A 195,747,435 vertex graph related to the Fischer group Fi23, part III

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A 195,747,435 VERTEX GRAPH RELATED TO THE FISCHER GROUP Fi_{23} ,III

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Introduction

This paper, together with our earlier work in [3] and [4], finishes our anatomical description of the 195,747,435 vertex graph $\mathcal{G}$. This graph is the point-line collinearity graph of a certain geometry Γ associated with the Fischer group Fi_{23} and is described in Section 2 of [3]. We shall continue the section numbering of [3] and [4]. Our main results are stated in Section 1 – also in Section 1 and the introduction of [4] there is a discussion of various aspects of this investigation of $\mathcal{G}$. We direct the reader to Section 2 for our notation as well as the all important line orbit data.

Again, a denotes a fixed point of $\mathcal{G}$. Our objective here is to complete the proofs of Theorems 9, 10, 11, 12, 13, 14 and 15. So, for the most part, we are interested in dissecting parts of $\Delta_4(a)$, the fourth disc of a. In particular we shall concentrate on the G_a-orbits $\Delta_4^3(a)$, $\Delta_4^5(a)$ and $\Delta_4^6(a)$. Sections 9, 11, 12 are mainly devoted to, respectively, $\Delta_4^3(a)$, $\Delta_4^5(a)$ and $\Delta_4^6(a)$. As a result of analysing these orbits we are able to fill in gaps in our knowledge of other G_a-orbits in $\Delta_4(a)$, namely $\Delta_4^1(a)$ and $\Delta_4^4(a)$. There are also some loose ends in the third disc concerning $\Delta_3^5(a)$ and $\Delta_3^6(a)$. For $\Delta_3^5(a)$ these are settled as easy consequences of certain results on $\Delta_4(a)$. While for $\Delta_3^6(a)$ much more work is needed, and most of Section 10 is given over to examining this orbit. We give no further information here of where specific results are stated. But remind the reader of Appendix B in which a detailed record of where the point distributions of particular line orbits are to be found.

9 A first look at $\Delta_4^5(a)$

We now fix a point x in $\Delta_4^5(a)$ and begin to look at the points incident with lines in $\Gamma_1(x)$.

Firstly we summarize the basic properties of $\Delta_4^5(a)$.

Lemma 9.1. *(i)* $|\Delta_4^5(a)| = 2^{15}.11.23$.

(ii) $|\Delta_1(x) \cap \Delta_3^4(a)| = 1$.

(iii) $G_{ax} \cong A_7$ *and* $|G_{ax} \cap Q(a)| = 1$.

Proof. We prove (iii) first. By the definition of $\Delta_4^5(a)$ (see (2.15)(xiii)), there exists $d \in \Delta_1(x) \cap \Delta_3^4(a)$ with $d + x \in \alpha_0(d, X(d, a))$. Since $\tau(X(d, a)) \in Q(d)_a \backslash G_x$ by Lemma 3.2, (2.6) and Lemma 5.2 imply that $\Gamma_0(d + x) \backslash \{d\} \subseteq \Delta_4^5(a)$ and $\Delta_1(d) \cap \Delta_4^5(a)$ is a G_{ad}-orbit of points. Hence $[G_{ad} : G_{adx}] = 352$ by (2.6) and using [1] together with Lemma 4.8(iv) yields $G_{adx} \cong A_7$. Let $g \in G_{ax} \cap Q(a)$. Then g fixes $X(d, a)$ and so $g \in G_{adx}$ by Lemma 5.3(i). Therefore $G_{ax} \cap Q(a) = 1$ because G_{adx} is simple.

We now assume $G_{adx} \neq G_{ax}$ and argue for a contradiction. Since $Q(a)_x = 1$, G_{ax} is isomorphic to a subgroup of M_{23}. Considering possible subgroups which contain A_7 in [1] we conclude that $G_{ax} \sim 2^4 : A_7$, A_8, M_{22} or M_{23}. If $G_{ax} \cong M_{22}$ or M_{23}, then $|\Delta_1(x) \cap \Delta_3^4(a)| = 176$ or 23×176 by Lemma 5.4, which is impossible by Lemma 5.3(ii). If $G_{ax} \sim 2^4 : A_7$ or A_8, then $|\Delta_1(x) \cap \Delta_3^4(a)| = 16$ or 8 respectively. Now $\Delta_1(x) \cap \Delta_3^4(a)$ is a G_{ax}-orbit and by considering the possible orbit sizes of G_{ax} on $\Gamma_1(x)$ given in (2.1) and (2.10) we see this is impossible. Hence $G_{ax} = G_{adx} \cong A_7$, which proves (iii). An immediate consequence of $G_{ax} = G_{adx}$ is that $|\Delta_1(x) \cap \Delta_3^4(a)| = 1$.

Since $\Delta_4^5(a)$ is a G_a-orbit by Lemma 5.2, part (iii) gives

$$|\Delta_4^5(a)| = \frac{|G_a|}{|G_{ax}|} = 2^{15}.11.23,$$

so completing the proof of the lemma.

□

Therefore G_{ax} has orbits on $\Gamma_1(x)$ as described in (2.13). For the rest of this section we fix $\{d\} := \Delta_1(x) \cap \Delta_3^4(a)$.

Lemma 9.2. *(i) Let $c \in \Delta_2^1(x) \cap \Delta_2^2(a) \cap \Delta_1(d)$ and $\{b\} = \{a, c\}^\perp$. Then $b \in \Delta_3^6(x)$.*

(ii) Let $l \in \alpha_3(x, x + d, +)$. Then $|\Gamma_0(l) \cap \Delta_3^6(a)| = |\Gamma_0(l) \cap \Delta_4^6(a)| = 1$.

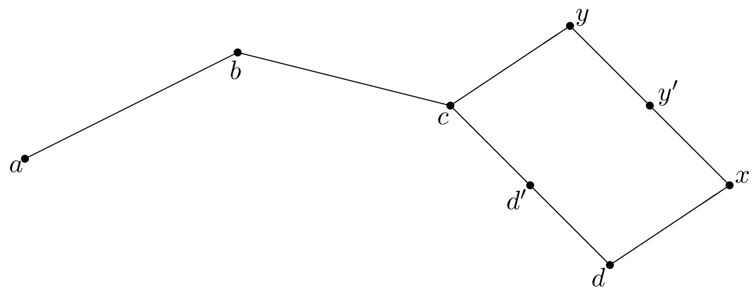
(iii) Let $l \in \alpha_3(x, x + d, -)$. Then $|\Gamma_0(l) \cap \Delta_3^5(a)| = |\Gamma_0(l) \cap \Delta_4^6(a)| = 1$. Furthermore, if $z \in \Gamma_0(l) \cap \Delta_3^5(a)$, then $z + x \in \alpha_3^{(2)}(z, z + c, -)$ where $\{c\} = \Delta_1(z) \cap \Delta_2^2(a)$.

Proof. By Theorem 4.13(vi), for every $l \in \Gamma_1(d, X(d, a))$, we have $\Gamma_0(l) \cap \Delta_2^2(a) \neq \emptyset$. Examining the MOG in [2] we see there are 35 possibilities for

$c \in \Delta_1(d) \cap \Delta_2^2(a)$ with $c \in \Delta_2^1(x)$. For any such c, $T(c,x)$ is a triad not incident with the hyperplane $X(c,a)$ in Ω_c because $\Gamma_3(a,x) = \emptyset$. Let $\{b\} = \{a,c\}^\perp$. Then $T(c,x) \cap (c+b) = \emptyset$ in Ω_c because $(c+d) \cap (c+b) = \{X(c,a)\}$ and so $b \in \Delta_3^6(x)$ by Lemma 4.6(ii). This proves part(i).

For each $y \in \{c,x\}^\perp \backslash \{d\}$, $c + y \notin \Gamma_1(X(c,a))$, otherwise $\Gamma_3(a,x) \neq \emptyset$ by Lemma 3.6 because $d \in \Gamma_0(X(c,a))$. Furthermore $d \in \Delta_2^2(b)$, whence $c+y \in \alpha_{3,0}(c, c+b, X(c,a))$ for one point $y \in \{c,x\}^\perp \backslash \{d\}$ and $c+y \in \alpha_{1,0}(c, c+b, X(c,a))$ for three points $y \in \{c,x\}^\perp \backslash \{d\}$ using (2.3) and Lemma 3.11(ii). Therefore (2.15)(vii),(viii) imply that $|\{c,x\}^\perp \cap \Delta_3^6(a)| = 1$ and $|\{c,x\}^\perp \cap \Delta_3^5(a)| = 3$.

Let $\{y\} = \{c,x\}^\perp \cap \Delta_3^6(a)$, $\Gamma_0(c+d) = \{c, d', d\}$ and $\Gamma_0(x+y) = \{x, y', y\}$. So we have



Then $d' \in \Delta_3^3(a)$ by Lemma 4.15(iii) and $y' \in \Delta_1(d')$ by Lemma 3.10. Hence $y' \in \Delta_4^4(a) \cup \Delta_4^6(a) \cup \Delta_3(a)$, using (2.3), (2.15)(xii),(xiv) and $x \in \Delta_4(a)$. However $X(d',a) = X(d,a)$ and so, since $\Gamma_3(a,x) = \emptyset$, $y' \notin \Gamma_0(X(d',a))$. Appealing to Lemma 4.15 yields $y' \in \Delta_4^4(a) \cup \Delta_4^6(a)$. Since $\{y',c\}^\perp > 1$, $d'+y' \in \alpha_3(d', d'+c)$ and we conclude that $y' \in \Delta_4^6(a)$ by (2.15)(xiv).

Next fix $z \in \{c,x\}^\perp \cap \Delta_3^5(a)$ and let $\Gamma_0(x+z) = \{x, z', z\}$. Using a similar argument as for y' we deduce that $z' \in \Delta_4^6(a)$.

We have $x+y$, $x+z \in \alpha_3(x, x+d, +) \cup \alpha_3(x, x+d, -)$ by (2.13) because $y, z \in \Delta_2^1(d)$. Let c vary through the 35 possibilities in $\Delta_2^2(a) \cap \Delta_2^1(x) \cap \Delta_1(d)$. This generates 35 distinct possibilities for y and 105 distinct possibilities for z (because $|\{c,x\}^\perp \cap \Delta_3^5(a)| = 3$). Examining the G_{ax}-orbit sizes in (2.13) we conclude that $x+y \in \alpha_3(x, x+d, +)$ and $x+z \in \alpha_3(x, x+d, -)$.

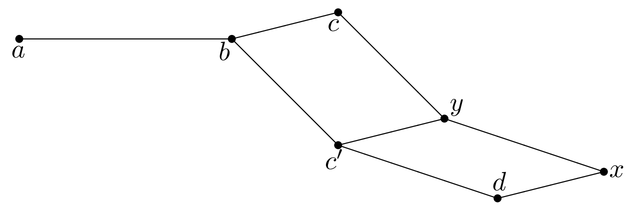
Finally, since $T(c,x) \cap (c+b) = \emptyset$ in Ω_c, $X(z,b) \notin \Gamma_3(x)$ (where $z \in \Delta_3^5(a) \cap$

$\{c,x\}^{\perp}$) and so by definition (see (2.7)), $z+x \in \alpha_3(z, z+c, -)$. Appealing to Lemma 5.7 (and definition of $\alpha_3^{(2)}(z, z+c, -)$ following that lemma) we obtain the result. $\square$

We can use Lemma 9.2 to extract information about certain lines incident with a point in $\Delta_3^6(a)$.

Lemma 9.3. *Let $l \in \alpha_3(x, x+d, +)$ and $y \in \Gamma_0(l) \cap \Delta_3^6(a)$ with $\{c\} = \Delta_1(y) \cap \Delta_2^1(a)$ and $\{b\} = \Delta_1(a) \cap \Delta_2^1(y)$. Then $y+x \in \alpha_1(y, y+c)$ and $(y+x) \cap T(y,b) = \emptyset$ in Ω_y.*

Proof. There exists $c' \in \{d,y\}^{\perp} \cap \Gamma_0(X(d,a))$ by Lemma 3.11(ii) and then $c' \in \Delta_2^2(a)$ by Theorem 4.13(vi) and Lemma 4.15. Let $\{b\} = \{a,c'\}^{\perp}$. By Lemma 9.2(i) $b \in \Delta_3^6(x)$. Moreover $b \in \{a,c\}^{\perp}$ by Lemma 4.10. Thus we have

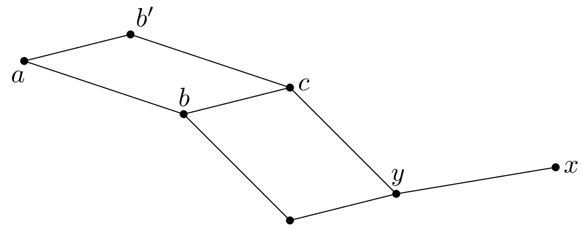


Applying Lemma 4.14(iii) to b and x, we have that $T(y,b) \cap T(c',x) = \emptyset$ in Ω_y. Hence $y+x \in \alpha_1(y, y+c)$ and $(y+x) \cap T(y,b) = \emptyset$ in Ω_y by Lemma 3.11(ii), as required. $\square$

At this stage we require the following symmetry result, which we now prove.

Lemma 9.4. $a \in \Delta_4^5(x)$.

Proof. By Lemma 9.3 there exists $y \in \Delta_1(x) \cap \Delta_3^6(a)$ with $y+x \in \alpha_1(y, y+c)$ and $(y+x) \cap T(y,b) = \emptyset$ in Ω_y where $\{c\} = \Delta_1(y) \cap \Delta_2^1(a)$ and $\{b\} = \Delta_1(a) \cap \Delta_2^1(y)$. By definition, $c \in \Delta_2^2(x)$. Further, by Lemma 3.11(ii), there exists $b' \in \{a,c\}^{\perp} \cap \Gamma_0(X(c,x))$. We have $b' \neq b$ because $(y+x) \cap T(y,b) = \emptyset$ in Ω_y and so $b \notin \Gamma_0(X(c,x))$.



Lemma 4.10 implies that $b' \in \Delta_2^2(y)$ and appealing to (2.15)(v),(vi) yields $b' \in \Delta_3^3(x) \cup \Delta_3^4(x)$.

Assume $b' \in \Delta_3^3(x)$. Then $a \in \Delta_4^4(x) \cup \Delta_4^6(x)$ by (2.15)(xii) and (xiv) because $\Gamma_3(a,x) = \emptyset$. Since $|\Delta_4^5(a)| \neq |\Delta_4^4(a)|$ by Lemmas 8.7(ii) and 9.1(i) we must have $a \in \Delta_4^6(x)$ with $|\Delta_4^6(x)| = |\Delta_4^6(a)| = |\Delta_4^5(a)| = 2^{15}.11.23$. Since $|\Delta_1(b') \cap \Delta_4^6(x)| = 160$ and $\Delta_1(b') \cap \Delta_4^6(x)$ is a $G_{xb'}$-orbit by Lemma 8.4(v), using Lemmas 5.4 and 4.8(iii) we get that

(9.4.1)

$$|\Delta_1(a) \cap \Delta_3^3(x)| = \frac{2^{10}.7.11.23.160}{2^{15}.11.23} = 35$$

and that

(9.4.2) $\Delta_1(a) \cap \Delta_3^3(x)$ is a G_{xa}-orbit.

We now switch our attention to $x \in \Delta_4^5(a)$ and let $z \in \Delta_1(x) \cap \Delta_4^6(a)$ with $x + z \in \alpha_3(x, x+d, +)$; by Lemma 9.2(i) such a z exists. Observe that, by (9.4.2), $\Delta_1(z) \cap \Delta_3^3(a)$ is a G_{az}-orbit. Hence $\{l \in \Gamma_1(z) \mid \Gamma_0(l) \cap \Delta_3^3(a) \neq \emptyset\}$ is a G_{az}-orbit of size 35 by (9.4.1). If $\Delta_4^5(a) = \Delta_4^6(a)$, then $a \in \Delta_4^6(x) = \Delta_4^5(x)$, and we are done. So we may assume that $\Delta_4^5(a) \neq \Delta_4^6(a)$. Therefore, by Lemma 5.5, $z + x$ is contained in a G_{za}-orbit of $\Gamma_1(z)$ of size 35 (since $|\Delta_4^5(a)| = |\Delta_4^6(a)|$). Now $G_{az} \cong A_7$ (because $G_{ax} \cong A_7$ and $a \in \Delta_4^6(x)$) and so, by (2.13), G_{az} has exactly one orbit on $\Gamma_1(z)$ of size 35. But, on the one hand, we have that the three points of $\Gamma_0(z+x)$ are in either $\Delta_4^5(a)$, $\Delta_3^6(a)$ or $\Delta_4^6(a)$ by Lemma 9.2(ii) and on the other $\Gamma_0(z+x) \cap \Delta_3^3(a) \neq \emptyset$. From this contradiction we infer that $b' \notin \Delta_3^3(x)$. Therefore $b' \in \Delta_3^4(x)$ and now $a \in \Delta_4^5(x)$ by (2.15)(xiii) because $\Gamma_3(a,x) = \emptyset$. □

In the next lemma we focus our attention on $\Delta_3^5(a)$.

Lemma 9.5. *Let $y \in \Delta_3^5(a)$ and $l \in \alpha_1(y, y+c, +)$ where $\{c\} = \Delta_1(y) \cap \Delta_2^2(a)$. Then $|\Gamma_0(l) \cap \Delta_4^4(a)| = |\Gamma_0(l) \cap \Delta_4^5(a)| = 1$.*

Proof. Let $\Gamma_0(l) = \{y, z, z'\}$. By definition (see (2.7)), $X(z,c) = X(y,b)$ where $\{b\} = \Delta_1(a) \cap \Delta_2^2(y)$. Therefore $y + z \in \alpha_{1,1}(y, y+c, X(y,b))$ and by Theorem 4 we may assume $z \in \Delta_3^3(b)$ and $z' \in \Delta_3^4(b)$. By Lemma 4.12, $b \in \Delta_3^3(z) \cap \Delta_3^4(z')$. We have $\Gamma_3(a,z) = \Gamma_3(a,z') = \emptyset$ by Lemmas 4.9(i), 4.15 and 5.1. This together with the fact that $a \in \Delta_2^2(c)$ yields $a \in \Delta_4^4(z) \cap \Delta_4^5(z')$ (see (2.15)). The result now follows by Lemmas 8.14 and 9.4. □

We now return to $x \in \Delta_4^5(a)$.

Lemma 9.6. *Let $l \in \alpha_1(x, x+d, +)$.*

(i) $|\Gamma_0(l) \cap \Delta_3^5(a)| = |\Gamma_0(l) \cap \Delta_4^4(a)| = 1.$

(ii) If $y \in \Gamma_0(l) \cap \Delta_3^5(a)$, then $y + x \in \alpha_1(y, y+c, +)(\{c\} = \Delta_1(y) \cap \Delta_2^2(a))$.

(iii) If $z \in \Gamma_0(l) \cap \Delta_4^4(a)$, then $z + x \in \alpha_5(z, \mathrm{END}, +)$.

Proof. By Lemmas 5.2 and 9.5 there exists $y \in \Delta_1(x) \cap \Delta_3^5(a)$ with $y + x \in \alpha_1(y, y+c, +)$. Since $\Gamma_0(y+x)\backslash\{y, x\} \subseteq \Delta_4^4(a)$ we can apply Lemma 5.5 to show

$$m = \frac{|\alpha_1(y, y+c, +)|.|\Delta_3^5(a)|}{|\Delta_4^5(a)|}$$

where m is the size of the G_{ax}-orbit on $\Gamma_1(x)$ containing $x + y$. Using (2.7), Lemmas 4.11(i) and 9.1(i) we get $m = 42$. Examining the possibilities in (2.13) we see that $x + y \in \alpha_1(x, x+d, +)$. Parts (i) and (ii) follow by Lemmas 5.5 and 9.5 together with the fact that $\alpha_1(x, x+d, +)$ is a G_{ax}-orbit.

For (iii) we again appeal to Lemma 5.5 to show that $z + x$ lies in a G_{az}-orbit on $\Gamma_1(z)$ of size

$$\frac{|\alpha_1(y, y+c, +)|.|\Delta_3^5(a)|}{|\Delta_4^4(a)|} = 11$$

(by Lemmas 4.11(i) and 8.7(ii)). Therefore (2.12) implies that $z+x \in \alpha_1(z, \mathrm{END}, +) \cup \alpha_1(z, \mathrm{END}, -) \cup \alpha_5(z, \mathrm{END}, -)$. The result now follows from Lemmas 8.8 and 8.13.

□

Lemma 9.7. $\Delta_4^5(a) \neq \Delta_4^6(a)$.

Proof. Suppose $\Delta_4^5(a) = \Delta_4^6(a)$, and argue for a contradiction. Let $x \in \Delta_4^5(a)$. By (2.15)(xiv) there exists $d' \in \Delta_1(x) \cap \Delta_3^3(a)$ and, using Lemma 8.4(v), $\Delta_1(d') \cap \Delta_4^5(a)$ is a G_{ax}-orbit of size 160. Lemmas 4.8(iii), 5.4 and 9.1(i) imply that $\Delta_1(x) \cap \Delta_3^3(a)$ is a G_{ax}-orbit of size 35. So $\{l \in \Gamma_1(x) | \Gamma_0(l) \cap \Delta_3^3(a) \neq \emptyset\}$ is a G_{ax}-orbit of $\Gamma_1(x)$ of size 35. By (2.13) this orbit must be $\alpha_3(x, x+d, +)$. Then Lemma 9.2 forces $d' \in \Delta_3^6(a)$, a contradiction. Therefore $\Delta_4^5(a) \neq \Delta_4^6(a)$.

□

10 Finishing $\Delta_3^6(a)$

In this section we reconsider $\Delta_3^6(a)$. Fix $x \in \Delta_3^6(a)$, $\{b\} = \Delta_1(a) \cap \Delta_2^1(x)$ and $\{c\} = \Delta_2^1(a) \cap \Delta_1(x)$ for the whole of Section 10. Earlier in Lemma 4.11(ii) we showed that $|G_{ax}| = 2^9.3^2$. In order to make further headway with $\Delta_3^6(a)$ we need to determine G_{ax}^{*x}. The first result of this section does just that.

Lemma 10.1. *G_{ax}^{*x} is equal to the stabilizer in G_x^{*x} ($\cong M_{23}$) of the heptad $x+c$ and the three element subset $T(x,b)$ of $x+c$. Moreover $|G_{ax}^{*x}| = 2^7.3^2$ and $|G_{ax} \cap Q(x)| = 2^2$.*

Proof. From Lemmas 4.10 and 4.11(ii) we already know that G_{ax}^{*x} is contained in H, the stabilizer in G_{ax}^{*x} of $x+c$ and $T(x,b)$. Thus, as $|H| = 2^7.3^2$, the lemma will follow once we establish that $|G_{ax} \cap Q(x)| = 2^2$. Since $|G_{ax}| = 2^9.3^2$ we clearly have $|G_{ax} \cap Q(x)| \geq 2^2$. Assuming that $|G_{ax} \cap Q(x)| > 2^2$ we seek a contradiction. By Lemma 7.5 there exists $f_1 \in \Delta_1(x) \cap \Delta_4^2(a)$. From (2.10), Corollary 7.4 and Lemma 8.12 we deduce that $\Delta_1(f_1) \cap \Delta_3^6(a)$ is a G_{af_1}-orbit of size 70. Hence, by Lemmas 4.11(ii), 5.4 and 7.3(ii) $\Delta_1(x) \cap \Delta_4^2(a)$ is a G_{ax}-orbit of size 16. Now $G_{ax} \cap Q(x) \leq G_{axf_1}$ (recall that $\Delta_4^2(a) \neq \Delta_4^3(a)$ by (2.15)(xi) and Corollary 7.4) and G_{axf_1} contains a Sylow 3-subgroup of G_{af_1} ($\cong A_8$). Consequently, by our assumption and properties of A_8, $|G_{ax} \cap Q(x)| = 2^4$. Appealing to Lemma 9.2(ii) yields the existence of $f_2 \in \Delta_1(x) \cap \Delta_4^5(a)$. Since $\Delta_4^5(a) \neq \Delta_4^6(a)$ by Lemma 9.7, Lemma 5.5 implies that $\{x+f \in \Gamma_1(x) \mid f \in \Delta_4^5(a)$ and $f+x \in \alpha_3(f, f+d, +)$ where $\{d\} = \Delta_1(f) \cap \Delta_3^4(a)\}$ is a G_{ax}-orbit of size 64. Hence

$$\left\{ \Gamma_0(x+f) \cap \Delta_4^5(a) \;\middle|\; \begin{array}{l} x+f \in \Gamma_1(x) \text{ and } f+x \in \alpha_3(f, f+d, +) \\ \text{where } \{d\} = \Delta_1(f) \cap \Delta_3^4(a) \end{array} \right\}$$

is a G_{ax}-orbit of $\Delta_4^5(a)$ of size 64. But since $G_{ax} \cap Q(x) \leq G_{axf_2}$ and $[G_{ax} : G_{ax} \cap Q(a)] = 2^5.3^2$ we see such an orbit must have size dividing $2^5.3^2$, a contradiction. From this contradiction we infer that $G_{ax} \cap Q(x) = 2^2$, so proving the lemma.

□

With Lemma 10.1 to hand, (2.8) gives a listing of the G_{ax}-orbits on $\Gamma_1(x)$ (setting TRI $= T(x,b)$ and with c playing the role of b in (2.8)).

Lemma 10.2. *Let $y \in \Delta_1(x) \cap \Delta_4^2(a)$. Then,*

(i) $y + x \in \alpha_4(y, O(y, a))$*;*

(ii) $x + y \in \alpha_{3,0}(x, x + c, \mathrm{TRI})$*; and*

(iii) if $l \in \alpha_{3,0}(x, x + c, \mathrm{TRI})$*, then* $|\Gamma_0(l) \cap \Delta_4^2(a)| = |\Gamma_0(l) \cap \Delta_4^3(a)| = 1$*.*

Proof. By Corollary 7.4 and Lemma 7.5 and (2.10),

$$|\Delta_1(y) \cap \Delta_3^6(a)| = 70 + 168n \quad \text{for } n = 0,\ 1 \text{ or } 2$$

As a consequence of Corollary 7.4(ii) and (2.15) $\Delta_4^2(a) \neq \Delta_4^3(a)$. Therefore Lemma 5.5 implies that

$$\begin{aligned} |\Delta_1(x) \cap \Delta_4^2(a)| &= (70 + 168n).\frac{|\Delta_4^2(a)|}{|\Delta_3^6(a)|} \\ &= (70 + 168n)\frac{2^{12}.11.23}{2^9.5.7.11.23} \\ &= (70 + 168n)\frac{8}{35} \end{aligned}$$

(using Lemmas 4.11(ii) and 7.3). Hence $n = 0$ and (2.10) yields $y + x \in \alpha_4(x, O(x, a))$, giving part (i). For part (ii) we may use Lemma 5.5 again to show

$$m = \frac{|\Delta_4^2(a)||\alpha_4(x, O(x, a))|}{|\Delta_3^6(a)|}$$

where m is the size of the G_{ax}-orbit on $\Gamma_1(d)$ containing $x + y$. So

$$m = \frac{2^{12}.11.23.70}{2^9.5.7.11.23} = 16$$

by (2.10) and Lemmas 4.11(ii) and 7.3(ii). Perusing (2.8) we see that $x + y \in \alpha_{3,0}(x, x + c, \mathrm{TRI})$ and so we have part (ii).

Part (iii) follows from Lemma 7.5 and part (ii) because $\alpha_{3,0}(x, x + c, \mathrm{TRI})$ is a G_{ax}-orbit of $\Gamma_1(x)$.

□

Lemma 10.3. *If* $l \in \alpha_{1,0}(x, x + c, \mathrm{TRI})$*, then* $|\Gamma_0(l) \cap \Delta_4^5(a)| = |\Gamma_0(l) \cap \Delta_4^6(a)| = 1$*.*

Proof. Since $\Delta_3^6(a)$ and $\Delta_4^5(a)$ are G_a-orbits, there exists $y \in \Delta_1(x) \cap \Delta_4^5(a)$ with $y + x \in \alpha_3(y, y+d, +)$ by Lemma 9.2(ii) (where $\{d\} = \Delta_1(y) \cap \Delta_3^4(a)$). Using Lemma 9.3 and (2.8) we have that $x+y \in \alpha_{1,0}(x, x+c, \mathrm{TRI})$. The result now follows from Lemma 9.2(ii) again because $\alpha_{1,0}(x, x+c, \mathrm{TRI})$ is a G_{ax}-orbit of $\Gamma_1(x)$. □

Lemma 10.4. *Let* $l \in \alpha_{3,1}(x, x+c, \mathrm{TRI})$. *Then*

(i) $|\Gamma_0(l) \cap \Delta_4^1(a)| = 2$; *and*

(ii) if $y \in \Gamma_0(l) \cap \Delta_4^1(a)$ *it follows that* $y + x \in \alpha_{3,0}^{\mathfrak{L}}(y, y+d, \mathrm{DUAD})$ *where* $\{d\} = \Delta_1(y) \cap \Delta_3^1(a)$.

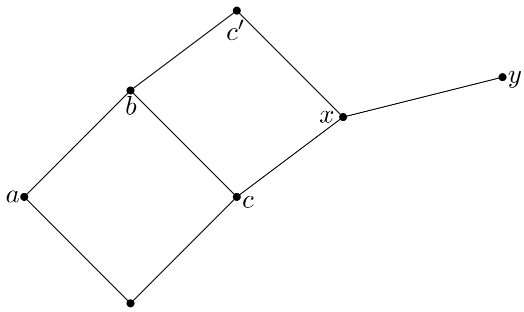
Proof. Let $\mathrm{TRI} = \{X_1, X_2, X_3\}$. Then $X_i \in \Gamma_3(x, b)$ for $i = 1, 2, 3$. Since $a \in \Delta_3^6(x)$ by Lemma 4.12, $a \notin \Gamma_0(X_i)$ for $i = 1, 2, 3$. Hence $\tau(X_i)$ interchanges a and the point in $\Gamma_0(a+b)\backslash\{a, b\}$ for each i by Lemma 3.2. Appealing to Lemma 6.6(iii) and the fact that $\Delta_3^6(a)$ is a G_a-orbit, there exists $y \in \Delta_1(x) \cap \Delta_4^1(a)$ with $y + x \in \alpha_{3,0}^{\mathfrak{L}}(y, y+d, \mathrm{DUAD})$ where $\{d\} = \Delta_1(y) \cap \Delta_3^1(a)$. Using (2.8) together with Lemmas 4.10, 10.3 and 10.2(iii) we must have $x+y \in \alpha_{1,1}(x, x+c, \mathrm{TRI}) \cup \alpha_{3,2}(x, x+c, \mathrm{TRI}) \cup \alpha_{3,1}(x, x+c, \mathrm{TRI})$. If $|(x+y) \cap \mathrm{TRI}| = 1$ in Ω_x we may assume $X_1 \in \Gamma_3(x+y)$ and $X_2, X_3 \notin \Gamma_3(x+y)$. If $|(x+y) \cap \mathrm{TRI}| = 2$ we may assume $X_1, X_2 \in \Gamma_3(x+y)$ and $X_3 \notin \Gamma_3(x+y)$. In either case let $\tau := \tau(X_1)\tau(X_3)$. Then $\tau \in G_{ax}$ and τ interchanges y and y' by Lemma 3.2, where $\Gamma_0(x+y) = \{x, y', y\}$. Therefore we may apply Lemma 5.6 to show that $x+y$ lies in a G_{ax}-orbit of $\Gamma_1(x)$ of size

$$m = \frac{|\Delta_4^1(a)||\alpha_{3,0}^{\mathfrak{L}}(y, y+d, \mathrm{DUAD})|}{2|\Delta_3^6(a)|}.$$

By (2.9) and Lemmas 4.11(ii) and 6.2(i) we have $m = 72$. Now (2.8) implies that $x+y \in \alpha_{3,1}(x, x+c, \mathrm{TRI})$. Since $\tau \in G_a$, $y' \in \Delta_4^1(a)$ and the proof of the lemma is complete. □

Lemma 10.5. *Let* $l \in \alpha_{1,1}(x, x+c, \mathrm{TRI})$. *Then* $|\Gamma_0(l) \cap \Delta_4^3(a)| = 2$.

Proof. Let $y \in \Gamma_0(l)\backslash\{x\}$. By Lemma 3.11(ii) there exists $c' \in \{b, x\}^{\perp}$ with $x + c' \in \alpha_3(x, l)$. Therefore $c' \in \Delta_2^1(y)$. Since $l \in \alpha_1(x, x+c)$, $c' \neq c$ and so Lemma 4.10 implies that $c' \in \Delta_2^2(a)$.



Since $(x+y) \cap \mathrm{TRI} = (x+y) \cap (x+c)$ is a singleton subset of Ω_x, there exists a unique hyperplane $X \in \Gamma_3(b, c', x, y)$. Moreover $X \neq X(c', a)$ because $\Gamma_3(a, x) = \emptyset$. Let $d \in \{c', y\}^\perp \cap \Gamma_0(X(c', a))$ (such a point exists by Lemma 3.11(ii)). Then $d \in \Gamma_0(X) \cap \Gamma_0(X(c', a))$, whence $d \in \Delta_2^1(b)$ by Lemma 3.8(i). Since $\Delta_1(d) \cap \Delta_2^2(a) \neq \emptyset$ and $\Gamma_3(d, a) \neq \emptyset$ we deduce that $d \in \Delta_3^2(a)$ using Theorem 4.13. We have $d + c' \in \alpha_{4,1}(d, O(d, a), X(d, a))$ by Theorem 4.13(iv)(b). Also, since $b \in \Delta_1(a)$, $\Delta_2^1(a) \cap \{b, d\}^\perp \neq \emptyset$ by Lemma 3.11(ii), which, together with Theorem 4.13(iv)(a) implies that $T(d, b) \cap O(d, a) = \emptyset$ in Ω_d. We already know that $T(d, x) \cap T(d, b) = \{X\}$, whence $|T(d, x) \cap O(d, a)| = 2$ because $T(d, x) \cup T(d, b) \subseteq d + c'$ as subsets of Ω_d. Hence, in Ω_d, $|(d + z) \cap O(d, a)| = 2$ for two points z in $\{d, x\}^\perp$ and $|(d + z) \cap O(d, a)| = 4$ for three points z in $\{d, x\}^\perp$. Using (2.15)(ix) and (xi) we have

(10.5.1) $\{d, x\}^\perp$ consists of c', two points in $\Delta_4^1(a)$ and two points in $\Delta_4^3(a)$.

Since $l \in \alpha_{1,1}(x, x+c, \mathrm{TRI})$ we have $|T(x, d) \cap T(x, b)| = 1$ with $T(x, b) \subseteq x + c$ in Ω_x. Therefore (2.8) implies $x + z \in \alpha_{3,1}(x, x+c, \mathrm{TRI})$ for two points $z \in \{x, d\}^\perp \backslash \{c'\}$ and $x + z \in \alpha_{1,1}(x, x+c, \mathrm{TRI})$ for two points $z \in \{x, d\}^\perp \backslash \{c'\}$. Appealing to Lemma 10.4 we have $z \in \Delta_4^1(a)$ for each $z \in \{x, d\}^\perp$ with $x + z \in \alpha_{3,1}(x, x+c, \mathrm{TRI})$. Therefore (10.5.1) yields that $z \in \Delta_4^3(a)$ for each $z \in \{x, d\}^\perp$ with $x + z \in \alpha_{1,1}(x, x+c, \mathrm{TRI})$. Since y is any point in $\Gamma_0(l) \backslash \{x\}$, the lemma now follows. □

Lemma 10.6. *Let* $l \in \alpha_{3,2}(x, x+c, \mathrm{TRI})$. *Then* $|\Gamma_0(l) \cap \Delta_4^1(a)| = 2$.

Proof. Let $y \in \Gamma_0(l) \backslash \{x\}$. Since $\mathrm{TRI} = T(x, b)$ (where $\{b\} = \Delta_1(a) \cap \Delta_2^1(x)$), by the definition of $\alpha_{3,2}(x, x+c, \mathrm{TRI})$, $|\Gamma_3(y, b)| \geq 2$. Hence $b \in \Delta_1(y) \cup \Delta_2^1(y) \cup \Delta_3^1(y)$ by Theorem 4.13. We have $y \notin \Delta_2(a) \cup \Delta_3^i(a)$ for $i = 1, \ldots, 5$ by (2.8) and Lemmas 4.10 and 4.15. Therefore $b \notin \Delta_1(y)$. If $b \in \Delta_2^1(y)$, then since $b \in \Delta_2^1(x)$, b is collinear with a point in $\Gamma_0(l)$ using Theorem 3. However we then have $l \in \Gamma_1(\mathrm{TRI})$ which is impossible. Hence $b \in \Delta_3^1(y)$.

Since $\Gamma_3(a, y) = \emptyset$, (2.15)(ix) implies that $a \in \Delta_4^1(y)$. Then $y \in \Delta_4^1(a)$ by Lemma 6.4. As y was an arbitrary point in $\Gamma_0(l)\backslash\{x\}$, the lemma is proved. □

Combining together all the results in this section with Lemma 4.10 and Theorem 4.13(vii),(viii) we see that Theorem 10 is now proven.

11 A first look at $\Delta_4^3(a)$

We are now in a position to consider the G_a-orbit $\Delta_4^3(a)$. Let x be a fixed point in $\Delta_4^3(a)$.

The first result in this section summarizes some of the main properties of $\Delta_4^3(a)$.

Lemma 11.1. *(i)* $|G_{ax}| = 2^6.3^2$ *and* $Q(x) \cap G_a = 1$*;*

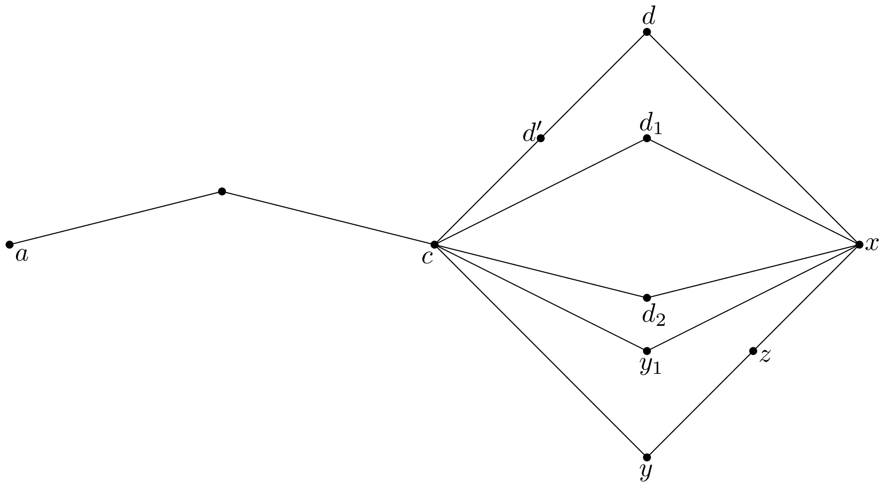
(ii) $|\Delta_4^3(a)| = 2^{12}.5.7.11.23$*;*

(iii) $|\Delta_1(x) \cap \Delta_4^2(a)| = 2$*;*

(iv) $|\Delta_1(x) \cap \Delta_3^2(a)| = 3$*; and*

(v) G_{ax} *stabilizes a disjoint triad* TRI *and octad* OCT *in* Ω_x *together with a partition of* OCT *into two 4-sets, each of whose union with* TRI *forms a heptad in* $\Gamma_1(x)$*.*

Proof. Since $\Delta_4^3(a)$ is a G_a-orbit by Lemma 5.2, Lemma 7.5 implies there exists $l \in \Gamma_1(x)$ where $\Gamma_0(l) = \{x, y, z\}$ with $y \in \Delta_3^6(a)$ and $z \in \Delta_4^2(a)$. Appealing to Lemma 10.2(ii), we have $l \in \alpha_{3,0}(y, y + c, \mathrm{TRI})$ where $\{c\} = \Delta_1(y) \cap \Delta_2^1(a)$. Thus $c \in \Delta_2^1(x)$. Since $\Gamma_3(a, x) = \emptyset$ by Lemma 5.1, we have $T(c, a) \cap T(c, x) = \emptyset$ in Ω_c. For each $d \in \{c, x\}^\perp$, $d \notin \Delta_3^1(a)$ by Theorem 5 because $x \notin \Delta_4^1(a)$ by (2.15)(xi) and Lemma 6.3(ii). Therefore Lemma 3.11(ii) and Theorem 3 imply there exist $d, d_1, d_2, y_1 \in \{c, x\}^\perp\backslash\{y\}$ with d, d_1, d_2 being distinct points in $\Delta_3^2(a)$ and $y_1 \in \Delta_3^6(a)$. Note that $d + x \in \alpha_{4,0}(d, O(d, a), X(d, a))$ by Lemma 6.3(ii) and Corollary 7.4(ii). Let $\Gamma_0(d+c) = \{d, d', c\}$. So the state of play is as follows.



Then $d' \in \Delta_1(z) \cap \Delta_3^2(a)$ by Theorem 3 and Lemma 3.10. Viewing l from z, Lemma 10.2(i) implies $l = z + x \in \alpha_4(x, O(x,a))$. Set $P := Q(x) \cap G_a$.

(11.1.1) Either $P = 1$ or $P \cong 2^2$.

Now P fixes l and hence $P \leq G_{azl}$. By Lemma 7.3(i) $G_{az} \cong A_8$. Using (2.10) we note that $P \leq O_2(G_{azl}) \cong 2^2$. Since $|\alpha_4(x, O(x,a))| = 70$, G_{azl} will contain a Sylow 3-subgroup S of G_{az}. Because $C_{G_{az}}(S) = S$ we must have $P = 1$ or $P \cong 2^2$.

(11.1.2) $P \leq Q(d)$.

Since $x \notin \Gamma_0(X(d,a))$, $\Gamma_0(x+d)\backslash\{d\} \subseteq \Delta_4^3(a)$ by Lemma 3.2. So P fixes d and fixes $X(d,a)$. Consequently P fixes the 8 elements of Ω_d in $\Gamma_3(d+x) \cup \{X(d,a)\}$. Hence $P \leq Q(d)$ because involutions in G_d^{*d} ($\cong M_{23}$) fix exactly 7 elements of Ω_d. This proves (11.1.2).

By (2.5) and Lemma 4.8(ii), $|G_{adx}| = 2^6.3$ with $|Q(d)_a| = 2^3$ and $G_{adx}^{*d} \cap O_2(G_{ad}^{*d}) = 1$. So, by (11.1.2), $G_{adx} \cap Q(a) = P \cap Q(a) = 1$. Therefore $|G_{adx}^{*a}| = 2^6.3$ and $P^{*a} \trianglelefteq G_{adx}^{*a}$. Now $|P| = |P^{*a}| = 1, 2$ or 2^3 by (2.5) which, together with (11.1.1), yields $P = 1$.

By (2.8) and Lemma 4.11(ii) we have $|G_{acyx}| = 2^5.3^2$ because $l \in \alpha_{3,0}(y, y+c, \mathrm{TRI})$ and $G_{al} = G_{ax}$. For each $c' \in \Delta_1(d) \cap \Delta_2^1(a)$, $d+c' \in \alpha_{0,1}(d, O(d,a), X(d,a))$. Since $d + x \in \alpha_{4,0}(d, O(d,a), X(d,a))$, c is the unique point in $\Delta_2^1(a) \cap \Delta_1(d) \cap \Delta_2^1(x)$. Hence $G_{acdx} = G_{adx}$ and so $|G_{acdx}| = 2^6.3$.

Put $H := \langle G_{acyx}, G_{acdx} \rangle$. Clearly $2^6.3^2 \mid |H|$.

In Ω_x define TRI $= (x+y) \cap (x+y_1)$ and OCT $= (x+y) \oplus (x+y_1)$ ($\oplus$ is symmetric difference). Then $|\mathrm{TRI}| = 3$ and so OCT is an octad in Ω_x. Furthermore H fixes (setwise) TRI and OCT and leaves invariant the partition of OCT into the tetrads OCT$\cap(x+y)$ and OCT$\cap(x+y_1)$. Since $H \leq G_{ax}$ and $Q(x) \cap G_a = P = 1$, $2^6.3^2 \mid |H^{*x}|$, and therefore we conclude that H^{*x} is the stabilizer in M_{23} of an octad and a partition of the octad into two tetrads and that H has order $2^6.3^2$. Furthermore the G_{ax}-orbits on $\Gamma_1(x)$ are certain unions of the H-orbits described in (2.11). Note that $\alpha_{3,4|0}(x, \mathrm{TRI}, \mathrm{OCT}) = \{x+y, x+y_1\}$ and that each of these lines has one point in $\Delta_4^3(a)$, one point in $\Delta_4^2(a)$ and one point in $\Delta_3^6(a)$. For any $k \in \alpha_{3,0}(x, \mathrm{TRI}, \mathrm{OCT})$, $|\Gamma_0(k) \cap \Delta_3^2(a)| = 1$ because $x+d \in \alpha_{3,0}(x, \mathrm{TRI}, \mathrm{OCT})$. This together with (2.11) and Lemma 5.3(ii) implies that $|\Delta_1(x) \cap \Delta_3^2(a)| = 3$ or 15. If $|\Delta_1(x) \cap \Delta_3^2(a)| = 15$, then, using Lemma 4.8(ii),

$$\begin{aligned} |\Delta_4^3(a)| &= \frac{|\Delta_3^2(a)|.112}{15} \\ &= 2^{12}.7.11.23. \end{aligned}$$

Thus $\Delta_4^3(a) \neq \Delta_4^4(a)$ since $|\Delta_4^4(a)| = 2^{16}.3.7.23$. Then Theorem 12 and Lemma 7.3(ii) yield that

$$\begin{aligned} |\Delta_1(x) \cap \Delta_4^2(a)| &= \frac{|\Delta_4^2(a)|.70}{|\Delta_4^3(a)|} \\ &= 10. \end{aligned}$$

This contradicts (2.11) and so we must have $|\Delta_1(x) \cap \Delta_3^2(a)| = 3$ with $|\Delta_4^3(a)| = 2^{12}.5.7.11.23$, $|\Delta_1(x) \cap \Delta_4^2(a)| = 2$ and $G_{ax} =$ H. This completes the proof of the lemma. $\square$

Lemma 11.2. *(i) Let $l \in \alpha_{3,4|0}(x, \mathrm{TRI}, \mathrm{OCT})$. Then $|\Gamma_0(l) \cap \Delta_3^6(a)| = |\Gamma_0(l) \cap \Delta_4^2(a)| = 1$. Moreover, if $y \in \Gamma_0(l) \cap \Delta_3^6(a)$ and $z \in \Gamma_0(l) \cap \Delta_4^2(a)$, then $l \in \alpha_{3,0}(y, y+c, \mathrm{TRI}) \cap \alpha_4(z, O(z,a))$ (where $\{c\} = \Delta_1(y) \cap \Delta_2^1(a)$).*

(ii) Let $l \in \alpha_{3,0}(x, \mathrm{TRI}, \mathrm{OCT})$. Then $|\Gamma_0(l) \cap \Delta_3^2(a)| = 1$ and $|\Gamma_0(l) \cap \Delta_4^3(a)| = 2$.

(iii) *Let* $l \in \alpha_{1,0}(x, \mathrm{TRI}, \mathrm{OCT})$*. Then* $|\Gamma_0(l) \cap \Delta_3^6(a)| = 1$ *and* $|\Gamma_0(l) \cap \Delta_4^3(a)| = 2$*. Moreover, if* $y \in \Gamma_0(l) \cap \Delta_3^6(a)$ *then* $l \in \alpha_{1,1}(y, y + c, \mathrm{TRI})$ *(where* $\{c\} = \Delta_1(y) \cap \Delta_2^1(a)$*).*

(iv) *Let* $l \in \alpha_{1,2|2}(x, \mathrm{TRI}, \mathrm{OCT})$*. Then* $|\Gamma_0(l) \cap \Delta_3^5(a)| = |\Gamma_0(l) \cap \Delta_4^1(a)| = 1$*. Moreover, if* $y \in \Gamma_0(l) \cap \Delta_3^5(a)$ *and* $z \in \Gamma_0(l) \cap \Delta_4^1(a)$*, then* $l \in \alpha_3(y, y + c, +) \cap \alpha_{1,1}(z, z+d, \mathrm{DUAD})$ *(where* $\{c\} = \Delta_1(y) \cap \Delta_2^2(a)$ *and* $\{d\} = \Delta_1(z) \cap \Delta_3^1(a)$*).*

Proof. Parts (i) and (ii) are a consequence of (2.5), (2.8), (2.10), (2.11), Lemmas 4.8(ii), 5.6, 7.3(ii), 7.5, 10.2 and 11.1(ii).

For part (iii), by Theorem 10, if $y \in \Delta_1(x) \cap \Delta_3^6(a)$, then $|\Delta_1(y) \cap \Delta_4^3(a)| = 112$. Therefore

$$\begin{aligned}
|\Delta_1(x) \cap \Delta_3^6(a)| &= \frac{|\Delta_3^6(a)|}{|\Delta_4^3(a)|}.112 \\
&= \frac{2^9.5.7.11.23.112}{2^{12}.5.7.11.23} \\
&= 14
\end{aligned}$$

by Lemmas 4.11(ii) and 11.1(ii). Considering the orbit sizes in (2.11) we have $x + y \in \alpha_{3,4|0}(x, \mathrm{TRI}, \mathrm{OCT}) \cup \alpha_{1,0}(x, \mathrm{TRI}, \mathrm{OCT})$. Therefore part (iii) follows from part (i) and Lemma 10.5. □

Turning to (iv), we obtain the result from Lemma 5.5 using the orbit sizes in (2.7), (2.9), (2.11) and Lemmas 4.11(i), 6.2(i) and 11.1(ii), together with the information about $\alpha_{1,1}(z, z + d, \mathrm{DUAD})$ given in Lemma 6.8(i).

Lemma 11.3. *Let* $y \in \Delta_1(x) \cap \Delta_3^5(a)$*. Then*

(i) $|\Delta_1(y) \cap \Delta_4^3(a)| = 140$*.*

(ii) $x + y \in \alpha_{1,2|2}(x, \mathrm{TRI}, \mathrm{OCT}) \cup \alpha_{0,1|1}(x, \mathrm{TRI}, \mathrm{OCT}) \cup \alpha_{2,1|1}(x, \mathrm{TRI}, \mathrm{OCT})$*.*

(iii) $|\Gamma_0(l) \cap \Delta_3^5(a)| = 1$ *and* $|\Gamma_0(l) \cap \Delta_4^3(a)| = 2$ *for some line* $l \in \alpha_{0,1|1}(x, \mathrm{TRI}, \mathrm{OCT}) \cup \alpha_{2,1|1}(x, \mathrm{TRI}, \mathrm{OCT})$*. Moreover, if* $y \in \Gamma_0(l) \cap \Delta_3^5(a)$*, then* $l \in \alpha_3^{(1)}(y, y + c, -)$ *(where* $\{c\} = \Delta_1(y) \cap \Delta_2^2(a)$*).*

Proof. By Theorem 4.13(vii), Lemmas 5.7, 6.7 and 9.5, if $y \in \Delta_1(x) \cap \Delta_3^5(a)$ and $k \in \Gamma_1(y)$, then $|\Gamma_0(k) \cap \Delta_4^3(a)| = 2$ whenever $k \in \alpha_3^{(1)}(y, y+c, -)$; $|\Gamma_0(k) \cap \Delta_4^3(a)| = 1$ whenever $k \in \alpha_3(y, y+c, +)$; and $\Gamma_0(k) \cap \Delta_4^3(a) = \emptyset$ whenever $k \in \{y+c\} \cup \alpha_1(y, y+c, +) \cup \alpha_3^{(2)}(y, y+c, -)$. So $|\Delta_1(y) \cap \Delta_4^3(a)| = 80+60+96n$ where $n = 0, 1$ or 2 by (2.11), depending on the point distribution of $\alpha_1(y, y+c, -)$. Using Lemmas 4.11(i) and 11.1(ii) we have

$$\begin{aligned} |\Delta_1(x) \cap \Delta_3^5(a)| &= \frac{|\Delta_3^5(a)|(140+96n)}{|\Delta_4^3(a)|} \\ &= \frac{2^{12}.3.7.11.23}{2^{12}.5.7.11.23}(140+96n). \end{aligned}$$

For this to be an integer we must have $n = 0$, whence $|\Delta_1(y) \cap \Delta_4^3(a)| = 140$. This proves part (i). Also we have $|\Delta_1(x) \cap \Delta_3^5(a)| = 84$. Using (2.11) together with Lemma 11.2 yields that $\{l \in \Gamma_1(x) \mid \Gamma_0(l) \cap \Delta_3^5(a) \neq \emptyset\} = \alpha_{1,2|2}(x, \mathrm{TRI}, \mathrm{OCT}) \cup \alpha_{i,1|1}(x, \mathrm{TRI}, \mathrm{OCT})$ for some $i = 0$ or 2. Therefore we have part (ii).

For part (iii) we know that $y+x \in \alpha_3^{(1)}(y, y+c, -) \cup \alpha_3(y, y+c, +)$. Suppose $y+x \in \alpha_3(y, y+c, +)$. Then Lemma 6.8(i) implies $|\Gamma_0(y+x) \cap \Delta_3^5(a)| = |\Gamma_0(y+x) \cap \Delta_4^1(a)| = |\Gamma_0(y+x) \cap \Delta_4^3(a)| = 1$. In this case Lemma 5.5 yields that $x+y$ lies in a G_{ax}-orbit of size 36. Using Lemma 5.5 again we have $x+y \in \alpha_{1,2|2}(x, \mathrm{TRI}, \mathrm{OCT})$ if and only if $y+x \in \alpha_3(y, y+c, +)$. Thus if $y+x \in \alpha_3^{(1)}(y, y+c, -)$, then $x+y \in \alpha_{0,1|1}(x, \mathrm{TRI}, \mathrm{OCT}) \cup \alpha_{2,1|1}(x, \mathrm{TRI}, \mathrm{OCT})$. Now Lemma 5.7(i) (and the remark immediately following that lemma) gives the result.

□

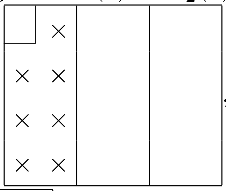
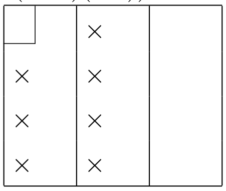
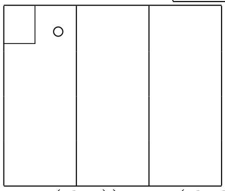
12 A first look at $\Delta_4^6(a)$

We now turn our attention to the final G_a-orbit in the fourth disc of G. So let x be a fixed point in $\Delta_4^6(a)$. By (2.15)(xiv), $\Delta_1(x) \cap \Delta_3^3(a) \neq \emptyset$. Our first task, achieved in Lemma 12.2, is to determine the size of $\Delta_1(x) \cap \Delta_3^3(a)$.

Lemma 12.1. *Let $d \in \Delta_1(x) \cap \Delta_3^3(a)$ and put $U_d := \{z \in \Delta_1(x) \cap \Delta_3^3(a) \cap \Delta_2^1(d) | \{d, z\}^\perp \cap \Delta_3^1(a) \neq \emptyset\} \cup \{d\}$. Then the following hold.*

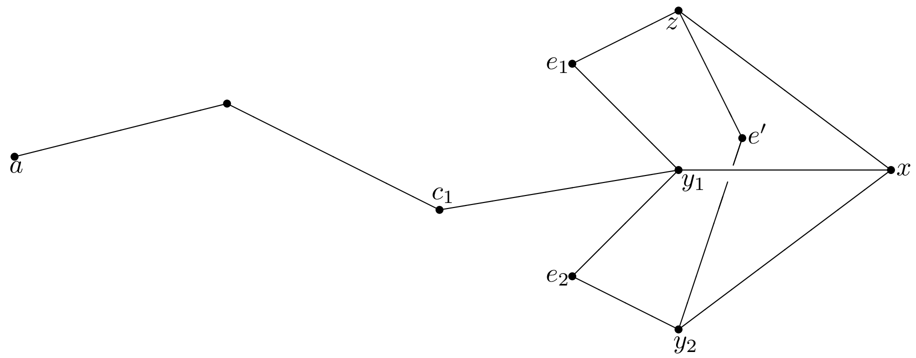
(i) $|U_d| = 5$ *and for each* $z \in U_d \backslash \{d\}$*,* $\Delta_3^3(a) \cap \{e, x\}^\perp = \{d, z\}$*, where* $\{e\} = \{d, z\}^\perp \cap \Delta_3^1(a)$*.*

(ii) $5 \mid |\Delta_1(x) \cap \Delta_3^3(a)|$.

Proof. Let $d \in \Delta_1(x) \cap \Delta_3^3(a)$. Then $d + x \in \alpha_{3,0}(d, d + c, X(d, a))$ by Lemma 8.4, where $\{c\} = \Delta_1(d) \cap \Delta_2^2(a)$ (see also (2.15)(xiv)). Using Appendix A with $d + c =$ , $d + x =$  and $X(d, a) =$  in Ω_d, we see that there are precisely 4 heptads in $\alpha_{1,1}(d, d + c, X(d, a)) \cap \alpha_3(d, d + x)$ (namely h_4, h_7, h_{11}, h_{15}). Lemma 4.15(i) together with (2.15) then imply that there are precisely four points in $S_d := \Delta_3^1(a) \cap \Delta_1(d) \cap \Delta_2^1(x)$.

For any $e \in S_d$, $T(e, x) \cap D(e, a) = \emptyset$ in Ω_e because $\Gamma_3(a, x) = \emptyset$ by Lemma 5.1(ii). Hence $\{e, x\}^\perp$ consists of d, one point z ($\neq d$) in $\Delta_3^3(a)$ and three points in $\Delta_4^1(a)$. Moreover, $\{d, z\}^\perp \cap \Delta_3^1(a) = \{e\}$ because $\Gamma_3(a, x) = \emptyset$. Hence, letting e vary throughout S_d we obtain four distinct points in $(\Delta_1(x) \cap \Delta_3^3(a)) \backslash \{d\}$. Therefore $U_d = 5$ and $\Delta_3^3(a) \cap \{e, x\}^\perp = \{d, z\}$, so giving (i).

To prove part (ii) it is enough to show that if y_1 and y_2 are distinct points in $\Delta_1(x) \cap \Delta_3^3(a)$ and $y_2 \in U_{y_1}$, then $U_{y_1} = U_{y_2}$. Assume $y_2, z \in U_{y_1}$ with $z \neq y_2$. By the definition of U_{y_1} there exist $e_1, e_2 \in \Delta_3^1(a)$ with $e_1 \in \{z, y_1\}^\perp$ and $e_2 \in \{y_2, y_1\}^\perp$. If $e_1 = e_2$, then $z \in U_{y_2}$ by definition and we are done. So we may suppose that $e_1 \neq e_2$, and then



where $\{c_1\} = \Delta_1(y_1) \cap \Delta_2^2(a)$. (We define e' shortly.)

In Ω_{y_1}, the heptads $y_1 + e_1$ and $y_1 + e_2$ intersect $y_1 + c_1$ in exactly $X(y_1, a)$ by Lemma 4.15(i). Moreover $|(y_1 + e_1) \cap (y_1 + e_2) \cap (y_1 + x)| = 2$. Thus $|T(y_1, z) \cap T(y_1, y_2)| = 2$ in Ω_{y_1}, and so $|\Gamma_3(z, y_1, y_2)| = 2$. Note that $y_2 \in \Delta_1(z)$ would imply $D(a, z) = D(a, y_2)$ by Lemma 4.16 which, by Lemma 3.6, yields the impossible $\Gamma_3(a, x) \neq \emptyset$. Therefore $y_2 \in \Delta_2^1(z)$ by Lemma 3.8(i). Appealing to Lemma 3.11(ii) we can choose $e' \in \{y_2, z\}^\perp$ with $\Gamma_3(a, e') \neq \emptyset$. Suppose $e' \in \Delta_2(a)$. Then, by Theorem 7, $e' \in \Delta_2^2(a)$ and consequently $\Gamma_3(a, z, y_2) \neq \emptyset$ which again, by Lemma 3.6, gives the untenable $\Gamma_3(a, x) \neq \emptyset$. Hence $e' \in \Delta_3(a)$ and so $e' \in \Delta_3^i(a)$ for $i = 1, 2, 3$ or 4. If $e' \in \Delta_3^i(a)$ for $i = 2, 3$ or 4, then Lemmas 4.15 and 4.16 imply that $y_2, z \in \Gamma_0(X(e', a))$ and thus $x \in \Gamma_0(X(e', a))$ by Lemma 3.6, a contradiction. Therefore $e' \in \Delta_3^1(a)$ and so $z \in U_{y_2}$. Thus $U_{y_1} = U_{y_2}$, as required, and the proof of the lemma is complete. □

Lemma 12.2. *(i)* $|\Delta_1(x) \cap \Delta_3^3(a)| = 5$.

(ii) $|\Delta_4^6(a)| = 2^{15}.7.11.23$.

(iii) $G_{ax} \cong G_{ax}^{*x} \sim (3 \times A_5)2$ *and* G_{ax}^{*x} *is the stabilizer in* G_x^{*x} *of a triad* TRI *and a 5-element subset* FIX *in* Ω_x *with* $\mathrm{TRI} \cap \mathrm{FIX} = \emptyset$.

(iv) *If* $l \in \alpha_{0,4}(x, \mathrm{TRI}, \mathrm{FIX})$, *then* $|\Gamma_0(l) \cap \Delta_3^3(a)| = 1$ *and* $|\Gamma_0(l) \cap \Delta_4^6(a)| = 2$.

(v) *If* $e_1, e_2 \in \Delta_1(x) \cap \Delta_3^3(a)$ *with* $e_1 \neq e_2$, *then* $e_1 \in \Delta_2^1(e_2)$ *and there exists* $e \in \{e_1, e_2\}^\perp$ *with* $e \in \Delta_3^1(a)$.

Proof. Let $d \in \Delta_3^3(a) \cap \Delta_1(x)$. As in Lemma 12.1, set

$$S_d = \Delta_3^1(a) \cap \Delta_1(d) \cap \Delta_2^1(x) \quad \text{and}$$
$$U_d = \{z \in \Delta_1(x) \cap \Delta_3^3(a) \cap \Delta_2^1(d) \mid \{d, z\}^\perp \cap \Delta_3^1(a) \neq \emptyset\} \cup \{d\}.$$

Recall that $|S_d| = 4$ and $|U_d| = 5$, and note that $X(a, z_1) \neq X(a, z_2)$ for $z_1, z_2 \in \Delta_1(x) \cap \Delta_3^3(a)$, $z_1 \neq z_2$. Since $G_{ad} \sim 2^5 A_6$, $G_{ad}^{*d} \sim 2^4 A_6$ and $|\Delta_1(d) \cap \Delta_4^6(a)| = 160$ by Lemmas 4.8(ii) and 8.4(v), $G_{adx} \sim 2^3 3^2 \geq H \cong 3 \times A_4$ with $G_{adx} \cong G_{adx}^{*d}$ being the stabilizer in G_{ad}^{*d} of the line $d + x$. Hence, by (2.3), there exists $T \trianglelefteq G_{adx}$ with $|T| = 3$ and T fixing each of the four lines in $\alpha_3(d, d+x) \cap \alpha_{1,1}(x, d+c, X(d, a))$ ($\{c\} = \Delta_1(d) \cap \Delta_2^2(a)$). Consequently T fixes each point in S_d. Since by Lemma 12.1(i), for each $z \in U_d \backslash \{d\}$, $\Delta_3^3(a) \cap \{e, x\}^\perp = \{d, z\}$ ($e \in \{d, z\}^\perp \cap \Delta_3^1(a)$) we deduce that T fixes each point in U_d and hence

(12.2.1) T fixes each of the 5 hyperplanes $X(a, z)$, $z \in U_d$.

Put $n = |\Delta_1(x) \cap \Delta_3^3(a)|$.

(12.2.2) $n = 5$ or 20.

By Lemmas 5.3(ii) and 12.1(ii) $n = 5$, 10, 15 or 20. Now $U_d \subseteq \Delta_1(x) \cap \Delta_3^3(a)$, and $\{X(a, z) \mid z \in \Delta_1(x) \cap \Delta_3^3(a)\}$ is a T-invariant subset of Ω_a of size n. Since an element of order 3 in G_a^{*a} ($\cong M_{23}$) fixes exactly 5 elements of Ω_a, by (12.2.1) T fixes each element in $\{X(a, z) \mid z \in U_d\}$ and has orbits of length 3 on the remaining elements of Ω_a. This rules out $n = 10$ or 15, so proving (12.2.2).

(12.2.3) $\{x + y \mid y \in \Delta_1(x) \cap \Delta_3^3(a)\}$ is a G_{ax}-orbit of $\Gamma_1(x)$ of length n.

This follows from Lemmas 5.4 and 8.4(iii),(v).

Using (12.2.2) and Lemmas 4.8(iii) and 8.4(v) gives

(12.2.4) $|G_{ax}| = 2^3.3^2.5$ (if $n = 5$) or $2^5.3^2.5$ (if $n = 20$).

Putting $\overline{G}_x = G_x^{*x}$ and using the bar notation for subgroups of G_x, we next prove

(12.2.5) $\overline{R} \trianglelefteq \overline{G}_{ax}$ where $|\overline{R}| = 3$.

By (12.2.4) $|\overline{G}_{ax}| = 2^i.3^2.5$ where $i \leq 5$. Let $\overline{N}$ be a minimal normal subgroup of $\overline{G}_{ax}$. First we consider the case when $\overline{N}$ is abelian. Since $|C_{M_{23}}(\vartheta)| = 15$ for ϑ an element of order 5, $|\overline{N}| = 3$ and 2^4 are the only possibilities. Suppose $\overline{N} = 2^4$ were to hold. Then $\overline{G}_{ax} \leq 2^4(3 \times A_5)2$ or 2^4A_7, and hence $\overline{G}_{ax}/\overline{N} \leq (3 \times A_5)2$ or A_7. Now $\overline{G}_{ax}/\overline{N}$ must contain a Hall subgroup of order $3^2.5$. However neither $(3 \times A_5)2$ nor A_7 contain a subgroup of order $3^2.5$, which rules out the possibility $\overline{N} = 2^4$. So (12.2.5) holds with $\overline{R} = \overline{N}$.

Now suppose $\overline{N}$ is not abelian. Then the order of $\overline{G}_{ax}$ implies that $\overline{N} \cong A_5$ or A_6. If the latter holds, then $\overline{G}_{ax}/\overline{N}$ is a 2-group and hence $\overline{N}$ contains $\overline{H} \cong 3 \times A_4$ (note that $\overline{H} \cong 3^2$ would give $2^2 \cong H \cap Q(x) \leq Z(G_{ax})$, whereas $Z(H) = 3$). Since A_6 contains no such subgroups we must have $\overline{N} \cong A_5$. Then, by centralizers of 5-elements in M_{23} and $Out(A_5) = 2$ we obtain $C_{\overline{G}_{ax}}(\overline{N}) = 3$. Taking $\overline{R} = C_{\overline{G}_{ax}}(\overline{N})$ in this case, proves (12.2.5).

(12.2.6) (i) $n = 5$; and

(ii) $G_{ax} \cong \overline{G}_{ax} \sim (3 \times A_5)2$.

By (12.2.5), $\overline{G}_{ax} \leq N_{\overline{G}_x}(\overline{R}) \sim (3 \times A_5)2$. Since $\overline{G}_{ax}$ intersects $3 \times A_5$ in a subgroup of order at least $3^2.5$, we deduce that $\overline{G}_{ax} \cong 3 \times A_5$ or $(3 \times A_5)2$. As a consequence, the size of a G_{ax}-orbit on $\Gamma_1(x)$ will be either one of the sizes or half of one of the sizes given in (2.14). Consulting (2.14) reveals that G_{ax} cannot have an orbit on $\Gamma_1(x)$ of length 20 and therefore (12.2.2) and (12.2.3) force $n = 5$. Hence $|G_{ax}| = 2^3.3^2.5$ by (12.2.4). Since $G_{adx} \sim 2^3.3^2$ has no central elements of order 2, we must have $G_{ax} \cong \overline{G}_{ax} \sim (3 \times A_5)2$, so yielding (12.2.6).

By (12.2.6)(ii) $|\Delta_4^6(a)| = [G_a : G_{ax}] = 2^{15}.7.11.23$ and so we have proved parts (i) and (ii). Because $T \trianglelefteq G_{adx}$ $(\sim 2^3.3^2) \leq G_{ax}$ $(\sim (3 \times A_5)2)$ we see that $T \trianglelefteq G_{ax}$ and so $G_{ax}^{*x} = N_{G_{ax}^{*x}}(T)$. Let FIX denote the set of (5 elements) of Ω_x fixed by T and set

$$\mathrm{TRI} = \Omega_x \backslash \bigcup \{l \in \Gamma_1(x) \mid |l \cap \mathrm{FIX}| = 4\}.$$

Then (see (2.14)) G_{ax}^{*x} is the stabilizer in G_x^{*x} of TRI and FIX which proves part (iii).

We now prove parts (iv) and (v). Clearly, by part (i) and Lemma 12.1(i), $U_d = \Delta_1(x) \cap \Delta_3^3(a)$. As previously observed in Lemma 12.1(i) for each $z \in U_d \backslash \{d\}$, $\Delta_3^3(a) \cap \{e, x\}^\perp = \{d, z\}$ ($e \in \{d, z\}^\perp \cap \Delta_3^1(a)$). Looking at the list in (2.14), and noting that all the lines in $\alpha_{3,1}(x, \mathrm{TRI}, \mathrm{FIX})$ lie in the same diamond, we deduce that $\{x + y \mid y \in \Delta_1(x) \cap \Delta_3^3(a)\} = \alpha_{0,4}(x, \mathrm{TRI}, \mathrm{FIX})$, from which part (iv) follows. Since $U_{e_1} = \Delta_1(x) \cap \Delta_3^3(a)$, Lemma 12.1(i) gives part (v). □

Lemma 12.3. *Let $y \in \Delta_1(x) \cap \Delta_4^1(a)$ with $y + x \in \alpha_{3,0}^{\mathcal{L}^c}(y, y+d, \mathrm{DUAD})$ (where $\{d\} = \Delta_1(y) \cap \Delta_3^1(a)$). Suppose $x + y$ lies in the G_{ax}-orbit $\mathcal{O}_x$ on $\Gamma_1(x)$. Then*

(i) $\mathcal{O}_x$ is $\alpha_{2,2}(x, \mathrm{TRI}, \mathrm{FIX})$ or $\alpha_{1,3}(x, \mathrm{TRI}, \mathrm{FIX})$.

(ii) For all $l \in \mathcal{O}_x$, $|\Gamma_0(l) \cap \Delta_4^1(a)| = |\Gamma_0(l) \cap \Delta_4^4(a)| = |\Gamma_0(l) \cap \Delta_4^6(a)| = 1$.

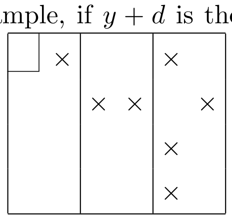
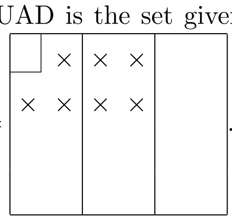
Proof. The result is a consequence of Lemma 5.5, using the orbit information given in (2.9), (2.14) and Lemmas 6.2(i), 8.10(i) and 12.2(ii). □

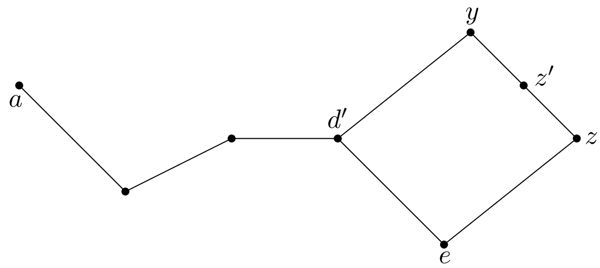
Lemma 12.4. *Let $y \in \Delta_4^1(a)$ and $l \in \alpha_{1,0}(y, y+d, \mathrm{DUAD})$ where $\{d\} = \Delta_1(y) \cap \Delta_3^1(a)$. Then*

(i) $|\Gamma_0(l) \cap \Delta_4^6(a)| = 2$.

(ii) If $z \in \Gamma_0(l) \cap \Delta_4^6(a)$, then $l \in \alpha_{0,2}(z, \mathrm{TRI}, \mathrm{FIX})$.

(iii) For every $m \in \alpha_{0,2}(x, \mathrm{TRI}, \mathrm{FIX})$, $|\Gamma_0(m) \cap \Delta_4^6(a)| = 2$ and $|\Gamma_0(m) \cap \Delta_4^1(a)| = 1$.

Proof. By Lemma 6.6(i) we may choose $d' \in \Delta_3^2(a) \cap \Delta_1(y)$ with $y + d' \in \alpha_3(y, l)$. For example, if $y + d$ is the standard heptad, DUAD is the set given in (2.9) and $l =$  we can choose $y + d' =$ . We have



where $\Gamma_0(l) = \{y, z', z\}$ and $d' \in \Delta_2^1(z)$. By Lemma 3.11(ii) there exists $e \in \{d', z\}^\perp \cap \Gamma_0(X(d', a))$, whence $e \in \Delta_3^i(a) \cap \Delta_2(a)$ for $i = 2, 3$ by Lemma 4.15. Assume $e \in \Delta_3^2(a) \cup \Delta_2(a)$. Then z or z' is collinear with a point in $\Delta_2(a) \cap \Gamma_0(e + d')$ by Lemma 3.10. Therefore z or $z' \in \Delta_3(a)$ and Theorems 7,8 and 10 together with Lemmas 6.1, 6.3(ii) and 6.6 imply that z or $z' \in \Delta_3^5(a)$. We may suppose $z \in \Delta_3^5(a)$ (the argument is similar if $z' \in \Delta_3^5(a)$). Appealing to (2.9) and Lemmas 6.1, 6.6, 6.8(i) and 8.10 we have $|\Delta_1(y) \cap \Delta_3^5(a)| = 42 + 56$ or $42 + 112$. This, together with Lemmas 4.11(i) and 6.2(i) gives $|\Delta_1(z) \cap \Delta_4^1(a)| =$ 140 or 220. Now (2.7) and Lemma 5.7 yield a contradiction.

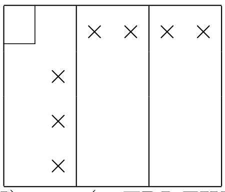
Therefore $e \in \Delta_3^3(a)$, and Lemma 4.15(ii) implies that $\Gamma_0(e + d') \backslash \{e, d'\} \subseteq \Delta_3^3(a)$. Thus by (2.15)(xii),(xiv) we have z, $z' \in \Delta_4^4(a) \cup \Delta_4^6(a)$. If z, $z' \in \Delta_4^4(a)$, then Lemmas 6.2(i) and 8.7(ii) imply that $z + y$ lies in a G_{az}-orbit on $\Gamma_1(z)$ of size 110. From (2.12) we must have $z + y \in \alpha_3(z, \mathrm{END}, -)$. However Lemma 8.10(ii) now yields the contradiction $l = y + z \in \alpha_{3,1}(y, y + c, \mathrm{DUAD})$.

Thus we may assume that $z \in \Delta_4^6(a)$ (and $z' \in \Delta_4^4(a) \cup \Delta_4^6(a)$). Using Lemmas 4.11(i), 6.6, 6.8, 8.10 and 12.2(ii) we have

$$\begin{aligned} |\Delta_1(z) \cap \Delta_4^1(a)| &= \frac{|\Delta_1(y) \cap \Delta_4^6(a)||\Delta_4^1(a)|}{|\Delta_4^6(a)|} \\ &= \frac{(56 + 56n).2^{13}.3.5.11.23}{2^{15}.7.11.23} \end{aligned}$$

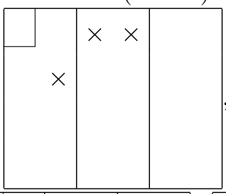
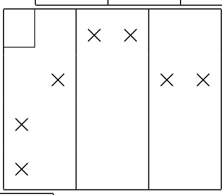
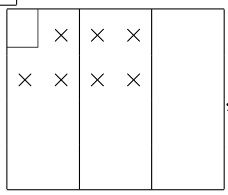
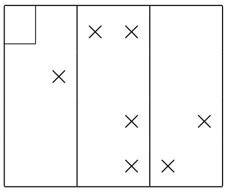
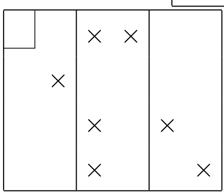
where $n = |\Gamma_0(l) \cap \Delta_4^6(a)| = 1$ or 2. So

(12.4.1) $|\Delta_1(z) \cap \Delta_4^1(a)| = 30(1 + n)$.

By Lemma 12.3 there exists $k \in \alpha_{2,2}(z, \mathrm{TRI}, \mathrm{FIX}) \cup \alpha_{1,3}(z, \mathrm{TRI}, \mathrm{FIX})$ with $|\Gamma_0(k) \cap \Delta_4^1(a)| = 1$. Therefore, considering the possibilities in (2.14), we conclude that $z + y \in \alpha_{2,2}(z, \mathrm{TRI}, \mathrm{FIX}) \cup \alpha_{0,2}(z, \mathrm{TRI}, \mathrm{FIX}) \cup \alpha_{1,3}(z, \mathrm{TRI}, \mathrm{FIX})$. Assume TRI and FIX are the subsets of Ω_z described in (2.14). Without loss of generality we may assume $z+e =$  by Lemma 12.2(i),(iv).

We now show $z+y \in \alpha_{2,2}(z, \mathrm{TRI}, \mathrm{FIX}) \cup \alpha_{1,3}(z, \mathrm{TRI}, \mathrm{FIX})$ is untenable. Assume that $z+y \in \alpha_{2,2}(z, \mathrm{TRI}, \mathrm{FIX}) \cup \alpha_{1,3}(z, \mathrm{TRI}, \mathrm{FIX})$.

(12.4.2) $\{z+f \mid f \in \{d', z\}^{\perp} \backslash \{e\}\}$ consists of one line in $\alpha_{2,2}(z, \mathrm{TRI}, \mathrm{FIX})$, one line in $\alpha_{1,3}(z, \mathrm{TRI}, \mathrm{FIX})$ and two lines in $\alpha_{0,2}(z, \mathrm{TRI}, \mathrm{FIX})$.

First suppose $z+y \in \alpha_{2,2}(z, \mathrm{TRI}, \mathrm{FIX})$. If $|T(z, d') \cap \mathrm{FIX}| = 0, 1$ or 3, then none of the heptads in $\Gamma_1(T(z, d'))$ lie in $\alpha_{2,2}(z, \mathrm{TRI}, \mathrm{FIX})$. So we must have $|T(z, d') \cap \mathrm{FIX}| = 2$ and then (12.4.2) follows from (2.14) in this case. (For example if $T(z, d') =$ , then the four heptads in $\{z+f \mid f \in \{d', z\}^{\perp} \backslash \{e\}\}$ are , ,  and .) Next assume $z+y \in \alpha_{1,3}(z, \mathrm{TRI}, \mathrm{FIX})$. So $|T(z, d') \cap \mathrm{FIX}| = 2$ or 3. If $T(z, d') \subseteq \mathrm{FIX}$, then $\{d', z\}^{\perp}$ contains a point e' $(\neq e)$ $\in \Delta_3^3(a)$ by Lemma 12.2(iv). However Lemma 12.2(v) now implies that there exists $d_1 \in \{e, e'\}^{\perp} \cap \Delta_3^1(a)$. Thus $\Gamma_3(a, d_1, d', e) \neq \emptyset$ by Lemma 4.15, whence $\Gamma_3(a, z) \neq \emptyset$. With this contradiction we conclude that $|T(z, d') \cap \mathrm{FIX}| = 2$ and so (12.4.1) follows from (2.14) again.

Let $\{d', z\}^{\perp} = \{e, y, y_1, y_2, y_3\}$ where $z+y$ lies in one of the orbits $\alpha_{2,2}(z, \mathrm{TRI}, \mathrm{FIX})$ and $\alpha_{1,3}(z, \mathrm{TRI}, \mathrm{FIX})$, and $z+y_1$ lies in the other orbit. Also $z+y_2, z+y_3 \in \alpha_{0,2}(z, \mathrm{TRI}, \mathrm{FIX})$ by (12.4.1). Since $y+z \in \alpha_{1,0}(y, y+d, \mathrm{DUAD})$, Lemma 12.3 implies that $y_1 + z \in \alpha_{3,0}^{\mathcal{L}^c}(y_1, y_1 + d'', \mathrm{DUAD})$ (where $\{d''\} = \Delta_1(y_1) \cap \Delta_3^1(a)$)

and $y_1 \in \Delta_4^1(a) \cup \Delta_4^4(a)$. We have $d' \in \Delta_3^2(a)$ and therefore $y_1 \in \Delta_4^1(a)$ by Theorem 6. In $\Omega_{d'}, d'+y, d'+y_1$ and $d'+e$ intersect $O(d',a)$ in two elements by Lemmas 4.15(ii) and 6.3(ii). So $|T(d',z) \cap O(d',a)| = 0, 1$ or 2. If $T(d',z) \cap O(d',a) = \emptyset$, then $\{d',z\}^{\perp} \cap \Delta_4^2(a) \neq \emptyset$ by (2.15)(x). This is impossible because $\Delta_1(z) \cap \Delta_4^2(a) = \emptyset$ by Theorem 12. Also $|T(d',z) \cap O(d',a)| \neq 2$, otherwise $|\{d',z\}^{\perp} \cap \Delta_4^3(a)| = 3$ by (2.15)(xi) which contradicts the fact that $y, y_1, e \notin \Delta_4^3(a)$. Hence $|T(d',z) \cap O(d',a)| = 1$ in $\Omega_{d'}$. Thus, we may suppose that $d'+y_2 \in \alpha_{2,0}(d', O(d',a), X(d',a))$ and $d'+y_3 \in \alpha_{4,0}(d', O(d',a), X(d',a))$. Therefore $y_2 \in \Delta_4^1(a)$ and $y_3 \in \Delta_4^3(a)$ by (2.15)(xi) and Lemma 6.3(ii).

Since $z+y_2$ and $z+y_3$ both lie in $\alpha_{0,2}(z, \mathrm{TRI}, \mathrm{FIX})$ by (12.4.2), if $m \in \alpha_{0,2}(z, \mathrm{TRI}, \mathrm{FIX})$, then $|\Gamma_0(m) \cap \Delta_4^1(a)| = |\Gamma_0(m) \cap \Delta_4^3(a)| = |\Gamma_0(m) \cap \Delta_4^6(a)| = 1$. However we have $l \in \alpha_{1,0}(y, y+d, \mathrm{DUAD})$ with $\Gamma_0(l) \cap \Delta_4^3(a) = \emptyset$. This, together with Lemmas 6.1, 6.6, 6.8 and 8.10, implies that there are no lines in $\Gamma_1(y)$ incident with points in both $\Delta_4^3(a)$ and $\Delta_4^6(a)$. Since $\Delta_4^1(a)$ is a G_a-orbit there are no lines in Γ_1 incident with points in each of $\Delta_4^1(a)$, $\Delta_4^3(a)$ and $\Delta_4^6(a)$. This contradicts the point distribution of $\Gamma_0(m)$.

Hence our assumption that $z+y \in \alpha_{2,2}(z, \mathrm{TRI}, \mathrm{FIX}) \cup \alpha_{1,3}(z, \mathrm{TRI}, \mathrm{FIX})$ was false and we have $z+y \in \alpha_{0,2}(z, \mathrm{TRI}, \mathrm{FIX})$ as required. To show that $z' \in \Delta_4^6(a)$, notice that $|\Delta_1(z) \cap \Delta_4^1(a)| = 90$ by (2.14), whence $n = |\Gamma_0(l) \cap \Delta_4^6(a)| = 2$ by (12.4.1).

Finally part (iii) follows from (i) and (ii) because $\Delta_4^6(a)$ is a G_a-orbit. This completes the proof of the lemma.

□

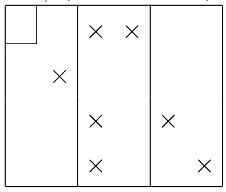
Theorem 11 has now been proved.

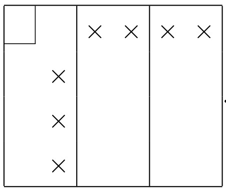
Lemma 12.5. *(i)* $|\Gamma_0(l) \cap \Delta_4^1(a)| = |\Gamma_0(l) \cap \Delta_4^4(a)| = 1$ *for all* l *in exactly one of* $\alpha_{2,2}(x, \mathrm{TRI}, \mathrm{FIX})$ *and* $\alpha_{1,3}(x, \mathrm{TRI}, \mathrm{FIX})$.

(ii) $|\Gamma_0(l) \cap \Delta_4^3(a)| = |\Gamma_0(l) \cap \Delta_4^4(a)| = 1$ *for all* l *in exactly one of* $\alpha_{2,2}(x, \mathrm{TRI}, \mathrm{FIX})$ *and* $\alpha_{1,3}(x, \mathrm{TRI}, \mathrm{FIX})$.

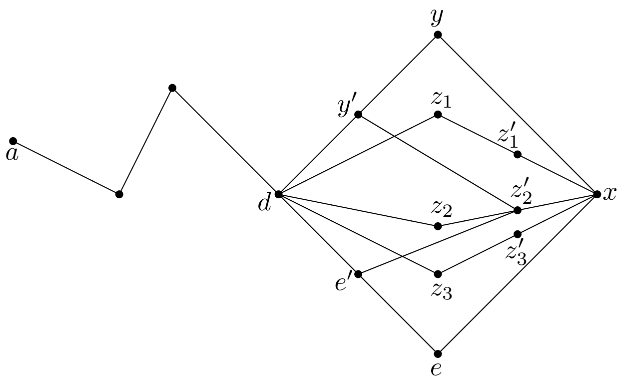
Proof. Put $\mathcal{O} = \alpha_{2,2}(x, \mathrm{TRI}, \mathrm{FIX}) \cup \alpha_{1,3}(x, \mathrm{TRI}, \mathrm{FIX})$. Let $y \in \Delta_1(x) \cap \Delta_4^1(a)$ with $x+y \in \alpha_{0,2}(x, \mathrm{TRI}, \mathrm{FIX})$ and $y+x \in \alpha_{1,0}(y, y+c, \mathrm{DUAD})$, where $\{c\} = \Delta_1(y) \cap \Delta_3^1(a)$ (such a y exists by Lemma 12.4). Then we can choose $e \in$

$\Delta_1(x) \cap \Delta_3^3(a)$ with $x+e \in \alpha_3(x, x+y)$ and $|T(e,y) \cap \mathrm{FIX}| = 2$ in Ω_x because every 2-element subset of FIX is contained in some heptad in $\alpha_{0,4}(x, \mathrm{TRI}, \mathrm{FIX})$.

For example if TRI and FIX are as in (2.14) and $x+y =$ 

then we can choose $x+e$ to be .

There exists $d \in \{e,y\}^{\perp} \cap \Gamma_0(X(e,a))$ by Lemma 3.11(ii). Since $y \in \Delta_4^1(a)$, Theorem 11 implies that $d \in \Delta_3^2(a)$ because $y+x \in \alpha_1(y, y+c)$ and so $c \neq d$. In $\Gamma_1(x)$ the set $\{x+z \mid z \in \{x,d\}^{\perp}\}$ consists of $x+e$, two lines in $\alpha_{0,2}(x, \mathrm{TRI}, \mathrm{FIX})$ (one of which is $x+y$), one line in $\alpha_{1,3}(x, \mathrm{TRI}, \mathrm{FIX})$ and one line in $\alpha_{2,2}(x, \mathrm{TRI}, \mathrm{FIX})$. Let $x+z_1 \in \alpha_{1,3}(x, \mathrm{TRI}, \mathrm{FIX}), x+z_2 \in \alpha_{2,2}(x, \mathrm{TRI}, \mathrm{FIX})$ and $x+z_3 \in \alpha_{0,2}(x, \mathrm{TRI}, \mathrm{FIX})$ where $\{x,d\}^{\perp} = \{z_1, z_2, z_3, e, y\}$.



Below we introduce the points e' , y' , z_1', z_2' and z_3'.

We now consider $\{x,d\}^{\perp}$ from the viewpoint of d. In Γ_d, $d+y \in \alpha_{2,0}(d, O(d,a), X(d,a))$ and $d+e \in \alpha_{2,1}(d, O(d,a)X(d,a))$ by Theorem 6. There are three possible cases. In Ω_d either

(1) $|T(d,x) \cap O(d,a)| = 0$, so we have $|\{d,x\}^{\perp} \cap \Delta_4^1(a)| = 3$ and $|\{d,x\}^{\perp} \cap \Delta_4^2(a)| = 1$; or

(2) $|T(d,x) \cap O(d,a)| = 1$, so we have $|\{d,x\}^{\perp} \cap \Delta_4^1(a)| = 3$ and $|\{d,x\}^{\perp} \cap$

$\Delta_4^3(a)| = 1$; or

(3) $|T(d,x) \cap O(d,a)| = 2$, so we have $|\{d,x\}^\perp \cap \Delta_4^1(a)| = 1$ and $|\{d,x\}^\perp \cap \Delta_4^3(a)| = 3$.

(using (2.15)(x),(xi) and Lemma 6.3(ii)).

Case (1) is impossible by Theorem 12. Using Lemma 12.4(iii) and the fact that $x+y$, $x+z_3 \in \alpha_{0,2}(x, \mathrm{TRI}, \mathrm{FIX})$ we see that case (3) cannot occur either. Therefore case (2) holds, whence, using Lemma 12.4(iii), $\Gamma_0(l) \cap \Delta_4^1(a) \neq \emptyset$ for all l in one of the orbits in $\mathcal{O}$ and $\Gamma_0(l) \cap \Delta_4^3(a) \neq \emptyset$ for all l in the other orbit in $\mathcal{O}$.

Let $\Gamma_0(d+e) = \{d, e', e\}$, $\Gamma_0(y+d) = \{y, y', d\}$ and $\Gamma_0(x+z_i) = \{x, z_i', z_i\}$ for $i = 1,2,3$. To complete the proof we need to show that $z_i' \in \Delta_4^4(a)$ for $i = 1, 2$. By Lemma 4.15(ii) $e' \in \Delta_3^3(a)$. Appealing to (2.15)(xii),(xiv) and Lemma 3.10 yields that $z_i' \in \Delta_4^4(a) \cup \Delta_4^6(a)$ for $i = 1, 2$. We know $z_j \in \Delta_4^1(a)$ for some $j \in \{1, 2\}$, whence $z_j + x \in \alpha_{1,0}(z_j, z_j + d', \mathrm{DUAD}) \cup \alpha_{3,0}^{\mathcal{L}^c}(z_j, z_j + d', \mathrm{DUAD})$ by Theorem 11 (where $\{d'\} = \Delta_1(z_j) \cap \Delta_3^1(a)$). However Lemma 12.4(ii) now yields $z_j + x \in \alpha_{3,0}^{\mathcal{L}^c}(z_j, z_j + d', \mathrm{DUAD})$ because $x + z_j \notin \alpha_{0,2}(x, \mathrm{TRI}, \mathrm{FIX})$. Therefore $z_j' \in \Delta_4^4(a)$ by Lemma 8.10(i). Using (2) and Lemma 12.4(iii), $z_3 \in \Delta_4^1(a)$ and $z_3' \in \Delta_4^6(a)$. So by Lemma 3.10 we have

(12.5.1) $\{e', y'\}^\perp \cap \Delta_4^i(a) \neq \emptyset$ for $i = 4$ and 6.

We now consider $T(e', y') \cap (e' + c')$ as a subset of $\Omega_{e'}$ (where $\{c'\} = \Delta_1(e') \cap \Delta_2^2(a)$). If $|T(e', y') \cap (e' + c')| = 2$ or 3 then $\{e', y'\}^\perp \subseteq \Delta_4^6(a) \cup \Delta_3(a)$ by Theorem 7 which contradicts (12.5.1). Also if $T(e', y') \cap (e' + c') = \emptyset$, then $\{e', y'\}^\perp \backslash \{d\} \subseteq \Delta_4^4(a)$ which is impossible by (12.5.1) again. Hence $|T(e', y') \cap (e' + c')| = 1$ and thus $\{e', y'\}^\perp \backslash \{d\}$ consists of two points in each of $\Delta_4^4(a)$ and $\Delta_4^6(a)$. Since $|\Gamma_0(x+y) \cap \Delta_4^6(a)| = |\Gamma_0(x+z_3) \cap \Delta_4^6(a)| = 2$ by Lemma 12.4(iii) we obtain the required result that z_1', $z_2' \in \Delta_4^4(a)$. □

Lemma 12.6. *Let $y \in \Delta_4^4(a)$ and $l \in \alpha_5(y, \mathrm{END}, -)$. Then $|\Gamma_0(l) \cap \Delta_4^3(a)| = |\Gamma_0(l) \cap \Delta_4^6(a)| = 1$. Moreover if $f \in \Gamma_0(l) \cap \Delta_4^6(a)$, then $l \in \alpha_{2,2}(f, \mathrm{TRI}, \mathrm{FIX}) \cup \alpha_{1,3}(f, \mathrm{TRI}, \mathrm{FIX})$.*

Proof. By Lemma 12.5 there exists $f \in \Delta_1(y) \cap \Delta_4^6(a)$ with $f+y \in \alpha_{2,2}(f, \mathrm{TRI}, \mathrm{FIX}) \cup \alpha_{1,3}(f, \mathrm{TRI}, \mathrm{FIX})$. Since the point in $\Gamma_0(f+y) \backslash \{f, y\}$ does not lie in $\Delta_4^4(a) \cup$

$\Delta_4^6(a)$ by Lemma 12.5 again, we may appeal to Lemma 5.5 to show that $y+f$ lies in a G_{ay}-orbit $\mathcal{O}_y$ of $\Gamma_1(y)$ of size

$$\frac{|\Delta_4^6(a)|}{|\Delta_4^4(a)|}.30$$

(since $|\alpha_{2,2}(f,\mathrm{TRI},\mathrm{FIX})| = |\alpha_{1,3}(f,\mathrm{TRI},\mathrm{FIX})| = 30$ by (2.14)). Using Lemmas 8.7(ii) and 12.2(ii) we conclude that

$$|\mathcal{O}_y| = \frac{2^{15}.7.11.23.30}{2^{16}.3.7.23} = 55.$$

So $y+f \in \alpha_5(y,\mathrm{END},-)$ by (2.12). Hence Theorem 11 and Lemmas 8.10(i) and 12.5 imply that $|\Gamma_0(y+f)\cap\Delta_4^3(a)| = |\Gamma_0(y+f)\cap\Delta_4^6(a)| = 1$. The lemma now follows using Lemma 5.5 because G_{ay} is transitive on $\alpha_5(y,\mathrm{END},-)$ by (2.12).

□

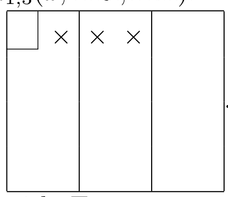
We remark that Theorem 14 has now been proved (see Lemmas 8.8, 8.9(i),(ii), 8.13, 9.6(i),(iii) and 12.6).

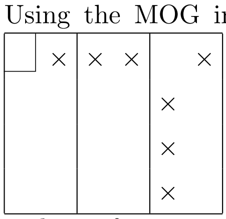
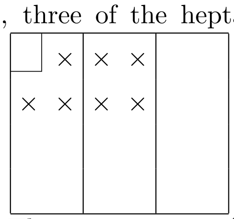
Lemma 12.7. *Let* $l \in \alpha_{1,3}(x,\mathrm{TRI},\mathrm{FIX})$. *Then*

(i) $|\Gamma_0(l)\cap\Delta_4^1(a)| = |\Gamma_0(l)\cap\Delta_4^4(a)| = 1$.

(ii) If $y \in \Gamma_0(l)\cap\Delta_4^4(a)$ *then* $l \in \alpha_3(y,\mathrm{END},+)$.

Proof. By Lemma 12.5 there exists $l \in \alpha_{1,3}(x,\mathrm{TRI},\mathrm{FIX}) \cup \alpha_{2,2}(x,\mathrm{TRI},\mathrm{FIX})$ with $|\Gamma_0(l)\cap\Delta_4^1(a)| = |\Gamma_0(l)\cap\Delta_4^4(a)| = 1$. We need to show that $l \in \alpha_{1,3}(x,\mathrm{TRI},\mathrm{FIX})$.

Let TRI and FIX be the subsets of Ω_x described in (2.14) and let $T =$ .

Using the MOG in [2], three of the heptads in $\Gamma_1(x)$, incident with T are  and  which we denote by $x+e_1, x+e_2$ and $x+f$ respectively. Then $x+e_i \in \alpha_{0,4}(x,\mathrm{TRI},\mathrm{FIX})$ $(i=1,2)$ and $x+f \in \alpha_{1,3}(x,\mathrm{TRI},\mathrm{FIX})$. By Lemma 12.2(iv),(v) we may assume that $e_i \in \Delta_3^3(a)$ for each i and that $\{e\} = \{e_1, e_2\}^{\perp} \cap \Delta_3^1(a)$. Since $T(x, e) = T$, we have that $f \in \{e, x\}^{\perp}$. Therefore $f \in \Delta_2^1(a) \cup \Delta_3^1(a) \cup \Delta_3^3(a) \cup \Delta_4^1(a)$ by Theorem 5. But Lemma 12.5 implies that $f \in \Delta_4^1(a)$ and part (i) follows by Lemma 12.5(i).

For part (ii) we appeal to Lemmas 5.5, 8.7(ii) and 12.2(ii) to show that l lies in a G_{ay}-orbit of $\Gamma_1(y)$ of size 55. Thus (2.12) and Lemma 12.6 imply that $l \in \alpha_3(y, \mathrm{END}, +)$ as required. □

Lemma 12.8. *Let $l \in \alpha_{2,2}(x, \mathrm{TRI}, \mathrm{FIX})$. Then*

(i) $|\Gamma_0(l) \cap \Delta_4^3(a)| = |\Gamma_0(l) \cap \Delta_4^4(a)| = 1$.

(ii) If $y \in \Gamma_0(l) \cap \Delta_4^4(a)$, then $l \in \alpha_5(y, \mathrm{END}, -)$.

(iii) If $z \in \Gamma_0(l) \cap \Delta_4^3(a)$, then $l \in \alpha_{2,1|1}(z, \mathrm{TRI}, \mathrm{OCT})$.

Proof. Parts (i) and (ii) follow immediately from Lemmas 12.5, 12.6 and 12.7. We now look at part (iii). Let $\Gamma_0(l) = \{x, y, z\}$ where $y \in \Delta_4^4(a)$ and $z \in \Delta_4^3(a)$. By Lemmas 5.5, 11.1(ii) and 12.2(ii) $z + x$ lies in a G_{az}-orbit of $\Gamma_1(z)$ of size 48. So $z + x \in \alpha_{0,1|1}(z, \mathrm{TRI}, \mathrm{OCT}) \cup \alpha_{2,1|1}(z, \mathrm{TRI}, \mathrm{OCT})$ by (2.11).

(12.8.1) (i) If $z + x \in \alpha_{0,1|1}(z, \mathrm{TRI}, \mathrm{OCT})$, then there exists a unique point in $\Delta_1(z) \cap \Delta_2^1(x) \cap \Delta_3^2(a)$.

(ii) If $z + x \in \alpha_{2,1|1}(z, \mathrm{TRI}, \mathrm{OCT})$, then there exist three points in $\Delta_1(z) \cap \Delta_2^1(x) \cap \Delta_3^2(a)$.

This follows because $z + d \in \alpha_{3,0}(z, \mathrm{TRI}, \mathrm{OCT})$ for all $d \in \Delta_1(z) \cap \Delta_3^2(a)$.

Since $x + z \in \alpha_{2,2}(x, \mathrm{TRI}, \mathrm{FIX})$, there exist three points in $S := \Delta_1(x) \cap \Delta_2^1(z) \cap \Delta_3^3(a)$ because three lines in $\alpha_{0,4}(x, \mathrm{TRI}, \mathrm{FIX})$ lie in $\alpha_3(x, x + z)$. For each $e \in S$, there exists $d \in \{e, z\}^{\perp}$ with $d \in \Delta_3^2(a)$ because $z \notin \Delta_1(f)$ for any $f \in \Delta_2^2(a) \cup \Delta_3^i(a)$ (i = 1,3,4) by Theorems 3,4 and 7. If, for two distinct elements e_1, $e_2 \in S$, there exists $d \in \Delta_3^2(a)$ with $d \in \{e_1, z\}^{\perp} \cap \{e_2, z\}^{\perp}$, then $X(e_1, a) = X(e_2, a)$ by Lemma 4.16, whence $\Gamma_3(x, a) \neq \emptyset$ by Lemma 3.6. This is impossible and so we have three distinct points in $\Delta_1(z) \cap \Delta_2^1(x) \cap \Delta_3^2(a)$. Now (12.8.1) implies that $z + x \in \alpha_{2,1|1}(z, \mathrm{TRI}, \mathrm{OCT})$.

□

Lemma 12.9. *Let* $z \in \Delta_4^3(a)$ *and* $l \in \alpha_{0,1|1}(z, \mathrm{TRI}, \mathrm{OCT})$*. Then*

(i) $|\Gamma_0(l) \cap \Delta_3^5(a)| = 1$ *and* $|\Gamma_0(l) \cap \Delta_4^3(a)| = 2$*.*

(ii) If $y \in \Gamma_0(l) \cap \Delta_3^5(a)$*, then* $l \in \alpha_3^{(1)}(y, y+c, -)$ *(where* $\{c\} = \Delta_1(y) \cap \Delta_2^2(a)$*).*

Proof. Let $y \in \Delta_1(z) \cap \Delta_3^5(a)$. By Lemma 11.3(i) $|\Delta_1(y) \cap \Delta_4^3(a)| = 140$. Hence Lemmas 4.11(i) and 11.1(ii) imply that $|\Delta_1(z) \cap \Delta_3^5(a)| = 84$. Thus Lemma 11.2 and (2.11) imply that $\{z + y \mid y \in \Delta_1(z) \cap \Delta_3^5(a)\} = \alpha_{1,2|2}(z, \mathrm{TRI}, \mathrm{OCT}) \cup \alpha_{i,1|1}(z, \mathrm{TRI}, \mathrm{OCT})$ for $i = 0$ or 2. However $z + y \notin \alpha_{2,1|1}(z, \mathrm{TRI}, \mathrm{OCT})$ for all $y \in \Delta_1(z) \cap \Delta_3^5(a)$ by Lemma 12.8(iii) and so $\{z + y \mid y \in \Delta_1(z) \cap \Delta_3^5(a)\} = \alpha_{1,2|2}(z, \mathrm{TRI}, \mathrm{OCT}) \cup \alpha_{0,1|1}(z, \mathrm{TRI}, \mathrm{OCT})$. The result now follows by Lemma 11.3(iii). □

We finish this section by proving two results which pull together information about the orbits $\Delta_3^5(a), \Delta_3^6(a)$ and $\Delta_4^5(a)$.

Lemma 12.10. *Let* $l \in \alpha_{3,1}(x, \mathrm{TRI}, \mathrm{FIX})$*. Then*

(i) $|\Gamma_0(l) \cap \Delta_3^6(a)| = |\Gamma_0(l) \cap \Delta_4^5(a)| = 1$*.*

(ii) If $y \in \Gamma_0(l) \cap \Delta_3^6(a)$*, then* $l \in \alpha_{1,0}(y, y + c, \mathrm{TRI})$ *(where* $\{c\} = \Delta_1(y) \cap \Delta_2^1(a)$*).*

(iii) If $z \in \Gamma_0(l) \cap \Delta_4^5(a)$*, then* $l \in \alpha_3(z, z+d, +)$ *(where* $\{d\} = \Delta_1(z) \cap \Delta_3^4(a)$*).*

Proof. By Lemmas 5.2 and 9.2(ii) we can find $z \in \Delta_1(x) \cap \Delta_4^5(a)$ with $z + x \in \alpha_3(z, z+d, +)$ ($\{d\} = \Delta_1(z) \cap \Delta_3^4(a)$). Let $\Gamma_0(z+x) = \{x, y, z\}$. Then $y \in \Delta_3^6(a)$ by Lemma 9.2(ii) again. Lemma 5.5, together with the orbit sizes $|\Delta_3^6(a)|$, $|\Delta_4^5(a)|$ and $|\Delta_4^6(a)|$ given in Lemmas 4.11(ii), 9.1(i) and 12.2(ii) respectively, imply that $y + z$ $(= x + z)$ lies in a G_{ay}-orbit of $\Gamma_1(y)$ of size 64 and a G_{ax}-orbit of $\Gamma_1(x)$ of size 5. By (2.14) and Lemma 12.2(iv), $x + z \in \alpha_{3,1}(x, \mathrm{TRI}, \mathrm{FIX})$. The result now follows from (2.8) and Lemma 5.5 because $\alpha_{3,1}(x, \mathrm{TRI}, \mathrm{FIX})$ is a G_{ax}-orbit. □

Lemma 12.11. *Let* $l \in \alpha_{0,0}(x, \mathrm{TRI}, \mathrm{FIX})$*. Then*

(i) $|\Gamma_0(l) \cap \Delta_3^5(a)| = |\Gamma_0(l) \cap \Delta_4^5(a)| = 1$*.*

(ii) If $y \in \Gamma_0(l) \cap \Delta_3^5(a)$*, then* $l \in \alpha_2^{(3)}(y, y+c, -)$ *(where* $\{c\} = \Delta_1(y) \cap \Delta_2^2(a)$*).*

(iii) If $z \in \Gamma_0(l) \cap \Delta_4^5(a)$, then $l \in \alpha_3(z, z+d, -)$ (where $\{d\} = \Delta_1(z) \cap \Delta_3^4(a)$).

Proof. Let $\Gamma_0(l) = \{x, y, z\}$. By Lemma 9.2(iii) and the fact that $\Delta_4^6(a)$ is a G_a-orbit, we may assume that $y \in \Delta_3^5(a)$ and $z \in \Delta_4^5(a)$ with $y + x = l \in \alpha_3^{(2)}(y, y+c, -)$ and $z + x = l \in \alpha_3(z, z+d, -)$ (where $\{c\} = \Delta_1(y) \cap \Delta_2^2(a)$ and $\{d\} = \Delta_1(z) \cap \Delta_3^4(a)$). Appealing to Lemmas 5.5, 9.1(i) and 12.2(ii) we have that $l = x + z$ lies in a G_{ax}-orbit on $\Gamma_1(x)$ of size 15. Thus (2.14) and Lemma 9.2(iii) yield the result.

□

13 And finally...

In this, our last section, we stitch up a few loose ends so as to complete the proofs of Theorems 9, 13, 15 and 16 and thus, as a consequence, prove Theorem 1. Firstly we focus our attention on $\Delta_4^5(a)$.

Lemma 13.1. *Let $x \in \Delta_4^5(a)$.*

(i) $\Delta_1(x) \cap \Delta_4^i(a) = \emptyset$ for $i = 1$ and 2.

(ii) If $l \in \alpha_1(x, x+d, -)$, then $\Gamma_0(l) \subseteq \Delta_4^3(a) \cup \Delta_4^5(a)$ (where $\{d\} = \Delta_1(x) \cap \Delta_3^4(a)$).

Proof. Part (i) is a consequence of Theorems 11 and 12. Turning to (ii), let $y \in \Gamma_0(l) \backslash \{x\}$. First suppose $y \in \Delta_3(a)$. By Theorems 5–8 and 10 and Lemmas 9.1 and 9.2(ii) we must have $y \in \Delta_3^5(a)$. Then (2.13) and Lemmas 9.1(ii), 9.2(ii),(iii) and 9.6(i) imply that

$$|\Delta_1(x) \cap \Delta_3^5(a)| = 147 + 70 \quad \text{or} \quad 147 + 140.$$

Hence

$$|\Delta_1(y) \cap \Delta_4^5(a)| = \frac{|\Delta_4^5(a)|}{|\Delta_3^5(a)|}.n$$

with $n = 217$ or 287 because $\Delta_3^5(a)$ and $\Delta_4^5(a)$ are G_a-orbits. Appealing to Lemmas 4.11(i) and 9.1(i) yields

$$|\Delta_1(y) \cap \Delta_4^5(a)| = \frac{248}{3} \quad \text{or} \quad \frac{328}{3}$$

which is clearly impossible. Therefore $y \notin \Delta_3(a)$. By part (i), $y \in \Delta_4^i(a)$ for $i = 3, 4, 5$ or 6. We have $y \notin \Delta_4^4(a)$, otherwise Theorem 14 and Lemma 9.5 imply that $l \in \alpha_1(x, x+d, +)$. If $y \in \Delta_4^6(a)$, then

$$|\Delta_1(x) \cap \Delta_4^6(a)| = 140 + 70 \quad \text{or} \quad 140 + 140$$

by (2.13) and Lemmas 9.1(ii), 9.2(ii),(iii) and 9.6(i). Using the orbit sizes given in Lemmas 9.1(i) and 12.2(ii) we obtain

$$|\Delta_1(y) \cap \Delta_4^5(a)| = 30 \quad \text{or} \quad 40.$$

However this contradicts the fact that

$$|\Delta_1(y) \cap \Delta_4^5(a)| = 20 + 18p + 90q \qquad \text{for some } p, q \in \{0, 1, 2\}$$

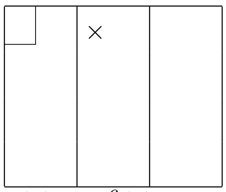
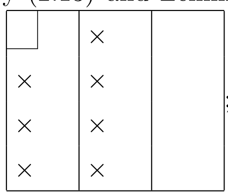
by (2.14) and Lemmas 12.2(iv), 12.4(iii), 12.7(i), 12.8(i), 12.10(i) and 12.11(i). This completes the proof of part (ii) and the lemma. □

Lemma 13.2. *Let $x \in \Delta_4^5(a)$ and $l \in \alpha_1(x, x+d, -)$ (where $\{d\} = \Delta_1(x) \cap \Delta_3^4(a)$). Then*

(i) $|\Gamma_0(l) \cap \Delta_4^3(a)| = 2$;

(ii) for $y \in \Gamma_0(l) \cap \Delta_4^3(a)$ we have $l \in \alpha_{0,3|1}(y, \mathrm{TRI}, \mathrm{OCT})$; and

(iii) for all $y \in \Delta_4^3(a)$ and $l \in \alpha_{0,3|1}(y, \mathrm{TRI}, \mathrm{OCT})$ we have $|\Gamma_0(l) \cap \Delta_4^3(a)| = 2$ and $|\Gamma_0(l) \cap \Delta_4^5(a)| = 1$.

Proof. Let $x+d$ be the standard heptad and X be the element  of Ω_x. By (2.13) and Lemma 9.2(ii) we may choose $e \in \Delta_1(x) \cap \Delta_3^6(a)$ with $x + e =$ ; so $x + e \in \alpha_3(x, x+d, +)$. We have $e + x \in$

$\alpha_{1,0}(e, e+c, \mathrm{TRI})$ by (2.8) and Lemma 9.3 (where $\{c\} = \Delta_1(e) \cap \Delta_2^1(a)$). Let $y \in \Delta_1(x) \cap \Delta_2^1(e)$ with $x+y = \begin{array}{|cc|cc|cc|}\hline \square & \times & & \times & \times & \times \\ & & \times & & & \\ & & \times & & & \\ & & \times & & & \\ \hline \end{array}$. Then $x+y \in \alpha_1(x, x+d, -)$.

Let $\Gamma_0(x+y) = \{x, y, y'\}$. By Lemma 13.1(ii), $y, y' \in \Delta_4^3(a) \cup \Delta_4^5(a)$. Suppose that $y, y' \in \Delta_4^5(a)$ holds. Then Lemma 3.11(ii) implies that there exists $z \in \{e, y\}^\perp$ with $z \in \Delta_2^1(c)$. Hence $e+z \in \alpha_{3,i}(e, e+c, \mathrm{TRI})$ for some $i = 0, 1, 2, 3$ (see (2.8)). So $z \neq x$. Using Theorem 10 we must have y or y' collinear with a point in $\Delta_2^2(a) \cup \Delta_4^1(a) \cup \Delta_4^2(a)$. This contradicts Lemma 13.1(i) and the fact that y, $y' \in \Delta_4(a)$. Therefore at least one of y and y' lies in $\Delta_4^3(a)$; suppose that $y \in \Delta_4^3(a)$.

Lemmas 9.1(ii), 9.2(ii),(iii) and 9.6(i) together with (2.13) imply that

$$|\Delta_1(x) \cap \Delta_4^3(a)| = 70 \quad \text{or} \quad 140.$$

Thus

$$|\Delta_1(y) \cap \Delta_4^5(a)| = 16 \quad \text{or} \quad 32$$

by Lemmas 5.2, 9.1(i) and 11.1(ii). Considering the possibilities in (2.11) we must have $y + x \in \alpha_{0,3|1}(y, \mathrm{TRI}, \mathrm{OCT})$ with $|\Gamma_0(y+x) \cap \Delta_4^5(a)| = 1$ and $|\Gamma_0(y+x) \cap \Delta_4^3(a)| = 2$. The lemma now follows because $\alpha_1(x, x+d, -)$ and $\alpha_{0,3|1}(y, \mathrm{TRI}, \mathrm{OCT})$ are orbits under the action of G_{ax} and G_{ay} respectively. □

We have now proved Theorem 15 (see Lemmas 9.1(ii), 9.2(ii),(iii), 9.6(i) and 13.2(i)).

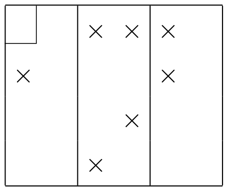
Lemma 13.3. *Let* $x \in \Delta_4^3(a)$ *and* $l \in \alpha_{1,2|0}(x, \mathrm{TRI}, \mathrm{OCT})$. *Then*

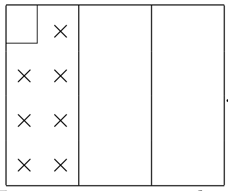
(i) $|\Gamma_0(l) \cap \Delta_4^6(a)| = 2$;

(ii) for $y \in \Gamma_0(l) \cap \Delta_4^6(a)$, $l \in \alpha_{1,1}(y, \mathrm{TRI}, \mathrm{FIX})$; *and*

(iii) for all $y \in \Delta_4^6(a)$ *and* $l \in \alpha_{1,1}(y, \mathrm{TRI}, \mathrm{FIX})$ *we have* $|\Gamma_0(l) \cap \Delta_4^3(a)| = 1$ *and* $|\Gamma_0(l) \cap \Delta_4^6(a)| = 2$.

Proof. Let $\Gamma_0(l) = \{x, y, y'\}$ and $d \in \Delta_1(x) \cap \Delta_3^2(a)$. With TRI and OCT as in (2.11) we may suppose that

$l =$ 

and, by Lemma 11.2(ii), that $x + d =$ .

By Theorems 5,6,7,8 and 10 and Lemmas 11.2 and 11.3(i) we have $y, y' \in \Delta_4(a)$. Lemma 3.11(ii) implies that there exists $e \in \{d, y\}^\perp$ with $e \in \Gamma_0(X(d, a))$ and thus $e \in \Delta_2(a) \cup \Delta_3^2(a) \cup \Delta_3^3(a)$ by Theorem 6. If $e \in \Delta_2(a) \cup \Delta_3^2(a)$, then by Theorem 6 again and Lemma 3.10, y or y' is collinear with a point in $\Delta_2(a)$. Since $y, y' \in \Delta_4(a)$ this is impossible, whence $e \in \Delta_3^3(a)$. If $\Gamma_0(d+e) = \{d, e', e\}$, then $e' \in \Delta_3^3(a)$ by Lemma 4.15(ii). Appealing to Theorem 7 and Lemma 3.10 yields y, $y' \in \Delta_4^4(a) \cup \Delta_4^6(a)$. If, say, $y \in \Delta_4^4(a)$, then Theorem 14 implies that $y + x \in \alpha_5(y, \mathrm{END}, -)$. But then Lemma 12.8(ii),(iii) forces $x + y = l \in \alpha_{2,1|1}(x, \mathrm{TRI}, \mathrm{OCT})$, whereas $l \in \alpha_{1,2|0}(x, \mathrm{TRI}, \mathrm{OCT})$ by assumption. Thus $y \notin \Delta_4^4(a)$ and likewise $y' \notin \Delta_4^4(a)$. Hence $y, y' \in \Delta_4^6(a)$, so giving part (i).

For part (ii), by (2.11) and Lemmas 11.2, 11.3(ii), 12.8(iii) and 13.2 together with part (i),

$$|\Delta_1(x) \cap \Delta_4^6(a)| = 192.$$

Therefore

$$|\Delta_1(y) \cap \Delta_4^3(a)| = \frac{|\Delta_4^3(a)|.192}{|\Delta_4^6(a)|} = \frac{2^{12}.5.7.11.23.192}{2^{15}.7.11.23} = 120$$

by Lemmas 11.1(i) and 12.2(ii). Using Lemmas 12.2(iv), 12.4(iii), 12.7(i), 12.8(i), 12.10(i) and 12.11(i) together with (2.14) we must have

$$\{y + z \mid z \in \Delta_4^3(a)\} = \alpha_{2,2}(y, \mathrm{TRI}, \mathrm{FIX}) \cup \alpha_{1,1}(y, \mathrm{TRI}, \mathrm{FIX}).$$

Since $l \notin \alpha_{2,2}(y, \mathrm{TRI}, \mathrm{FIX})$ by Lemma 12.8(iii) we conclude that $l \in \alpha_{1,1}(y, \mathrm{TRI}, \mathrm{FIX})$.

Part (iii) follows from (i) and (ii) because $\Delta_4^6(a)$ is a G_a-orbit. □

We have now verified Theorem 13 (see Lemmas 11.2, 12.8(i),(iii), 12.9(i), 13.2(iii) and 13.3(i)).

We turn our attention to the final pair of line orbits still to be considered, emanating from points in $\Delta_3^5(a)$ and $\Delta_4^6(a)$.

Lemma 13.4. *Let* $x \in \Delta_3^5(a)$ *and* $l \in \alpha_1(x, x+c, -)$ *where* $\{c\} = \Delta_1(x) \cap \Delta_2^2(a)$. *Then* $\Gamma_0(l) \subseteq \Delta_3^5(a) \cup \Delta_4^6(a)$.

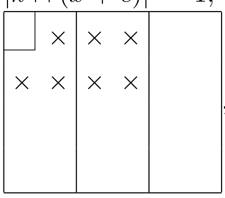
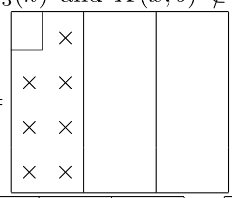
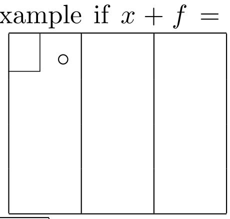
Proof. This follows from the point distributions given in Theorems 2–8 and 10–15 together with Lemmas 6.8(ii), 9.2(iii), 9.6(ii) and 12.9(ii). □

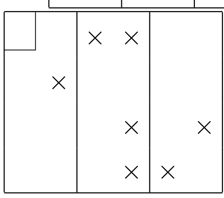
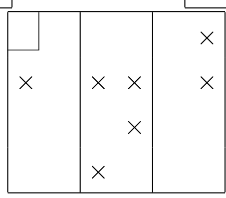
Lemma 13.5. *Let* $x \in \Delta_3^5(a)$ *and* $l \in \alpha_1(x, x+c, -)$ *where* $\{c\} = \Delta_1(x) \cap \Delta_2^2(a)$. *Then*

(i) $|\Gamma_0(l) \cap \Delta_4^6(a)| = 2$*; and*

(ii) for $y \in \Gamma_0(l) \cap \Delta_4^6(a)$, $l \in \alpha_{2,0}(y, \mathrm{TRI}, \mathrm{FIX})$.

Proof. We first show that $\Gamma_0(l) \cap \Delta_4^6(a) \neq \emptyset$. Let $f \in \Delta_1(x) \cap \Delta_4^1(a)$ with $x+f \in \alpha_3(x, x+c, +)$ (such a point f exists by Lemma 6.7). Then Lemmas 5.5 and 6.8 imply that $f+x \in \alpha_{1,1}(f, f+d, \mathrm{DUAD})$ where $\{d\} = \Delta_1(f) \cap \Delta_3^1(a)$. Let X be the element of Ω_f in $\mathrm{DUAD} \cap (f+x)$. Then $X \in \Gamma_3(f+x) \backslash \Gamma_3(f+d)$ by (2.9). Let $\{b\} = \Delta_1(a) \cap \Delta_2^2(x)$ and $\{b'\} = \Delta_1(a) \cap \Delta_3^1(f)$. Since $\mathrm{DUAD} = D(f, b')$, we have $X \in \Gamma_3(b')$. We now show that $X \neq X(x, b)$. Assume $X = X(x, b)$. Then $b' = b$ otherwise Lemma 3.6 implies that $a \in \Gamma_0(X)$, contrary to the fact that $x \in \Delta_4(a)$. Therefore $f \in \Delta_3^1(b)$ and $x \in \Delta_1(f) \cap \Delta_2^2(b)$. This is impossible by Theorem 5 and so we conclude that $X \neq X(x, b)$.

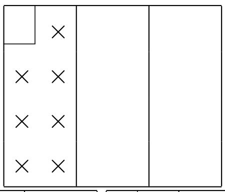
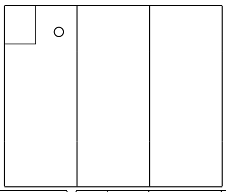
Thus, in Ω_x we may choose a heptad $k \in \Gamma_1(x)$ with $|k \cap (x+f)| = 3$, $|k \cap (x+c)| = 1$, $X \in \Gamma_3(k)$ and $X(x, b) \notin \Gamma_3(k)$. For example if $x+f =$ , $x+c =$  and $X(x, b) =$ 

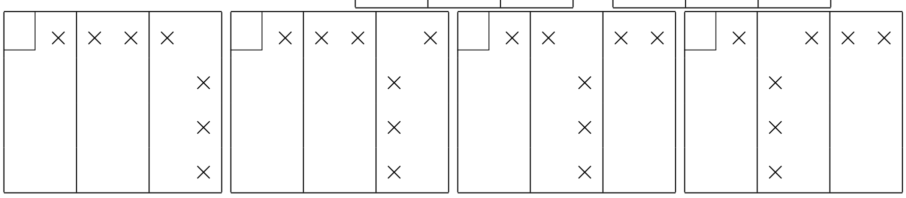
we may choose k to be  or , depending on which element of $(x+f)\backslash\{X(x,b)\}$ is X. By (2.7), $k \in \alpha_1(x, x+c, -)$. Let T be the unique triad incident with $x+f$ and k. Then $T \in \Gamma_2(X)$. Let $k' \in \Gamma_1(f,T)$ be such that , in Ω_f, DUAD lies in the heptad k' (k' exists by Lemma 3.11(ii) because X lies in (DUAD$\cap T$) in Ω_f). So $k' \in \alpha_{i,2}(f, f+d, \mathrm{DUAD})$ for $i = 1$ or 3. Appealing to Lemma 6.6(i),(iv), there exists $z' \in \Gamma_0(k')$ with $z' \in \Delta_3^2(a) \cup \Delta_3^6(a)$. Furthermore $z' \in \Delta_1(z)$ for some $z \in \Gamma_0(k)$ by Lemma 3.10. Using Lemma 13.4 together with Lemma 4.15 yields $z \in \Delta_4^6(a)$. Since $\alpha_1(x, x+c, -)$ is a G_{ax}-orbit of $\Gamma_1(x)$, we have proved that $\Gamma_0(l) \cap \Delta_4^6(a) \neq \emptyset$.

Let $y \in \Gamma_0(l) \cap \Delta_4^6(a)$. Then Lemmas 12.2(iv), 12.4(iii), 12.7(i), 12.8(i), 12.10(i), 12.11(i) and 13.3(iii) imply that $l \in \alpha_{2,0}(y, \mathrm{TRI}, \mathrm{FIX})$. We have $|\Delta_1(x) \cap \Delta_4^6(a)| = 40 + 96n$ for $n = 1$ or 2 and $|\Delta_1(y) \cap \Delta_3^5(a)| = 15 + 18m$ for $m = 1$ or 2 by (2.7), (2.14), Lemmas 5.7, 12.11(i) and the fact that $\Gamma_0(l) \cap \Delta_3^5(a) \neq \emptyset$ and $\Gamma_0(l) \cap \Delta_4^6(a) \neq \emptyset$. Taking this together with the orbit sizes of $\Delta_3^5(a)$ and $\Delta_4^6(a)$ given in Lemmas 4.11(i) and 12.2(ii) yields $n = 1$ and $m = 2$. Therefore $|\Gamma_0(l) \cap \Delta_3^5(a)| = 2$ and $|\Gamma_0(l) \cap \Delta_4^6(a)| = 1$ as required. This completes the proof of the lemma. □

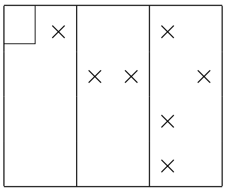
Thus we have now proved Theorems 9 and 16 (see Theorem 4.13(vii) and Lemmas 5.7, 6.7, 9.5 and 13.5(i) for Theorem 9; and Lemmas 12.2(iv), 12.4(iii), 12.7(i), 12.8(i), 12.10(i), 12.11(i), 13.3(iii) and 13.5(i),(ii) for Theorem 16).

Appendix A

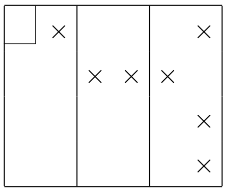
We list the 16 heptads meeing  in  :-



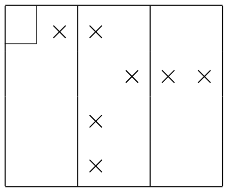
h_1 h_2 h_3 h_4



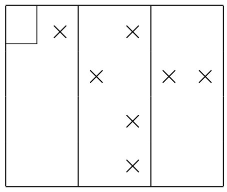
h_5



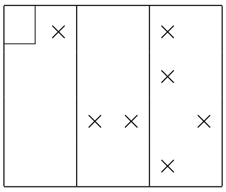
h_6



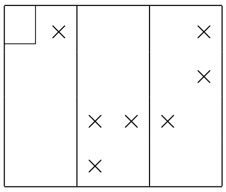
h_7



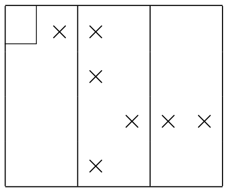
h_8



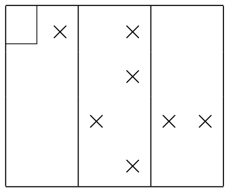
h_9



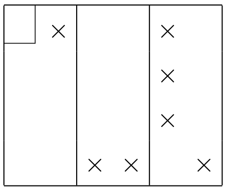
h_{10}



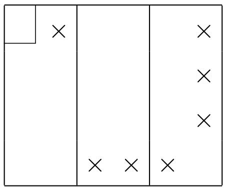
h_{11}



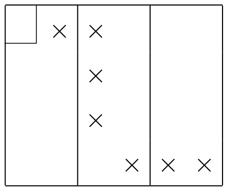
h_{12}



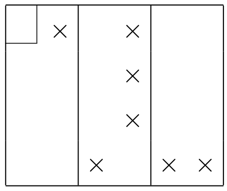
h_{13}



h_{14}



h_{15}



h_{16}

Appendix B

Here we list the results in which we determine the point distribution of the G_{ax}-orbits on lines in $\Gamma_1(x)$ for $x \in \Delta_3(a) \cup \Delta_4(a)$.

(1) $x \in \Delta_3^1(a)$

$\alpha_2(x, D(x,a))$	Theorem 4.13(iii)
$\alpha_1(x, D(x,a))$	Lemma 4.15(i)
$\alpha_0(x, D(x,a))$	(2.15)(ix) and Lemma 3.2

(2) $x \in \Delta_3^2(a)$

$\alpha_{0,1}(x, O(x,a), X(x,a))$	Theorem 4.13(iv)
$\alpha_{0,0}(x, O(x,a), X(x,a))$	(2.15)(x) and Lemma 3.2
$\alpha_{4,1}(x, O(x,a), X(x,a))$	Theorem 4.13(iv)
$\alpha_{2,1}(x, O(x,a), X(x,a))$	Lemma 4.15(ii)
$\alpha_{4,0}(x, O(x,a), X(x,a))$	(2.15)(xi) and Lemma 3.2
$\alpha_{2,0}(x, O(x,a), X(x,a))$	Lemma 6.6(i),(ii)

(3) $x \in \Delta_3^3(a)$

$\{x+b\}$	Theorem 4.13(v)
$\alpha_{1,1}(x, x+b, X(x,a))$	Lemma 4.13(i)
$\alpha_{3,1}(x, x+b, X(x,a))$	Lemma 4.13(ii)
$\alpha_{3,0}(x, x+b, X(x,a))$	(2.15)(xiv) and Lemma 3.2
$\alpha_{1,0}(x, x+b, X(x,a))$	(2.15)(xii) and Lemma 3.2

(4) $x \in \Delta_3^4(a)$

$\alpha_1(x, X(x,a))$	Theorem 4.13(vi)
$\alpha_0(x, X(x,a))$	(2.15)(xiii) and Lemma 3.2

(5) $x \in \Delta_3^5(a)$

$\{x+b\}$	Theorem 4.13(vii)
$\alpha_1(x, x+b, +)$	Lemma 9.5
$\alpha_3^{(1)}(x, x+b, -)$	Lemma 5.7(i)
$\alpha_3^{(2)}(x, x+b, -)$	Lemma 5.7(ii)
$\alpha_3(x, x+b, +)$	Lemma 6.7
$\alpha_1(x, x+b, -)$	Lemma 13.5(i)

(6) $x \in \Delta_3^6(a)$

$\{x+b\}$	Theorem 4.13(viii)
$\alpha_{3,3}(x, x+b, \mathrm{TRI})$	Lemma 4.10 and Theorem 4.13(viii)
$\alpha_{3,0}(x, x+b, \mathrm{TRI})$	Lemma 10.2(ii),(iii)
$\alpha_{1,1}(x, x+b, \mathrm{TRI})$	Lemma 10.5
$\alpha_{3,2}(x, x+b, \mathrm{TRI})$	Lemma 10.6
$\alpha_{1,0}(x, x+b, \mathrm{TRI})$	Lemma 10.3
$\alpha_{3,1}(x, x+b, \mathrm{TRI})$	Lemma 10.4(i)

(7) $x \in \Delta_4^1(a)$

$\{x+b\}$	Lemma 6.1
$\alpha_{3,2}(x, x+b, \mathrm{DUAD})$	Lemma 6.6(i)
$\alpha_{1,2}(x, x+b, \mathrm{DUAD})$	Lemma 6.6(iv)
$\alpha_{3,0}^{\mathcal{L}}(x, x+b, \mathrm{DUAD})$	Lemma 6.6(iii)
$\alpha_{1,1}(x, x+b, \mathrm{DUAD})$	Lemma 6.8(i)
$\alpha_{1,0}(x, x+b, \mathrm{DUAD})$	Lemma 12.4(i)
$\alpha_{3,0}^{\mathcal{L}^c}(x, x+b, \mathrm{DUAD})$	Lemma 8.10(i)
$\alpha_{3,1}(x, x+b, \mathrm{DUAD})$	Lemma 8.10(ii)

(8) $x \in \Delta_4^2(a)$

$\alpha_0(x, O(x,a))$	Corollary 7.4
$\alpha_4(x, O(x,a))$	Lemma 7.5
$\alpha_2(x, O(x,a))$	Lemma 8.12

(9) $x \in \Delta_4^3(a)$

$\alpha_{3,4\|0}(x, \mathrm{TRI}, \mathrm{OCT})$	Lemma 11.2(i)
$\alpha_{3,0}(x, \mathrm{TRI}, \mathrm{OCT})$	Lemma 11.2(ii)
$\alpha_{1,0}(x, \mathrm{TRI}, \mathrm{OCT})$	Lemma 11.2(iii)
$\alpha_{0,3\|1}(x, \mathrm{TRI}, \mathrm{OCT})$	Lemma 13.2(iii)
$\alpha_{1,2\|2}(x, \mathrm{TRI}, \mathrm{OCT})$	Lemma 11.2(iv)
$\alpha_{0,1\|1}(x, \mathrm{TRI}, \mathrm{OCT})$	Lemma 12.9(i)
$\alpha_{2,1\|1}(x, \mathrm{TRI}, \mathrm{OCT})$	Lemma 12.8(i),(iii)
$\alpha_{1,2\|0}(x, \mathrm{TRI}, \mathrm{OCT})$	Lemma 13.3(i)

(10) $x \in \Delta_4^4(a)$

$\alpha_1(x, \mathrm{END}, +)$	Lemma 8.13
$\alpha_1(x, \mathrm{END}, -)$	Lemma 8.8
$\alpha_5(x, \mathrm{END}, +)$	Lemma 9.6(i),(iii)
$\alpha_3(x, \mathrm{END}, +)$	Lemma 8.9(ii)
$\alpha_5(x, \mathrm{END}, -)$	Lemma 12.6
$\alpha_3(x, \mathrm{END}, -)$	Lemma 8.9(i)

(11) $x \in \Delta_4^5(a)$

$\{x+b\}$	Lemma 9.1(ii)
$\alpha_3(x, x+b, +)$	Lemma 9.2(ii)
$\alpha_1(x, x+b, +)$	Lemma 9.6(i)
$\alpha_1(x, x+b, -)$	Lemma 13.2(i)
$\alpha_3(x, x+b, -)$	Lemma 9.2(iii)

(12) $x \in \Delta_4^6(a)$

$\alpha_{0,4}(x, \mathrm{TRI}, \mathrm{FIX})$	Lemma 12.2(iv)
$\alpha_{3,1}(x, \mathrm{TRI}, \mathrm{FIX})$	Lemma 12.10(i)
$\alpha_{0,0}(x, \mathrm{TRI}, \mathrm{FIX})$	Lemma 12.11(i)
$\alpha_{2,0}(x, \mathrm{TRI}, \mathrm{FIX})$	Lemma 13.5(i),(ii)
$\alpha_{1,3}(x, \mathrm{TRI}, \mathrm{FIX})$	Lemma 12.7(i)
$\alpha_{2,2}(x, \mathrm{TRI}, \mathrm{FIX})$	Lemma 12.8(i)
$\alpha_{0,2}(x, \mathrm{TRI}, \mathrm{FIX})$	Lemma 12.4(iii)
$\alpha_{1,1}(x, \mathrm{TRI}, \mathrm{FIX})$	Lemma 13.3(iii)

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