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DEFINABLE ADDITIVE CATEGORIES

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Abstract. This is essentially the talk I gave on definable additive categories; I define these categories, say where they came from, describe some of what is around them and then point out the 2-category which they form.

Definable additive categories

Definable additive categories give a context, generalising that of the category of modules over a ring, in which much of the structure and many of the arguments familiar from modules are still available.

We say that a full subcategory $\mathcal{D}$ of the category Mod-R of (right) modules over a ring R is a **definable subcategory** if it is closed under direct products, direct limits and pure subobjects, where we say that an embedding $A \to B$ is **pure** if some ultrapower of it is split. There are many equivalent definitions of purity but we use this one because it needs only direct limits and products; let us recall this.

Let $(M_i)_{i\in I}$ be an indexed set of modules (more generally, of objects of a category with direct products and direct limits). An **filter** on the index set I is a set $\mathcal{F}$ of subsets of I such that $I \in \mathcal{F}$, $\emptyset \notin \mathcal{F}$, $\mathcal{F}$ is closed under finite intersections and, if $J \in \mathcal{F}$ and $J \subseteq J' \subseteq I$ then $J' \in \mathcal{F}$. The products $M_J = \prod_{i\in J} M_i$, with $J \in \mathcal{F}$, together with the canonical projection maps $M_{J'} \to M_J$, whenever $J \subseteq J'$, form a directed system and the direct limit of this system is the **reduced power**, $M^I/\mathcal{F}$, of the M_i with respect to $\mathcal{F}$. Clearly it is the quotient of the full product $M_I = \prod_{i\in I} M_i$ by the submodule consisting of those tuples $(m_i)_{i\in I}$ which are 0 on a set of indices belonging to $\mathcal{F}$. In the case that $\mathcal{F}$ is an **ultrafilter**, that is, a maximal filter, equivalently for every subset J of I either J or $I \setminus J$ is in $\mathcal{F}$, we use the term **ultraproduct**. In the case that all the component structures M_i are isomorphic, to M say, we refer to a **reduced power** and an **ultrapower**, and write $M^I/\mathcal{F}$.

A more down-to-earth definition of purity is that an inclusion $A \leq B$ between R-modules is pure if every finite system of R-linear equations $\sum_i x_i r_{ij} = a_j$ $(j = 1, \dots, m)$ with constants a_j from A, the $r_{ij} \in R$ and with a solution in B already has a solution in A. That these are equivalent follows because there is an index set I and an ultrafilter $\mathcal{F}$ on I such that for every R-module A, $A^I/\mathcal{F}$ is pure-injective (a result of Sabbagh, [22,

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Thm. 1]); and, for the other direction, use the fact that the embedding of a module into any of its reduced powers is pure.

The definition of purity *via* systems of linear equations might seem less widely applicable since it makes reference to the elements of the structures involved; nevertheless, definable categories always are realisable as definable subcategories of locally finitely presented abelian categories and it follows that "elementwise" definitions can be made and arguments using "elements" usually translate from modules to this generality. Of course in the general case, one loses some features, in particular the existence of enough projective objects.

To obtain the general notion of definable subcategory we replace the ring R by any skeletally small preadditive category $\mathcal{R}$: that is, $\mathcal{R}$ is a category with, up to isomorphism, only a set of objects and with each hom-set having an abelian group structure such that composition is bilinear. If $\mathcal{R}$ has just one object then it is a ring (the endomorphisms of the single object being the ring in the usual sense). We denote by Mod-$\mathcal{R}$ the abelian, Grothendieck category $(\mathcal{R}^{\mathrm{op}}, \mathbf{Ab})$ of contravariant additive functors from $\mathcal{R}$ to the category $\mathbf{Ab}$ of abelian groups. Then we say that a preadditive category $\mathcal{D}$ is **definable** if it is equivalent to a definable subcategory of Mod-$\mathcal{R}$, meaning a full subcategory closed under products, direct limits and pure subobjects.

We will see later that every definable category may be realised as a definable subcategory of Mod-$\mathcal{R}$ for some skeletally small abelian category $\mathcal{R}$; in this case Mod-$\mathcal{R}$ is locally coherent.

On the other hand, a preadditive category is said to be **finitely accessible** if it has direct limits, if every object is a direct limit of finitely presented objects and if there is, up to isomorphism, just a set of finitely presented objects (in this generality the definition of an object C being finitely presented is that the representable functor $(C, -)$ commutes with direct limits). If $\mathcal{C}$ is a finitely accessible additive category with products then we may make the same definition of "definable subcategory" but it can be seen that the collection of definable categories is not enlarged by this, because each such category $\mathcal{C}$ is itself a definable subcategory of a category of the form Mod-$\mathcal{R}$. One also needs the observation that a definable subcategory of a definable subcategory is, fairly immediately from the definitions (noting that purity has an "internal" definition), a definable subcategory. One also needs to note that if $\mathcal{D}$ is a definable subcategory of some category $\mathcal{C}$ then $\mathcal{D}$ has products and direct limits which coincide with the restrictions of those in $\mathcal{C}$ to $\mathcal{D}$.

We give some examples.

First consider the category Mod-$\mathbb{Z} = \mathbf{Ab}$ of abelian groups. Examples of definable subcategories are the category of torsionfree abelian groups (not an abelian category, note); the category of divisible abelian groups (a category with no finitely presented object apart from 0); the category of abelian groups of exponent bounded by some integer n. On the other hand the category of those abelian groups with bounded exponent is not definable since it does not have products. The category of torsion abelian groups *is* definable - it is a locally finitely presented abelian category, in particular an accessible additive category with products - but certainly not a definable subcategory of Mod-$\mathbb{Z}$. There are, in fact, uncountably many definable subcategories of Mod-$\mathbb{Z}$, essentially for the same reasons that we will see in the case of artin algebras of infinite representation type.

One of the first results about definable categories (though the terminology was different) was that of Eklof and Sabbagh [5, 3.16], showing that if R is a ring then the class of absolutely pure right modules is definable iff R is right coherent (and that the class of injective modules is definable iff R is right noetherian). A module M is **absolutely pure** if every embedding $M \to N$ into any R-module N is pure; equivalently M is **fp-injective**, meaning that it is injective over any embedding with finitely presented cokernel. The same authors showed [23, Thm. 4] that the class of flat *left* R-modules is definable iff R is right coherent and the class of projective left modules is definable iff R is left perfect and right coherent.

If M is a module of finite length over its endomorphism ring then Add(M), the category of direct summands of arbitrary direct sums of copies of M, is a definable subcategory (see, e.g. [19, 4.3.30].

Apart from module, and more general functor, categories, examples of definable categories include certain categories of comodules [4] (in particular the category of comodules over a k-coalgebra where k is a field) and certain categories of sheaves (in particular the category of sheaves of modules over a locally noetherian ringed space [20]).

A completely different route to these categories was noticed by Herzog (for modules) and Krause [14] (in general): they are precisely the categories (equivalent to one) of the form $\mathrm{Ex}(\mathcal{A}, \mathbf{Ab})$ where $\mathcal{A}$ is a skeletally small abelian category and $\mathrm{Ex}(\mathcal{A}, \mathcal{B})$ denotes the category of exact additive functors from $\mathcal{A}$ to $\mathcal{B}$ - the **exactly definable categories**.

Model theory and the Ziegler spectrum

If you look at the papers of Eklof and Sabbagh referred to then you will see, not the statements that certain categories are definable, rather that certain classes of modules are "elementary" - a notion belonging to mathematical logic but, in somewhat more algebraic terms, meaning closed under ultraproducts and elementary subobjects. "Elementary subobject" is not an algebraic concept but, in the context of modules, if the class is closed under direct summands (as those mentioned above are) then being closed under elementary subobjects is equivalent to being closed under pure submodules. Indeed, an alternative characterisation of definable subcategories is those subcategories closed under ultraproducts, pure submodules and (finite) direct sums.

The importance of definable categories for the model theory of modules (the fact that they are in some sense "typical" rather than just giving many important and nice examples) was, first, a consequence of the pp-elimination of quantifiers for modules and then was driven home by Ziegler's work [24] on the model theory of modules; in particular Ziegler showed that these categories are in bijection with the closed subsets of the topological space which he introduced. The pp-elimination of quantifiers theorem, due independently to Baur, Monk and others (see [16, p. 36] for references), says that formulas in the first-order model theory of modules reduce to statements which may be made using "pp formulas" - and these are just projections of systems of linear equations, so have clear algebraic meaning; in particular, formulas with many alternations of universal and existential quantifiers are not needed. This gives rise to another characterisation of definable subcategories: those which are defined by the equivalence of a set of pairs of pp formulas.

In Ziegler's paper, which built on earlier work of Garavaglia [8] but went far beyond and which transformed the landscape of the subject, a topology was defined on the set of indecomposable pure-injectives (by "indecomposable" we mean direct-sum indecomposable but not 0). A module N is **pure-injective** if N is injective over pure embeddings, equivalently if every pure embedding $N \to M$ into any module M is split. There is a structure theorem, which appears first (to my knowledge) in the work [6, 7.21] of Fisher, and which states that every pure-injective is the direct sum of a "discrete" part - the pure-injective hull of a direct sum of indecomposable pure-injectives - and a "superdecomposable" part - a module without any indecomposable direct summand. Ziegler showed that every module is elementarily equivalent to (satisfies the same first-order sentences as) some direct sum of indecomposable pure-injectives (it was known already from Sabbagh [22, Cor. 4 to Thm. 4] that every module is elementarily equivalent to its pure-injective hull). This meant that theories (in the technical model-theoretic sense) of modules were strongly reflected by the indecomposable pure-injective direct summands of their models. In particular this means that every definable subcategory is determined by the indecomposable pure-injective objects that it contains. So Ziegler defined a topology on the set, pinj_R, of isomorphism-types of indecomposable pure-injectives, as follows: a basis of open sets consists of the

- $(\phi/\psi) = \{N \in \mathrm{pinj}_R : \phi(N) > \psi(N)\}$ as ϕ/ψ ranges over pairs of pp formulas (i.e. projected systems of linear equations) such that $\phi \geq \psi$ in the sense that, in any module, the solution set to ϕ contains the solution set to ψ.

This does give a basis for a topology and the resulting space, denoted Zg_R, is termed the (right) **Ziegler spectrum** of R. Ziegler showed that the above sets are, provided one allows pp formulas with any finite number of free variables, exactly the compact open sets and there is a bijection between the closed subsets and the elementary classes closed under direct sums and direct summands (that is, the definable subcategories). Before describing this bijection we give alternative ways, which do not involve model theory, of defining the topology.

One makes use of an associated functor category: denote by mod-R the full subcategory of finitely presented modules and set Fun-$R = (\text{mod-}R, \mathbf{Ab})$ to be the category of additive functors from mod-R to $\mathbf{Ab}$. Denote by fun-$R = (\text{mod-}R, \mathbf{Ab})^{\text{fp}}$ the category of finitely presented such functors. For $F \in$ fun-R set:

• $(F) = \{N \in \text{pinj}_R : \overrightarrow{F}N \neq 0\}$ where $\overrightarrow{F}$ denotes the unique extension of F (a functor on finitely presented modules) to a functor on Mod-R which commutes with direct limits (since every module is a direct limit of finitely presented modules it is obvious how to define $\overrightarrow{F}$ and that it is well-defined is easy to check).

Then these sets, as F ranges over fun-R, give a basis for the same topology, indeed the same basis since, as was shown by Burke [2], the category of finitely presented functors is equivalent to the category of pp-pairs (given a pp formula ϕ the assignment $M \mapsto \phi(M)$ defines a functor which may be seen to be in fun-R, so pp pairs ϕ/ψ also define finitely presented functors, and conversely ([16, p. 251] or, better, [19, §10.2.5])).

Independently Auslander [1], and Gruson and Jensen [9] showed that there is a duality $d : (\text{mod-}R, \mathbf{Ab})^{\text{fp}} \simeq ((R\text{-mod}, \mathbf{Ab})^{\text{fp}})^{\text{op}}$, that is, fun-$R \simeq$ (fun-(R^{op}))$^{\text{op}}$. We use the notation fun$^{\text{d}}$-R for fun-(R^{op}). Note that this is a strong result: there is nothing like a duality between the categories of right and left modules (in general not even if we restrict to finitely presented modules) but if we step up one level of representation then we obtain a perfect duality. This allows another presentation of this basis of open sets:

• $(dF) = \{N \in \text{pinj}_R : (dF, N \otimes_R -) \neq 0\}$ where $N \otimes_R -$ is regarded as a functor from R-mod to $\mathbf{Ab}$, where dF ranges over arbitrary (by the duality) functors in fun$^{\text{d}}$-R. This follows immediately from the formula $(dF, N \otimes_R -) \simeq \overrightarrow{F}N$ [17] (see [19, 10.3.5]).

There is yet another presentation, pointed out by Crawley-Boevey [3], which makes use of neither model theory nor the functor category:

• $(f) = \{N \in \text{pinj}_R : (f, N) \text{ is not epi}\}$ where $f : A \to B$ ranges over morphisms in mod-R and $(f, N) : (B, N) \to (A, N)$ is the obvious map.

This follows from the fact that every finitely presented functor from mod-R to $\mathbf{Ab}$ has a presentation of the form $(B, -) \to (A, -) \to F \to 0$ where, by Yoneda, the morphism between the projective functors $(B, -)$ and $(A, -)$ has the form $(f, -)$ for some such f.

This space was used by Ziegler to resolve a number of questions in the model theory of modules; at the same time it opened new avenues for exploration. In particular it gives rise to a notion of support of a module M, namely the set of indecomposable pure-injective direct summands of modules elementarily equivalent to M; at least that was the original formulation but it may be said otherwise:

• $\mathrm{supp}(M) = \{N \in \mathrm{pinj}_R : \forall F \in \text{fun-}R,\ \overrightarrow{F}N = 0 \text{ whenever } \overrightarrow{F}M = 0\}$.

Alternatively the support of M is the set of those $N \in \mathrm{pinj}_R$ such that $(N \otimes_R -)$ is torsionfree for the finite type torsion theory on $(R\text{-mod}, \mathbf{Ab})$ which is congenerated by the injective hull of the functor $M \otimes_R -$. The functor $M \otimes_R -$ is an absolutely pure object of the functor category but the torsion theory cogenerated by its injective hull $E(M \otimes -)$ ($\simeq H(M) \otimes -$ where $H(M)$ denotes the pure-injective hull of M) need not be of finite type (unless M is a so-called elementary cogenerator), so we mean the finite type torsion theory whose torsion class is contained in that determined by $M \otimes -$ and which is largest such.

Therefore to a closed subset X of Zg_R we may assign $\mathcal{X} = \{M \in \text{Mod-}R : \mathrm{supp}(M) \subseteq X\}$. By [24, 4.7, 4.10] this gives a bijection between closed subsets of Zg_R and those full subcategories of Mod-R closed under ultraproducts, direct sums and pure submodules; that is, between closed subsets of Zg_R and definable subcategories of Mod-R. Proofs of the equivalences of the various definitions of definable subcategories may be found (done in some generality) in [21]. The other direction of the correspondence is given by taking a definable subcategory to its intersection with pinj_R.

None of this depends on our having started with the category of modules over a ring: if we replace the ring R by a skeletally small preadditive category $\mathcal{R}$ then everything works, and with few changes. The only point which needs some consideration is what we mean by an "element" of an object of (for instance) a functor category: in fact if $\mathcal{C}$ is any finitely accessible category then by an "element" of an object C of $\mathcal{C}$ one should mean a morphism from a finitely presented object $A \in \mathcal{C}$ to C. There is room for flexibility, in that a restricted set of generating (in some sense) finitely presented objects may be chosen. For instance if $\mathcal{C} = \text{Mod-}R$ then usually we choose just the single finitely presented object R_R and then the equivalence $(R_R, M) \simeq M$ encapsulates the relation between this notion of "element" and the usual one. But one could have, for each finitely presented module A, the notion of an A-element of M, meaning a morphism from A to M. In fact, regarding Mod-R as a definable subcategory (namely that consisting of the exact functors, given by evaluation) of $(\text{fun-}R, \mathbf{Ab}) = (\text{fun-}R)\text{-Mod}$, we could take an even wider notion of element, meaning, after unwinding the definition, that an F-element of M, where $F \in \text{fun-}R$, is just an element of the result, $\overrightarrow{F}M$, of evaluation of F (rather, $\overrightarrow{F}$) at M. Model theory handles this by using a multi-sorted language with a sort for each element of the chosen generating set of finitely presented objects (see, for example, [19, Appx. B]).

The bijection between closed subsets of $\mathrm{Zg}_{\mathcal{R}}$ and definable subcategories of Mod-$\mathcal{R}$ is computationally very useful, especially in those cases where the space $\mathrm{Zg}_{\mathcal{R}}$ has been given a reasonably clear description [19, Chpt. 8].

For instance if R is an artin algebra (a ring with artinian centre and which is finitely generated as a module over its centre, for example any algebra finite-dimensional over a central subfield), then every indecomposable module of finite length is a point of Zg_R. Moreover, the existence of Auslander-Reiten sequences gives (in fact, is more or less equivalent to) the fact that every such point is isolated (=open) in the space. Furthermore these finite-dimensional points are dense in the space (see, e.g., [19, 5.3.36] for this). Typically "natural" sets of finite-length modules give rise to interesting definable subcategories. One sees in particular that, if R is not of finite representation type, hence has an infinite set of isolated points in Zg_R, then there are at least continuum many definable subcategories of Mod-R: take any set of isolated=finite-length points of Zg_R, form the closure, noting that no new isolated points get in, and take the corresponding definable subcategory. The same argument applies to $\mathrm{Zg}_{\mathbb{Z}}$.

Associated Structures

Now we turn to the structure surrounding a definable category: first, the associated functor category.

Suppose that $\mathcal{D}$ is a definable subcategory of Mod-$\mathcal{R}$. Define $\mathcal{S}_{\mathcal{D}} = \{F \in$ fun-$\mathcal{R} : F\mathcal{D} = 0\}$ where $F\mathcal{D} = 0$ is short for $\overrightarrow{F}D = 0$ for every $D \in \mathcal{D}$. It is easily checked that $\mathcal{S}_{\mathcal{D}}$ is a Serre subcategory of fun-$\mathcal{R}$: it is closed under subobjects, quotient objects and extensions. Indeed, every Serre subcategory of fun-$\mathcal{R}$ arises in this way and the correspondence between definable subcategories of Mod-$\mathcal{R}$ and Serre subcategories of fun-$\mathcal{R}$ is bijective. If we define a hereditary torsion theory, $\tau_{\mathcal{D}}$ say, on Fun-$\mathcal{R}$ = (mod-$R, \mathbf{Ab}$) by declaring the torsion class to be that generated by $\mathcal{S}_{\mathcal{D}}$ then we obtain a torsion theory of finite type (one definition of this is by the condition that the torsionfree class be closed under direct limits) and every torsion theory on Fun-$\mathcal{R}$ of finite type arises in this way. Since Fun-$\mathcal{R}$ is locally coherent it follows (see [19, 11.1.33] for a proof and references) that the localisation of Fun-$\mathcal{R}$ with respect to $\tau_{\mathcal{D}}$ is again locally coherent and has (up to equivalence), for its subcategory of finitely presented = coherent objects the quotient category fun-$\mathcal{R}/\mathcal{S}_{\mathcal{D}}$. We set fun$(\mathcal{D})$ = fun-$\mathcal{R}/\mathcal{S}_{\mathcal{D}}$ and refer to this as the **functor category** of $\mathcal{D}$: since it is fun-$\mathcal{R}$ modulo those functors which are 0 when restricted to $\mathcal{D}$ (and also since it has a parallel model-theoretic interpretation) this seems reasonable; but is it well-defined? That is, suppose that $\mathcal{D}$ is equivalent, as a category, to the definable subcategory $\mathcal{D}'$ of Mod-$\mathcal{R}'$; then is fun-$\mathcal{R}/\mathcal{S}_{\mathcal{D}} \simeq$ fun-$\mathcal{R}'/\mathcal{S}_{\mathcal{D}'}$?

This is in fact the case, and follows from results of Krause [14, §2] which give a characterisation in terms of the category of all pure-injective objects of $\mathcal{D}$. But there is a simpler characterisation, namely ([18, 12.10]) fun$(\mathcal{D}) \simeq (\mathcal{D}, \mathbf{Ab})^{\prod \rightarrow}$ - the category of those additive functors from $\mathcal{D}$ to $\mathbf{Ab}$ which commute with direct products and direct limits. All the rest of the associated structure that we go on to mention can be defined in terms of $\mathcal{D}$ and this functor category (and the actions of the one on the other) so the issue of well-definedness is resolved.

For example, associated to the definable subcategory $\mathcal{D}$ is the closed subset $D = \mathcal{D} \cap \mathrm{Zg}_{\mathcal{R}}$ of $\mathrm{Zg}_{\mathcal{R}}$. Clearly the set of points can be defined from $\mathcal{D}$ and, from one of the descriptions of the topology, we see that the topology can be recovered using $\mathrm{fun}(\mathcal{D})$. Equally it can be recovered using the model theory of $\mathcal{D}$, and that is completely contained in the category $\mathrm{fun}(\mathcal{D})$, regarded as a category of pp pairs, together with its action on $\mathcal{D}$.

The representation of $\mathcal{D}$ as an exactly definable category is $\mathcal{D} \simeq \mathrm{Ex}(\mathrm{fun}(\mathcal{D}), \mathbf{Ab})$ where we regard $\mathcal{D}$ as a category of functors on $\mathrm{fun}(\mathcal{D})$ simply by evaluation: $D \mapsto \mathrm{ev}_D : F \mapsto FD$ for $D \in \mathcal{D}$.

There is also, associated to a definable category $\mathcal{D}$, a "dual" definable category which bears the same relation to $\mathcal{D}$ as does $\mathcal{R}\text{-Mod} = (\mathcal{R}, \mathbf{Ab})$ to $\text{Mod-}\mathcal{R} = (\mathcal{R}^{\mathrm{op}}, \mathbf{Ab})$. Namely, if $\mathcal{D}$ is a definable subcategory of Mod-$\mathcal{R}$ then take the associated Serre subcategory $\mathcal{S}_{\mathcal{D}}$ of fun-$\mathcal{R}$ and use the aforementioned duality between the functor categories to obtain $\mathcal{S}_{\mathcal{D}}^{\mathrm{d}} = \{dF : F \in \mathcal{S}_{\mathcal{D}}\}$. This is a Serre subcategory of $\mathrm{fun}^{\mathrm{d}}\text{-}\mathcal{R} = \mathrm{fun}(\mathcal{R}\text{-Mod})$ so to it corresponds the definable subcategory $\{L \in \mathcal{R}\text{-Mod} : \overrightarrow{G}L = 0 \ \forall G \in \mathcal{S}_{\mathcal{D}}^{\mathrm{d}}\}$. One may check that this category, which we denote $\mathcal{D}^{\mathrm{d}}$ and call the (**elementary**) **dual** of $\mathcal{D}$, is well-defined. It follows from a result of Herzog [10, 4.4] that the Ziegler spectra of $\mathcal{D}$ and its dual $\mathcal{D}^{\mathrm{d}}$ are "homeomorphic at the level of topology", meaning that there is a (canonical) isomorphism of the lattices of open sets which also preserves infinite unions. It is not known whether or not these spaces are homeomorphic at the level of points. Given $D \in \mathcal{D}$ and $E \in \mathcal{D}^{\mathrm{d}}$ one may make some sense of $D \otimes E$ (just as one makes sense of $M \otimes_R L$ when $M \in \text{Mod-}\mathcal{R}$ and $L \in \mathcal{R}\text{-Mod}$) though, if taken literally, this does depend on the representations chosen: one may use the canonical representations with $\mathcal{R} = \mathrm{fun}(\mathcal{D})$ but, for instance, whether or not $D \otimes E = 0$ can be measured using any matching representations.

There is another space associated with a definable category $\mathcal{D}$: one takes its Ziegler spectrum $\mathrm{Zg}_{\mathcal{D}}$ and then constructs a new topology by taking, for a basis of open sets, the complements of compact (quasi-compact in the terminology of some) open sets in the Ziegler topology. This is the **rep-Zariski spectrum**; the name arises because it may be regarded as the Gabriel-Zariski topology of the category $\mathrm{Fun}(\mathcal{D})$; that, in turn, is obtained by applying to $\mathrm{Fun}(\mathcal{D})$ that definition of the classical Zariski spectrum of a commutative noetherian ring which is obtained when it is re-phrased in terms just of the category of modules.

This rep-Zariski space carries a presheaf of categories, namely, above the basic open set $(F)^{\mathrm{c}}$ ($^{\mathrm{c}}$ denoting complement), one places the small abelian category $\mathrm{fun}(\mathcal{D})/\langle F \rangle$ where $\langle F \rangle$ is the Serre subcategory generated by the object $F \in \mathrm{fun}(\mathcal{D})$. Note that this is the functor category of the definable subcategory corresponding to the Ziegler-closed set $(F)^{\mathrm{c}}$. The restriction morphisms are just the relevant localisation maps (the situation is sufficiently rigidly-defined that we can stay with the language of presheaves and avoid moving to that of fibred categories and stacks). This is seldom a sheaf but sometimes significant parts of it do satisfy the glueing condition (see, e.g., [19, Chpt. 14]). In the case that $\mathcal{D}$ is a definable subcategory of a category Mod-R of modules over a ring R it also contains within it the thread consisting of the localisations of the forgetful functor $(R_R, -)$, rather the endomorphism rings of these objects, and thus one obtains a presheaf of rings which are, in a weak sense, localisations of R. The ring, $\mathrm{End}((R_R, -)_{\tau_{\mathcal{D}}})$, corresponding to $\mathcal{D}$ is referred to as the **ring of definable scalars** of $\mathcal{D}$ because of its model-theoretic interpretation as exactly that - the ring of pp-definable maps on objects of $\mathcal{D}$. For more on this ring and some applications, see [19, Chpt. 6].

We emphasise again that a definable category $\mathcal{D}$ carries an internal theory of purity which coincides with that induced by any definable category of which it is a definable subcategory. The same applies to the model theory of objects of $\mathcal{D}$; for example elementary duality of pp formulas is defined and takes pp formulas for $\mathcal{D}$ to those of its elementary dual $\mathcal{D}^{\mathrm{d}}$.

2-categories

Finally we turn to the 2-categories involved. The first, $\mathbb{ABEX}$, is that whose objects are the skeletally small abelian categories, whose 1-arrows are the exact functors between these and whose 2-arrows are the natural transformations. The second, $\mathbb{DEF}$, is that whose objects are the definable additive categories, whose 1-arrows are the functors which preserve direct products and direct limits and whose 2-arrows are the natural transformations.

Note that if $\mathcal{C}$ and $\mathcal{D}$ are definable categories and if $I_0 : \mathrm{fun}(\mathcal{D}) \to \mathrm{fun}(\mathcal{C})$ is an exact functor then, since $\mathcal{C} \simeq \mathrm{Ex}(\mathrm{fun}(\mathcal{C}), \mathbf{Ab})$ and $\mathcal{D} \simeq \mathrm{Ex}(\mathrm{fun}(\mathcal{D}), \mathbf{Ab})$, I_0 induces, by composition, a functor $I : \mathcal{C} \to \mathcal{D}$ which, one may check, preserves direct products and direct limits. The converse is much harder, was proved by Krause in the case that $\mathcal{C}$ is locally finitely presented [14, 7.2] and by Prest in the general case [18, 12.10]: every exact functor from $\mathrm{fun}(\mathcal{D})$ to $\mathrm{fun}(\mathcal{C})$ is induced by a functor from $\mathcal{C}$ to $\mathcal{D}$ which preserves direct products and direct limits. These functors also turn out to be precisely the pp-interpretation functors in the sense of model theory. From this one derives a pair of contravariant 2-equivalences between $\mathbb{ABEX}$ and $\mathbb{DEF}$.

In fact, it turns out that this can be derived also from results of Makkai [15, 5.1] and Hu [12, 5.10(ii)] who work in the general context of Barr-exact categories. Their methods are very different and, indeed, it was necessary, at least for me, to work quite hard to understand their language before comprehending the relation between their results and those just stated.

Finally, we point out something of the richness of this context of definable categories.

It is easily observed that the complexity of a ring R *qua* ring is quite unrelated to the complexity of its category of modules; on the other hand the functor category fun-R much better reflects this complexity. There are, for instance, a number of dimensions which reflect the complexity of Mod-R and which are defined in terms of fun-R (see, e.g., [11], [13], [19, Chpt. 13]). This is also a good point at which to note that the functor category fun-R is equivalent to the free abelian category, $\mathbf{Ab}(R)$, of R, more accurately, of its opposite: fun-$R \simeq (\mathbf{Ab}(R)^{\mathrm{op}}) \simeq \mathbf{Ab}(R^{\mathrm{op}})$ (see, e.g., [19, §10.2.7] for references). This category, introduced (in a more general setting) by Freyd [7, 4.1] is defined by the requirement that any additive functor from R (regarded as a category and, of course, all this applies with $\mathcal{R}$ in place of R) to an abelian category $\mathcal{B}$ factors uniquely-to-natural-equivalence through the canonical map $R \to \mathbf{Ab}(R)$.

The context we have described also contains affine algebraic geometry in the following sense. Suppose that R is commutative; replace R by $\mathbf{Ab}(R)$ (one may check that the centre of any ring R may be recovered as the centre of the category $\mathbf{Ab}(R)$ - the ring of natural transformations from $\mathbf{Ab}(R)$ to itself). To this, equally to Mod-R, one has the associated rep-Zariski spectrum with its presheaf of small abelian categories; restrict this presheaf to the subspace based on the set $\mathrm{inj}_R \subseteq \mathrm{pinj}_R$ of indecomposable injective modules and pick out the thread of "localisations" of R described earlier. In the case that R is also noetherian there is a bijection between the primes of R and the indecomposable injective modules so this gives us a presheaf of rings over Spec(R) which, one may check, is exactly the structure sheaf of this affine variety. Morphisms between rings induce morphisms between these presheaves so one sees, contained in the anti-equivalence between $\mathbb{ABEX}$ and $\mathbb{DEF}$, the well-known anti-equivalence between the category of commutative rings and affine algebraic varieties, at least restricted to noetherian rings (for more general commutative rings the link between primes of R and indecomposable injective modules is weaker, see, e.g., [19, §14.4]).

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