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2004

MIMS EPrint: **2008.47**

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ISSN 1749-9097

RANK ANALYSIS OF THE ANISOTROPIC INVERSE CONDUCTIVITY PROBLEM

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ABSTRACT: Anisotropic conductivity arises in bio-medical EIT, for example in muscle tissue, however the anisotropic inverse conductivity problem can be solved uniquely only up to a diffeomorphism which fixes all points on the boundary. Although a scalar conductivity can be reconstructed from current and voltage measurements, extra information must be provided to uniquely recover the full symmetric tensor field in the anisotropic case. Rank analysis of the sensitivity matrix reveals that fixing the mesh a-priori selects one of the infinity of possible diffeomorphisms, but may lead to a wrong solution. Hence one should not attempt solving the problem for a fixed mesh, and we suggest a procedure to resolve the ambiguity by imposing some constraints which restore uniqueness of the solution.

Keywords: Anisotropy, Dirichlet-to-Neumann mapping, Constraints, Jacobian.

1. INTRODUCTION

Anisotropic conductivity is present in bio-medical EIT, for example in muscle or bone tissue which are highly anisotropic. The equation that models the electrical potential ϕ in a domain Ω with conductivity σ is Laplace's equation

$$\sum_{i,j=1}^3 \frac{\partial}{\partial x_i} \left(\sigma_{ij} \frac{\partial \phi}{\partial x_j} \right) = 0 \quad (1)$$

where for an anisotropic medium σ_{ij} is a symmetric positive definite tensor field.

The inverse problem is to determine σ in Ω from the Dirichlet-to-Neumann (DtN) mapping or voltage-to-current relation

$$\Lambda_\sigma : \phi|_{\partial\Omega} \rightarrow \mathbf{n} \cdot \sigma_{ij} \frac{\partial \phi}{\partial x_j} \quad (2)$$

where \mathbf{n} is the unit outward normal to the surface $\partial\Omega$.

Uniqueness for the isotropic inverse conductivity problem was first proved by Kohn and Vogelius in [3] for real analytic conductivity and extended to piecewise analytic conductivity in [2]. On the other hand, for an anisotropic analytic conductivity the DtN mapping is invariant under a diffeomorphism Ψ from $\bar{\Omega}$ to $\bar{\Omega}$ fixing points on the boundary [4], that is $\Lambda_\sigma = \Lambda_{\tilde{\sigma}}$, where $\tilde{\sigma}$ is the transformed conductivity under Ψ and is given by the relation

$$\tilde{\sigma} = \frac{(D\Psi)\sigma(D\Psi)^T}{|D\Psi|} \mathcal{M}^{-1}, \quad (3)$$

where $D\Psi$ is the Jacobian of the diffeomorphism Ψ .

The aim of this paper is to analyse numerically the non-uniqueness of the anisotropic inverse problem. We verify the invariance of the DtN mapping (3) for a finite element (FE) approximation, showing that there are two different conductivities that satisfy the same boundary data. To get a unique solution some

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constraints must be imposed. Rank analysis of the Jacobian shows that solving the problem for a fixed mesh constrains the problem, if a particular diffeomorphism is selected, and may lead to a wrong solution. From this analysis we suggest a procedure to recover a unique solution by removing the uncertainty in the diffeomorphism.

2. MODEL

The forward problem has been solved using a modified version of EIDORS-3D toolkit [5] which uses first order tetrahedral finite elements. Some changes are necessary to extend isotropic EIDORS-3D for anisotropic media, specifically the representation of the conductivity tensor, the calculation of the system matrix, the Jacobian and the Regularisation matrix [1].

Since only six conductivity coefficients are independent, we can represent the conductivity tensor as the vector $\bar{\sigma} = [\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{yz}, \sigma_{zx}]$. The Jacobian measures the sensitivity of the voltage measurements to changes in the conductivity. An efficient way to compute the Jacobian (given in [5]) is to calculate the tensor $J_{d,m}^k$ for each element Ω^k as the tensor product of the driven and the measurement fields (with boundary current patterns I_d and I_m respectively)

$$J_{d,m}^k = - \int_{\Omega^k} \nabla \phi(I_d) \otimes \nabla \phi(I_m) dx dy dz, \quad (4)$$

which has only six independent components so we can denote the Jacobian for each element Ω^k by

$$\bar{J} = [J_{xx}, J_{yy}, J_{zz}, J_{xy}, J_{yz}, J_{zx}]. \quad (5)$$

Each entry is the sensitivity of a voltage measurement with respect to each of the coefficients of $\bar{\sigma}$. Note that for an isotropic formulation the tensor product in (4) becomes a scalar product and (5) a scalar.

To solve the inverse problem using a nonlinear method one may compute on each step the correction on the conductivity as $\delta\sigma = J^\diamond \delta V$, where δV is the difference between predicted and measured voltages and J^\diamond is the regularised inverse of the Jacobian. Therefore, a rank analysis of the Jacobian quantifies how much information is available for the reconstruction of the conductivity. The most reliable way to examine the rank of the Jacobian is the singular value decomposition.

3. METHODOLOGY

We study the non-uniqueness of the anisotropic inverse problem, first by verifying numerically the invariance of the DtN mapping, and second by looking at the rank of the Jacobian.

The invariance of the DtN mapping was first noted in [3]. The non-injectivity of the DtN mapping can be verified for the FE case by showing that there are two different conductivity distributions σ and $\bar{\sigma}$, related by a diffeomorphism that produce the same voltages at the boundary. It shows that once the mesh is fixed, there is a unique conductivity distribution consistent with the DtN mapping, since fixing the mesh selects one of the infinitely many possible diffeomorphisms. To study the effect of fixing the mesh we look at the rank of the Jacobian for both the isotropic and the anisotropic formulations.

For the rank-analysis, in a n -elements mesh we simulate a current injection which consists of l voltage measurements where only m of them are independent. Hence, J is a $(l \times 6n)$ matrix for the anisotropic formulation, while for the isotropic case J is a matrix of dimensions $(l \times n)$. Our rank analysis is performed on the Jacobian by identifying the first large step reduction in the singular values. We denote by $r = \text{rank}(J)$ the (effective numerical) rank of the Jacobian. This gives the minimum value between m and the number of unknowns that can be recovered from the given noise free data.

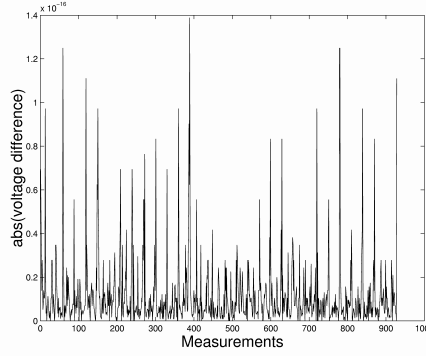
What we want to show is that while for an isotropic formulation $r = n$, when $m \geq n$, for an anisotropic conductivity we expect $n \leq r \leq 6n$.

4. RESULTS

4.1. Invariance of the DtN mapping

The invariance of the DtN mapping is verified numerically by applying a known diffeomorphism in Ω and comparing the voltages at the boundary given by $\bar{\sigma}$ and $\tilde{\sigma}$. Let $\bar{\sigma}$ be an anisotropic conductivity defined in a cylindrical mesh FE, the vector of surface voltages V is computed by solving the Forward Problem F such that $F(\bar{\sigma}) = V$. The diffeomorphism Ψ is simulated by moving randomly all interior nodes, avoiding a big distortion of the elements and leaving the boundary unchanged, such that we get a new mesh and a known diffeomorphism. The transformed conductivity $\tilde{\sigma}$ is calculated using (3) for each of the elements. Then, by solving the Forward Problem for $\tilde{\sigma}$ in the new mesh, that is $F(\tilde{\sigma}) = \tilde{V}$, we get that $|\tilde{V} - V| \leq 10^{-16}$. Figure 1 shows the voltage difference $(\tilde{V} - V)$ for a numerical simulation using an adjacent current pattern with 32 electrodes.

Figure 1: Boundary voltage difference $(\tilde{V} - V)$



4.2. Rank Analysis. Fixing the mesh

For an isotropic σ , the number of independent measurements m gives the rank r of the Jacobian. The goals of this paper are to analyse r of anisotropic σ for a fixed mesh and to estimate the maximum number of unknowns we can recover for an anisotropic conductivity tensor given the DtN mapping.

First, for a spherical mesh of $n = 148$ elements and for an adjacent current pattern, we vary the number of measurements l and calculate r for $\sigma = 1$ by both the isotropic and the anisotropic formulations, where $r_I(\sigma_I)$ and $r_A(\sigma_I)$ are the ranks calculated by the isotropic and the anisotropic formulation respectively. For $m \leq n$, $r_I(\sigma_I)$ and $r_A(\sigma_I)$ gives the number of independent measurements m , which provides the number of unknowns we can recover. We also calculate $r_A(\sigma_I)$ for σ_I a non-homogeneous isotropic matrix and get the same results, so then the rank does not depend on the σ values, as we expected. To analyse the effect of fixing the mesh and the ambiguity of the diffeomorphism we must study r for the case $m \geq n$, where r_I is limited by the number of elements n , that is, $r_I = n$, and r_A is expected to be in the range $n \leq r_A \leq 6n$. The maximum number of independent measurements that we can get depends on the number and the position of the electrodes, the current pattern, the geometry of the object and the discretisation. We look at r_A for different values of these parameters for showing the maximum number of unknowns we can recover for a given mesh. Table 1 shows the maximum r_A achievable, for a sphere ($n = 148$), a cylinder ($n = 828$) and a cube

($n = 384$), by adding as many electrodes as possible, changing their position and implementing opposite and adjacent current-patterns.

Table 1: \mathbf{r}_I and \mathbf{r}_A , Jacobian ranks calculated by the isotropic and the anisotropic formulation respectively for isotropic and anisotropic conductivities σ_I and σ_A .

Geometry	$\mathbf{r}_I(\sigma_I)/n$	$\mathbf{r}_A(\sigma_I)/n$	$\mathbf{r}_A(\sigma_A)/n$	$\frac{\# \text{surfaces}}{\# \text{elements}}$	$\frac{\# \text{vertices}}{\# \text{elements}}$
Sphere	1	1.73	3.1	0.71	0.33
Cube	1	1.1	2.5	0.5	0.33
Cylinder	1	1.3	2.5	0.45	0.3

Having considered different geometries we found that $n \leq r_A \leq 3n$ for anisotropic conductivity σ_A , starting from anisotropic conductivity close to isotropic with the calculation of $r_A(\sigma_I)$ to non-homogeneous anisotropic conductivity with $r_A(\sigma_I)$.

5. CONCLUSIONS

We have checked first that for a given mesh there is a unique discrete conductivity distribution that agrees with the voltage data and that choosing the mesh is equivalent to selecting a diffeomorphism by representing the conductivity tensor on a particular reference coordinate system. We have gone on to show that for an anisotropic formulation that the rank of the Jacobian is in the range $n \leq r_A \leq 3n$, when the mesh is fixed. While for the continuous case it is known that we are able to recover the anisotropic tensor only up to a diffeomorphism and then by providing the diffeomorphism we expect to recover the complete conductivity tensor, there is not yet theoretical explanation for piecewise constant conductivity FE case. For the Finite Element case by fixing the mesh we select one of the many possible diffeomorphisms by fixing the position of the vertices, however since there are approximately three times more elements than vertices, then we are not able to recover more than $3n$ unknowns.

The problem is that by selecting the mesh we provide information of a specific diffeomorphism, which may lead to a wrong solution since we are selecting a representation on the conductivity that may differ from the physical solution we are interested in. Since the anisotropic inverse conductivity problem is uniquely reconstructed only up to a diffeomorphism fixing the boundary, the idea is to provide a constraint that allows the recovery of the unknown diffeomorphism. We suggest a procedure for resolving the ambiguity in the diffeomorphism and recovering a unique conductivity by imposing some constraints on the solution. First, get an estimate of the conductivity for a fixed mesh, and then find the diffeomorphism that makes $\tilde{\sigma}$ satisfy the constraints.

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