MANCHESTER
1824

The University
of Manchester

A note on quantum chaology and gamma approximations to eigenvalue spacings for infinite random matrices

Dodson, CTJ

2008

MIMS EPrint: **2008.2**

Manchester Institute for Mathematical Sciences
School of Mathematics

The University of Manchester

Reports available from: http://eprints.maths.manchester.ac.uk/
And by contacting: The MIMS Secretary
School of Mathematics
The University of Manchester
Manchester, M13 9PL, UK

ISSN 1749-9097

CHAOS 2008 Charnia, Crete 3-6 June 2008

A note on quantum chaology and gamma approximations to eigenvalue spacings for infinite random matrices

C.T.J. Dodson

School of Mathematics
University of Manchester, UK

Abstract

Quantum counterparts of certain classical systems exhibit chaotic spectral statistics of their energy levels; eigenvalues of infinite random matrices model irregular spectra.

Eigenvalue spacings for the Gaussian orthogonal ensemble (GOE) of infinite random real symmetric matrices admit a gamma distribution approximation, as do the hermitian unitary (GUE) and quaternionic symplectic (GSE) cases.

Then chaotic and non chaotic cases fit in the information geometric framework of the manifold of gamma distributions, which has been the subject of recent work on neighbourhoods of randomness for general stochastic systems.

Regular and Irregular Spectra

Quantum chaology, Berry [8]: semiclassical but non-classical behaviour of systems whose classical motion exhibits chaos, illustrated by the statistics of energy levels.

Regular spectrum of bound system with $n \geq 2$ degrees of freedom and n constants of motion: energy levels labelled by n quantum numbers, but quantum numbers of nearby energy levels may be very different.

For irregular spectrum, quantum number labelling fails; use energy level spacing distributions for comparisons among different spectra [9]. Regular systems are negative exponential, that is Poisson random.

Energy spacing levels of complex nuclei and atoms with n large modelled by the spacings of eigenvalues of random matrices, Porter [22], well fitted by Wigner distribution [28].

Eigenvalues of (Gaussian) Random Matrices

GOE: A random real symmetric $n \times n$ matrix belongs to the Gaussian orthogonal ensemble (GOE) if the diagonal and upper triangular elements are independent random variables with Gaussian distributions of zero mean and standard deviation 1 for the diagonal and $\frac{1}{\sqrt{2}}$ for the upper triangular elements.

GUE: The corresponding random hermitian complex case belongs to the Gaussian unitary ensemble.

GSE: The corresponding random hermitian case with real quaternionic elements belongs to the Gaussian symplectic ensemble.

Distributions of Random Matrices

The matrices in these GOE, GUE and GSE ensembles are respectively invariant under the appropriate orthogonal, unitary and symmetric transformation groups, and moreover in each case the joint density function of all independent elements is controlled by the trace of the matrices and is of form [16]

$$p(X) = A_n \, e^{-\frac{1}{2} Tr X^2} \tag{1}$$

where A_n is a normalizing factor.

Approximating Eigenvalue Spacing Distributions

Wigner [26, 27, 28] Approximation for GOE unit mean

$$w(s) = \frac{\pi}{2} s\, e^{-\frac{\pi s^2}{4}} \quad \text{For eigenvalue spacing } s > 0.$$

Gamma Distribution Approximation

The family of gamma probability density functions, $\kappa, \nu > 0$,

$$p(s;\nu,\kappa) = \nu^{\kappa} \frac{s^{\kappa-1}\, e^{-s\nu}}{\Gamma(\kappa)}. \text{ (Exponential, ie Poisson case : } \kappa = 1).$$

Important uniqueness property:

Theorem (Hwang and Hu [17])

For independent positive random variables with a common probability density function f, having independence of the sample mean and the sample coefficient of variation is equivalent to f being the gamma distribution.

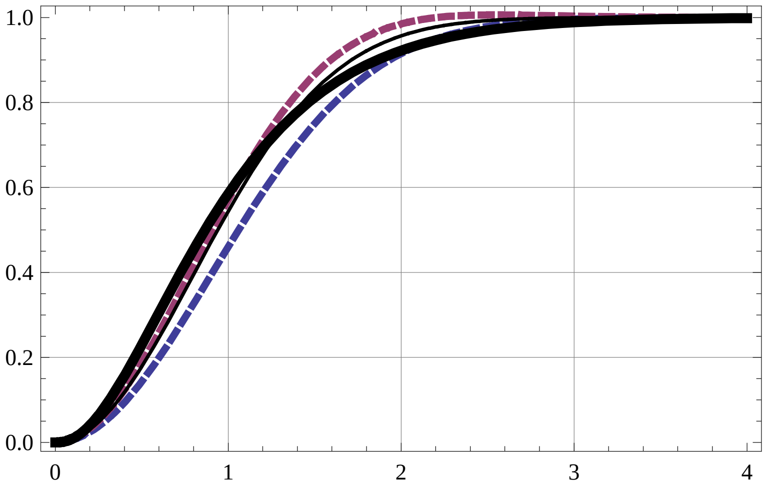


Figure: *The bounds on eigenvalue spacings cdf for GOE random matrices (dashed), the Wigner surmise (thin solid) and unit mean gamma fit to true GOE from Mehta [19] (thick solid).*

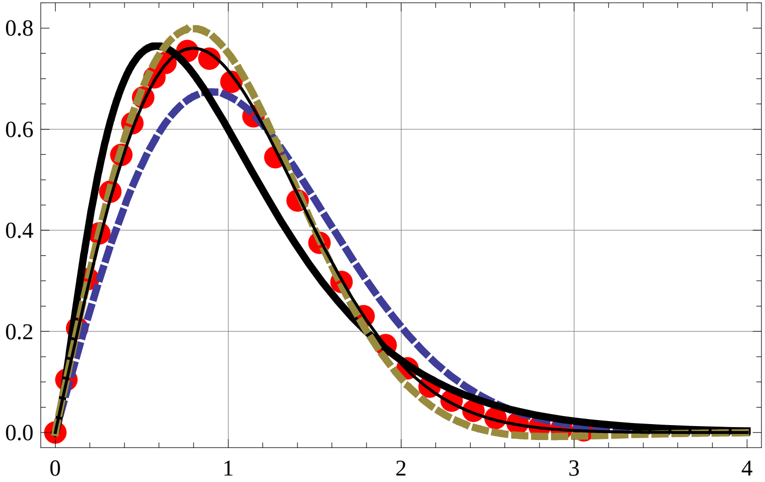


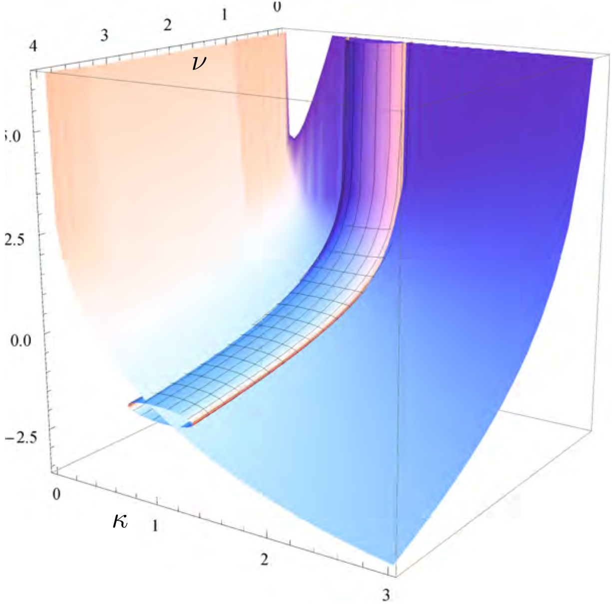
Figure: *Unit mean gamma pdf fit (thick solid) to true GOE (points), Wigner (thin solid) and bounds (dashed) for spacings between eigenvalues.*

Gamma Distribution Properties
Contains exponential distribution as a special case.
Contains uniform distribution of logarithm.
Approximates GOE, also GUE, GSE, away from origin.

Non-chaotic case has Poisson random energy spacings
The sum of n iid exponentials (ie n Poisson random cases) follows a gamma distribution and the sum of n iid gammas follows a gamma distribution. Product of gamma distributions approximated by a gamma distribution.

Tractable Information Geometry
The gamma family has a well-understood and tractable information geometry [3, 4, 14], defining a Riemannian 2-manifold using the information metric, with arc length function

$$d\ell^2 = \frac{\kappa}{\nu^2}\, d\nu^2 - \frac{2}{\nu}\, d\nu d\kappa + \frac{d^2 \log(\Gamma)}{d\kappa^2}\, d\kappa^2.$$



Affine Embedding in $\mathbb{R}^3$ of Gamma Manifold

Generalized Gamma Distribution Approximation [Caër [12]]

$$g(s;\beta,\omega) = a(\beta,\omega)\, s^{\beta}\, e^{-b(\beta,\omega)s^{\omega}} \quad \text{for } \beta,\omega > 0$$

$$a(\beta,\omega) = \frac{\omega[\Gamma((2+\beta)/\omega)]^{\beta+1}}{[\Gamma((1+\beta)/\omega)]^{\beta+2}} \quad \text{and} \quad b(\beta,\omega) = \left[\frac{\Gamma((2+\beta)/\omega)}{\Gamma((1+\beta)/\omega)}\right]^{\omega}.$$

Best fit parameters for generalized gamma:

Ensemble	β	ω	Variance
Exponential	0	1	1
GOE	1	1.886	0.2856
GUE	2	1.973	0.1868
GSE	4	2.007	0.1100

Unfortunately, unlike for gamma distributions, the information geometry is intractable for generalized gamma distributions.

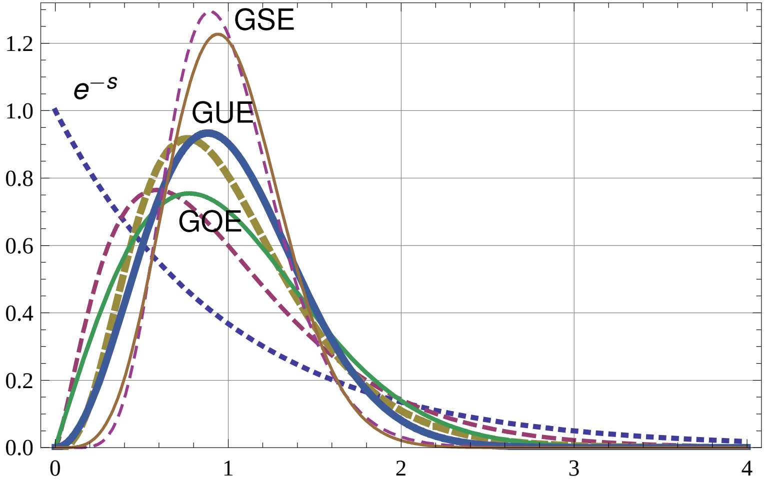


Figure: *Unit mean gamma pdfs (dashed) and generalized gamma (solid) fits to true variances for left to right the GOE , GUE and GSE cases. Both coincide in the exponential case, e^{-s}, shown dotted.*

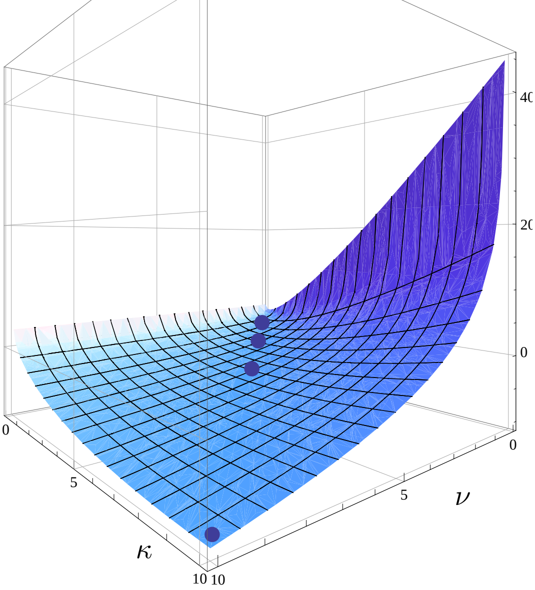


Figure: *The unit mean gamma distribution best fits to GOE, GUE and GSE cases, as points on the 2-manifold of gamma distributions.*

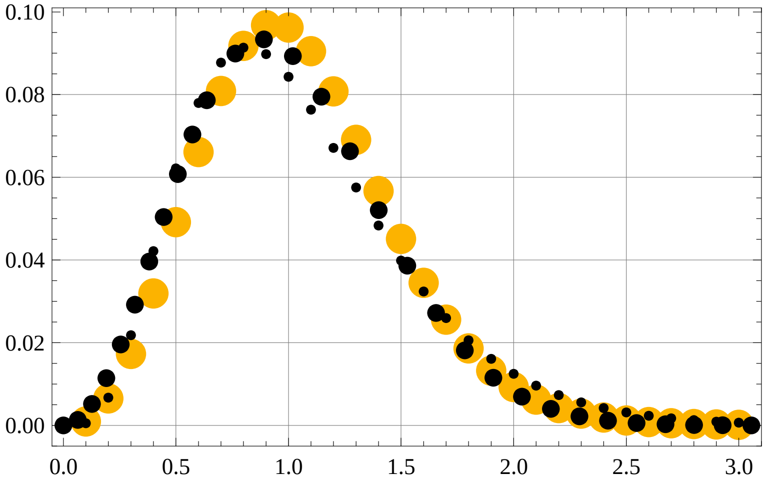


Figure: *Probability plot with unit mean for the spacings between the first 2,001,052 zeros of the Riemann zeta function (large points), true GUE distribution (medium points) and gamma fit to the true GUE (small points).*

Important uniqueness property of gamma distributions:

Theorem (Hwang and Hu [17])

For independent positive random variables with a common probability density function f, having independence of the sample mean and the sample coefficient of variation is equivalent to f being the gamma distribution.

So, having the ratio of sample standard deviation to sample mean independent of sample mean indicates a gamma process.

This is approximately true in a surprisingly wide range of real statistical processes. In particular, quite often sample standard deviation is approximately proportional to sample mean over ranges of practical interest.

We can see in the following tables that even the normalized (to unit mean) spacings of zeros of the Riemann zeta function have coefficient of variation quite stable under variations of location and size of samples.

Block	*Mean*	*Variance*	*CV*	κ
1	1.232360	0.276512	0.426697	5.49239
2	1.072330	0.189859	0.406338	6.05654
3	1.025210	0.174313	0.407240	6.02974
4	0.996739	0.165026	0.407563	6.02019
5	0.976537	0.158777	0.408042	6.00607
6	0.960995	0.154008	0.408367	5.99651
7	0.948424	0.150136	0.408544	5.99131
8	0.937914	0.147043	0.408845	5.98250
9	0.928896	0.144285	0.408926	5.98014
10	0.921034	0.142097	0.409276	5.96991

Table: **Effect of location on sample coefficient of variation CV:** *Statistical data for spacings in the first ten consecutive blocks of 200,000 zeros of the Riemann zeta function normalized with unit grand mean from the tabulation of Odlyzko [21].*

m	Mean	Variance	CV	κ
1	1.23236	0.276511	0.426696	5.49242
2	1.15234	0.239586	0.424765	5.54246
3	1.10997	0.221420	0.423934	5.56421
4	1.08166	0.209725	0.423384	5.57869
5	1.06064	0.201303	0.423018	5.58833
6	1.04403	0.194799	0.422748	5.59548
7	1.03037	0.189538	0.422527	5.60133
8	1.01881	0.185161	0.422357	5.60584
9	1.00882	0.181418	0.422207	5.60983
10	1.00004	0.178180	0.422094	5.61282

Table: **Effect of size on sample coefficient of variation CV:** *Statistical data for spacings in ten blocks of increasing size* $200,000m$, $m = 1, 2, \ldots, 10$, *for the first 2,000,000 zeros of the Riemann zeta function, normalized with unit grand mean, from the tabulation of Odlyzko [21].*

Further Remarks on Information Geometry

We have given a large number of results on the information geometry of spaces of gamma and log-gamma distributions, also bivariate versions including Gaussians, with many applications; see the book Arwini and Dodson [4].

Characterization of Perturbations
These results include explicit information geometric representations with distance measures, of neighbourhoods for each of these important states for statistical processes:

- (Poisson) randomness,
- independence,
- uniformity.

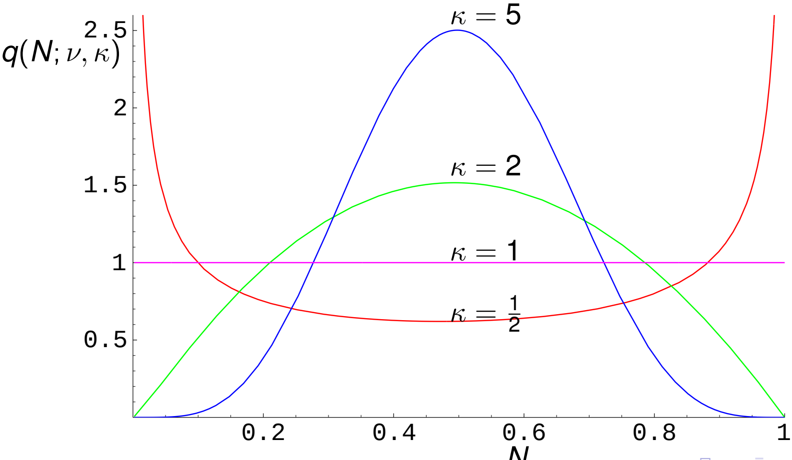
Such results are significant theoretically because they are very general, and significant practically because they are topological and so therefore stable under perturbations.

Log-gamma ($\log N = -s$) probability density functions

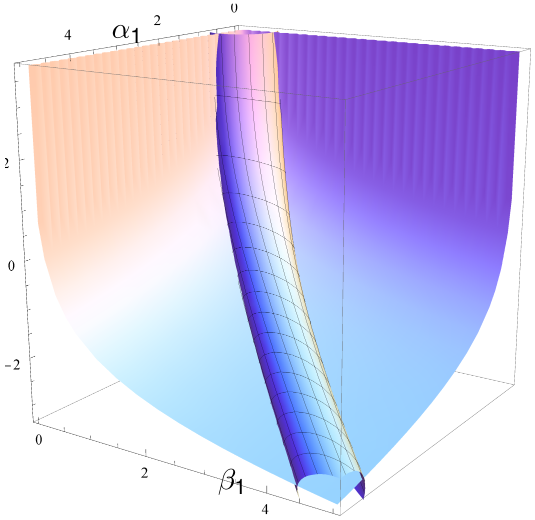
$\kappa = 1$: Uniform distribution on $[0, 1]$, complementary to the exponential case for gamma distribution.

$\kappa >> 1$: Approximate truncated Gaussians.

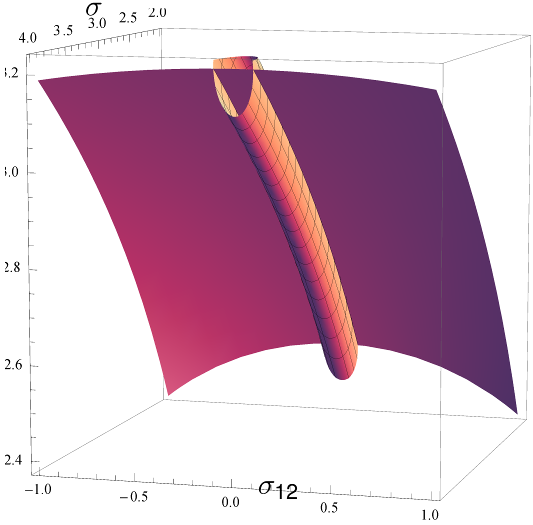
$\kappa < 1$: Clustering.



Tubular neighbourhood of independent Poisson random processes in Freund manifold of bivariate exponentials.



Tubular neighbourhood of independent ($\sigma_{12} = 0$) identical Gaussians



Bibliography

- H. Alt, C. Dembrowski, H.D. Graf, R. Hofferbert, H. Rehfield, A. Richter and C. Schmit. Experimental versus numerical eigenvalues of a Bunimovich stadium billiard: A comparison. *Phys. Rev. E* 60, 3 (1999) 2851-2857.
- S-I. Amari and H. Nagaoka. **Methods of Information Geometry**, American Mathematical Society, Oxford University Press, 2000.
- Khadiga Arwini and C.T.J. Dodson. Information geometric neighbourhoods of randomness and geometry of the McKay bivariate gamma 3-manifold. *Sankhya: Indian Journal of Statistics* 66, 2 (2004) 211-231.
- Khadiga Arwini and C.T.J. Dodson. **Information Geometry Near Randomness and Near Independence.** Lecture Notes in Mathematics, Springer-Verlag, New York, Berlin 2008.

- O.E. Barndorff-Nielsen, R.D. Gill and P.E. Jupp. On Quantum Statistical Inference. Preprint (2001).
- A.H. Barnett. `http://math.dartmouth.edu/~ahb/pubs.html`
- M.V. Berry. Private communication. 2008.
- M.V. Berry. Quantum Chaology. *Proc. Roy. Soc. London A* 413, (1987) 183-198.
- M.V. Berry and M. Tabor. Level clustering in the regular spectrum. *Proc. Roy. Soc. London A* 356, (1977) 373-394.
- M.V. Berry and M. Robnik. Semiclassical level spacings when regular and chaotic orbits coexist. *J. Phys. A Math. General* 17, (1984) 2413-2421.
- O. Bohigas, M.J. Giannoni and C. Schmit. Characterization of Chaotic Quantum Spectra and Universality of Level Fluctuation Laws. *Phys. Rev. Lett.* 52, 1 (1984) 1-4.

G. Le Caër, C. Male and R. Delannay. Nearest-neighbour spacing distributions of the β-Hermite ensemble of random matrices. *Physica A* (2007) 190-208. Cf. also their Erratum: *Physica A* 387 (2008) 1713.

P. Deift. Some open problems in random matrix theory and the theory of integrable systems. Preprint, arXiv:arXiv:0712.0849v1 6 December 2007.

C.T.J. Dodson. Quantifying galactic clustering and departures from randomness of the inter-galactic void probablity function using information geometry. http://arxiv.org/abs/astro-ph/0608511 (2006).

C.T.J. Dodson and Hiroshi Matsuzoe. An affine embedding of the gamma manifold. *InterStat*, January 2002, 2 (2002) 1-6.

P. J. Forrester, Log-Gases and Random Matrices, Chapter 1 Gaussian matrix ensembles. Book manuscript, `http://www.ms.unimelb.edu.au/~matpjf/matpjf.html` 2007.

T-Y. Hwang and C-Y. Hu. On a characterization of the gamma distribution: The independence of the sample mean and the sample coefficient of variation. *Annals Inst. Statist. Math.* 51, 4 (1999) 749-753.

S. Kullback. **Information and Statistics**, J. Wiley, New York, 1959.

Madan Lal Mehta. **Random Matrices** 3^{rd} Edition, Academic Press, London 2004.

Steven J. Miller and Ramin Takloo-Bighash. **An Invitation to Modern Number Theory** Princeton University Press, Princeton 2006. Cf. also the seminar notes:

- Steven J. Miller: Random Matrix Theory, Random Graphs, and L-Functions: How the Manhatten Project helped us understand primes. Ohio State University Colloquium 2003.
  `http://www.math.brown.edu/~sjmiller/math/talk`
  .

- Steven J. Miller. Random Matrix Theory Models for zeros of L-functions near the central point (and applications to elliptic curves). Brown University Algebra Seminar 2004.
  http://www.math.brown.edu/~sjmiller/math/talk

A. Odlyzko. Tables of zeros of the Riemann zeta function.
http://www.dtc.umn.edu:80/~odlyzko/zeta_tables/i

C.F. Porter. **Statistical Theory of Spectra: Fluctuations** Edition, Academic Press, London 1965.

Z. Rudnick. Private communication. 2008. Cf. also Z. Rudnick. What is Quantum Chaos? *Notices A.M.S.* 55, 1 (2008) 33-35.

M.R. Schroeder. **Number Theory in Science and Communication. With Applications in Cryptography, Physics, Digital Information, Computing, and Self-Similarity**. Springer Series in Information Science, 3^{rd} edition, Springer, Berlin 1999.

- A. Soshnikov. Universality at the edge of the spectrum in Wigner random matrices. *Commun. Math. Phys.* 207 (1999) 697-733.
- E.P. Wigner. Characteristic vectors of bordered matrices with infinite dimensions. *Annals of Mathematics* 62, 3 (1955) 548-564.
- E.P. Wigner. On the distribution of the roots of certain symmetric matrices. *Annals of Mathematics* 67, 2 (1958) 325-327.
- E.P. Wigner. Random matrices in physics. *SIAM Review* 9, 1 (1967) 1-23.