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Borovik, Alexandre and Cohen, Arjeh

2008

MIMS EPrint: **2008.45**

Manchester Institute for Mathematical Sciences

School of Mathematics

The University of Manchester

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Alexandre V. Borovik and Arjeh Cohen

ABSTRACT. This is a manuscript last dated 30 June 1994; it was mostly written during the first author's visit to Eindhoven in December 1992. This approach to study of finite subgroups of simple algebraic groups could still be of some interest.

1. Introduction.

This preprint contains two results related to the variety $\mathrm{Rep}(F, G)$ of representations of a finite groups F with values in a simple algebraic group G (i.e. homomorphisms $\chi : F \longrightarrow G$). We use the following notation. Let F be a finite group of order $|F| = n$, G a reductive algebraic group over an algebraically closed field K and $\chi : F \longrightarrow G$ a representation. It will be convenient to denote the image of an element $f \in F$ under χ by x_f. If $f_1, \ldots, f_n$ are all elements of F, then we can identify χ with a point $(x_{f_1}, \ldots, x_{f_n})$ of G^n. Obviously these points fill in the variety $R = \mathrm{Rep}(F, G) \subset G^n$ given by the equations

$$x_f x_h = x_{fh}$$

for all $f, h \in F$.

The group G acts on G^n by simultaneous conjugation and this action obviously leaves invariant the variety $\mathrm{Rep}(F, G)$. So there is an obvious one-to-one correspondence between the conjugacy classes of homomorphisms ('representations') $\chi : F \longrightarrow G$ and the G-orbits on $\mathrm{Rep}(F, G)$.

Following R. W. Richardson [R2], we call a representation χ and the corresponding point $\bar{x} \in R$ *strongly reductive* if the subgroup $\chi(F)$ is not contained in any proper parabolic subgroup of $C_G(T)$ for a maximal torus $T \leq C_G(\chi(F))$. It is clear that the definition does not depend on the choice of the maximal torus T of $C_G(\chi(F))$.

The importance of this definition is explained by the following theorem.

THEOREM 1 (R. W. Richardson [R2], Theorem 16.4). *Let $\bar{x} \in \mathrm{Rep}(F, G)$. The orbit $G \cdot \bar{x}$ is closed if and only if $\bar{x}$ is strongly reducible.*

In characteristic zero, or, more, generally, if $(|F|, \mathrm{char}\, K) = 1$, the situation is very simple, as the following Lemma shows.

Department of Mathematics, UMIST, United Kingdom.
University of Technology Eindhoven, Netherlands.

LEMMA 1 (See Lemma 3.6 in [**Bor3**]). *Assume that* $\operatorname{char} K = 0$ *or* $(|F|, \operatorname{char} K) = 1$. *Then every representation of a finite group F into G is strongly reductive.*

Proof: Assume by way of contradiction that $X = \chi(F)$ lies in a proper parabolic subgroup P of $H = N_G(T)$, where T is a maximal torus in $C_G(X)$. Notice that H is a connected reductive group. Let $P = QL$, where $Q = R_u(P)$ is the unipotent radical of P and L is a Levi complement. Since X consists of semisimple elements, $X \cap Q = 1$. Now take a subgroup $Y < L$ such that $QX = QY$. If $\operatorname{char} K = 0$ then every factor $\overline{Q}$ of the central series of Q is a KX-module and $H^1(X, \overline{Q}) = 0$ by Theorem 3.10.2 in [**Brown**], hence all complements of Q in the group QX are conjugate [**Brown**, Proposition 4.2.3]. Thus $Y^g = X$ for some $g \in P$ and $C_P(X)$ contains the nontrivial torus $Z(L)^g$. This contradicts to our choice of T as a maximal torus in $C_G(X)$. □

However, in characteristic p the situation is more complicated.

LEMMA 2 (See Lemma 16.2 in [**R2**]). *A representation*

$$\chi : F \longrightarrow GL(V)$$

is strongly reductive if and only if it is completely reducible.

A representation $\chi : F \longrightarrow G$ and the corresponding point $\bar{x} \in \operatorname{Rep}(F, G)$ are called *irreducible*, if $\chi(F)$ is not contained in any proper parabolic subgroup of G. Obviously, this definition is an immediate generalization of the notion of irreducible linear representation $\chi : F \longrightarrow GL(V)$.

THEOREM 2 (R. W. Richardson [**R2**], Proposition 16.7). *A representation $\chi : F \longrightarrow G$ is irreducible if and only if $\bar{x}$ is a stable point of G^n.*

Recall that a point $\bar{x} \in G^n$ is *stable* if the orbit $G \cdot \bar{x}$ is closed and the $|C_G(\bar{x}) : Z(G)|$ is finite. This form of definition is due to R. W. Richardson [**R2**] and slightly generalizes a more traditional one [**MF**]. A point $\bar{x}$ is stable in the sense of Richardson if it is stable in the sense of Mumford [**MF**] for the action of $G/Z(G)$ on G^n.

2. Rigidity Theorem

THEOREM 3. *Let G be a simple algebraic group over the algebraic closure K_p of the prime field $\mathbb{F}_p$ of order p and F a finite group. Assume that $(|F|, p) = 1$ Then $\operatorname{Rep}(F, G)$ has only finitely many G-orbits. The number of G-orbits on $\operatorname{Rep}(F, G)$ is bounded by a number which depends only on F and G and does not depend on p. Moreover, is σ is a Steinberg endomorphism of G, then the number of G_σ-conjugate classes of homomorphisms $F \longrightarrow G_\sigma$ is bounded by a constant that depends only on F and G and does not depend on p and σ.*

Proof: Repetition of the proof of Proposition 3.1 in [**Bor3**], which, in its turn, repeats Richardson's proof [**R1**] of Corollary I.5.2 of [**SpSt**].

Let $G \leq GL_n(K_p)$. Set $G_1 = GL_n(K_p)$. Since $(|F|, p) = 1$ then by a well-known fact of the Representation Theory $\operatorname{Rep}(F, G_1)$ has only finitely many G_1-orbits. Let $|F| = n$. We can imbed G^n into G_1^n and correspondingly $\operatorname{Rep}(F, G)$ into $\operatorname{Rep}(F, G_1)$.

We shall prove now that

> *every G_1-orbit of $\operatorname{Rep}(F, G)$ meets $\operatorname{Rep}(F, G)$ in finitely many G-orbits.*

Let $\bar{x} \in \operatorname{Rep}(F, G)$. Let C_1 be the G_1-orbit of $\bar{x}$, C the G-orbit of $\bar{x}$, and Z the irreducible component of the variety $C_1 \cap G^n = C_1 \cap \operatorname{Rep}(F, G)$ containing C. Consider the mapping

$$f : \operatorname{diag}(G_1^n) \longrightarrow C_1 \bar{x}^{-1}$$

defined by $f(\bar{y}) = \bar{y}\bar{x}\bar{y}^{-1}\bar{x}^{-1}$. It is clear that f fixes the identity element of the group G_1^n.

For a point v of a variety V we write $T(V)_v$ for the tangent space to V at v. Let $L \le L_1$ be the Lie algebras of groups $G \le G_1$, correspondingly.

Notice that the differential $(df)_e$ of f at the point e has the property that the map

$$(df)_e : \operatorname{diag}(L_1^n) \longrightarrow T(C_1 \bar{x}^{-1})_e$$

is surjective. Indeed, since

$$\dim T(C_1 \bar{x}_e^{-1} = \dim G_1 - \dim C_G(\bar{x}),$$

it suffices to show that $\ker(df)_e$ and $C_G(\bar{x})$ have the same dimension. The first of these varieties is an associative algebra consisting of those $\overline{X} \in \operatorname{diag}(L_1^n)$ for which $\bar{x}\overline{X}\bar{x}^{-1} = \overline{X}$. The second variety consists of the invertible elements of this algebra, which form an open subset, and therefore ha the same dimension.

Consider now the following cycle of inclusions:

$$
\begin{aligned}
T(Z\bar{x}^{-1})_e &\le & T(C_1\bar{x}^{-1})_e \cap T(G^n)_e = (1 - \operatorname{ad}\bar{x})\operatorname{diag}(L_1^n) \cap L^n = \\
&= & (1 - \operatorname{ad}\bar{x})\operatorname{diag}(L^n) \le T(C\bar{x}^{-1})_e \le T(Z\bar{x}^{-1})_e.
\end{aligned}
$$

Here the first inclusion holds because $Z\bar{x}^{-1} \subseteq C_1\bar{x}^{-1} \cap G^n$, the second because, by the previous remark,

$$T(C_1\bar{x}^{-1})_e = (df)_e \operatorname{diag}(L_1^n) = (1 - \operatorname{ad}\bar{x})\operatorname{diag}(L_1^n),$$

the third because F acts on L_1 completely reducibly and L_1^n can be written as $L_1^n = L^n \oplus M^n$ for some F-invariant subspace $M \le L_1$, therefore

$$(1 - \operatorname{ad}\bar{x})L_1^n = (1 - \operatorname{ad}\bar{x})L^n \oplus (1 - \operatorname{ad}\bar{x})M^n$$

and

$$(1 - \operatorname{ad}\bar{x})\operatorname{diag}(L_1^n) \cap L^n = (1 - \operatorname{ad}\bar{x})\operatorname{diag}(L^n);$$

the fourth holds because

$$(1 - \operatorname{ad}\bar{x})\operatorname{diag}(L^n) = (df)_e\operatorname{diag}(L^n),$$

the fifth because $C \subseteq Z$. It follows that all terms of the cycle are equal, in particular $T(C)_{\bar{x}} = T(Z)_{\bar{x}}$. Thus C contains an open part of Z, and $C = Z$. Since there are finitely many possibilities for Z, $C_1 \cap G^n$ consists of finitely many G-orbits.

Now we want to prove that

> *the number of G-orbits on $\operatorname{Rep}(F, G)$ is uniformly bounded by a constant which does not depend on p.*

For these purposes we shall vary the characteristic p of a ground field. For a moment we consider G as a group scheme over $\mathbb{Z}$, then $G(K_p)$ is the groups of points of G over the algebraic closures of finite fields K_p [**Borel**]. Obviously the variety $\operatorname{Rep}(F, G)$ is defined over $\mathbb{Z}$ and by the previous discussion $G(K_p)$-orbits on $\operatorname{Rep}(F, G)$ are irreducible components of $\operatorname{Rep}(F, G)$. We are in a position now to apply Theorem 2.10(v) of [**vdDS**] which states that the number of the irreducible components over K_p of a variety defined over $\mathbb{Z}$ is bounded by a constant which does not depend on p. This proves our claim.

If now $\bar{x} \in \mathrm{Rep}(F,G) \cap G_\sigma^n$, then by Theorem I.2.7 in [**SpSt**] the G-orbit of $\bar{x}$ in G_σ^n splits into

$$|H^1(\sigma, C_G(\bar{x})/C_G(\bar{x})^0)| \leq |C_G(\bar{x}) : C_G(\bar{x})^0|$$

G_σ-orbits. Combining all these facts together, we conclude that the number of G_σ-conjugacy classes of homomorphisms

$$F \longrightarrow G_\sigma$$

is uniformly bounded by some constant d which does not depend on p and σ. $\qquad \square$

3. Projective cone over $\mathrm{Rep}(F,G)$

In this section G is a semisimple algebraic group over an algebraically closed field K, $L = \mathrm{Lie}\,(G)$ its Lie algebra and F a finite group. We assume that the Killing form $<,>$ on L is non-degenerate. Under this restrictions we will construct a certain compactification the variety $\mathrm{Rep}(F,G)$. It was introduced in [**Bor2, Bor4**].

Consider the affine space $V = (\mathrm{End}\,L)^n \times \mathbb{A}^1$. We denote an arbitrary point of V by $(x_{f_1}, \ldots, x_{f_n}, t)$, where $f_1, \ldots, f_n$ are all elements of F, $x_{f_i} \in \mathrm{End}\,L = \mathrm{Mat}\,_n(K)$, $i = 1, 2, \ldots, n$, and $t \in \mathbb{A}^1$. We can imbed

$$R = \mathrm{Rep}(F,G) \subset G^n \subset (\mathrm{End}\,L)^n \times \mathbb{A}^1$$

via

$$(x_{f_1}, \ldots, x_{f_n}) \mapsto (x_{f_1}, \ldots, x_{f_n}, 1).$$

Obviously R is given by the equations

$$\begin{aligned} x_f[a,b] &= [x_f a, x_f b] \\ x_f x_h &= x_{fh} \end{aligned}$$

for all $a, b \in L$ and $f, h \in F$.

Let $C \subset (\mathrm{End}\,L)^n \times \mathbb{A}^1$ be the closed projective cone over R. Every homogeneous equation in variables x_f, $f \in F$, and t which holds on R also holds on C. In particular, C satisfies the following equations, where for $a, b, c \in L$ we shorthand $[a,b,c] = [[a,b],c]$, e denotes the identity element of the group F and f, h run through all the elements of F.

(1)
$$x_f[a, x_h b] = [x_f a, x_{fh} b]$$

(2)
$$[x_f a, x_f b] = t x_f[a, b]$$

(3)
$$x_f[x_{f^{-1}} a, b, c] = [a, x_f b, x_f c]$$

(4)
$$< x_f a, x_f b > = t^2 < a, b >$$

(5)
$$< x_f a, b > = < a, x_{f^{-1}} b >$$

(6)
$$x_e = t \mathrm{Id}\,_L$$

(7)
$$x_f x_h = t x_{fh}$$

(8)
$$< x_f a, x_h b > = t < a, x_{g^{-1}h} b > .$$

let Q be the intersection of C with the hyperplane $t = 0$. Then the points on Q satisfy the equations:

$$(9) \qquad\qquad x_f[a, x_h b] = [x_f a, x_{fh} b]$$

$$(10) \qquad\qquad [x_f a, x_f b] = 0$$

$$(11) \qquad\qquad x_f[x_{f^{-1}}, b, c] = [a, x_f b, x_f c]$$

$$(12) \qquad\qquad < x_f a, x_f b >= 0$$

$$(13) \qquad\qquad < x_f a, b >=< a, x_{f^{-1}} b >$$

$$(14) \qquad\qquad x_e = 0$$

$$(15) \qquad\qquad x_f x_h = 0$$

$$(16) \qquad\qquad < x_f a, x_h b >= 0$$

Now denote by I_f the image of $x_f \in \operatorname{End} L$ in L.

A subspace I of a Lie algebra L is called an *inner ideal*, if $[[I, L], I] \leq I$.

LEMMA 3. *If* $\bar{x} = ((x_f)_{f \in F}, 0) \in Q$, *then all* I_f *are inner ideals of* L, *i.e.*

$$[[L, I_f], I_f] \leq I_f.$$

Proof: An immediate consequence of Equation 11. $\qquad\qquad\square$

In what follows $\bar{x} = ((x_f)_{f \in F}, 0) \in Q$.

LEMMA 4. *Under these assumptions we have*

$$[I_f, I_f] = 0$$

and

$$[I_f, I_h] \leq I_f \cap I_h.$$

In particular,

$$I = \langle I_f, \ f \in F \rangle$$

is a nilpotent subalgebra of L *and consists of nilpotent elements.*

Proof: Equation 10 immediately yields $[I_f, I_f] = 0$. We also have from Equation 9 that

$$x_h[x_{h^{-1}} f a, b] = [x_f a, x_h b] = x_f[a, x_{f^{-1}h} b],$$

which means that $[I_f, I_h] \leq I_f \cap I_h$. Next, by Equation 16 we have $< I_f, I_h >= 0$, so the restriction of the Killing form on I is trivial. So I is nilpotent and consists of nilpotent elements. $\qquad\qquad\square$

Now denote $K_f = \ker x_f$.

LEMMA 5.
$$I_f^{\perp} \leq K_{f^{-1}}.$$

Proof: By Equation 13
$$< x_f a, b > = < a, x_{f^{-1}} b > .$$
If $b \in I_f^{\perp}$ then $< x_f a, b > = 0$ and $< a, x_{f^{-1}} b > = 0$ for all $a \in L$. But then, since the Killing form $< , >$ is non-degenerate, $x_{f^{-1}} b = 0$ and $b \in K_{f^{-1}}$. $\qquad\square$

Since I is a nilpotent subalgebra it lies in a maximal nilpotent subalgebra N which, in its turn, lies in a Borel subalgebra B.

LEMMA 6. *For all $f \in F$*
$$I_f \leq N < B \leq K_f.$$

Proof: By a well-known property of simple Lie algebras $B = N^{\perp}$. Therefore $B \leq I_f^{\perp} \leq K_{f^{-1}}$ for all $f \in F$. So
$$B \leq \cap_{f \in F} K_{f^{-1}},$$
and $B \leq K_f$ for all $f \in F$. $\qquad\square$

Now consider the action of G on V given by
$$g \cdot ((x_f)_{f \in F}, t) \mapsto ((g^{-1} x_f g) f \in F, t),$$
where g stands for an arbitrary element of G.

LEMMA 7. *Every point $\bar{x} \in Q$ is unstable under the action of G on V, i.e. the closure $\overline{G \cdot \bar{x}}$ of the G-orbit of $\bar{x}$ contains 0.*

Proof: Let $\bar{x} = ((x_f)_{f \in F}, 0) \in Q$. Take a Borel subgroup B as in Lemma 6, then
$$I_f \leq N < B \leq K_f$$
for all $f \in F$. We can chose a Chevalley basis in L agreed with B. Now let Λ be a one-parameter subgroup in the torus $H = \langle h_r(\lambda), r \in \Pi \rangle$ of the form
$$h(\lambda) = h_{r_1}(\lambda) \cdots h_{r_k}(\lambda), \ \ r_i \in \Pi$$
(here Π is the system of fundamental roots). Then, if $s \in \Phi^+$,
$$h(\lambda) \cdot e_s = \lambda^{\sum A_{rs}} e_s,$$
where r runs tough Π and
$$A_{rs} = \frac{2(rs)}{(rr)} \geq 0.$$
At least one of the coefficients A_{rs} is positive, so for $s \in \Phi^+$ we have
$$h(\lambda) \cdot e_s = \lambda^{C_s} e_s$$
for $C_s > 0$.

Analogously for $s \in \Phi^-$ we have
$$h(\lambda) \cdot e_s = \lambda^{C_s} e_s$$
with $C_s < 0$.

Now, since $B = \langle h_r, e_s, r \in \Pi, s \in \Phi^+ \rangle$, and $I_f \leq N < B \leq K_f$, a linear transformation $x_f \in \text{End } L = L \otimes L^*$ has a form
$$x_f = \sum_{r \in \Phi^+, s \in \Phi^-} \kappa_{rs} e_r \otimes e_s^*.$$

Therefore

$$
\begin{aligned}
h(\lambda) \cdot x_f &= \sum_{r \in \Phi^+, s \in \Phi^-} \kappa_{rs} h(\lambda) e_r \otimes h(\lambda)^* e_s^* \\
&= \sum_{r \in \Phi^+, s \in \Phi^-} \kappa_{rs} \lambda^{C_r} \cdot \lambda^{-C_s} \cdot e_r \otimes e_s^* \\
&= \sum_{r \in \Phi^+, s \in \Phi^-} \kappa_{rs} \lambda^{C_r - C_s} \cdot e_r \otimes e_s^*
\end{aligned}
$$

Notice that all coefficients $C_r - C_s$ are strictly positive, so sending λ to 0, we get 0 as the limit point of $h(\lambda)x_f$. Thus the Λ-orbit of $\bar{x} = ((x_f)_{f \in F}, 0)$ has 0 in its closure and $\bar{x}$ is unstable. $\qquad\square$

4. Wide subgroups

Let G be a reductive algebraic group and F a finite group. We say that F is *wide in respect to G* (in characteristic char K) if every nontrivial representation $F \longrightarrow G$ is irreducible.

Let $\mathrm{Rep}^*(F, G)$ denotes the subvariety of $\mathrm{Rep}(F, G)$ whose points

$$
\bar{x} = (x_{f_1}, \dots, x_{f_n})
$$

have the property that at least one component x_{f_i} is a non-trivial semisimple element. Since conjugacy classes of semisimple elements in G are closed and the number of conjugacy classes of semisimple elements of order $\leq |F|$ is finite, $\mathrm{Rep}^*(F, G)$ is a closed subset. Moreover, it is obvious that if F is wide with respect to G then F is generated by elements of order coprime to char K and thus

$$
\mathrm{Rep}(F, G) = \{1\} \cup \mathrm{Rep}^*(F, G).
$$

THEOREM 4. *Assume that a finite group F is wide with respect to a reductive algebraic group G.*

Assume also that the Lie algebra $L = \mathrm{Lie}\,(G)$ of the group G has a non-degenerate Killing form $<,>$. Then $\mathrm{Rep}(F, G)$ has only finitely many G-orbits.

Proof: Obviously it is enough to consider the case of adjoint group G, then $G = (\mathrm{Aut}\,L)^\circ \subset \mathrm{End}\,L$ and G is semisimple. We can use the notation and results of Section 3

By Theorem 2 the orbits of G on $R = \mathrm{Rep}^*(F, G)$ are stable. Consider now the action of G on C. Every point of C either lies over a point of R and thus stable or lies over Q and thus unstable by the previous lemma. By [**Sesh**] the quotient variety R/G is projective. But, since all orbits of G on R are closed, the algebra of invariants $K[R]^G$ distinguishes the points of R/G, therefore R/G is the affine variety determined by this algebra. So the variety R/G, being affine and projective, is finite. $\qquad\square\square$

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