The n-valued groups: results and prospectives

Buchstaber, Victor M.

2005

MIMS EPrint: **2005.43**

ISSN 1749-9097

The n-valued groups: results and prospectives.

Victor M. Buchstaber

Steklov Mathematical Institute,
Russian Academy of Sciences,
Moscow

School of Mathematics,
University of Manchester,
Manchester

Abstract.

We shall give a survey of the most important results of the theory of n-valued groups and their applications. Main directions of advanced research will be discussed.

We start with the basic definitions. Further exposition follows a sequence of instructive examples that originated from various branches of Mathematics: Topology, Analysis, Algebra, and Dynamical Systems.

The talk will be accessible to a broad audience.

Introduction.

In various fields of research one encounters a natural multiplication on a space, say, X under which the product of a pair of points is a subset of X (e.g., a finite subset).

The literature on multivalued groups and their applications is very large and includes titles from 19th century, mainly in the context of hypergroups.

In 1971 S. Novikov and the author introduced a construction, suggested by the theory of characteristic classes of vector bundles, in which the product of each pair of elements is **an n-multiset**, the set of n points with multiplicities.

The construction lead to the notion of n-**valued group**.

The condition of n-valuedness is in fact very strong, so, initially it seemed that the supply of interesting examples of n-valued groups is not very rich.

Soon after the author developed the theory of formal, or local, n-valued Lie groups, which appeared to be rich of contents and have found important applications.

Since 1990, E. Rees and the author collaborate
on the topological and algebraical theory of n-valued groups. The methods of the theory lead in particular
to the solution of the problem about the ring of functions on a symmetric powers of a space.

The theory of n-valued groups has seminal connections with a number of classic and modern fields of research.

Symmetric product of a space.

If X is a topological space, let $(X)^n$ denote its n-fold symmetric product, i.e.,
$(X)^n = X^n/\Sigma_n$ where
the symmetric group Σ_n acts by permuting the co-ordinates.

An element of $(X)^n$ is called an n-subset of X or just an n-set; it is a subset with multiplicities of total cardinality n.

Example. The spaces $(\mathbb{C})^n = \mathbb{C}^n/\Sigma_n$ and $\mathbb{C}^n$ are identified using the map

$$\mathcal{S} : \mathbb{C}^n \to \mathbb{C}^n$$

whose components are given by

$$(z_1, z_2, \ldots, z_n) \to e_r(z_1, z_2, \ldots, z_n),\ 1 \le r \le n,$$

where e_r is the rth elementary symmetric polynomial.

The **projectivisation** of the map $\mathcal{S}$ induces a homeomorphism between $(\mathbb{CP}^1)^n$ and $\mathbb{CP}^n$.

n-valued group structure.

An n-**valued multiplication** on X is a map

$$\mu : X \times X \to (X)^n.$$

$\mu(x, y) = x * y = [z_1, z_2, \dots, z_n], \quad z_k = (x * y)_k.$

Associativity. The n^2-sets:

$$[x * (y * z)_1, x * (y * z)_2, \dots, x * (y * z)_n],$$
$$[(x * y)_1 * z, (x * y)_2 * z, \dots, (x * y)_n * z]$$

are equal for all $x, y, z \in X$.

Unit. An element $e \in X$ such that

$$e * x = x * e = [x, x, \dots, x]$$

for all $x \in X$.

Inverse. A map $\mathrm{inv}\colon X \to X$ such that

$$e \in \mathrm{inv}(x) * x \text{ and } e \in x * \mathrm{inv}(x)$$

for all $x \in X$.

The map μ defines an n-valued group structure on X
if it is associative, has a unit and an inverse.

First results

Lemma. For each $m \in \mathbb{N}$, an n-valued group on X, with the multiplication μ, can be regarded as
an mn-valued group by using as the multiplication
the composition

$$X \times X \xrightarrow{\mu} (X)^n \xrightarrow{(D)^m} (X)^{mn},$$

where D is diagonal.

Definition. A map $f\colon X \to Y$ is **a homomorphism**
of n-valued groups if

$$f(e_X) = e_Y, \quad f(\mathrm{inv}_X(x)) = \mathrm{inv}_Y(f(x)))$$

for all $x \in X$ and

$$\mu_Y(f(x), f(y)) = (f)^n \mu_X(x, y)$$

for all $x, y \in X$.

Lemma. Let $f\colon X \to Y$ be a homomorphism of n-valued groups. Then

$$\mathrm{Ker}(f) = \{x \in X \mid f(x) = e_Y\}$$

is an n-valued group.

A 2-valued group structure on $\mathbb{Z}_+$.

Consider the semigroup of nonnegative integers $\mathbb{Z}_+$.

Define the multiplication

$$\mu\colon \mathbb{Z}_+ \times \mathbb{Z}_+ \to (\mathbb{Z}_+)^2$$

by the formula

$$x * y = [x + y, |x - y|].$$

The unit: $e = 0$.

The inverse: $\mathrm{inv}(x) = x$.

The associativity:

one has to verify that the 4-subsets of $\mathbb{Z}_+$

$$[x + y + z, |x - y - z|, x + |y - z|, |x - |y - z||]$$

and

$$[x + y + z, |x + y - z|, |x - y| + z, ||x - y| - z|]$$

are equal for all nonnegative integers x, y, z.

Additive n-valued group structure on $\mathbb{C}$.

Define the multiplication

$$\mu\colon \mathbb{C}\times\mathbb{C}\to(\mathbb{C})^n$$

by the formula

$$x*y=[(\sqrt[n]{x}+\epsilon^r\sqrt[n]{y})^n,\quad 1\le r\le n],$$

where ϵ is a primitive nth root of unity.

The unit: $e=0$.

The inverse: $\mathrm{inv}(x)=(-1)^n x$.

The multiplication is described by the polynomials

$$p_n=\prod_{k=1}^{n}(z-(\mathrm{inv}(x)*\mathrm{inv}(y))_k).$$

It turns out that p_n are **symmetric** in x,y,z. For instance,

$$p_1=x+y+z,\quad p_2=(x+y+z)^2-4(xy+yz+zx).$$

For small n:

$$p_3 = \mathsf{e}_1^3 - 3^3\,\mathsf{e}_3,$$

$$p_4 = \mathsf{e}_1^4 - 2^3\,\mathsf{e}_1^2\,\mathsf{e}_2 + 2^4\,\mathsf{e}_2^2 - 2^7\,\mathsf{e}_1\,\mathsf{e}_3,$$

$$p_5 = \mathsf{e}_1^5 - 5^4\,\mathsf{e}_1^2\,\mathsf{e}_3 + 5^5\,\mathsf{e}_2\,\mathsf{e}_3,$$

$$\begin{aligned} p_6 = {} & \mathsf{e}_1^6 - 2^2\cdot 3\,\mathsf{e}_1^4\,\mathsf{e}_2 + 2^4\cdot 3\,\mathsf{e}_1^2\,\mathsf{e}_2^2 - 2^6\,\mathsf{e}_2^3 \\ & - 2\cdot 3^4\cdot 17\,\mathsf{e}_1^3\,\mathsf{e}_3 - 2^3\cdot 3^4\cdot 19\,\mathsf{e}_1\,\mathsf{e}_2\,\mathsf{e}_3 \\ & + 3^3\cdot 19^3\,\mathsf{e}_3^3, \end{aligned}$$

$$\begin{aligned} p_7 = {} & \mathsf{e}_1^7 - 5\cdot 7^4\,\mathsf{e}_1^4\,\mathsf{e}_3 + 2\cdot 7^6\,\mathsf{e}_1^2\,\mathsf{e}_2\,\mathsf{e}_3 - 7^7\,\mathsf{e}_2^2\,\mathsf{e}_3 \\ & + 7^8\,\mathsf{e}_1\,\mathsf{e}_3^2, \end{aligned}$$

where

$$\begin{aligned} \mathsf{e}_1 &= x + y + z, \\ \mathsf{e}_2 &= xy + yz + zx, \\ \mathsf{e}_3 &= xyz. \end{aligned}$$

Coset groups.

Let G be a (1-valued) group with the multiplication μ_0.

Let A be a group of automorphisms of G with $\#A = n$.

Let $\pi\colon G \to X$ be the quotient map. Define

$$\mu\colon X \times X \to (X)^n$$

by the formula

$$\mu(x, y) = [\pi(\mu_0(u, v^{a_i})], \quad 1 \leq i \leq n, \quad a_i \in A,$$

where $u \in \pi^{-1}(x)$ and $v \in \pi^{-1}(y)$.

Theorem. μ defines an n-valued group structure
on the orbit space $X = G/A$ called **a coset group** with:

the unit $e_X = \pi(e_G)$

the inverse

$$\mathsf{inv}_X(x) = \pi(\mathsf{inv}_G(u)),$$

where $u \in \pi^{-1}(x)$.

Examples of the coset groups.

(1) The 2-valued group on $\mathbb{Z}_+$.

(2) The additive n-valued group on $\mathbb{C}$.

(3) Let G be the infinite dihedral group

$$G = \{a, b \mid a^2 = b^2 = e\}.$$

The interchange of a and b generates the automorphism group A, $\#A = 2$. Then

$$X = G/A = \{u_{2n}, u_{2n+1}\}, \quad n \geq 0,$$

where

$$u_{2n} = \{(ab)^n, (ba)^n\},$$
$$u_{2n+1} = \{b(ab)^n, a(ba)^n\}.$$

Then the multiplication is given by the formula

$$u_k * u_\ell = [u_{k+\ell}, u_{|k-\ell|}].$$

Thus X is isomorphic to the 2-valued group on $\mathbb{Z}_+$.

(4) Let G be a finite group, $\#G = n$.
Let $A = G$ acts by inner automorphisms

$$g^a = a^{-1} g a, \quad g \in G,\ a \in A.$$

We have

$$X = G/A.$$

Thus the **set of characters** of G is an n-valued
coset group.

Consider $G = \Sigma_3$.
Then $X = \{e, x_1, x_2\}$ is a 6-valued group:

$$x_1 * x_1 = [e, e, e, x_1, x_1, x_1],$$
$$x_1 * x_2 = x_2 * x_1 = [x_2, x_2, x_2, x_2, x_2, x_2],$$
$$x_2 * x_2 = [e, e, x_1, x_1, x_1, x_1].$$

Note, that this 6-valued group on three elements
is impossible to reduce to a group of lesser multiplicity.

The n-valued deformations of a finite group.

Let G be a finite group, $\#G = m$.
Denote by X the set of elements of G:

$$X = \{x_0 = e, x_1, \dots, x_{m-1}\}; \quad X^0 = X \backslash e.$$

Lemma. Let $\ell \in \mathbb{N}$ and $k \in \mathbb{Z}_+$.
Set $n = \ell + k(m-1)$.
Using the group operation $x_i x_j \in G$, define

$$x_i * x_j = [\underbrace{x_i x_j, \dots, x_i x_j}_{\ell}, \underbrace{X^0, \dots, X^0}_{k}].$$

Then we obtain an n-valued group on X, with unit e and $\mathrm{inv}(x_j) = x_j^{-1}$, which is denoted by $X(G, \ell, k)$
and called **the n-valued deformation of G**.

Clearly, $X(G, 1, 0) = G$.
Also note that $X(G, r\ell, rk)$ is obtained from $X(G, \ell, k)$ by diagonal map $(D)^r$.

Example. $X(\mathbb{Z}_3, 1, 1)$ is the 3-valued **coset** group $\mathbb{Z}_7/A$. Here A is generated by multiplication by 2
on $\mathbb{Z}/7$ and $\#A = 3$.

A family of non-coset groups.

Consider the $(2k+1)$-valued group $X(\mathbb{Z}_3, 1, k)$.

The multiplication is given by the formulae

$$x_1 * x_1 = [\underbrace{x_1, \ldots, x_1}_{k}, \underbrace{x_2, \ldots, x_2}_{k+1}],$$

$$x_1 * x_2 = x_2 * x_1 = [e, \underbrace{x_1, \ldots, x_1}_{k}, \underbrace{x_2, \ldots, x_2}_{k}],$$

$$x_2 * x_2 = [\underbrace{x_1, \ldots, x_1}_{k+1}, \underbrace{x_2, \ldots, x_2}_{k}].$$

Theorem. Suppose that $4k+3 = p\,q$, where $q > p$ are prime numbers. Then the above $(2k+1)$-valued group is **non-coset**.

Note, that any pair of **twin primes**, like $(3,5), (5,7), (11,13), (17,19), \ldots$, defines a non-coset group from the family.

Proof.

Suppose $X(\mathbb{Z}_3, 1, k) = G/A$ with $\#A = 2k + 1$.

Since the $(2k+1)$-set $x_i * x_i$, $i = 1, 2$, does not contain e, the orbit $\pi^{-1}(x_i)$ does not contain simultaneously g and g^{-1} for all $g \in G \backslash e$.

Since the $(2k+1)$-set $x_1 * x_2$ contains only one e,
all elements of the orbit $\pi^{-1}(x_i)$ have multiplicity one.

Thus

$\#G = 1 + (2k+1) + (2k+1) = 4k + 3 = p\,q$.

Since $q > p$, by Sylow theorem, the q-subgroup of G
is **normal and invariant** with respect to all automorphisms of G.

Thus, $q - 1 = 2k + 1$, which implies that q is even.
The contradiction.

Local 2-valued groups on $\mathbb{C}$.

Consider the equation

$$z^2 - \Theta_1(x,y)z + \Theta_2(x,y) = 0$$

with respect to z, where $\Theta_1(x,y)$ and $\Theta_2(x,y)$ are formal series in (x,y) near $(0,0) \in \mathbb{C} \times \mathbb{C}$.

Suppose:

(1) $\Theta_1(x,0) = 2x$ and $\Theta_2(x,0) = x^2$.

(2) There exists a series $\varphi(x)$ such that

$$\Theta_2(x, \varphi(x)) = 0.$$

(3) Let $z_+(x,y)$ and $z_-(x,y)$ be symbols such that

$$z_+(x,y) + z_-(x,y) = \Theta_1(x,y),$$
$$z_+(x,y) \cdot z_-(x,y) = \Theta_2(x,y).$$

Set $X_\pm = z_\pm(u,v)$ and $Y_\pm = z_\pm(v,w)$.

(3 continued)
The following equalities of formal series in u, v, w hold:

$$\Theta_1(u, Y_+) + \Theta_1(u, Y_-) = \Theta_1(X_+, w) + \Theta_1(X_-, w);$$

$$\Theta_2(u, Y_+)+\Theta_2(u, Y_-)+\Theta_1(u, Y_+)\Theta_1(u, Y_-) = \Theta_2(X_+, w)+\Theta_2(X_-, w)+\Theta_1(X_+, w)\Theta_1(X_-, w);$$

$$\Theta_2(u, Y_+)\Theta_1(u, Y_-) + \Theta_1(u, Y_+)\Theta_2(u, Y_-) = \Theta_2(X_+, w)\Theta_1(X_-, w)+\Theta_1(X_+, w)\Theta_2(X_-, w);$$

$$\Theta_2(u, Y_+)\Theta_2(u, Y_-) = \Theta_2(X_+, w)\Theta_2(X_-, w).$$

Definition. When the conditions 1–3 are satisfied,
we say that the equation

$$z^2 - \Theta_1(x, y)z + \Theta_2(x, y) = 0$$

defines a **formal** 2-**valued group law on** $\mathbb{C}$.

Relation to the classic formal group law.

Suppose that the equation

$$z - F(x, y) = 0,$$

where $F(x, y)$ is a formal series, defines the classic formal group law.

This means that the series $F(x, y)$ satisfies the conditions:

(1) $F(x, 0) = x$.

(2) $F(x, F(y, z)) = F(F(x, y), z)$.

Then the equation

$$z^2 - 2F(x, y)z + F(x, y)^2 = 0$$

defines a formal 2-valued group law on $\mathbb{C}$.

Suppose the equation

$$(*) \qquad z^2 - \Theta_1(x,y)z + \Theta_2(x,y) = 0$$

defines a formal 2-valued group law on $\mathbb{C}$.

Theorem. If the series $z = z(x,y)$ is a solution of $(*)$ satisfying the condition
$z(x,0) = x$,
then

$$\Theta_1(x,y) = 2z(x,y), \quad \Theta_2(x,y) = z(x,y)^2,$$

and the equation

$$z - z(x,y) = 0$$

defines the classic formal group law.

The type of a 2-valued group.

Consider the 2-valued group law

$$z^2 - \Theta_1(x,y)z + \Theta_2(x,y) = 0.$$

Lemma. $\boxed{\left.\dfrac{\partial^2 \Theta_2(x,y)}{\partial x \partial y}\right|_{(0,0)} = \pm 2.}$

Definition. A formal 2-valued group is called a **1st type group** when

$$\left.\frac{\partial^2 \Theta_2(x,y)}{\partial x \partial y}\right|_{(0,0)} = -2;$$

and is called a **2nd type group** otherwise.

The elementary 2-valued groups.

1st type: $\boxed{z^2 - 2(x+y)z + (x-y)^2 = 0.}$

It is precisely the equation $p_2 = \mathrm{e}_1^2 - 4\mathrm{e}_2 = 0$.

$$z_\pm(x,y) = (\sqrt{x} \pm \sqrt{y})^2.$$

2nd type: $\boxed{(z - (x+y))^2 = 0.}$

$$z_\pm(x,y) = x + y.$$

The strong isomorphism.

Consider the 2-valued group laws

$$\text{(A)} \qquad z^2 - \Theta_1(x,y)z + \Theta_2(x,y) = 0,$$
$$\text{(B)} \qquad z^2 - \widehat{\Theta}_1(x,y)z + \widehat{\Theta}_2(x,y) = 0.$$

Definition. The group laws (A) and (B) are strongly isomorphic if there exists a power series

$$\psi(x) = x(1 + O(x))$$

(the *regular change of coordinate*) such that

$$z_\pm(x,y) = \psi^{-1}(\widehat{z}_\pm(\psi(x), \psi(y))).$$

Lemma. Strong isomorphism preserves the type.

Note, that an *irregular coordinate change*

$$\Psi(x) = x^2(1 + O(x))$$

takes a 1st type group law to the 2nd type.

The exponential and the logarithm of a 2-valued group.

Definition. The series

$$\psi(x) = x(1 + O(x))$$

defining the strong isomorphism of a 2-valued group with the elementary group is called **the logarithm of the group.**

The inverse series

$$\psi^{-1}(x) = x(1 + O(x)),$$

that is $\psi^{-1}(\psi(x)) = x$, is called **the exponential of the group.**

Theorem. Each 2-valued formal group on $\mathbb{C}$ has the logarithm.

Sketch of a proof.

Consider the differential operator

$$\mathcal{D}_x = \alpha_1(x)\frac{d}{dx} + \alpha_2(x)\frac{d^2}{dx^2},$$

where $\alpha_1(x), \alpha_2(x) \in \mathbb{C}[[x]]$ and $\alpha_1(0) = 1$, $\alpha_2(0) = 0$.

Lemma. Let $\varphi(x) \in \mathbb{C}[[x]]$.
Then the problems

$$\boxed{\begin{gathered} \mathcal{D}_x u(x,y) = \mathcal{D}_y u(x,y) \\ u(x,0) = \varphi(x) \end{gathered}} \qquad \boxed{\begin{gathered} \mathcal{D}_x u(x) = 1 \\ u(x) = 0 \end{gathered}}$$

have unique solutions

$$u(x,y) \in \mathbb{C}[[x,y]] \quad \text{and} \quad u(x) \in \mathbb{C}[[x]]$$

iff $\boxed{-\dfrac{1}{\alpha_2'(0)} \notin \mathbb{N}}$.

Generalized shift.

Suppose $-\dfrac{1}{\alpha_2'(0)} \notin \mathbb{N}$.

Then the linear map

$$T_x^y \colon \mathbb{C}[[x]] \to \mathbb{C}[[x,y]]$$

is defined by the formula

$$T_x^y \varphi(x) = u(x,y), \quad \text{where } u(x,0) = \varphi(x).$$

Lemma. The operator T_x^y is **a generalized shift**,
that is:

(1) the operator T_x^0 is identity;

(2) the operator

$$T_y^z T_x^y - T_x^y T_x^z \colon \mathbb{C}[[x]] \to \mathbb{C}[[x,y,z]]$$

is zero.

Consider the equation

$$(*) \qquad z^2 - Q_1(x,y)z + Q_2(x,y) = 0.$$

From the axioms of 2-valued group follows, that for $(*)$ to define a 2-valued group it is **necessary** that

$$Q_1(x,y) = 2(x+y) + \text{higher terms},$$
$$Q_2(x,y) = (x \pm y)^2 + \text{higher terms}.$$

Define $P_k(x,y)$, $k \in \mathbb{Z}_+$, by the generating function

$$\sum_{k\geq 0} \frac{P_k(x,y)}{t^{k+1}} = \frac{2t - Q_1(x,y)}{2(t^2 - Q_1(x,y)t + Q_2(x,y))}.$$

Introduce a linear map

$$L_x^y \colon \mathbb{C}[[x]] \to \mathbb{C}[[x,y]]$$

by the formula $L_x^y x^k = P_k(x,y)$.

Let $\alpha_1(x) = \phi_1(x)/2$ and $\alpha_2(x) = \phi_2(x)/8$, where

$$\phi_1(x) = \frac{\partial\, Q_1(x,y)}{\partial y}\bigg|_{y=0},$$

$$\phi_2(x) = \frac{\partial\,(Q_1(x,y)^2 - 4Q_2(x,y))}{\partial y}\bigg|_{y=0}.$$

By the above necessary conditions we have

$$\alpha_1(0) = 1, \quad \alpha_2(0) = 0,$$

$$\alpha_2'(0) = \begin{cases} 2, & \text{1st type} \\ 0, & \text{2nd type} \end{cases}$$

So, the generalized shift T_x^y is defined.

Theorem. If

$$Q_1(x,y) = 2(x+y) + \text{higher terms},$$
$$Q_2(x,y) = (x \pm y)^2 + \text{higher terms}.$$

Then $z^2 - Q_1(x,y)z + Q_2(x,y) = 0$ defines a 2-valued group iff

$$L_x^y x^k = T_x^y x^k$$

for $k = 1, 2, 3, 4$.

The 1st type case.

Consider the 1st type 2-valued group law

$$z^2 - \Theta_1(x,y)z + \Theta_2(x,y) = 0.$$

Lemma.

$$\phi_2(x) = 8\int\limits_0^x \phi_1(x)dx,$$

and $\quad \phi_2(0) = 0, \quad \phi_2'(0) = 16.$

Introduce $\Phi(x) = \dfrac{\phi_2(x)}{16x} = 1 + O(x).$

Theorem. The formula

$$\psi(x) = \Big(\int\limits_0^{\sqrt{x}} \frac{dt}{\sqrt{\Phi(t^2)}}\Big)^2$$

defines the series $\psi(x) = x(1 + O(x))$ such that

$$\mathcal{D}\psi(x) = 1.$$

The series $\psi(x)$ is **the logarithm**.

The 2nd type case.

Consider the 2nd type 2-valued group law

$$z^2 - \Theta_1(x,y)z + \Theta_2(x,y) = 0.$$

Theorem. The formula

$$\psi(x) = 2\int_0^x \frac{dt}{\phi_1(t)}$$

defines the series $\psi(x) = x(1 + O(x))$ such that

$$\mathcal{D}\psi(x) = 1.$$

The series $\psi(x)$ is **the logarithm**.

An algebraic 2-valued group structure on $\mathbb{C}$.

Consider the series

$$\varphi(x) = -\frac{1}{\wp(\sqrt{-x}; g_2, g_3)} = x(1 + O(x)),$$

where $\wp(z; g_2, g_3)$ is the Weierstrass elliptic function with the invariants g_2 and g_3.

Theorem. $\varphi(x)$ is **the exponential** of the 2-valued group on $\mathbb{C}$ defined by the equation

$$\left(x + y + z + \frac{g_2}{4}xyz\right)^2 - (4 + g_3xyz)(xy + yz + zx) = 0.$$

Action on a space

An n-valued group X **acts on a space** Y if there is a mapping

$$\phi\colon X \times Y \to (Y)^n,$$

also denoted $x \circ y = \phi(x, y)$,
such that the two n^2-subsets of Y

$$x_1 \circ (x_2 \circ y) \;\text{ and }\; (x_1 * x_2) \circ y$$

are equal for all $x_1, x_2 \in X$ and $y \in Y$;
and also

$$e \circ y = [y, y, \ldots, y]$$

for all $y \in Y$.

The coset construction of an action.

Let G be some (usual) group; and A a finite group
of automorphisms of G, $\#A = n$.

Suppose that G and A act on some space V such that

$$a(g(v)) = a(g)(a(v)),$$
$$a \in A, \quad g \in G, \quad v \in V$$

In other words; the action of G on V is **equivariant** with respect to the action of A on V and the diagonal action of A on $G \times V$.

Let us consider the canonical projections

$$\pi : G \to X = G/A \quad \text{and} \quad p : V \to Y = V/A.$$

As we know already X has the structure of an n-valued group.

Proposition. There is a natural action of an n-valued group X on the space Y.

Algebraic action

For a given action

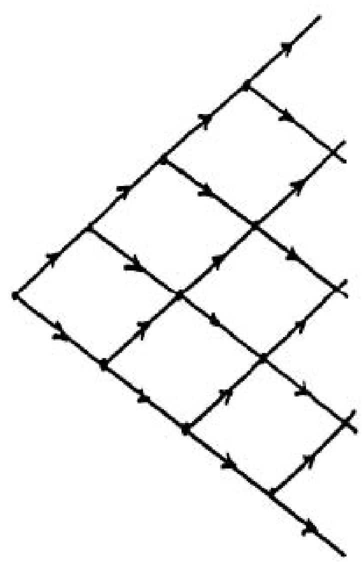
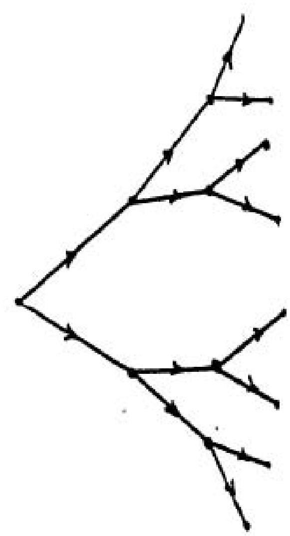
$$\phi\colon X \times Y \to (Y)^n,$$

define Γ_x, the **graph of the action** of an arbitrary element $x \in X$, as the subset of $Y \times Y$, which consists of the pairs (y_1, y_2) such that $y_2 \in \phi(x, y_1)$.

Definition. The action of an n-valued group X on an algebraic variety M is called **algebraic** if the action of any element of X is determined by an algebraic correspondence, i.e., its graph is an **algebraic subset** in $M \times M$.

Multivalued dynamics

Any equation $T(x, y) = 0$, where T is an order n polynomial in y defines an n-valued map (or a multivalued dynamics) $\mathbb{C} \to \mathbb{C}$ under which
x is taken to the set of roots $[y_1, y_2, \ldots, y_n]$ of $T(x, y)$.
In general case the number of different images
of a point grows exponentially with the number
of iterations of the map. In exceptional cases the growth is polynomial.
The following picture demonstrates the difference between exceptional and general situations.

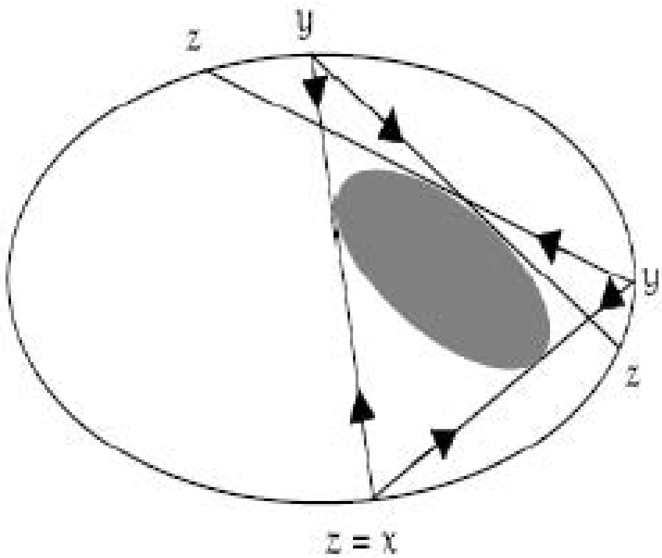
The Euler-Chasles correspondence.

The polynomial

$$T(x,y) = Ax^2y^2 + Bxy(x+y) \\ + C(x^2+y^2) + Dxy + E(x+y) + F.$$

defines the 2-valued dynamics, in which the number of different images after the kth iteration is $k+1$, but not 2^k as one could expect.

The picture explains this fact as the curve $T(x,y) = 0$ describes the geometric situation in the famous Poncelet porism for two conics on the plane.


z
y
y
z
z = x

It is known that for Euler-Chasles correspondence there exists an even elliptic function $f(z)$
of the degree 2, such that if $x = f(z)$ then $[y_1, y_2] = [f(z+a), f(z-a)]$ for some a.

This means that the Euler-Chasles correspondence
is the projection of the mapping $z \to z + a$
of the elliptic curve E into itself
to the projective line $\mathbb{CP}^1$ which is a coset space $E/\mathbb{Z}_2$, where $\mathbb{Z}_2$ is acting on E as $z \to -z$.

Thus, we have the representation of the two-valued group $\mathbb{Z}_+ = \mathbb{Z}/\mathbb{Z}_2$ with the multiplication

$$x * y = [x + y, |x - y|].$$

Theorem. All algebraic actions of the two-valued group $\mathbb{Z}_+$ on $\mathbb{CP}^1$ are generated either
by the Euler-Chasles correspondence
or by a reducible correspondence.

Main directions of advanced research.

- n-valued groups as deformations of usual groups;

- group algebras of n-valued groups as combinatorial algebras;

- n-Hopf algebras, their duals and n-quantum algebras;

- representations of n-valued groups on graphs;

- algebraic representations of n-valued groups;

- functional-algebraic theory of symmetric products of spaces.

References

[1] Buchstaber, V. M.; Novikov, S. P. Formal groups, power systems and Adams operators. (Russian) Mat. Sb. (N.S.) 84(126) (1971), 81–118.

[2] Bukhshtaber, V.M. Characteristic cobordism classes and topological applications of the theories of one-valued and two-valued formal groups. J. Sov. Math. 11, 815-921 (1979).

[3] Buchstaber, V. M. Functional equations that are associated with addition theorems for elliptic functions, and two-valued algebraic groups. Russian Math. Surveys 45 (1990), no. 3, 213–215

[4] Buchstaber, V. M.; Rees, E. G. Multivalued groups and Hopf n-algebras. Russian Math. Surveys 51 (1996), no. 4, 727–729.

[5] Buchstaber, V. M.; Veselov, A. P. Integrable correspondences and algebraic representations of multivalued groups. Internat. Math. Res. Notices 1996, no. 8, 381–400.

[6] Buchstaber, V. M.; Vershik, A. M.; Evdokimov, S. A.; Ponomarenko, I. N. Combinatorial algebras and multivalued involutive groups. Funct. Anal. Appl. 30 (1996), no. 3, 158–162 (1997)

[7] Buchstaber, V. M.; Rees, E. G. Multivalued groups, their representations and Hopf algebras. Transform. Groups 2 (1997), no. 4, 325–349.

[8] Buchstaber, V. M.; Rees, E. G. Multivalued groups, n-Hopf algebras and n-ring homomorphisms. Lie groups and Lie algebras, 85–107, Math. Appl., 433, Kluwer, 1998.

Further works

[9] Buchstaber, V. M.; Rees, E. G. The Gelfand map and symmetric products. Selecta Math. (N.S.) 8 (2002), no. 4, 523–535.

[10] Buchstaber, V. M.; Monastyrsky, M. I. Generalized Kramers-Wannier duality for spin systems with non-commutative symmetry. J. Phys. A 36 (2003), no. 28, 7679–7692.

[11] Buchstaber, V.M.; Rees, E.G. Rings of continuous functions, symmetric products, and Frobenius algebras. Russian Math. Survey 59, No.1, 125-145 (2004).

[12] Panov, T. E. On the structure of the 2-Hopf algebra in the cohomology of four-dimensional manifolds. Russian Math. Surveys 51 (1996), no. 1, 155–157.

[13] Yagodovskiĭ, P. V. Deformations of multivalued groups. Russian Math. Surveys 52 (1997), no. 3, 623–624.

[14] Yagodovskiĭ, P. V. Homogeneous deformations of discrete groups. J. Math. Sci. (N. Y.) 113 (2003), no. 5, 728–730.

[15] Yagodovskiĭ, P. V. Linear deformation of discrete groups, and constructions of multivalued groups. Izv. Math. 64 (2000), no. 5, 1065–1089

[16] Yagodovskiĭ, P. V. Representations of multivalued groups on graphs. Russian Math. Surveys 57 (2002), no. 1, 173–174

[17] Yagodovskiĭ, P. V. Bicoset groups and symmetric graphs. J. Math. Sci. (N. Y.) 126 (2005), no. 2, 1133–1139