of d_{X_M}. The resulting *Witten-Hodge-Laplacian* is $\Delta_{X_M} = (\mathsf{d}_{X_M} + \delta_{X_M})^2 = \mathsf{d}_{X_M}\delta_{X_M} + \delta_{X_M}\mathsf{d}_{X_M} : \Omega^{\pm}_G \to \Omega^{\pm}_G$.

When the manifold in question M is closed we define the X_M-cohomology which is the cohomology of the complex $(\Omega^*_G, \mathsf{d}_{X_M})$ where $\mathsf{d}^2_{X_M} = 0$ because the forms are invariant (see [1] for details) and we denote it by $H^{\pm}_{X_M}(M)$. In this setting, Witten [4] introduces the definition of the *X_M-harmonic forms* (as we call in [1]) which we denote in this paper by $\mathrm{Harm}^*_{X_M}(M) = \mathrm{Harm}^+_{X_M}(M) + \mathrm{Harm}^-_{X_M}(M)$; then it is the kernel of the Witten-Hodge-Laplacian operator Δ_{X_M} (following [1]), i.e.

$$\mathrm{Harm}^{\pm}_{X_M}(M) = \ker \Delta_{X_M} \cap \Omega^{\pm}_G = \{\omega \in \Omega^{\pm}_G \mid \Delta_{X_M}\omega = 0\}.$$

Clearly, $\mathrm{Harm}^{\pm}_{X_M}(M) \subset \Omega^{\pm}_G$, but Δ_{X_M} and d_{X_M} commute which means that the coboundary operator d_{X_M} preserves the X_M-harmonicity of invariant forms. i.e.

$$\mathrm{Harm}^{\pm}_{X_M}(M) \xrightarrow{\mathsf{d}_{X_M}} \mathrm{Harm}^{\mp}_{X_M}(M)\cdot$$

Hence, $(\mathrm{Harm}^*_{X_M}(M), \mathsf{d}_{X_M})$ is a subcomplex of the $\mathbb{Z}_2$-graded complex $(\Omega^*_G, \mathsf{d}_{X_M})$. Therefore, we can compute the X_M-cohomology of this complex which we call the X_M-harmonic cohomology and denote by $H^{\pm}(\mathrm{Harm}^*_{X_M}(M), \mathsf{d}_{X_M})$.

In the boundaryless case, we have proved that the space of *X_M-harmonic fields* $\mathcal{H}^{\pm}_{X_M} = \ker \mathsf{d}_{X_M} \cap \ker \delta_{X_M}$ equal to the space of X_M-harmonic forms [1], i.e.

$$\mathrm{Harm}^{\pm}_{X_M}(M) = \mathcal{H}^{\pm}_{X_M}$$

Thus, we can conclude that all of the maps in the subcomplex $(\mathrm{Harm}^*_{X_M}(M), \mathsf{d}_{X_M})$ are zero which means that

$$H^{\pm}(\mathrm{Harm}^*_{X_M}(M), \mathsf{d}_{X_M}) = \mathrm{Harm}^{\pm}_{X_M}(M) = \mathcal{H}^{\pm}_{X_M}.$$

But, proposition 2.6 of [1] asserts that $H^{\pm}_{X_M}(M) \cong \mathcal{H}^{\pm}_{X_M}$, hence,

$$H^{\pm}(\mathrm{Harm}^*_{X_M}(M), \mathsf{d}_{X_M}) \cong H^{\pm}_{X_M}(M). \tag{1.1}$$

From another hand, eq.(1.1) is no longer true when the manifold in question has a boundary because the space of X_M-harmonic forms $\mathrm{Harm}^{\pm}_{X_M}(M)$ no longer coincides with the space of X_M-harmonic fields $\mathcal{H}^{\pm}_{X_M}$ [1]. Therefore, the main purpose of this paper is to study the X_M-harmonic cohomology when the manifold in question has a boundary and the result is theorem 2.3.

In the remainder of this introduction we recall necessary results from [1] and [2] when $\partial M \neq \emptyset$. In [1], we define two types of X_M-cohomology, the absolute X_M-cohomology $H^{\pm}_{X_M}(M)$ and the relative X_M-cohomology $H^{\pm}_{X_M}(M, \partial M)$. The first is the cohomology of the complex $(\Omega^*_G, \mathsf{d}_{X_M})$, while the second is the cohomology of the subcomplex $(\Omega^*_{G,D}, \mathsf{d}_{X_M})$, where $\omega \in \Omega^{\pm}_{G,D}$ if it satisfies $i^*\omega = 0$

(the D is for Dirichlet boundary condition). One also defines $\Omega^{\pm}_{G,N}(M) = \left\{\alpha \in \Omega^{\pm}_{G}(M) \mid i^*(\star\alpha) = 0\right\}$ (Neumann boundary condition). Clearly, the Hodge star provides an isomorphism

$$\star : \Omega^{\pm}_{G,D} \xrightarrow{\sim} \Omega^{n-\pm}_{G,N}$$

where we write $n - \pm$ for the parity (modulo 2) resulting from subtracting an even/odd number from n. Furthermore, because d_{X_M} and i^* commute, it follows that d_{X_M} preserves Dirichlet boundary conditions while δ_{X_M} preserves Neumann boundary conditions. In fact, the space $\mathcal{H}^{\pm}_{X_M}(M)$ is infinite dimensional and so is much too big to represent the X_M-cohomology, hence, we restrict $\mathcal{H}^{\pm}_{X_M}(M)$ into each of two finite dimensional subspaces, namely $\mathcal{H}^{\pm}_{X_M,D}(M)$ and $\mathcal{H}^{\pm}_{X_M,N}(M)$ with the obvious meanings (Dirichlet and Neumann X_M-harmonic fields, respectively). There are therefore two different candidates for X_M-harmonic representatives when the boundary is present. This construction firstly leads us to present the X_M-Hodge-Morrey decomposition theorem which states that

$$\Omega^{\pm}_{G}(M) = \mathcal{E}^{\pm}_{X_M}(M) \oplus \mathcal{C}^{\pm}_{X_M}(M) \oplus \mathcal{H}^{\pm}_{X_M}(M) \tag{1.2}$$

where $\mathcal{E}^{\pm}_{X_M}(M) = \{\mathsf{d}_{X_M}\alpha \mid \alpha \in \Omega^{\mp}_{G,D}\}$ and $\mathcal{C}^{\pm}_{X_M}(M) = \{\delta_{X_M}\beta \mid \beta \in \Omega^{\mp}_{G,N}\}$. This decomposition is orthogonal with respect to the L^2-inner product given above.

In addition, we present the X_M-Friedrichs Decomposition Theorem which states that

$$\mathcal{H}^{\pm}_{X_M}(M) = \mathcal{H}^{\pm}_{X_M,D}(M) \oplus \mathcal{H}^{\pm}_{X_M,\mathrm{co}}(M) \tag{1.3}$$

$$\mathcal{H}^{\pm}_{X_M}(M) = \mathcal{H}^{\pm}_{X_M,N}(M) \oplus \mathcal{H}^{\pm}_{X_M,\mathrm{ex}}(M) \tag{1.4}$$

where $\mathcal{H}^{\pm}_{X_M,\mathrm{ex}}(M) = \{\xi \in \mathcal{H}^{\pm}_{X_M}(M) \mid \xi = \mathsf{d}_{X_M}\sigma\}$ and $\mathcal{H}^{\pm}_{X_M,\mathrm{co}}(M) = \{\eta \in \mathcal{H}^{\pm}_{X_M}(M) \mid \eta = \delta_{X_M}\alpha\}$. These give the orthogonal *X_M-Hodge-Morrey-Friedrichs* decomposition [1],

$$\begin{aligned}\Omega^{\pm}_{G}(M) &= \mathcal{E}^{\pm}_{X_M}(M) \oplus \mathcal{C}^{\pm}_{X_M}(M) \oplus \mathcal{H}^{\pm}_{X_M,N}(M) \oplus \mathcal{H}^{\pm}_{X_M,\mathrm{ex}}(M) & (1.5)\\ &= \mathcal{E}^{\pm}_{X_M}(M) \oplus \mathcal{C}^{\pm}_{X_M}(M) \oplus \mathcal{H}^{\pm}_{X_M,D}(M) \oplus \mathcal{H}^{\pm}_{X_M,\mathrm{co}}(M) & (1.6)\end{aligned}$$

The two decompositions are related by the Hodge star operator. The consequence for X_M-cohomology is that each class in $H^{\pm}_{X_M}(M)$ is represented by a unique X_M-harmonic field in $\mathcal{H}^{\pm}_{X_M,N}(M)$, and each relative class in $H^{\pm}_{X_M}(M, \partial M)$ is represented by a unique X_M-harmonic field in $\mathcal{H}^{\pm}_{X_M,D}(M)$. The Hodge star operator $\star$ induces an isomorphism

$$H^{\pm}_{X_M}(M) \cong H^{n-\pm}_{X_M}(M, \partial M). \tag{1.7}$$

We call eq.(1.7) the X_M-Poincaré-Lefschetz duality.

In order to prove the results for next section we will need the following theorem which is proved in [2].

Theorem 1.2 *[2]. Let M be a compact, oriented smooth Riemannian manifold of dimension n with boundary and with an action of a torus G which acts by isometries on M. If an X_M-harmonic field $\lambda \in \mathcal{H}^{\pm}_{X_M}(M)$ vanishes on the boundary ∂M, then $\lambda \equiv 0$, i.e.*

$$\mathcal{H}^{\pm}_{X_M,N}(M) \cap \mathcal{H}^{\pm}_{X_M,D}(M) = \{0\} \tag{1.8}$$

As a consequence of Theorem 1.2, we obtain the following result.

Corollary 1.3 *[2]*

$$\mathcal{H}^{\pm}_{X_M}(M) = \mathcal{H}^{\pm}_{X_M,\mathrm{ex}}(M) + \mathcal{H}^{\pm}_{X_M,\mathrm{co}}(M) \tag{1.9}$$

where "$+$" is not a direct sum.

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2 Main results

In this section, we use the symbol $+$ between the spaces to indicate a direct sum whereas we reserve the symbol $\oplus$ for an orthogonal (with respect to L^2-inner product) direct sum unless otherwise indicated.

We begin with the following remark.

Remark 2.1 We need to define the following subspaces:

$$E^{\pm}_{X_M}(M) = \{\mathsf{d}_{X_M}\alpha \mid \alpha \in \Omega^{\mp}_G(M)\}$$

and

$$cE^{\pm}_{X_M}(M) = \{\delta_{X_M}\alpha \mid \alpha \in \Omega^{\mp}_G(M)\}.$$

But, the X_M-Hodge-Morrey decomposition (1.2) implies the following decompositions:

$$E^{\pm}_{X_M} = E^{\pm}_{X_M}(M) = \mathcal{E}^{\pm}_{X_M}(M) \oplus \mathcal{H}^{\pm}_{X_M,\mathrm{ex}}(M)$$

and

$$cE^{\pm}_{X_M} = cE^{\pm}_{X_M}(M) = \mathcal{C}^{\pm}_{X_M}(M) \oplus \mathcal{H}^{\pm}_{X_M,\mathrm{co}}(M).$$

2.1 The image of the *Witten-Hodge-Laplacian* operator

The image of the *Witten-Hodge-Laplacian* operator Δ_{X_M} will be most important to obtain our main theorem 2.3. We therefore need first to prove the following lemma 2.2.

Lemma 2.2 *Let M be a compact, connected, oriented smooth Riemannian manifold of dimension n with boundary and with an action of a torus G which acts by isometries on M. Then the* Witten-Hodge-Laplacian *operator* $\Delta_{X_M} = \mathsf{d}_{X_M}\delta_{X_M} + \delta_{X_M}\mathsf{d}_{X_M} : \Omega^{\pm}_G(M) \longrightarrow \Omega^{\pm}_G(M)$ *is surjective.*

PROOF: We need to prove that $\Delta_{X_M}(\Omega^{\pm}_G(M)) = \Omega^{\pm}_G(M)$. Clearly, $\Delta_{X_M}(\Omega^{\pm}_G(M)) \subset \Omega^{\pm}_G(M)$, so we only need to prove the converse. To do so, we will first compute the image of Δ_{X_M} on each summand of the X_M-Hodge-Morrey decomposition (1.2).

It is clear that

$$\Delta_{X_M}(\mathcal{E}^{\pm}_{X_M}(M)) = \mathsf{d}_{X_M}\delta_{X_M}(\mathcal{E}^{\pm}_{X_M}(M)) \subset E^{\pm}_{X_M}.$$

Now, let $\beta \in E^{\pm}_{X_M}$ then $\beta = \mathsf{d}_{X_M}\alpha$ and by applying the X_M-Hodge-Morrey decomposition (1.2) on α we get $\alpha = \mathsf{d}_{X_M}\sigma + \delta_{X_M}\rho + \lambda$, so

$$\beta = \mathsf{d}_{X_M}\alpha = \mathsf{d}_{X_M}\delta_{X_M}\rho$$

but also by (1.2), ρ can be written as $\rho = \mathsf{d}_{X_M}\varepsilon + \delta_{X_M}\pi + \kappa$ which implies that

$$\beta = \mathsf{d}_{X_M}\alpha = \mathsf{d}_{X_M}\delta_{X_M}\rho = \mathsf{d}_{X_M}\delta_{X_M}\mathsf{d}_{X_M}\varepsilon \in \Delta_{X_M}(\mathcal{E}^{\pm}_{X_M}(M)).$$

Hence, $\Delta_{X_M}(\mathcal{E}^{\pm}_{X_M}(M)) = E^{\pm}_{X_M}$. Likewise, $\Delta_{X_M}(\mathcal{C}^{\pm}_{X_M}(M)) = cE^{\pm}_{X_M}$. Clearly, $\Delta_{X_M}(\mathcal{H}^{\pm}_{X_M}(M)) = 0$. Using, the above equations together with remark 2.1, we obtain

$$\begin{aligned}\Delta_{X_M}(\Omega^{\pm}_G(M)) &= E^{\pm}_{X_M} + cE^{\pm}_{X_M} \\ &= (\mathcal{E}^{\pm}_{X_M}(M) \oplus \mathcal{H}^{\pm}_{X_M,\mathrm{ex}}(M)) + (\mathcal{C}^{\pm}_{X_M}(M) \oplus \mathcal{H}^{\pm}_{X_M,\mathrm{co}}(M)).\end{aligned} \tag{2.1}$$

where " + " is not a direct sum.

Finally, let $\omega \in \Omega^{\pm}_G(M)$ then the X_M-Hodge-Morrey decomposition (1.2) together with corollary 1.3 assert that ω can be decomposed as

$$\omega = \mathsf{d}_{X_M}\alpha_\omega + \delta_{X_M}\beta_\omega + (\mathsf{d}_{X_M}\rho_\omega + \delta_{X_M}\sigma_\omega) \in \mathcal{E}^{\pm}_{X_M}(M) \oplus \mathcal{C}^{\pm}_{X_M}(M) \oplus (\mathcal{H}^{\pm}_{X_M,\mathrm{ex}}(M) + \mathcal{H}^{\pm}_{X_M,\mathrm{co}}(M)) \quad (2.2)$$

Rearranging eq.(2.2), we get that eq.(2.1) shows that $\omega \in \Delta_{X_M}(\Omega^{\pm}_G(M))$ as desired. Thus, Δ_{X_M} is surjective.

❒

Now, it is time to present the following fundamental theorem which is analogues to theorem 1.1.

Theorem 2.3 *Let M be a compact, connected, oriented smooth Riemannian manifold of dimension n with boundary and with an action of a torus G which acts by isometries on M. Then the (even or odd) X_M-harmonic cohomology of the subcomplex $(\mathrm{Harm}^*_{X_M}(M), \mathsf{d}_{X_M})$ completely determines the total X_M-cohomology of the complex $(\Omega^*_G, \mathsf{d}_{X_M})$ and it is given by the direct sum:*

$$H^{\pm}(\mathrm{Harm}^*_{X_M}(M), \mathsf{d}_{X_M}) \cong H^{\pm}_{X_M}(M) + H^{\mp}_{X_M}(M) = H^*_{X_M}(M) \quad (2.3)$$

PROOF: Applying the definition of the X_M-cohomology of the subcomplex $(\mathrm{Harm}^{\pm}_{X_M}(M), \mathsf{d}_{X_M})$, we obtain that

$$H^{\pm}(\mathrm{Harm}^*_{X_M}(M), \mathsf{d}_{X_M}) = \frac{\ker \mathsf{d}_{X_M} |_{\mathrm{Harm}^{\pm}_{X_M}(M)}}{\mathsf{d}_{X_M}(\mathrm{Harm}^{\mp}_{X_M}(M))}$$

where $\ker \mathsf{d}_{X_M} |_{\mathrm{Harm}^{\pm}_{X_M}(M)} = \ker \mathsf{d}_{X_M} \cap \mathrm{Harm}^{\pm}_{X_M}(M)$. But, the X_M-Hodge-Morrey-Friedrichs decomposition (1.5) implies the following decomposition

$$\ker \mathsf{d}_{X_M} |_{\mathrm{Harm}^{\pm}_{X_M}(M)} = \mathcal{E}^{\pm}_{X_M}(M) \oplus \mathcal{H}^{\pm}_{X_M,N}(M) \oplus \mathcal{H}^{\pm}_{X_M,\mathrm{ex}}(M) = \mathcal{H}^{\pm}_{X_M,N}(M) \oplus E_{X_M}\,\mathrm{Harm}^{\pm}_{X_M}(M)$$

where $E_{X_M}\,\mathrm{Harm}^{\pm}_{X_M}(M) = E^{\pm}_{X_M}(M) \cap \mathrm{Harm}^{\pm}_{X_M}(M)$. But $\mathsf{d}_{X_M}(Harm^{\mp}_{X_M}(M)) \subset \ker \mathsf{d}_{X_M} |_{\mathrm{Harm}^{\pm}_{X_M}(M)}$, then we obtain a direct sum decomposition

$$H^{\pm}(\mathrm{Harm}^*_{X_M}(M), \mathsf{d}_{X_M}) = \frac{\ker \mathsf{d}_{X_M} |_{\mathrm{Harm}^{\pm}_{X_M}(M)}}{\mathsf{d}_{X_M}(\mathrm{Harm}^{\mp}_{X_M}(M))} = \mathcal{H}^{\pm}_{X_M,N}(M) + \frac{E_{X_M}\,\mathrm{Harm}^{\pm}_{X_M}(M)}{\mathsf{d}_{X_M}(\mathrm{Harm}^{\mp}_{X_M}(M))}$$

However, the X_M-Hodge isomorphism theorem [1] asserts that $H^{\pm}_{X_M}(M) \cong \mathcal{H}^{\pm}_{X_M,N}(M)$. Hence, we only need to prove that

$$\frac{E_{X_M}\,\mathrm{Harm}^{\pm}_{X_M}(M)}{\mathsf{d}_{X_M}(\mathrm{Harm}^{\mp}_{X_M}(M))} \cong \frac{\ker \mathsf{d}_{X_M}}{\mathsf{d}_{X_M}\Omega^{\pm}_G} \cong H^{\mp}_{X_M}(M).$$

We define the map $\overline{\delta}_{X_M}$ as follows :

$$\overline{\delta}_{X_M}([\varphi]) = [\delta_{X_M}\varphi] \in H^{\mp}_{X_M}(M), \quad \forall [\varphi] \in \frac{E_{X_M}\,\mathrm{Harm}^{\pm}_{X_M}(M)}{\mathsf{d}_{X_M}(\mathrm{Harm}^{\mp}_{X_M}(M))}$$

To prove $\overline{\delta}_{X_M}$ is a well-defined:

Let $\theta_1 - \theta_2 = \mathsf{d}_{X_M}\beta$, for some $\beta \in \mathrm{Harm}^{\mp}_{X_M}(M)$. i.e. $\Delta_{X_M}\beta = (\mathsf{d}_{X_M}\delta_{X_M} + \delta_{X_M}\mathsf{d}_{X_M})\beta = 0$.

Then

$$\begin{aligned}\delta_{X_M}\theta_1-\delta_{X_M}\theta_2 &= \delta_{X_M}\mathsf{d}_{X_M}\beta \\ &= -\mathsf{d}_{X_M}\delta_{X_M}\beta \\ &= \mathsf{d}_{X_M}(-\delta_{X_M}\beta)\in \mathsf{d}_{X_M}\Omega_G^{\pm} \qquad (2.4)\end{aligned}$$

Moreover, $\delta_{X_M}\beta$ is X_M-harmonic as $\Delta_{X_M}(\delta_{X_M}\beta)=\delta_{X_M}\mathsf{d}_{X_M}\delta_{X_M}\beta=\delta^2_{X_M}(\theta_1-\theta_2)=0$. It means that $\delta_{X_M}(\theta_1-\theta_2)\in \mathsf{d}_{X_M}\,\mathrm{Harm}^{\mp}_{X_M}$. Thus, $\overline{\delta}_{X_M}$ is a well-defined.

Next, we prove $\overline{\delta}_{X_M}$ is one-to-one. To this end, let $\varphi\in E_{X_M}\,\mathrm{Harm}^{\pm}_{X_M}(M)$ and $\overline{\delta}_{X_M}\varphi\in \mathsf{d}_{X_M}\Omega_G^{\pm}$. We only need to prove $\varphi\in \mathsf{d}_{X_M}(\mathrm{Harm}^{\mp}_{X_M}(M))$. So, $\varphi=\mathsf{d}_{X_M}\beta$, and therefore

$$\Delta_{X_M}\beta=(\mathsf{d}_{X_M}\delta_{X_M}+\delta_{X_M}\mathsf{d}_{X_M})\beta=\mathsf{d}_{X_M}\delta_{X_M}+\delta_{X_M}\varphi\in \mathsf{d}_{X_M}\Omega_G^{\pm}$$

Thus, $\Delta_{X_M}\beta=\mathsf{d}_{X_M}\eta$ for some $\eta\in\Omega_G^{\pm}$, but Δ_{X_M} is onto by lemma (2.2) then we can write $\eta=\Delta_{X_M}\sigma$. Hence, $\Delta_{X_M}\beta=\mathsf{d}_{X_M}\eta=\mathsf{d}_{X_M}\Delta_{X_M}\sigma=\Delta_{X_M}\mathsf{d}_{X_M}\sigma$ which implies that $\beta-\mathsf{d}_{X_M}\sigma\in\mathrm{Harm}^{\mp}_{X_M}(M)$. Hence, we can rewrite $\varphi=\mathsf{d}_{X_M}\beta$ as follows, $\varphi=\mathsf{d}_{X_M}(\beta-\mathsf{d}_{X_M}\sigma)\in \mathsf{d}_{X_M}(\mathrm{Harm}^{\mp}_{X_M}(M))$.

Finally, to prove $\overline{\delta}_{X_M}$ is onto. Given $\alpha\in\ker \mathsf{d}_{X_M}$, we should find $\varphi\in E_{X_M}\,\mathrm{Harm}^{\pm}_{X_M}(M)$ such that $\delta_{X_M}\varphi-\alpha\in \mathsf{d}_{X_M}\Omega_G^{\pm}$. Applying lemma (2.2) on α, then we can write $\alpha=\Delta_{X_M}\beta$ and then we take $\varphi=\mathsf{d}_{X_M}\beta$. one should notice that $\Delta_{X_M}\varphi=\Delta_{X_M}\mathsf{d}_{X_M}\beta=\mathsf{d}_{X_M}\Delta_{X_M}\beta=\mathsf{d}_{X_M}\alpha=0$, so $\alpha\in\ker\mathsf{d}_{X_M}$. Thus, $\varphi\in E_{X_M}\,\mathrm{Harm}^{\pm}_{X_M}(M)$. Now,

$$\delta_{X_M}\varphi=\delta_{X_M}\mathsf{d}_{X_M}\varphi=\Delta_{X_M}\beta-\mathsf{d}_{X_M}\delta_{X_M}\beta=\alpha-\mathsf{d}_{X_M}\delta_{X_M}\beta$$

So, $\delta_{X_M}\varphi-\alpha\in \mathsf{d}_{X_M}\Omega_G^{\pm}$, as desired. Hence $\overline{\delta}_{X_M}$ is bijection map. So, eq.(2.3) holds. ❒

In addition, Δ_{X_M} and δ_{X_M} commute. Hence, the coboundary operator δ_{X_M} preserves the X_M-harmonicity of invariant forms. i.e.

$$\mathrm{Harm}^{\pm}_{X_M}(M)\xrightarrow{\delta_{X_M}}\mathrm{Harm}^{\mp}_{X_M}(M)$$

Thus, $(\mathrm{Harm}^{*}_{X_M}(M),\delta_{X_M})$ is a subcomplex of the $\mathbb{Z}_2$-graded complex $(\Omega^*_G,\delta_{X_M})$. Therefore, we can compute the X_M-cohomology of this complex which we denote by $H^{\pm}(\mathrm{Harm}^{*}_{X_M}(M),\delta_{X_M})$. So, applying the Hodge star to the isomorphism given by theorem 2.3 and replace $n-(\pm)$ by $\pm$ and then using X_M-Poincaré-Lefschetz duality (1.7) to obtain the following corollary.

Corollary 2.4

$$H^{\pm}(\mathrm{Harm}^{*}_{X_M}(M),\delta_{X_M})\cong H^{\pm}_{X_M}(M,\partial M)+H^{\mp}_{X_M}(M,\partial M)=H^{*}_{X_M}(M,\partial M)$$

3 Conclusions

In [1], we elucidate the connection between the X_M-cohomology groups and the relative and absolute equivariant cohomology groups (i.e. $H^{\pm}_G(M)$ and $H^{\pm}_G(M,\partial M)$) which are modules over $\mathbb{R}[u_1,\dots,u_\ell]$ and the result is the following theorem.

Theorem 3.1 *[1]. Let $\{X_1,\dots,X_\ell\}$ be a basis of the Lie algebra $\mathfrak{g}$ and $\{u_1,\dots,u_\ell\}$ the corresponding coordinates and let $X=\sum_j s_jX_j\in\mathfrak{g}$. If the set of zeros $N(X_M)$ of the corresponding vector field X_M is equal to the fixed point set F for the G-action then*

$$H^{\pm}_{X_M}(M,\partial M)\cong H^{\pm}_G(M,\partial M)/\mathfrak{m}_X H^{\pm}_G(M,\partial M)\cong H^{\pm}(F,\partial F), \tag{3.1}$$

and

$$H^{\pm}_{X_M}(M)\cong H^{\pm}_G(M)/\mathfrak{m}_X H^{\pm}_G(M)\cong H^{\pm}(F) \tag{3.2}$$

where $\mathfrak{m}_X=\langle u_1-s_1,\dots,u_l-s_l\rangle$ is the ideal of polynomials vanishing at X.

We conclude that theorem 3.1, theorem 2.3 and corollary 2.4 prove the following theorem:

Theorem 3.2 *With the hypotheses of the theorem 3.1. Then the (even or odd) X_M-harmonic cohomology of the subcomplexes $(\mathrm{Harm}^*_{X_M}(M),\mathrm{d}_{X_M})$ and $(\mathrm{Harm}^*_{X_M}(M),\delta_{X_M})$ completely determine the free part of the absolute and relative equivariant cohomology groups, i.e.*

$$H^{\pm}(\mathrm{Harm}^*_{X_M}(M),\mathrm{d}_{X_M})\cong H^*_G(M)/\mathfrak{m}_X H^*_G(M)\cong H^*(F)$$

and

$$H^{\pm}(\mathrm{Harm}^*_{X_M}(M),\delta_{X_M})\cong H^*_G(M,\partial M)/\mathfrak{m}_X H^*_G(M,\partial M)\cong H^*(F,\partial F).$$

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