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Quotient spaces and critical points of invariant functions for C*-actions

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Abstract

Consider a linear action of the group $\mathbf{C}^*$ on $X = \mathbf{C}^{n+1}$. We study the fundamental algebraic properties of the sheaves of invariant and basic differential forms for such an action, and use these to define an algebraic notion of multiplicity for critical points of functions which are invariant under the $\mathbf{C}^*$-action. We also prove a theorem relating the cohomology of the Milnor fibre of the critical point on the quotient space with this algebraic multiplicity.

Contents

Introduction

If an analytic function germ f on $X = \mathbf{C}^{n+1}$ has an isolated critical point at 0, then under any 1-parameter deformation f_t of f this critical point decomposes into finitely many (simpler) critical points. For a generic deformation, the simpler critical points are all non-degenerate and in this case the number of critical points can be computed algebraically as the dimension of the Jacobian algebra, $\dim_{\mathbf{C}}(\mathcal{O}_X/Jf)$, where Jf is the Jacobian ideal generated by the partial derivatives of f. The essential reasons for this are that the partial derivatives $(\partial f/\partial x_i)$ form a regular sequence, and that for a non-degenerate critical point the Jacobian algebra has dimension 1.

Consider now a linear action of a finite group G on X and let f be an invariant function with an isolated critical point at 0. If f_t is an invariant deformation of f, then G acts by permuting the critical points of f_t. Moreover, if the critical points are non-degenerate (which is the case generically if the action is real) then the associated permutation representation of G is isomorphic to the representation of G on $(\mathcal{O}_X/Jf)$. Consequently, the number of group orbits of critical points is equal to $\dim_{\mathbf{C}}[\mathcal{O}_X/Jf]^G$, (where $[M]^G$ denotes the fixed point space of the G-space M). If the critical points in the deformation remain degenerate, then the permutation representation must be counted with appropriate multiplicites. For further details see [29] and [21].

If G is an infinite (reductive) group then invariant critical points are no longer isolated, and $(\mathcal{O}_X/Jf)$ is accordingly no longer finite dimensional. Furthermore, $[\mathcal{O}_X/Jf]^G$, which is finite dimensional, does not behave well in a deformation: its dimension is in general only upper semicontinuous. Mark Roberts has conjectured that for complexifications of representations of compact Lie groups on $\mathbf{R}^{n+1}$ this number is well behaved and determines the multiplicity of a degenerate invariant critical point [7].

An alternative approach is to use differential forms. If f has an isolated critical point, then the complex $(\Omega^{\cdot}_X, df\wedge)$ of differential forms on X

$$0 \to \mathcal{O}_X \xrightarrow{df\wedge} \Omega^1_X \xrightarrow{df\wedge} \Omega^2_X \xrightarrow{df\wedge} \cdots \xrightarrow{df\wedge} \Omega^n_X \xrightarrow{df\wedge} \Omega^{n+1}_X \to 0, \tag{0.1}$$

is exact except for $H^{n+1}(f) := H^{n+1}(\Omega^{\cdot}_X, df\wedge) = \Omega^{n+1}_X/df\wedge\Omega^n_X$. The complex is therefore a free resolution of this $\mathcal{O}_X$-module, and it follows that in a family of functions f_t the sum $\sum_x \dim_{\mathbf{C}} H^{n+1}(f)_x$ is constant. (This is really the same reason as given in the first paragraph: the partial derivatives forming a regular sequence. Any isomorphism $\mathcal{O}_X \to \Omega^{n+1}_X$ takes Jf onto $df\wedge\Omega^n_X$, and the complex $(\Omega^{\cdot}_X, df\wedge)$ is isomorphic to the Koszul complex on the partial derivatives.)

If the function is invariant under a finite group G, then one can also consider $H^{n+1}_G(f) := [H^{n+1}(f)]^G = \underline{\Omega}^{n+1}_X/df\wedge\underline{\Omega}^n_X$, where $\underline{\Omega}^p_X$ denotes *invariant* differential forms. This also behaves well under deformations and so defines a multiplicity of the isolated critical point, though it does not necessarily agree with the multiplicity defined by $[\mathcal{O}_X/Jf]^G$.

This approach has the advantage that it does generalize to the infinite groups, and the main purpose of this paper is to establish this for $G = \mathbf{C}^*$, the simplest infinite reductive group. We expect that the results on multiplicity hold in greater generality — the basic feature here is that for $\mathbf{C}^*$ all the computations can be done explicitly. If the $\mathbf{C}^*$-action is the complexification of an $\mathbf{S}^1$-action on $\mathbf{R}^{n+1}$, then $\underline{\Omega}_X^{n+1}/df\wedge\underline{\Omega}_X^n \cong [\mathcal{O}_X/Jf]^{\mathbf{C}^*}$, and consequently the latter behaves well under a deformation of f, supporting Roberts' conjecture.

The paper is organized as follows.

Section 1 consists of background material on quotients by $\mathbf{C}^*$-actions and their natural stratifications by orbit type; most, if not all, of this section is well-known.

Sections 2 and 3 aim at understanding the $\mathbf{C}^*$-equivariant analogues of (0.1). In Section 2 we consider the two classes of "equivariant" differential forms, the *invariant* forms and the *basic* forms. The first are forms on X which are invariant under the group action, while the latter are those invariant forms which annihilate vector fields tangent to the firbres of the quotient map, and so are more properly forms on the quotient space Y. Accordingly, there are two equivariant analogues of (0.1), which are intimately linked. These complexes are both studied in Section 3, where it is seen that in contrast to the ordinary case, they are not in general acyclic, although their low cohomology groups depend more on the $\mathbf{C}^*$-action than on the critical point in question. Section 3 concludes with a brief discussion of the implication of local duality for the cohomology groups of the analogue of (0.1) using basic forms.

Most of Sections 2 and 3 are written with the simplifying assumption that the origin in X is an isolated fixed point of the $\mathbf{C}^*$-action. The modifications for the general case are described in Remarks 2.10 and 3.8.

The top cohomology group of (0.1) gives the multiplicity of an isolated critical point. In the same way, the top cohomology group of the equivariant counterparts can be used to define a multiplicity of an invariant critical point. Section 4 uses the results of Section 3 to show that this multiplicity behaves well in a deformation, so can indeed be called a multiplicity. We also give some estimates on the multiplicity of generic critical points away from the fixed point set of the $\mathbf{C}^*$-action. In Section 5 we compare the multiplicity defined in Section 4 with the Jacobian algebra approach described above.

In Section 6 we use techniques due to Malgrange to show that the cohomology of the Milnor fibres in the quotient space of an invariant function with an isolated critical point is given by the cohomology of the analogue of (0.1) with basic forms. We also relate the cohomology of this quotient Milnor fibre to the Chern class of the quotient map, which is an extension of a theorem of Duistermaat & Heckman.

The paper concludes with an appendix containing an account of some simple basic facts on local cohomology which are relied on heavily in Sections 2 and 5. Although most of the material contained in the appendix is well-known to experts, it also serves to establish some notation which facilitates the spectral sequence calculations performed in Section 2.

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1 $\mathbf{C}^*$-actions and their quotient spaces

We will be considering linear actions of $\mathbf{C}^*$ on $X = \mathbf{C}^{n+1}$. Any such action can be diagonalized, so that the action is determined solely by a list of $n+1$ integers, the *weights*. We can assume that the highest common factor of the weights is 1. It will be convenient to use a notation which distinguishes between the positive weights, the negative weights and the zero weights and their respective coordinates. Let a be the number of positive weights, b the number of negative weights and c the number of zero weights. Thus, $n+1 = a+b+c$. Let $\lambda_1,\dots,\lambda_a$ be the positive weights and $\mu_1,\dots,\mu_b$ be the negative weights. We denote the corresponding coordinates by $x_1,\dots,x_a$, $y_1,\dots,y_b$, and $z_1,\dots,z_c$. We assume that $a,b>0$ (otherwise the invariant functions would just be functions on the fixed point set $F = \mathbf{C}^c$). We also assume that $a \geq b$, for the involution of $\mathbf{C}^*$ given by $t \mapsto t^{-1}$ changes the signs of all the weights, but leaves the invariant theory invariant! In this notation, $t \in \mathbf{C}^*$ acts by

$$t\cdot(x_1,\dots,x_a,y_1,\dots,y_b,z_1,\dots,z_c) = (t^{\lambda_1}x_1,\dots,t^{\lambda_a}x_a,t^{\mu_1}y_1,\dots,t^{\mu_b}y_b,z_1,\dots,z_c).$$

The vector field which generates this $\mathbf{C}^*$-action is,

$$\vartheta = \sum_{i=1}^{a} \lambda_i x_i \tfrac{\partial}{\partial x_i} + \sum_{j=1}^{b} \mu_j y_j \tfrac{\partial}{\partial y_j}. \tag{1.2}$$

The quotient space

The $\mathbf{C}^*$ orbits are all 1-dimensional except those in the fixed point set $F = \mathbf{C}^c$. The orbits which are not closed lie in the "bad planes" (or *null cones*)

$$\mathcal{B}_+ = \{(x,0,z)\} \qquad \text{and} \qquad \mathcal{B}_- = \{(0,y,z)\}.$$

Each orbit in the bad set $\mathcal{B} = \mathcal{B}_+ \cup \mathcal{B}_-$ contains a unique point of F in its closure. The *quotient space* Y as a set is defined to be the set of closed orbits. The quotient map $\pi : X \to Y$ associates to each point $x \in X$ the unique closed orbit in the closure of the orbit through x. The restriction $\pi|_F : F \to \pi(F)$ is clearly an isomorphism, so we denote $\pi(F)$ by F as well. Note then that $\pi^{-1}(F) = \mathcal{B}$. The topology on Y is the finest such that π is continuous.

The algebraic structure on Y is given by the ring of invariant polynomials on X, denoted by R. The invariant polynomials separate the closed orbits (but not the others, of course). The ring R is finitely generated by, say, $\pi_1,\dots,\pi_l$ (which can be chosen to be monomials) and the quotient map π can be identified with $(\pi_1,\dots,\pi_l) : X \to \mathbf{C}^l$. It is easy to see that $l \geq ab+c$, since for each pair (i,j), with $1 \leq i \leq a$, $1 \leq j \leq b$, there must be a generator of the form $x_i^r y_j^s$ for some r,s. Furthermore, since $\dim(Y) = n = a+b+c-1$ it follows that Y is never smooth unless $b = 1$. In Section 2, we show that if $b \neq 1$ then Y is not even isomorphic to a finite quotient of $\mathbf{C}^n$.

The following result is well-known.

Proposition 1.1 *The quotient space is a normal, Cohen-Macaulay variety.*

PROOF That it is Cohen-Macaulay follows from a general theorem of [15], see also [16], which states that the quotient space for any reductive group action on a smooth space is Cohen-Macaulay. It also follows from the local cohomology computations we do in Section 2. It is easy to see that the quotient by a reductive group of any normal space is normal: just take the invariant part of any monic polynomial in the definition of normality. □

It is not true in general that the quotient of a Cohen-Macaulay space by a reductive group is again Cohen-Macaulay, unlike the case for finite groups. A simple example can be found in Remark 5.7. However, [5] has established that the quotient by a reductive group action of a variety with only rational singularities itself has only rational singularities.

We now give a brief account of the geometry of Y. Let $X_0 = \mathbf{C}^{a+b} \subset X$, so that $X = X_0 \times F$. Let Y_0 be the quotient of X_0 by $\mathbf{C}^*$, so that $Y = Y_0 \times F$. Now, there is another action of $\mathbf{C}^*$ on X_0 which commutes with the given one, namely $s \in \mathbf{C}^*$ acts by

$$s \cdot (x_i, y_j) = (s^{\lambda_i} x_i, s^{-\mu_j} y_j).$$

Note that all the weights are positive. We denote this copy of $\mathbf{C}^*$ by $\mathbf{C}^*_+$. The action of $\mathbf{C}^*_+$ passes down to an action on Y_0 whose only fixed point is $0 \in Y_0$. Consider $(Y_0 \setminus 0)/\mathbf{C}^*_+$. This is isomorphic to $(X \setminus \mathcal{B})/(\mathbf{C}^* \times \mathbf{C}^*_+)$. Now, $\mathbf{C}^* \times \mathbf{C}^*_+$ acts by

$$(t,s) \cdot (x_i, y_j) = ((ts)^{\lambda_i} x_i, (ts^{-1})^{\mu_j} y_j).$$

The epimorphism $\phi : \mathbf{C}^* \times \mathbf{C}^*_+ \to T^2,\ (t,s) \mapsto (ts, ts^{-1}) = (u,v)$ (where T^2 is the complex 2-torus) induces an action of T^2 on $X \setminus \mathcal{B}$ by

$$(u,v) \cdot (x_i, y_j) = (u^{\lambda_i} x_i, v^{\mu_j} y_j).$$

The quotient $(X \setminus \mathcal{B})/T^2$ is thus isomorphic to the product of two weighted projective spaces, one is $\mathbf{P}(\lambda_1, \ldots, \lambda_a)$, the quotient of $\mathbf{C}^a \setminus \{0\}$ by the $\mathbf{C}^*$-action with weights $(\lambda_1, \ldots, \lambda_a)$, and the other is $\mathbf{P}(\mu_1, \ldots, \mu_b)$, the quotient of $\mathbf{C}^b \setminus \{0\}$ by the $\mathbf{C}^*$-action with weights $(\mu_1, \ldots, \mu_b)$. It follows that $Y = Y_0 \times F$ and Y_0 is a 'weighted cone' on the product $\mathbf{P}(\lambda_1, \ldots, \lambda_a) \times \mathbf{P}(\mu_1, \ldots, \mu_b)$. (For details on weighted projective spaces see [9] and [11].)

The real link S of the origin in Y, which is the intersection of Y with a real $(2l-1)$-sphere surrounding 0, has real dimension $2n-1$. One can show that the rational homology is as follows: the betti numbers of S are 1 in all even degrees up to and including $2(b-1)$ and in all odd degrees from $2a-1$ up to $2n-1 = \dim S$; the other betti numbers are zero. We do not make any use of this fact so do not give a proof here.

Example 1.2 Consider the $\mathbf{C}^*$-action on $X = \mathbf{C}^{n+1}$ with $\lambda_1 = \cdots = \lambda_a = 1$ and $\mu_1 = \cdots = \mu_b = -1$. If $c = 0$, then this action is free outside $\{0\}$, and so Y has an isolated singular point. The quotient space is just the cone on $\mathbf{CP}^{a-1} \times \mathbf{CP}^{b-1}$, and if we write the invariants as $a_{ij} = x_i y_j$, then it is clear that the quotient space can be identified with the variety of $a \times b$ matrices (a_{ij}) of rank 1.

This action ‘covers’ the action with with same values of a,b,c but general values of λ_i and μ_j by the following diagram

$$\begin{array}{ccccc} \mathbf{C}^*\times X & \xrightarrow{\phi_1} & X & \xrightarrow{\pi_1} & Y_1 \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{C}^*\times X & \xrightarrow{\phi} & X & \xrightarrow{\pi} & Y \end{array}$$

where ϕ_1 is the action with weights ± 1 and Y_1 its quotient. The vertical arrows are the quotient maps for the action of the product of cyclic groups

$$G = \mathbf{Z}_{\lambda_1}\times\cdots\times\mathbf{Z}_{\lambda_a}\times\mathbf{Z}_{\mu_1}\times\cdots\times\mathbf{Z}_{\mu_b}$$

on X, with each factor acting on the appropriate coordinate, and its induced action on the quotient Y_1.

Let $U\subset Y$, and consider the ring of invariant analytic functions on $\pi^{-1}(U)$. This defines a presheaf of rings on Y, which can be sheafified to form the sheaf $\mathcal{O}_Y$ on Y of germs of invariant analytic functions on X. It is a coherent sheaf, by the general results of [22].

Let $x\in X\setminus\mathcal{B}$, and denote by H_x the isotropy subgroup at x (that is, the subgroup of $\mathbf{C}^*$ leaving x fixed). There exists a complex submanifold (germ) at x which is invariant under H_x and transverse to $\mathbf{C}^*.x$, the $\mathbf{C}^*$ orbit through x; it is denoted S_x and called the slice at x. One defines the *twisted product* $\mathbf{C}^*\times_{H_x}S_x$ to be the quotient of $\mathbf{C}^*\times S_x$ by the H_x-action $h\cdot(t,s)=(th^{-1},hs)$. $\mathbf{C}^*$ acts on this space by $t_1\cdot[t,s]=[t_1t,s]$ (where $[t,s]$ denotes the point corresponding to (t,s) in the twisted product). The quotient of this action is isomorphic to S_x/H_x. The action of H_x on S_x is called the *slice representation*.

Theorem 1.3 (The Slice Theorem: Luna, [17]) *Let x lie on a closed orbit. With notation as above, S_x can be chosen so that the $\mathbf{C}^*$ equivariant map*

$$\begin{array}{rcl} \phi:\mathbf{C}^*\times_{H_x}S_x & \to & X \\ {[t,s]} & \mapsto & t\cdot s \end{array}$$

is an isomorphism onto a $\mathbf{C}^*$-invariant neighbourhood U of x. It follows that ϕ passes down to an isomorphism $\overline{\phi}:S_x/H_x\to\pi(U)$, a neighbourhood of $\pi(x)\in Y$.

PROOF (Outline) It is easy to see that (i) ϕ is well-defined; (ii) that $d\phi_{(1,x)}$ is an isomorphism, and thus is an isomorphism at each point of $\mathbf{C}^*\times_{H_x}S_x$ if S_x is sufficiently small, and (iii) ϕ is a bijection, again if S_x is chosen to be suitably small. The result follows. Luna in fact proves this theorem in the algebraic category, where (iii) is considerably more subtle. □

The stratification of the quotient space

The quotient space Y comes equipped with a natural stratification: the *stratification by orbit type*. For each isotropy subgroup H of the $\mathbf{C}^*$-action, the associated stratum of Y consists of all closed orbits in X with isotropy group precisely H, and we denote this by $Y_{(H)}$. Let

$X_{(H)} = \pi^{-1}(Y_{(H)}) \setminus \mathcal{B}$, so that $X_{(H)}$ consists of points on closed orbits which have isotropy precisely H. Clearly, $X_{(H)}$ is a submanifold of X contained in $\mathrm{Fix}(X;H)$, the fixed point set of H. Moreover, the group $\mathbf{C}^*/H$ (which is either trivial or isomorphic to $\mathbf{C}^*$) acts freely on $X_{(H)}$, and so the restriction of π to $X_{(H)}$ is a submersion onto $Y_{(H)}$, which is therefore a manifold itself. The stratification of Y by orbit type is given by the collection of manifolds $Y_{(H)}$ as H varies through the isotropy subgroups of $\mathbf{C}^*$ including $Y_{(\mathbf{C}^*)} = F$.

A (closed) orbit $\xi \in Y$ is said to be *regular* if its isotropy subgroup is trivial, and the set of all regular closed orbits is denoted Y_{reg}. Because of the nature of the $\mathbf{C}^*$-action (recall we are assuming that the h.c.f. of all the weights is one) the non-regular orbits in X are contained in coordinate hyperplanes, so form a subspace of codimension at least 1 in both X and Y. Suppose $Y_{(H)}$ has codimension 1 in Y and that $\xi \in Y_{(H)}$. Then for $x \in \pi^{-1}(\xi)$, H acts on a neighbourhood of x by pseudoreflections (i.e. the generator of the cyclic group H has only one eigenvalue different from 1). The quotient is therefore smooth at any point in $Y_{(H)}$, and we see that Y is non-singular in codimension 1, in accordance with Proposition 1.1. The open subspace of Y consisting of regular points and these pseudoreflexion hyperplanes will be denoted U. Obviously, $U = Y_{\mathrm{reg}}$ if and only if the action is without pseudoreflexions.

This seems to be a convenient point to state the relationship between critical points of functions on Y and of their lift to X. (We use the same notation for a function on Y and its lift to X.) Recall first that a function on a stratified set has a *stratified critical point* at ξ if its restriction to the stratum through ξ has a critical point at ξ. Recall also the so-called *principle of symmetric criticality*, which states that a function on a smooth manifold X invariant under the action of a reductive group G has a critical point at x if and only if its restriction to $\mathrm{Fix}(H_x)$ has a critical point at x. A geometric proof of this principle is roughly that the H_x-invariant complement to $T_x(\mathrm{Fix}(H_x))$ in T_xX has no trivial component and so df_x restricted to this complement must be 0. There is a more algebraic statement and proof as follows:

Lemma 1.4 *Let H be a subgroup of $\mathbf{C}^*$, and let $V = \mathrm{Fix}(H;X)$. Let f and f' be invariant functions on X whose restrictions to V agree. Then $f - f' \in I(V)^2$. Consequently, $Jf + I(V) = Jf' + I(V)$, where Jf is the Jacobian ideal of f.*

PROOF We may assume that $V = \{x_1 = \cdots = x_r = y_1 = \cdots = y_s = 0\}$ for some $r \le a$, $s \le b$, and let W be the complementary coordinate subspace, so $X = V \oplus W$. It is enough to prove the assertion for $f' = f_V$, where f_V is defined by $f_V(v,w) = f(v,0)$.

Now, $f - f_V$ is a sum of monomials, and it is enough to show that each monomial is in $I(V)^2$. So, $x^\alpha y^\beta$ is invariant if and only if $(\alpha,\lambda)+(\beta,\mu) = 0$. If $H = \mathbf{Z}/\mathbf{Z}_\nu$ then reducing this equation modulo μ gives $\sum_{i=1}^r \alpha_i\lambda_i + \sum_{j=1}^s \beta_j\mu_j = 0 \pmod{\nu}$. Since the λ_i, μ_j occurring in this sum are non-zero modulo ν it is not possible for all but one of the α_i and β_j to vanish, with the remaining one being equal to 1. □

Lemma 1.5 *Let f be an analytic function (germ) at $\xi \in Y$. Then f has a stratified critical point at ξ if and only if f has a critical point at some (and hence any) point x in the closed orbit in $\pi^{-1}(\xi)$.*

PROOF Suppose $\xi \in Y_{(H)}$ and choose any x to lie on the closed orbit in $\pi^{-1}(\xi)$, so $x \in X_{(H)} \subset X^H$. Now, $\pi : X_{(H)} \to Y_{(H)}$ is a submersion, and f is constant along the fibres, so it follows that f has a critical point at ξ if and only if its restriction to $X_{(H)}$ has a critical pont at x. By the principle of symmetric criticality, this is in turn equivalent to f having a critical point at x. □

Examples

We end this section with a brief discussion of three special classes of actions, firstly where $b = 1$, secondly actions for which the sum of the weights is zero, so the representation is in $SL_{n+1}(\mathbf{C})$, and thirdly "real actions". We will be returning to each of these in later sections.

Actions with one negative weight Suppose the $\mathbf{C}^*$-action has only one negative weight μ, so $b = 1$. Thus, $X = \mathbf{C}^a \times \mathbf{C} \times F$. Now the cyclic group $\mathbf{Z}/\mu\mathbf{Z} \subset \mathbf{C}^*$ acts on $X_1 = \mathbf{C}^a \times F$ and trivially on $\mathbf{C}^b = \mathbf{C}$. Let R_1 denote the ring of polynomials on X_1 invariant under this group, then there is a ring homomorphism $R \to R_1$, $p(x,y,z) \mapsto p(x,1,z)$. This is clearly injective, as a $\mathbf{C}^*$ invariant function which vanishes on $\{y = 1\}$ must be identically 0, and is surjective as each term in any $\mathbf{Z}/\mu\mathbf{Z}$ invariant function must have weight in $\mu.\mathbf{Z}$ (with respect to the $\mathbf{C}^*$-action). The terms can then be multiplied by appropriate powers of y to make the weight 0.

Thus if $b = 1$ and the only negative weight is μ, then Y is isomorphic to the product of $\mathbf{C}^a/(\mathbf{Z}/\mu\mathbf{Z})$ and F, and so is a cyclic quotient singularity. In particular, if $\mu = -1$, then Y is smooth. (Y is also smooth if the action of $\mathbf{Z}/\mu\mathbf{Z}$ on $\mathbf{C}^a$ is by pseudoreflexions, which happens when all but one of the λ_i are multiples of μ.)

It is not hard to show that the stratifications by orbit type of Y and of $Y_1 = X_1/(\mathbf{Z}/\mu\mathbf{Z})$ coincide outside F. If, furthermore, $\mathrm{Fix}(\mathbf{Z}/\mu\mathbf{Z}; X_1) = F$, then the stratifications coincide completely. Indeed, if we identify X_1 with $\mathbf{C}^a \times \{y = 1\} \times F \subset X$, then X_1 is invariant under $\mathbf{Z}/\mu\mathbf{Z}$ and a map $Y \to Y_1$ can be defined by $[x,y,z] \mapsto [x,y,z] \cap X_1$, (square brackets means the $\mathbf{C}^*$ orbit through a point — note that $[x,y,z] \cap X_1$ is a $\mathbf{Z}/\mu\mathbf{Z}$-orbit in X_1). Let $Y_{(H)}$ be a stratum of Y. Then either $H = \mathbf{C}^*$ or $H = \mathbf{Z}/\nu\mathbf{Z}$ for some $\mathbf{Z}/\nu\mathbf{Z} \subset \mathbf{Z}/\mu\mathbf{Z}$ (otherwise $\mathrm{Fix}(H;X) \subset \mathcal{B}$). Clearly, then, $\mathrm{Fix}(\mathbf{Z}/\nu\mathbf{Z}; X_1) = \mathrm{Fix}(\mathbf{Z}/\nu\mathbf{Z}; X) \cap X_1$. So the image of $Y_{(H)}$ is $Y_{1(H)}$. The stratifications of $Y \setminus F$ and $Y_1 \setminus F$ therefore coincide. If $\mathrm{Fix}(\mathbf{Z}/\mu\mathbf{Z}; X_1) = F$ then F is a stratum of Y_1 as well as of Y. The fact that the stratifications coincide on the complement of F was already noticed by [30] for $\mathbf{C}^*$-actions on $\mathbf{C}^3$.

Actions with the sum of the weights equal to zero These actions have some particularly nice properties. We will see in Section 2 that the quotient space is Gorenstein. For now though, we will limit ourselves to noting that the $\mathbf{C}^*$-action contains no pseudoreflexions, because a pseudoreflection cannot have determinant 1.

If the sum of the weights is 0, and there is only one negative weight μ we have that Y and the cyclic quotient Y_1 are isomorphic as stratified varieties, since in this case $\mathrm{Fix}(\mathbf{Z}/\mu\mathbf{Z}; X_1) = F$.

Real actions A complex representation of a (reductive) group is said to be *real* if it is the complexification of a real representation of a (real reductive) group. This is particularly simple in the case of finite groups, as the complexification of a finite group is the group itself. On the other hand, $\mathbf{C}^*$ can be viewed as the complexification of the circle group $\mathbf{S}^1 = \mathbf{SO}(2;\mathbf{R})$.

Let the circle group $\mathbf{S}^1$ act on $\mathbf{R}^{n+1}$, with rotation speeds $\lambda_1,\dots,\lambda_a,0,\dots,0$ with each $\lambda_i > 0$ and $(n-2a)$ 0's (note that λ and $-\lambda$ give isomorphic actions). The complexification of this action is the action of $\mathbf{C}^*$ on $\mathbf{C}^{n+1}$ with weights

$$(\lambda_1,\dots,\lambda_a,-\lambda_1,\dots,-\lambda_a,0,\dots,0)$$

Thus a $\mathbf{C}^*$-action is real if and only if the weights occur in equal and opposite pairs.

It follows from this characterization that real actions have the property that the sum of the weights is zero, so there are no pseudoreflections.

In [25], there are the following characterizations of real actions which we will need in Section 4.

Proposition 1.6 (Schwarz, [25]) *The following are equivalent:*

1. *The $\mathbf{C}^*$-action is real,*
2. *Every slice representation is real, and*
3. *There is an invariant non-degenerate quadratic form.*

The proof in the $\mathbf{C}^*$ case is easy (Schwarz's theorem is for general reductive group actions). In particular, if the weights are as above, then $x_1y_1+\cdots+x_ay_a+z_1^2+\cdots+z_c^2$ is an invariant non-degenerate quadratic form.

2 Invariant and basic differential forms

In order to do analysis on singular spaces it is useful to have a notion of differential forms. Now, for any singular space, there are the Kähler differentials, but these do not usually have very nice properties. In our case, Y is a quotient space for a $\mathbf{C}^*$-action so it is natural to use differential forms related to the group action. There are two such classes of differential forms: the invariant forms and the basic forms. In this section, we define these two classes of forms and then discuss some fundamental properties.

On $X = \mathbf{C}^{n+1}$ we have the ordinary holomorphic differential forms, Ω^p_X. There are two operators on $\Omega^{\cdot}_X$: exterior differentiation,

$$d : \Omega^p_X \to \Omega^{p+1}_X$$

and contraction with ϑ, the vector field given by (1.2) generating the $\mathbf{C}^*$-action,

$$\iota_\vartheta : \Omega^p_X \to \Omega^{p-1}_X.$$

These can be combined to give the Lie derivative,

$$\mathcal{L}_\vartheta = \iota_\vartheta d + d\iota_\vartheta : \Omega^p_X \to \Omega^p_X,$$

which acts on a monomial form $\omega = z^\alpha dz^\beta$ as multiplication by its weight $w(\omega) = (\alpha+\beta, \lambda)$, where λ is the n-tuple of weights of the $\mathbf{C}^*$-action. For each integer k, there is a subset of Ω^p_X consisting of forms of weight k, which we denote by $[\Omega^p_X]_k$. Each of these weight spaces is a module over the ring R of invariants, and more generally the wedge product respects the weights:

$$[\Omega^p_X]_k \wedge [\Omega^q_X]_l \subset [\Omega^{p+q}_X]_{k+l}.$$

We put:

$$\underline{\Omega}^p_X := [\Omega^p_X]_0 = \{\omega \in \Omega^p_X \mid \mathcal{L}_\vartheta(\omega) = 0\}.$$

This R-module is called the module of *invariant differential p-forms*, because they satisfy $t^*\omega = \omega$ for all $t \in \mathbf{C}^*$. The elements of $\underline{\Omega}^p_X$ are not to be regarded as differential forms on Y, since they are not necessarily killed by vector fields along the fibres of the quotient map π, and moreover, $\underline{\Omega}^{n+1}_X$ is non-zero and torsion free, even though $\dim Y = n$. The module of *basic p-forms* is defined to be,

$$\Omega^p_Y = \ker[\iota_\vartheta : \underline{\Omega}^p_X \to \underline{\Omega}^{p-1}_X].$$

Note that $\iota_\vartheta : \underline{\Omega}^{n+1}_X \to \underline{\Omega}^n_X$ is injective, so that $\Omega^{n+1}_Y = 0$. Note also that Ω^p_Y, like $\underline{\Omega}^p_X$, is a torsion free but not necessarily free R-module.

The above constructions can be sheafified, and from now on we consider $\underline{\Omega}^p_X$ and Ω^p_Y to be sheaves of $\mathcal{O}_Y$ modules. By the theorem of Roberts [22] the sheaves $\underline{\Omega}^p_X$ are coherent, and it then follows that so are the Ω^p_Y.

Away from F, the basic forms can be identified with forms invariant under a finite group action:

Proposition 2.1 *Let $\xi \in Y \setminus F$, and $x \in \pi^{-1}(\xi)$. Let S_x be a slice to the group action at x, and H_x be the isotropy subgroup of x. Then H_x acts on the module $\Omega^p_{S_x}$ of p-forms on S_x and the stalk $\Omega^p_{Y,\xi}$ is isomorphic to the $\mathcal{O}_{Y,\xi}$-module of H_x-invariant forms $(\Omega^p_{S_x})^{H_x}_x$.*

PROOF We use the notation of the slice theorem (Theorem 1.3). Let $i : S_x \to U$ be the inclusion, and let $\omega \in \underline{\Omega}^p_X(U)$. Then $i^*\omega \in (\Omega^p_{S_x})^{H_x}$. Moreover the restriction of i^* to the basic forms $\Omega^p_Y(U)$ is injective. Its surjectivity is seen by using the slice theorem: one has the composite $\mathbf{C}^* \times S_x \to \mathbf{C}^* \times_{H_x} S_x \to U$ (where the first map is the quotient by the action of the finite group H_x, and the second map is ϕ). Let $\alpha \in (\Omega^p_{S_x})^{H_x}$. This p-form can be extended trivially to $\mathbf{C}^* \times S_x$ and the trivial extension is then $\mathbf{C}^* \times H_x$-invariant and lies in $\ker \iota_\vartheta$. □

Corollary 2.2 *For any $\xi \in Y \setminus F$, the stalks $\Omega^p_{Y,\xi}$ are Cohen-Macaulay $\mathcal{O}_{Y,\xi}$-modules.*

PROOF $\Omega^p(S_x)_x$ is a free, and hence Cohen-Macaulay, $\mathcal{O}_{S_x,x}$-module, and is therefore a Cohen-Macaulay $\mathcal{O}_{Y,\xi}$-module (since H_x is finite). Furthermore, $\Omega^p_{Y,\xi}$ is a direct summand of $\Omega^p(S_x)_x$, so it too is Cohen-Macaulay. □

Recall that the set of smooth points $U \subset Y$ consists of the regular orbits and the pseudoreflexion hyperplanes.

Corollary 2.3 *The restriction of Ω^p_Y to U is precisely the $\mathcal{O}_U$-module of holomorphic p-forms on U.*

There is therefore no ambiguity in writing Ω^p_U.

PROOF Firstly, let $Y_{\mathrm{reg}} \subset Y$ be the set of regular orbits (those with trivial isotropy). If $\xi \in Y_{\mathrm{reg}}$ then the result holds since π is a submersion over Y_{reg}. If $\xi \in Y_{(H)}$ with H acting by pseudoreflexions, then this follows from the proposition by a simple local computation. □

It should perhaps be emphasised that basic forms do not coincide with Kähler forms. If we denote the Kähler forms by $\hat{\Omega}^p$ then there is a map $\hat{\Omega}^p \to \Omega^p_Y$, which in general is neither injective nor surjective. We will show at the end of this section that $\Omega^p_Y = j_*\Omega^p_U$, where $j : U \hookrightarrow Y$ denotes the inclusion; in general the Kähler differentials do not have this nice property. It follows, in fact that Ω^p_Y is the sheaf of *Zariski forms* — the bidual of $\hat{\Omega}^p$.

Example 2.4 Consider the real $\mathbf{C}^*$-action on $X = \mathbf{C}^{n+1} = \mathbf{C}^{2a}$ with weights ± 1. The ring of invariants $\mathcal{O}_Y$ is generated by the a^2 monomials $x_i y_j$. The modules of invariant differential forms are $\mathcal{O}_Y$-modules with the following generators:

$$\begin{aligned}
\underline{\Omega}^{n+1}_X &: \quad \omega = dx_1 \wedge \ldots dx_a \wedge dy_1 \wedge \ldots \wedge dy_a, \\
\underline{\Omega}^{n}_X &: \quad x_i \frac{\omega}{dx_j}, \; y_i \frac{\omega}{dy_j}, \\
\underline{\Omega}^{n-1}_X &: \quad \frac{\omega}{dx_i \wedge dy_j}, \; x_i x_j \frac{\omega}{dx_k \wedge dx_l}, \; y_i y_j \frac{\omega}{dy_k \wedge dy_l},
\end{aligned}$$

$$
\begin{array}{rcl}
 & \vdots & \\
\underline{\Omega}^2_X & : & dx_i\wedge dy_j,\ x_ix_jdy_k\wedge dy_l,\ y_iy_jdx_k\wedge dx_l,\\
\underline{\Omega}^1_X & : & x_idy_j,\ y_idx_j.
\end{array}
$$

Here the notation $\frac{\omega}{dx_i}$ means $dx_1\wedge\ldots\wedge dx_{i-1}\wedge dx_{i+1}\wedge\ldots\wedge dx_a\wedge dy_1\wedge\ldots\wedge dy_a$, and similarly for other forms. The basic forms are then generated by:

$$
\begin{array}{rcl}
\Omega^n_Y & : & \sum_{i=1}^a (x_i\frac{\omega}{dx_i}-y_i\frac{\omega}{dy_i}),\\
\Omega^{n-1}_Y & : & x_i\sum_{k=1}^a (x_k\frac{\omega}{dx_k\wedge dx_j}-y_k\frac{\omega}{dy_k\wedge dx_j}),\ y_i\sum_{k=1}^a (x_k\frac{\omega}{dx_k\wedge dy_j}-y_k\frac{\omega}{dy_k\wedge dy_j}),\\
 & \vdots & \\
\Omega^1_Y & : & x_idy_j+y_jdx_i,\ x_ix_j(y_kdy_l-y_ldy_k),\ y_iy_j(x_kdx_l-x_ldx_k).
\end{array}
$$

Note that the Kähler one-forms $\hat{\Omega}^1$ on the quotient space are generated by $d(x_iy_j)=x_idy_j+y_jdx_i$ and so do not coincide with Ω^1_Y.

The following lemma is well-known, though we give a proof as there does not seem to be a good reference. The referee has pointed out to us that I. Naruki gives a proof in [19, Lemma 2.1.1], but only in the case that all the weights have the same sign (so that the Lie derivative $\mathcal{L}_\vartheta$ acts as an isomorphism on each Ω^p_X).

Lemma 2.5 *The homology of the complex* $(\underline{\Omega}^{\cdot}_X,\iota_\vartheta)$,

$$0\to\underline{\Omega}^{n+1}_X\xrightarrow{\iota_\vartheta}\underline{\Omega}^n_X\xrightarrow{\iota_\vartheta}\cdots\xrightarrow{\iota_\vartheta}\underline{\Omega}^1_X\xrightarrow{\iota_\vartheta}\mathcal{O}_Y\to 0$$

is given by

$$H_i(\underline{\Omega}^{\cdot}_X,\iota_\vartheta)\cong\Omega^i_F.$$

Here $\Omega^{\cdot}_F$ *is just differential forms on* F, $\Omega^0_F=\mathcal{O}_F$ *and if* $F=0$, *then* $\mathcal{O}_F=\mathbf{C}$. *The isomorphism is induced from restriction to* F *of differential forms:* $\underline{\Omega}^i_X\to\Omega^i_F$.

PROOF Suppose first that $F=0$, and consider the sheaf complex $(\Omega^{\cdot}_X,\iota_\vartheta)$, of all differential forms on X. In a neighbourhood of any $z\in X\setminus 0$, coordinates can be chosen so that $\vartheta=\frac{\partial}{\partial z_1}$. It is then clear that the complex is exact in a neighbourhood of z, and thus is exact on the complement of $\{0\}$. Now, the Ω^p_X are all free $\mathcal{O}_X$-modules, so by the acyclicity lemma (see, for example, the appendix) it follows that $H_i(\Omega^{\cdot}_X,\iota_\vartheta)=0$ for $i>0$. Using the form of ϑ given in (1.2), it is immediate that $\iota_\vartheta(\Omega^1_X)=m$ (the sheaf of functions vanishing at 0), so $H_0(\Omega^{\cdot}_X,\iota_\vartheta)=\mathbf{C}$.

The lemma now follows in the case that $F=0$ by taking invariant parts, an operation that commutes with ι_ϑ.

The general case follows since the ring of invariant differential forms $\underline{\Omega}^*_X$ is isomorphic to the tensor product of the pull-backs, $p_1^*\underline{\Omega}^*_{X_0}\otimes_{\mathcal{O}_X}p_2^*\Omega^*_F$, where $p_1:X\to X_0$ and $p_2:X\to F$ are the cartesian projections, and ι_ϑ is zero on the $p_2^*\Omega^*_F$ factor. □

Note that this lemma implies in particular that $\iota_\vartheta:\underline{\Omega}^{n+1}_X\to\Omega^n_Y$ is an isomorphism.

Local cohomology calculations

As usual, we suppose $\mathbf{C}^*$ acts linearly on $\mathbf{C}^{n+1}$, with a positive weights $\{\lambda_1, \ldots, \lambda_a\}$ and b negative weights $\{\mu_1, \ldots, \mu_b\}$, and we choose coordinates x_i, y_j and z_l accordingly (as in Section 1). The ring of invariant polynomials is denoted R. The ring $\mathbf{C}[x,y,z]$ is an R-module on which $\mathbf{C}^*$ acts in the obvious way. The submodules $\mathbf{C}[x,y,z]_k$ consist of polynomials of weight k with respect to this $\mathbf{C}^*$-action, and $\mathbf{C}[x,y,z]$ decomposes as a direct sum of these weight spaces.

For simplicity, in this subsection we consider only the case $F = 0$. Thus, $n = a+b-1$. The modifications necessary for the general case are described in Remark 2.10. For a discussion of local cohomology, see the Appendix.

Proposition 2.6 *For $i < n$, the local cohomology groups at $0 \in Y$ of $\mathbf{C}[x,y]$ (as an R-module) are given by:*

$$H^i_{\{0\}}(\mathbf{C}[x,y]) \cong \begin{cases} 0 & \text{if } i \neq a,b; \\ \mathbf{C}[y]A(x) & \text{if } i = a \neq b; \\ \mathbf{C}[x]A(y) & \text{if } i = b \neq a; \\ \mathbf{C}[x]A(y) \oplus \mathbf{C}[y]A(x) & \text{if } i = a = b. \end{cases}$$

Here

$$A(x) = \mathbf{C}[x_1^{-1}, \ldots, x_a^{-1}] \cdot \frac{1}{x_1 x_2 \ldots x_a}$$

and $A(y)$ is defined similarly. The isomorphism is an isomorphism of $R\mathbf{C}^*$-modules (in particular, it respects the weighting $[\cdot]_k$).

PROOF For this proof, we denote $\mathbf{C}[x,y]$ by S, and as usual $\pi : X \to Y$ is the quotient map. Since π is affine, π_* is exact and we have an isomorphism,

$$H^i_{\{0\}}(\pi_* S) \cong \pi_* H^i_{\mathcal{B}}(S),$$

where $\mathcal{B} = \pi^{-1}(0) = \mathcal{B}_+ \cup \mathcal{B}_-$, and $\mathcal{B}_+ = \{y = 0\}$, $\mathcal{B}_- = \{x = 0\}$. The result is then obtained by computing the local cohomology along the subspaces $\mathcal{B}_\pm$ and $\mathcal{B}_+ \cap \mathcal{B}_- = \{0\}$ (which is well-known, see Example A.5), and then using the Mayer-Vietoris sequence (see for example [12]) to deduce the local cohomology along $\mathcal{B}$. □

Recall that a module is maximal Cohen-Macaulay if it is Cohen-Macaulay and has full support.

Corollary 2.7 *Let $\lambda = \sum_{i=1}^a \lambda_i$, and $\mu = \sum_{j=1}^b \mu_j$ and suppose $-\lambda < k < -\mu$. Then $\mathbf{C}[x,y]_k$ is a maximal Cohen-Macaulay R-module. Furthermore, as $R\mathbf{C}^*$-modules,*

$$H^i_{\{0\}}(\mathbf{C}[x,y]_{-\lambda}) \cong \begin{cases} \mathbf{C} \cdot \frac{1}{x_1 \ldots x_a}, & \text{if } i = a \\ 0 & \text{otherwise,} \end{cases}$$

and

$$H^i_{\{0\}}(\mathbf{C}[x,y]_{-\mu}) \cong \begin{cases} \mathbf{C} \cdot \frac{1}{y_1 \ldots y_b}, & \text{if } i = b \\ 0 & \text{otherwise.} \end{cases}$$

A particular case of this corollary is that $R = \mathbf{C}[x,y]_0$ is itself Cohen-Macaulay.

We turn to the invariant differential forms $\underline{\Omega}^p_X$. Now,

$$\underline{\Omega}^p_X = [\Omega^p_X]_0 = \bigoplus_{|\alpha|+|\beta|=p} [\mathbf{C}[x,y]dx^\alpha \wedge dy^\beta]_0.$$

Here α and β are multi-indices of lengths a and b respectively. Since the weight of $dx^\alpha \wedge dy^\beta$ is $\sum \alpha_i\lambda_i + \sum \beta_j\mu_j = (\alpha,\lambda) + (\beta,\mu)$ it follows that, as R-modules,

$$[\mathbf{C}[x,y]dx^\alpha \wedge dy^\beta]_0 \cong \mathbf{C}[x,y]_{-(\alpha,\lambda)-(\beta,\mu)}\,.$$

Now, since all the entries in α and β are 0's and 1's, $-\lambda \leq -(\lambda,\alpha) - (\beta,\mu) \leq -\mu$, with the equalities occurring for $\alpha = (1,\dots,1)$, $\beta = (0,\dots,0)$ and vice-versa. Thus we have the following central result.

Theorem 2.8 *Suppose $F = 0$. The local comology groups $H^i_{\{0\}}(\underline{\Omega}^p_X)$ of the invariant differential forms for $i < n$ are as follows.*

$$H^i_{\{0\}}(\underline{\Omega}^p_X) = 0 \qquad \text{for } p \neq a,b.$$

In other words, for $p \neq a,b$, $\underline{\Omega}^p_X$ is maximal Cohen-Macaulay.

The local cohomology groups of $\underline{\Omega}^a_X$ and $\underline{\Omega}^b_X$ are all zero (for $i < n$) except for

$$H^a_{\{0\}}(\underline{\Omega}^a_X) \cong \mathbf{C}\frac{dx_1 \wedge \ldots \wedge dx_a}{x_1 \ldots x_a}, \qquad H^b_{\{0\}}(\underline{\Omega}^b_X) \cong \mathbf{C}\frac{dy_1 \wedge \ldots \wedge dy_b}{y_1 \ldots y_b},$$

for $a \neq b$, while if $a = b$,

$$H^a_{\{0\}}(\underline{\Omega}^a_X) \cong \mathbf{C}\frac{dx_1 \wedge \ldots \wedge dx_a}{x_1 \ldots x_a} \oplus \mathbf{C}\frac{dy_1 \wedge \ldots \wedge dy_b}{y_1 \ldots y_b}.$$

PROOF This follows quite simply from the Corollary, and the discussion above. □

We now derive from the local cohomology of $\underline{\Omega}^p_X$ the local cohomology groups for the basic forms Ω^p_Y.

Theorem 2.9 *Suppose $F = 0$. For $i < n$, the local cohomology groups at 0 of the modules of basic differential forms are given by,*

$$H^i_{\{0\}}(\Omega^p_Y) = \begin{cases} \mathbf{C} & \text{if } i = p+1 \text{ and } 1 \leq p < b \\ \mathbf{C} & \text{if } i = p \text{ and } p > a \\ 0 & \text{otherwise.} \end{cases}$$

This result is summarized pictorially in Figure 1.

PROOF We will use the truncations of the $(\underline{\Omega}^\cdot_X, \iota_\vartheta)$ complex,

$$\tau_{\leq p}: \qquad 0 \to \Omega^p_Y \to \underline{\Omega}^p_X \xrightarrow{\iota_\vartheta} \underline{\Omega}^{p-1}_X \to \cdots \to \underline{\Omega}^1_X \to \mathcal{O}_Y \to 0.$$

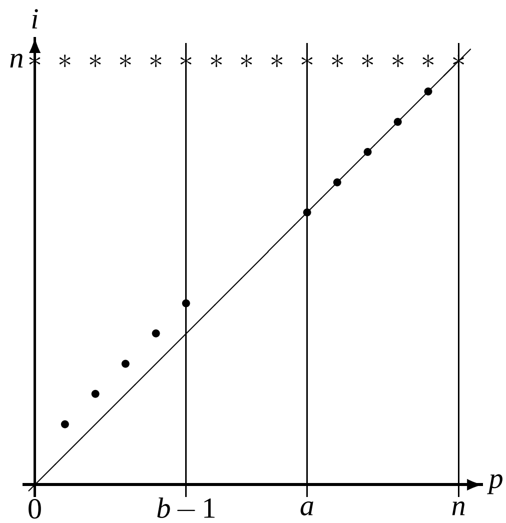

Figure 1. The local cohomology groups $H^i_{\{0\}}(\Omega^p_Y)$.
(Dots represent 1-dimensional groups, while stars represent infinite-dimensional groups)

By Lemma 2.5 this is acyclic, with $H_0(\tau_{\leq p}) = \mathbf{C}$. The proof proceeds by a spectral sequence argument on the Čech double complex $\check{C}^{\cdot}_{\{0\}}(\tau_{\leq p})$ over $\tau_{\leq p}$ (see Example A.5), together with Theorem 2.8. The result does not follow immediately, however, and it is necessary to analyse the higher differentials.

We use the Čech complexes associated to the covering of $\mathbf{C}^{n+1} \setminus \mathcal{B}$ by the open sets $W_{ij} = U_i \cap V_j$ where

$$\begin{aligned} U_i &= \{x_i \neq 0\}, \qquad i = 1, \ldots, a, \\ V_j &= \{y_j \neq 0\}, \qquad j = 1, \ldots, b. \end{aligned}$$

Thus $W_{ij} = \{\phi_{ij} \neq 0\}$, where $\phi_{ij} = x_i^{-\mu_j} y_j^{\lambda_i}$. To facilitate the computation we use the denominator symbols c_{ij}, $i = 1, \ldots, a$ and $j = 1, \ldots, b$, as introduced in the Appendix.

First consider the total untruncated double complex $\check{C}\underline{\Omega}_X = \oplus_{p,q} \check{C}^q(\underline{\Omega}^p_X)$. Elements of $\check{C}^q(\underline{\Omega}^p_X)$ are linear combination of terms of the form $c_I \omega_I$, where I is a q-tuple of pairs (i, j), and ω_I is an invariant p-form with denominators which are nowhere zero on $W_I = \cap_{(i,j)\in I} W_{ij}$. The double complex $\check{C}\underline{\Omega}_X$ is made into a graded-commutative algebra by giving all the generators c_{ij}, dx_i and dy_j degree 1, and letting them all anticommute. To remind us of this, we use the '$\wedge$' notation for the c_{ij} as well.

On $\check{C}\underline{\Omega}_X$ there are two differentials:

$$\begin{aligned} \iota_\vartheta : \check{C}^q(\underline{\Omega}^p_X) &\to \check{C}^q(\underline{\Omega}^{p-1}_X) \\ \mathbf{c} : \check{C}^q(\underline{\Omega}^p_X) &\to \check{C}^{q+1}(\underline{\Omega}^p_X). \end{aligned}$$

Note that $\iota_\vartheta(\alpha \wedge \beta) = \iota_\vartheta \alpha \wedge \beta + (-1)^{|\alpha|} \alpha . \iota_\vartheta \beta$, and so ι_ϑ and $\mathbf{c}$ anticommute. $D = (\iota_\vartheta + \mathbf{c})$ is the total differential.

Define a map $\mathrm{ev} : \check{C}\underline{\Omega}_X \to \mathbf{C}$ to be the composite

$$\check{C}\underline{\Omega}_X \to \mathcal{O}_Y \to \mathbf{C}$$

where the first map is the cartesian projection from the direct sum to one of its summands $\mathcal{O}_Y = \check{C}^0(\mathcal{O}_Y)$, and the second map is just evaluation at $0 \in X$. By the spectral sequence of Proposition A.2 we know that ev induces an isomorphism on homology,

$$\mathrm{ev}_* : H(\check{C}\underline{\Omega}_X, D) \to \mathbf{C}.$$

Consider the following elements of $\check{C}^1(\underline{\Omega}^1_X)$:

$$\begin{aligned} \xi_+ &= \sum_{i=1}^{a}\sum_{j=1}^{b} c_{ij}\frac{dx_i}{\lambda_i x_i} \\ \xi_- &= \sum_{i=1}^{a}\sum_{j=1}^{b} c_{ij}\frac{dy_j}{\mu_j y_j} \end{aligned}$$

As a Čech form, ξ_+ is just $dx_i/\lambda_i x_i$ on W_{ij} for each j, and ξ_- is analogous. It is immediate that $\iota_\theta(\xi_\pm) = -c$, where $c = \sum_{ij} c_{ij}$, and so

$$\iota_\vartheta\left(\frac{1}{p!}\xi_\pm^p\right) = -\mathbf{c}\left(\frac{1}{(p-1)!}\xi_\pm^{p-1}\right).$$

Thus, with

$$\eta_\pm = \exp(\xi_\pm),$$

one has

$$D\eta_\pm = (\iota_\vartheta + \mathbf{c})\eta_\pm = 0.$$

Consequently, we have two cycles η_+ and η_- in $H(\check{C}\underline{\Omega}_X, D)$ and both are non-trivial as $\mathrm{ev}(\eta_+) = \mathrm{ev}(\eta_-) = 1$. Moreover, it follows that the difference $\eta_+ - \eta_-$ is a boundary, say $\eta_+ - \eta_- = D\zeta$ for some $\zeta = \sum_{k=1}^n \zeta_k$, with $\zeta_k \in \check{C}^k(\underline{\Omega}_X^{k+1})$. (In fact $\zeta_1 = \sum_i \sum_j c_{ij}(dx_i/\lambda_i x_i) \wedge (dy_j/\mu_j y_j)$.)

With this much in hand, we now pass to the truncated double complexes $\check{C}\tau_{\leq p}$. Again one has

$$\mathrm{ev}_* : H(\check{C}\tau_{\leq p}, D) \xrightarrow{\cong} \mathbf{C}.$$

The computations depend to some extent on p, and we distinguish three cases.
Case 1: $0 < p < b$. In this range it follows at once from the 'first vertical' spectral sequence that, for $q < n$

$$H^q_{\{0\}}(\Omega^p_Y) = \begin{cases} \mathbf{C} & \text{if } q = p+1 \\ 0 & \text{otherwise.} \end{cases}$$

A representative of $H^{p+1}_{\{0\}}(\Omega^p_Y)$ can be taken taken as

$$\iota_\vartheta\left(\frac{1}{(p+1)!}\xi_\pm^{(p+1)}\right) = -\mathbf{c}\left(\frac{1}{p!}\xi_\pm^p\right).$$

Case 2: $b \le p < a$. In this range, the 'first vertical' spectral sequence allows the following two possibilities:

$$\text{(A)} \qquad H^q_{\{0\}}(\Omega^p_Y) = 0 \quad \text{for all } q < n$$

$$\text{(B)} \qquad H^q_{\{0\}}(\Omega^p_Y) = \begin{cases} \mathbf{C} & \text{if } q = p, p+1 \\ 0 & \text{otherwise.} \end{cases}$$

Now (B) would occur if there were a non-zero higher differential $H^p_{\{0\}}(\Omega^p_Y) \to H^b_{\{0\}}(\underline{\Omega}^b_X)$. But, because $\xi_-^{p+1} = 0$, the element $\eta_- = \exp(\xi_-)$ is a cycle in $\check{C}\tau_{\le p}$. The non-zero higher differential would imply that η_- is a boundary in $\check{C}\tau_{\le p}$, which contradicts $\mathrm{ev}(\eta_-) = 1$. Consequently (A) must hold.

Case 3: $a \le p < n+1$. Again there are *a priori* two possibilities:

$$\text{(C)} \qquad H^q_{\{0\}}(\Omega^p_Y) = \begin{cases} \mathbf{C} & \text{if } q = p \\ 0 & \text{otherwise,} \end{cases}$$

$$\text{(D)} \qquad H^q_{\{0\}}(\Omega^p_Y) = \begin{cases} \mathbf{C}^2 & \text{if } q = p \\ \mathbf{C} & \text{if } q = p+1 \\ 0 & \text{otherwise.} \end{cases}$$

We claim (D) cannot occur. Since $\xi_+^{p+1} = \xi_-^{p+1} = 0$ in this range, both η_+ and η_- are cycles in $\check{C}\tau_{\le p}$. Possibility (D) could occur if both η_+ and η_- were killed by higher differentials, but this is impossible since $\mathrm{ev}(\eta_\pm) = 1$.

Furthermore,

$$D\left(\sum_k^p \zeta_k + \iota_\vartheta \zeta_{p+1}\right) = \eta_+ - \eta_-,$$

so $\iota_\vartheta \zeta_{p+1}$ can be taken as a generator of $H^p_{\{0\}}(\Omega^p_Y)$ for p in this range. □

Remark 2.10 All computations of local cohomology for the general case $F \neq 0$ (for $i < n$) can be derived from the corresponding results for the case $F = 0$ by using Lemma A.4 in the Appendix. Write $X = X_0 \times F$ and $Y = Y_0 \times F$. The results corresponding to Theorem 2.8 are:

$$H^a_F(\underline{\Omega}^p_X) \cong H^a_{\{0\}}(\underline{\Omega}^a_{X_0}) \otimes \Omega^{p-a}_F \tag{2.3}$$

$$H^b_F(\underline{\Omega}^p_X) \cong H^b_{\{0\}}(\underline{\Omega}^b_{X_0}) \otimes \Omega^{p-b}_F \tag{2.4}$$

(where $\{0\} \subset X_0$) and for $i \neq a, b$, $i < n$, one has $H^i_F(\underline{\Omega}^p_X) = 0$. The local cohomology along $\{0\} \subset X$ is given by

$$H^{a+c}_{\{0\}}(\underline{\Omega}^p_X) \cong H^a_{\{0\}}(\underline{\Omega}^a_{X_0}) \otimes H^c_{\{0\}}(\Omega^{p-a}_F) \tag{2.5}$$

$$H^{b+c}_{\{0\}}(\underline{\Omega}^p_X) \cong H^a_{\{0\}}(\underline{\Omega}^b_{X_0}) \otimes H^c_{\{0\}}(\Omega^{p-b}_F) \tag{2.6}$$

(where on the left hand side $\{0\} \subset X$, while on the right hand side $\{0\} \subset X_0$), and again, for $i \neq a+c, b+c$ and $i < n$ all $H^i_{\{0\}}(\underline{\Omega}^p_X) = 0$.

For the basic forms one obtains similar statements with $\underline{\Omega}_X$ replaced by Ω_Y and $\underline{\Omega}_{X_0}$ by Ω_{Y_0}. More precisely, the analogues of Theorem 2.9 are

$$H^i_F(\Omega^p_Y) \cong \bigoplus_{r=0}^{c} H^i_{\{0\}}(\Omega^{p-r}_{Y_0}) \otimes \Omega^r_F \tag{2.7}$$

$$H^i_{\{0\}}(\Omega^p_Y) \cong \bigoplus_{r=0}^{c} H^{i-c}_{\{0\}}(\Omega^{p-r}_{Y_0}) \otimes H^c_{\{0\}}(\Omega^r_F) \tag{2.8}$$

Recall that $U \subset Y \setminus F$ is the set of smooth points in Y, and that $Y \setminus U$ has codimension at least 2 in Y. The following result does not assume $F = 0$.

Theorem 2.11 *1. Let $j : U \to Y$ denote the inclusion, and let Ω^p_U denote the usual p-forms on the smooth space U. Then,*

$$\Omega^p_Y = j_*\Omega^p_U.$$

2. Ω^n_Y is the dualizing sheaf of $\mathcal{O}_Y$.

3. If the the sum of the weights of the action is zero then Y is Gorenstein.

4. If $b > 1$ then Y is not isomorphic to a quotient of $\mathbf{C}^n$ by a finite group.

PROOF (1) Consider the inclusions $\alpha : U \hookrightarrow Y \setminus F$, and $\beta : Y \setminus F \hookrightarrow Y$, so that $j = \beta \circ \alpha$. Now, by Corollary 2.2, $\Omega^p_{Y\setminus F}$ is Cohen-Macaulay, so $\Omega^p_{Y\setminus F} = \alpha_*\Omega^p_U$. Secondly, by Theorem 2.9 (and Remark 2.10 if $F \neq 0$), $\beta_*\Omega^p_{Y\setminus F} = \Omega^p_Y$.

(2) On a smooth space, the sheaf of top differential forms is a dualizing module, so this holds for U. Thus (2) follows from the fact that dualizing sheaves and Ω^n_Y are both Cohen-Macaulay.

(3) This follows from (2) because if the sum of the weights is zero then there is an isomorphism $\mathcal{O}_Y \xrightarrow{\sim} \underline{\Omega}^{n+1}_X$ given by $f \mapsto f dx_1 \wedge \ldots \wedge dz_c$. Furthermore, as already pointed out, $\iota_\vartheta : \underline{\Omega}^{n+1}_X \to \Omega^n_Y$ is an isomorphism.

(4) If Y is a finite quotient of a smooth space then Ω^1_Y is Cohen-Macaulay. If $b > 1$ this contradicts Theorem 2.9. □

For any $\mathcal{O}_Y$-module M one sets $M^\vee = \mathrm{Hom}_{\mathcal{O}_Y}(M, \Omega^n_Y)$, since Ω^n_Y is the dualizing sheaf of $\mathcal{O}_Y$, and it follows that $\mathrm{depth}_{Y\setminus U} M^\vee \geq 2$. Since $\Omega^p_Y \wedge \Omega^{n-p}_Y \subset \Omega^n_Y$, it follows that there is a natural map $\Omega^{n-p}_Y \to (\Omega^p_Y)^\vee$. Now, on U this map is an isomorphism, and since both Ω^{n-p}_Y and $(\Omega^p_Y)^\vee$ have depth at least two we obtain the following:

Corollary 2.12 *For each p,*

$$(\Omega^p_Y)^\vee \cong \Omega^{n-p}_Y.$$

For similar reasons,

$$(\underline{\Omega}^p_X)^\vee \cong \underline{\Omega}^{n+1-p}_X.$$

3 Quasi-acyclicity of the $df\wedge$-complexes

Let f be an analytic function defined in a neighbourhood of $0 \in X = \mathbf{C}^{n+1}$ and invariant under the action of $\mathbf{C}^*$. Recall that the action has a positive weights, b negative weights and c zero weights, and without loss of generality we assume $a \geq b$. Then df is a 1-form which is not only invariant but also basic. Thus, for each p,

$$df \wedge \underline{\Omega}_X^p \subset \underline{\Omega}_X^{p+1} \qquad \text{and} \qquad df \wedge \Omega_Y^p \subset \Omega_Y^{p+1}.$$

We can therefore define two complexes of sheaves on Y with differentials $df\wedge$: the *invariant* $df\wedge$-complex,

$$(\underline{\Omega}_X^{\cdot}, df\wedge): \qquad 0 \to \mathcal{O}_Y \xrightarrow{df\wedge} \underline{\Omega}_X^1 \xrightarrow{df\wedge} \underline{\Omega}_X^2 \xrightarrow{df\wedge} \cdots \xrightarrow{df\wedge} \underline{\Omega}_X^{n+1} \to 0,$$

and the *basic* $df\wedge$-complex,

$$(\Omega_Y^{\cdot}, df\wedge): \qquad 0 \to \mathcal{O}_Y \xrightarrow{df\wedge} \Omega_Y^1 \xrightarrow{df\wedge} \Omega_Y^2 \xrightarrow{df\wedge} \cdots \xrightarrow{df\wedge} \Omega_Y^n \to 0.$$

We are interested in the cohomology of these complexes. In the classical case where there is no group acting on X and f has an isolated critical point, all cohomology groups are zero, except for $H^{n+1} = \Omega_X^{n+1}/df \wedge \Omega_X^n$, and the complex is said to be acyclic. Moreover, the multiplicity of the isolated critical point is given by the dimension of H^{n+1}. In our case, the lower cohomology groups are not all zero, though they depend only on the $\mathbf{C}^*$-action and not on the function f, provided it has an isolated critical point on Y, and the action of $\mathbf{C}^*$ has an isolated fixed point, and we say rather loosely that the complexes are *quasi-acyclic*.

We consider each of the complexes in turn, and use the *equivariant* $df\wedge$ complex to relate them. To simplify the exposition, we assume that the fixed point set $F = 0$. For the modifications necessary in the general case, see Remark 3.8.

The invariant $df\wedge$-complex

Lemma 3.1 *The cohomology of the invariant $df\wedge$-complex is supported on the critical locus of f.*

PROOF Let $\xi \in Y$ be a regular point of f, and let $x \in \pi^{-1}(\xi)$. Then f is non-singular at x by Lemma 1.5, and there is a neighbourhood of x on which the complex of ordinary (non-invariant) differential forms $(\Omega_X^{\cdot}, df\wedge)$ is exact. The result follows by taking invariant parts (which commutes with $df\wedge$). □

Since the $\underline{\Omega}_X^p$ are coherent sheaves, so are the cohomology sheaves of the above complex. It follows from the Lemma and the Nullstellensatz for coherent sheaves that if f has an isolated critical point on Y then the cohomology groups are finite dimensional.

We now show that the complex $(\underline{\Omega}_X^{\cdot}, df\wedge)$ is quasi-acyclic.

Proposition 3.2 *If f has an isolated critical point at $0 \in Y$ then, for $i < n$,*

$$H^i(\underline{\Omega}_X^{\cdot}, df\wedge) = \begin{cases} 0 & \textit{if } i \neq 2b, \\ \mathbf{C} & \textit{for } i = 2b \textit{ if } a > b+1. \end{cases}$$

PROOF Since we know the local cohomology groups of the $\underline{\Omega}^{\cdot}_X$, we can use the spectral sequence of Proposition A.2. By Theorem 2.8, we know that for $q < n$ and $a > b$,

$$E_1^{p,q} \cong \begin{cases} \mathbf{C} & \text{if } (p,q) = (a,a) \text{ or } (b,b) \\ 0 & \text{otherwise.} \end{cases}$$

If $a = b$ we get for $q < n$,

$$E_1^{p,q} \cong \begin{cases} \mathbf{C}^2 & \text{if } (p,q) = (a,a) \\ 0 & \text{otherwise.} \end{cases}$$

The spectral sequence therefore degenerates at E_1, and the result follows. □

Example 3.3 Consider the $\mathbf{C}^*$-action with weights ± 1 only. Recall that the local cohomology group $H^b(\underline{\Omega}^b_X)$ is generated over $\mathbf{C}$ by $\eta = \frac{dy_1 \wedge \dots \wedge dy_b}{y_1 \dots y_b}$. Suppose that f is a generic linear form on the quotient space, then after an equivariant change of coordinates, $f = x_1 y_1 + \cdots x_b y_b$. The element of $H^{2b}(\underline{\Omega}^{\cdot}_X, df\wedge)$ corresponding to η in the above proof is then $dx_1 \wedge \cdots \wedge dx_b \wedge dy_1 \wedge \cdots \wedge dy_b$. This is not hard to show using the spectral sequence, however it is simpler to observe that this form is indeed killed by $df\wedge$ though it is clearly not contained in $df \wedge \underline{\Omega}^{2b-1}_X$.

The complex $(\underline{\Omega}^{\cdot}_X, df\wedge)$ has two further cohomology groups, namely $H^n(\underline{\Omega}^{\cdot}_X, df\wedge)$ and $H^{n+1}(\underline{\Omega}^{\cdot}_X, df\wedge)$. We will see below that these two groups are in fact very closely related.

The basic $df\wedge$-complex

Lemma 3.4 *The cohomology of the basic $df\wedge$-complex is supported on the critical locus of f.*

PROOF Define the complex $({}'\Omega_Y, df\wedge)$ to coincide with $(\Omega_Y, df\wedge)$ except for replacing $\mathcal{O}_Y$ by m_Y in degree 0. There is then an exact sequence of complexes

$$0 \to (\Omega^{\cdot}_Y, df\wedge) \to (\underline{\Omega}^{\cdot}_X, df\wedge) \to ({}'\Omega^{\cdot -1}_Y, df\wedge) \to 0,$$

with the associated long exact sequence in cohomology,

$$\cdots \to H^i_Y \to \underline{H}^i \xrightarrow{\iota_\vartheta} {}'H^{i-1}_Y \to H^{i+1}_Y \to \cdots \tag{3.9}$$

(with the obvious notation). For $p > 1$, ${}'H^p_Y = H^p_Y$ while for $p = 1$ there is short exact sequence $0 \to \mathbf{C} \to {}'H^1_Y \to H^1_Y \to 0$. The result follows by induction on i, as $\underline{H}^i = 0$ off the critical locus. □

Proposition 3.5 *Let f be an invariant function, with an isolated critical point at $0 \in Y$, then for $i < n$,*

$$H^i(\Omega^{\cdot}_Y, df\wedge) = \begin{cases} \mathbf{C} & \text{if } 3 \le i \le 2b-1 \text{ and } i \text{ is odd} \\ 0 & \text{otherwise.} \end{cases}$$

PROOF As in the proof of Proposition 3.2 we use the spectral sequence of Proposition A.2. The details are left to the reader. □

The complex $(\Omega^{\cdot}_Y, df\wedge)$ has one more cohomology group $H^n(\Omega^{\cdot}_Y, df\wedge)$, which is in fact closely related to $H^{n+1}(\underline{\Omega}^{\cdot}_X, df\wedge)$. Indeed, ι_ϑ provides a map $\underline{H}^{n+1} \to H^n_Y$, as in (3.9), which is an isomorphism if $n > 1$, and is surjective with a 1-dimensional kernel if $n = 1$ (i.e. if $a = b = 1$).

The equivariant $df\wedge$-complex

For any group action there are the so-called equivariant cohomolgy groups, see for example the paper of Atiyah & Bott, [1]. In the case of a circle action, the de Rham model for the equivariant cohomology can be described by a complex $(\underline{\Omega}^{\cdot}_X[u], D)$, where the differential is $D = d + u.\iota_\vartheta$, and d is the exterior derivative, u is a formal variable which commutes with everything, and ι_ϑ is the contraction with the vector field generating the circle action.

We have found it useful to consider the analogous complex $(\underline{\Omega}^{\cdot}_X[u], D)$, with $D = df\wedge + u.\iota_\vartheta$, and again u is a formal variable commuting with everything. We can represent this complex as a double complex with terms $A^{p,q} = \underline{\Omega}^{p-q}_X.u^q$, and with horizontal differential $df\wedge$ and vertical differential ι_ϑ as follows:

$$
\begin{array}{cccccccccccc}
 & & & & & & \uparrow & & \uparrow & & \uparrow & \\
 & & & & \mathcal{O}_Y.u^2 & \to \cdots \to & \underline{\Omega}^{n-2}_X.u^2 & \to & \underline{\Omega}^{n-1}_X.u^2 & \to & \underline{\Omega}^{n}_X.u^2 & \to \\
 & & & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
 & & \mathcal{O}_Y.u & \to & \underline{\Omega}^{1}_X.u & \to \cdots \to & \underline{\Omega}^{n-1}_X.u & \to & \underline{\Omega}^{n}_X.u & \to & \underline{\Omega}^{n+1}_X.u & \\
 & & \uparrow u.\iota_\vartheta & & \uparrow & & \uparrow & & \uparrow & & & \\
\mathcal{O}_Y & \to & \underline{\Omega}^{1}_X & \xrightarrow{df\wedge} & \underline{\Omega}^{2}_X & \to \cdots \to & \underline{\Omega}^{n}_X & \to & \underline{\Omega}^{n+1}_X & & &
\end{array}
$$

The complex is a $\mathbf{C}[u]$-module, and since the differential commutes with u, the cohomology of the complex is also a $\mathbf{C}[u]$-module.

The homology of this complex can be computed by two spectral sequences. Comparison of the two limits gives a way of constructing explicit generators of $H^{2b}(\underline{\Omega}^{\cdot}_X, df\wedge)$ and $H^i(\Omega_Y, df\wedge)$ for $i < n$, as well as enabling us to compare the remaining groups $H^n(\underline{\Omega}^{\cdot}_X, df\wedge)$, $H^{n+1}(\underline{\Omega}^{\cdot}_X, df\wedge)$ and $H^n(\Omega^{\cdot}_Y, df\wedge)$. We will denote these three groups by $\underline{H}^n$, $\underline{H}^{n+1}$ and H^n_Y respectively. As usual, we assume $F = 0$ to simplify the exposition, see Remark 3.8 for the general case.

Computing the horizontal homology of this complex gives $H^{\cdot}(\underline{\Omega}^{\cdot}_X, df\wedge)$ on each row, most terms of which are 0 if the critical point f is isolated in Y. On the other hand, the vertical homology gives $\Omega^{\cdot}_Y$ along the bottom row, copies of $\mathbf{C}$ along the diagonal $E_1^{p,p}$, $p > 0$, and zeros elsewhere.

Consider as usual a function $f \in m_Y \subset \mathcal{O}_Y$ with an isolated critical point at $0 \in Y$. By the acyclicity of the $(\underline{\Omega}^{\cdot}_X, \iota_\vartheta)$ complex, there is an element $\alpha \in \underline{\Omega}^1_X$ satisfying

$$\iota_\vartheta(\alpha) = f. \tag{3.10}$$

Consequently, $df = d\iota_\vartheta(\alpha) = -\iota_\vartheta(d\alpha)$. Define the closed form $\omega = d\alpha \in \underline{\Omega}^2_X$, so

$$\iota_\vartheta(\omega) = -df.$$

(This is the same relationship as between a symplectic form and the hamiltonian function associated to a symplectic vector field — see Section 6.) Note that ω is only an invariant form, while $\omega \wedge df$ is a basic form, for $\iota_\vartheta(\omega \wedge df) = -df \wedge df = 0$.

Now consider, for $k = 1, 2 \ldots$, the elements

$$\sigma^{(2k)} := \sum_{l=0}^{k} u^{k-l} \left(\frac{\omega^l}{l!} \right) \in \underline{\Omega}^{\cdot}_X[u]. \tag{3.11}$$

We have

$$D(\sigma^{(2k)}) = (df \wedge + u\iota_\vartheta) \left(\sum_{l=0}^{k} u^{k-l} \left(\frac{\omega^l}{l!} \right) \right) = df \wedge \left(\frac{\omega^k}{k!} \right) \in \Omega^{2k+1}_Y. \tag{3.12}$$

It will be useful to consider a particular choice of α satisfying (3.10), which is defined as follows. Let

$$d_+ f = \sum_{i=1}^{a} \frac{\partial f}{\partial x_i} dx_i,$$

and $\vartheta_+ = \sum_{i=1}^{a} \lambda_i x_i \partial/\partial x_i$, then $\iota_\vartheta(d_+ f) = \vartheta_+(f)$. We can decompose f into its 'ϑ_+-homogeneous' parts:

$$f = \sum_{\rho > 0} f_\rho,$$

where f_ρ satisfies $\vartheta_+(f_\rho) = \rho . f_\rho$. Define

$$\alpha = \sum_{\rho > 0} \rho^{-1} d_+ f_\rho. \tag{3.13}$$

and one has $\iota_\vartheta(\alpha) = f$, as required. The form $\omega = d\alpha$ is then

$$\omega = \sum H_{ij} dy_j \wedge dx_i := \sum_{\rho > 0} \sum_{i,j} \rho^{-1} \frac{\partial^2 f_\rho}{\partial x_i \partial y_j} dy_j \wedge dx_i. \tag{3.14}$$

We call this ω the "weighted mixed Hessian" of f.

Proposition 3.6 *Let α satisfy (3.10) and let $\sigma^{(2k)}$ be as in (3.11) with $\omega = d\alpha$. Then the elements $\beta_k := D(\sigma^{(2k)}) = df \wedge \left(\frac{\omega^k}{k!} \right)$, are d-closed representatives of non-zero elements of $H^{2k+1}(\Omega^{\cdot}_Y, df \wedge)$, for $k = 1, 2, \ldots, b-1$.*

Moreover, with ω defined by (3.14), and the resulting σ in (3.11), one has

1. *$\frac{\omega^b}{b!}$ represents a non-zero cohomology class in $H^{2b}(\underline{\Omega}^{\cdot}_X, df \wedge)$.*

2. The elements

$$\sigma^{(2b)},\ u.\sigma^{(2b)} = \sigma^{(2b+2)},\ u^2\sigma^{(2b)}, \dots$$

are cycles in H^*_{eq} which are not boundaries, i.e.

$$\mathbf{C}[u].\sigma^{(2b)} \hookrightarrow H^*_{\mathrm{eq}}$$

as $\mathbf{C}[u]$-modules.

3. This choice of ω gives a splitting of $\mathbf{C}[u]$-modules

$$H^*_{\mathrm{eq}} = \mathbf{C}[u].\sigma^{(2b)} \oplus T.$$

Here T is the $\mathbf{C}[u]$-torsion part which is concentrated in degree n, thus

$$H^n_{\mathrm{eq}} = \begin{cases} T & \text{if } n \text{ is odd} \\ T \oplus \mathbf{C}.\sigma^{(n)} & \text{if } n \text{ is even.} \end{cases}$$

4. The groups T, $\underline{H}^{n+1}$ and $\underline{H}^n$ are related as follows:

$$\begin{aligned} a = b: &\qquad 0 \to \mathbf{C}.\left[\frac{\omega^b}{b!}\right] \longrightarrow \underline{H}^{n+1} \xrightarrow{\iota_\vartheta} \underline{H}^n \to 0; \quad T \cong \underline{H}^n \\ a = b+1: &\qquad 0 \to \underline{H}^{n+1} \xrightarrow{\iota_\vartheta} \underline{H}^n \longrightarrow \mathbf{C}.\left[\frac{\omega^b}{b!}\right] \to 0; \quad T \cong \underline{H}^{n+1} \\ a > b+1: &\qquad \underline{H}^{n+1} \xrightarrow{\cong} \underline{H}^n \cong T. \end{aligned}$$

PROOF By the first horizontal spectral sequence for the equivariant double complex we see that $H^k_{\mathrm{eq}} = 0$ for $k < 2b$. It follows then from the first vertical spectral sequence that the elements $\sigma^{(2k)}$ form a ‘ladder’ for the higher differentials, so the classes of $D(\sigma^{(2k)})$ generate H^{2k+1}_Y. Furthermore, because $\omega = d\alpha$ the forms β_k are d-closed.

Let ω now be given by (3.14).

1) First we show that $df \wedge \omega = 0$. Now, $\iota_\vartheta(\omega) = -df$ implies

$$\frac{\partial f}{\partial x_i} = \sum_j \mu_j y_j H_{ij}, \qquad \frac{\partial f}{\partial y_j} = \sum_i \lambda_i x_i H_{ij}.$$

The coefficent of $dx_I \wedge dy_1 \wedge \ldots \wedge dy_b$ in $df \wedge \omega^b$ is therefore a $(b+1)\times(b+1)$-minor of the $a \times (b+1)$ matrix

$$\begin{pmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_a} \\ \cdots & H_{ij} & \cdots \end{pmatrix}.$$

These minors are zero however, since the vector $(\partial f/\partial x_i)$ is a linear combination of the rows of the matrix (H_{ij}).

By (3.12), $\sigma^{(2b)}$ is a cycle in the equivariant $df\wedge$-complex. Moreover, the coefficient of u^b in $\sigma^{(2b)}$ is 1, so from the first vertical spectral sequence one sees that $\sigma^{(2b)}$ is not a boundary. The contibution of $\sigma^{(2b)}$ to the first horizontal spectral sequence is $\omega^b/b!$ which is therefore non-trivial in $\underline{H}^{2b}$.

2) For $k \geq b$, the element $\sigma^{(2k)}$ is a non-trivial cycle in the equivariant double complex by the same argument as given for $\sigma^{(2b)}$ in part (1).

3) This follows from the first vertical spectral sequence.

4) Consider the case $a > b+1$ (the other cases are similar). In the first horizontal spectral sequence, one has

$$\begin{aligned} E_2^{p+2b,p} &\cong \underline{H}^{2b}.u^{p-1}, \\ E_2^{1+k,n+k} &\cong \begin{cases} \underline{H}^n & \text{if } k=0 \\ Q.u^k & \text{if } k>0 \end{cases}, \\ E_2^{1+k,n+1+k} &\cong K.u^k \quad \text{for } k \geq 0, \end{aligned}$$

where K and Q are defined by

$$0 \to K \to \underline{H}^{n+1} \xrightarrow{\iota_\vartheta} \underline{H}^n \to Q \to 0.$$

Since the $\underline{H}^{2b}u^p$ are all non-trivial in H^*_{eq}, the higher differentials vanish which implies that $K = Q = 0$. □

So, in particular, all the groups $\underline{H}^{n+1}$, $\underline{H}^n$, H^n_Y, H^n_{eq} and T are essentially equal, differing in dimension by at most 1. (Recall that ι_ϑ induces an isomorphism $\underline{H}^{n+1} \to H^n_Y$ unless $a = b = 1$, in which case there is a 1-dimensional kernel.) The group T is always the smallest.

Example 3.7 Consider a real action of $\mathbf{C}^*$ on X with weights $\{\pm\lambda_1, \ldots, \pm\lambda_a\}$ and consider the invariant function $f = \sum x_i y_i$. This function has an isolated critical point at $0 \in X$ and so $df \wedge \Omega^n_X = m_X \Omega^{n+1}_X$. Taking invariant parts gives $df \wedge \underline{\Omega}^n_X = m_Y \underline{\Omega}^{n+1}_X$ and consequently,

$$H^i(\Omega^\cdot_Y, df\wedge) = \begin{cases} \mathbf{C} & \text{if i=3,5,\ldots,n} \\ 0 & \text{otherwise.} \end{cases}$$

Remark 3.8 So far in this section we have assumed that $F = \{0\}$, that is, $c = 0$. The modifications necessary for the general case are reasonably straightforward.

Firstly, Lemmas 3.1 and 3.4 are unchanged, with identical proofs. Propositions 3.2 and 3.5 remain very similar. The non-zero cohomology groups are all shifted to the right by c, though they are no longer 1-dimensional in general but depend on the restriction of f to F. Write f_F for the restriction of f to F, and put

$$\mathcal{M}_F(f) := \frac{\Omega^c_F}{df_F \wedge \Omega^{c-1}_F},$$

which measures the multiplicity of this restriction. Then Proposition 3.2 becomes, with the same hypotheses,

$$H^i(\underline{\Omega}^\cdot_X, df\wedge) \cong \begin{cases} 0 & \text{if } i \neq 2b+c, \\ \mathcal{M}_F(f) & \text{for } i = 2b+c \text{ if } a > b+1. \end{cases} \tag{3.15}$$

The cohomology of the basic $df\wedge$ complex, given for $F=0$ in Proposition 3.5, becomes

$$H^i(\Omega^{\cdot}_Y, df\wedge) = \begin{cases} \mathcal{M}_F(f) & \text{if } 3+c \le i \le 2b-1+c \text{ and } i-c \text{ is odd} \\ 0 & \text{otherwise.} \end{cases} \tag{3.16}$$

The proofs of these are very similar to those of the corresponding $F=0$ statements, except that the spectral sequence now degenerates at E_2 rather than E_1. The E_1 consists now of two horizontal complexes, each copies of the complex $(H^c_{\{0\}}(\Omega^{\cdot}_F), df_F\wedge)$. Since by hypothesis f_F has an isolated critical point at 0 we get that

$$\ker[H^c_{\{0\}}(\mathcal{O}_F) \xrightarrow{df\wedge} H^c_{\{0\}}(\Omega^1_F)] \cong \mathcal{M}_F(f),$$

and elsewhere these complexes are exact, by a spectral sequence argument on the Čech resolution of $(\Omega^{\cdot}_F, df\wedge)$, see Example A.6.

One still has $\iota_\vartheta : \underline{H}^{n+1} \xrightarrow{\cong} H^n_Y$ unless $a=b=1$. In this latter case there is a short exact sequence

$$0 \to \mathcal{M}_F(f) \to \underline{H}^{n+1} \xrightarrow{\iota_\vartheta} H^n_Y \to 0.$$

Representatives of the non-zero cohomology groups of the basic $df\wedge$ complex can be found as follows. Given f, define f' by

$$f'(x,y,z) = f(x,y,z) - f(0,0,z)$$

with the usual splitting of the coordinates into x,y and z. Note that $df'\wedge\nu = df\wedge\nu$ for $\nu \in \Omega^c_F$ (if we consider Ω^c_F as a subset of $\underline{\Omega}^c_X$).

Now, the restriction of f' to F is identically zero, so by Lemma 2.5 there is a 1-form α with

$$\iota_\vartheta(\alpha) = f'.$$

Let, as usual, $\omega = d\alpha \in \underline{\Omega}^2_X$, so $\iota_\vartheta(\omega) = df'$. The non-trivial representatives of $H^i(\Omega^{\cdot}_Y, df\wedge)$ for $i<n$ are given by

$$\{df\wedge\nu\wedge\omega^i \mid i=1,2,\ldots,b-1;\ \nu\in\mathcal{M}_F(f)\}.$$

Remark 3.9 (*Local Duality*)
Suppose $F=0$ (the modifications for the general case can be found easily). Using the Cartan-Eilenberg projective resolution of the complex $(\Omega^{\cdot}_Y, df\wedge)$, together with local duality in the form of the existence of natural pairings $\mathrm{Ext}(M,\Omega^n_Y)\times H_{\{0\}}(M)\to\mathbf{C}$, one can prove the existence of the following natural pairings:
For $a\neq b$:

$$H^{p+1}_{\{0\}}(\Omega^p_Y)\times H^{n-p}_{\{0\}}(\Omega^{n-p}_Y)\to\mathbf{C},$$
$$H^n_Y\times H^n_Y\to\mathbf{C},$$

for $p=1,\ldots,b-1$.
For $a=b$:

$$H^{p+1}_{\{0\}}(\Omega^p_Y)\times H^{n-p}_{\{0\}}(\Omega^{n-p}_Y)\to\mathbf{C},$$

for $p = 1, \ldots, b-2$, and a degenerate pairing on H^n_Y with a one-dimensional null-space:

$$0 \to \mathbf{C}.\omega^b \to H^n_Y \to (H^n_Y)^* \to (\mathbf{C}.\omega^b)^* \to 0$$

where $(\ )^*$ represents the $\mathbf{C}$-dual of a vector space.

For details on Cartan-Eilenberg resolutions, see [13, p.74] or [14, Lemma 9.4].

The argument is briefly as follows. Denote the Cartan-Eilenberg projective resolution of $(\Omega^{\cdot}_Y, df\wedge)$ by $P = (P^{\cdot,\cdot})$, so that for each p, the subcomplex $(P^{p,\cdot})$ is a projective resolution of Ω^p_Y. Now apply $\mathrm{Hom}(-,\Omega^n_Y)$, and call the new complex $Q = Q^{\cdot,\cdot}$; recall that Ω^n_Y is a dualizing module on Y. The homology of the associated single complex is the HyperExt of $(\Omega^{\cdot}_Y, df\wedge)$. This homology can be computed via two spectral sequences.

First horizontal spectral sequence: Use the fact that the P's are projective to see that $E_1(Q)$ is isomorphic to the Ω^n_Y-dual of $E_1(P)$, and the fact that it is Cartan-Eilenberg to show that $E_1^{p,\cdot}(P)$ is a projective resolution of H^p_Y. Thus,

$$\begin{aligned} E_\infty^{p,q} = E_2^{p,q} &= \mathrm{Ext}^q(H^p_Y, \Omega^n_Y) \\ &= \begin{cases} \mathrm{Hom}(H^p_Y, \mathbf{C}) & \text{if } q = n \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

First vertical spectral sequence: As the p-th column of P is a projective resolution of Ω^p_Y, one has that $E_1^{p,q} = \mathrm{Ext}^q(\Omega^p_Y, \Omega^n_Y)$. For $q = 0$ this is just Ω^{n-p}_Y by Corollary 2.12, while for $q > 0$ it is $\mathbf{C}$-dual to the local cohomology group $H^{n-q}_{\{0\}}(\Omega^p_Y)$ by local duality. If the original resolution P is written below the complex $(\Omega^{\cdot}_Y, df\wedge)$, then the picture is the same as that in Figure 1 (in Section 2), with the stars representing the Ω^{n-p}_Y and the dots the $\mathbf{C}$-duals of the local cohomology groups.

$E_2^{p,q}$ is the same as $E_1^{p,q}$ for $q > 0$ (though for $a = b$ one needs to be careful), and $E_2^{p,0} = H^{n-p}_Y$. Comparing this with the results from the other spectral sequence, one sees that all the higher differentials are isomorphisms (save that of $\mathrm{Ext}^{n-a}(\Omega^a_Y, \Omega^n_Y) \to H^n_Y$ for $a = b$, which must be injective).

Comparing the limits of the two spectral sequences gives the desired result. In the case that $a = b$, one obtains a 4-term exact sequence

$$0 \to H^a_{\{0\}}(\Omega^{a+1}_Y) \to H^n_Y \to (H^n_Y)^* \to H^a_{\{0\}}(\Omega^a_Y) \to 0,$$

and one can identify this (or its $\mathbf{C}$-dual) with the 4-term exact sequence given above.

It would be interesting to find an explicit formula for these natural pairings. We will return to these questions of local duality and the resulting natural pairings in greater detail and greater generality at a later date.

4 Deforming the critical point

Let f be an invariant analytic function defined on an open set in Y, and define

$$\mathcal{M}(f) = H^{n+1}(\underline{\Omega}^{\cdot}_X, df\wedge) = \underline{\Omega}^{n+1}_X / df \wedge \underline{\Omega}^n_X.$$

This is a coherent sheaf defined on the domain of definition of f. It was shown in Section 3, that the maps $\iota_\vartheta : \underline{\Omega}^p_X \to \Omega^{p-1}_Y$ induce an isomorphism $H^{n+1}(\underline{\Omega}^{\cdot}_X, df\wedge) \to H^n(\Omega^{\cdot}_Y, df\wedge)$ for $n-c > 1$. For $n-c = 1$ (i.e. $a = b = 1$) the map is surjective with kernel $\mathcal{M}_F(f)$, and in this case the quotient space is smooth. Recall that $\mathcal{M}_F(f)$ is the multiplicity of the critical point of the restriction of f to F which if $c = 0$ is just a 1-dimensional space. Thus, for $n-c > 1$,

$$\mathcal{M}(f) = \frac{\underline{\Omega}^{n+1}_X}{df \wedge \underline{\Omega}^n_X} \cong \frac{\Omega^n_Y}{df \wedge \Omega^{n-1}_Y}.$$

Let f_t be an invariant deformation of $f_0 = f$, with $t \in S$, a neighbourhood of $0 \in \mathbf{C}$. The purpose of this section is to show that $\mathcal{M}(f)$ is a multiplicity of the critical point in the sense that it is supported on the critical locus of f_t (Lemma 3.1) and viewed as a sheaf over S, $\mathcal{M}(f_t)$ is locally free. However, there are some cases where f_t can have a critical point but $\mathcal{M}(f_t) = 0$, as we shall see. In the case of real actions of $\mathbf{C}^*$, $\mathcal{M}$ does define a good multiplicity in the sense that the contribution from generic critical points is one. It follows from these results and Proposition 3.6(4) that the other homology group depending on f, $H^n(\underline{\Omega}^{\cdot}_X, df\wedge)$ also behaves well in a deformation. Note that if $F \neq 0$, the lower cohomology groups $\mathcal{M}_F(f)$ behave well in a deformation by the standard theory.

We need to consider sheaves of relative differential forms on $X \times S$ and $Y \times S$. These can be defined as

$$\Omega^p_{X\times S/S} = \frac{\Omega^p_{X\times S}}{dt \wedge \Omega^{p-1}_{X\times S}}.$$

The sheaves $\underline{\Omega}^p_{X\times S/S}$ and $\Omega^p_{Y\times S/S}$ on $Y \times S$ are defined similarly.

Let $F(x,t)$ be a $\mathbf{C}^*$-invariant analytic function defined on some neighbourhood of $(0,0)$ in $X \times S$, and let $f_t(x) = F(x,t)$. Now, $dF\wedge : \underline{\Omega}^p_{X\times S/S} \to \underline{\Omega}^{p+1}_{X\times S/S}$, and $dF\wedge : \Omega^p_{Y\times S/S} \to \Omega^{p+1}_{Y\times S/S}$. We define $\mathcal{M}(F)$ by

$$\mathcal{M}(F) := \tau_*[\underline{\Omega}^{n+1}_{X\times S/S} / dF \wedge \underline{\Omega}^n_{X\times S/S}],$$

where $\tau : X \times S \to S$ is the cartesian projection. If $n - c > 1$ then $\mathcal{M}(F) \cong \tau_*[\Omega^n_{Y\times S/S} / df \wedge \Omega^{n-1}_{Y\times S/S}]$. (There should be no confusion arising from the two uses of the symbol F.)

Theorem 4.1 *Let f be an invariant function with an isolated critical point at $0 \in Y$, and let f_t, $t \in S$ be in invariant deformation of $f = f_0$. Then $\mathcal{M}(F)$ is a free $\mathcal{O}_S$-module.*

PROOF We show that

$$0 \to \mathcal{M}(F) \xrightarrow{.t} \mathcal{M}(F) \to \mathcal{M}(f) \to 0 \tag{4.17}$$

is exact, for then $\mathcal{M}(F)$ is torsion free over S, and hence free. In the case that $a=b=1$, the quotient space is smooth, and it follows that $\tau_*[\Omega^n_{Y\times S/S}/df\wedge\Omega^{n-1}_{Y\times S/S}]$ is free by the standard theory, and hence so is $\mathcal{M}(F)$ (as the kernel of the map from one to the other is a free $\mathcal{O}_S$ module of rank 1. From now on assume $n-c>1$.

For each p, the following is clearly a short exact sequence:

$$0\to\Omega^p_{Y\times S/S}\xrightarrow{.t}\Omega^p_{Y\times S/S}\to\Omega^p_Y\to 0.$$

Since multiplication by t commutes with $dF\wedge$, it follows that

$$0\to(\Omega^{\cdot}_{Y\times S/S},dF\wedge)\xrightarrow{.t}(\Omega^{\cdot}_{Y\times S/S},dF\wedge)\to(\Omega^{\cdot}_Y,df\wedge)\to 0. \tag{4.18}$$

is exact.

Consider the long exact sequence of cohomology arising from (4.18):

$$0\to H^0(\Omega^{\cdot}_{Y\times S/S})\xrightarrow{.t}H^0(\Omega^{\cdot}_{Y\times S/S})\to H^0(\Omega^{\cdot}_Y)\to H^1(\Omega^{\cdot}_{Y\times S/S})\to\cdots$$

$$\to H^{n-1}(\Omega^{\cdot}_Y)\to\mathcal{M}(F)\xrightarrow{.t}\mathcal{M}(F)\to\mathcal{M}(f)\to 0.$$

(Here $H^i(\Omega^{\cdot}_{Y\times S/S})$ refers to the cohomology of the complex $(\Omega^{\cdot}_{Y\times S/S},dF\wedge)$, etc.)

By Proposition 3.5 (and Remark 3.8 if $c\neq 0$) one has exactness of

$$0\to H^{n-1}(\Omega^{\cdot}_{Y\times S/S})\xrightarrow{.t}H^{n-1}(\Omega^{\cdot}_{Y\times S/S})\to H^{n-1}(\Omega^{\cdot}_Y)\to 0.$$

(Note that in the case that $c\neq 0$ and $a=b+1$, we are using the exactness of $0\to\mathcal{M}_F(F)\to\mathcal{M}_F(F)\to\mathcal{M}_F(f)\to 0$, which follows from the standard theory as the space F is smooth.) The exactness of (4.17) follows. □

Generic multiplicities

The theorem above states that when an invariant function f with an isolated critical point is perturbed, the number of critical points in Y appearing in the deformation is equal to $\dim_{\mathbf{C}}\mathcal{M}(f)$, provided local multiplicities are taken into account. The local multiplicity of a critical point of f_t at $y\in Y$ is of course given by $\dim_{\mathbf{C}}(\mathcal{M}(f_t)_y)$. It is therefore important to know what local multiplicities to expect for generic critical points. The answer depends on the local geometry of Y, that is, on the stabilizer of an orbit.

Proposition 4.2 *The minimal multiplicity for any stratum of a real action is 1. The minimal multiplicities for the strata of low codimension are as follows:*

1. *1 for the open stratum;*
2. *0 for the codimension 1 strata (i.e. the pseudoreflexion hyperplanes);*
3. $e-2$ *for the codimension* 2 *strata (where e is the embedding dimension of the quotient singularity).*

PROOF For the real actions, the result follows from Schwarz' Theorem, given as Theorem 1.6 above. 1) and 2) are straightforward, since at such points Y is smooth and the modules Ω_Y^p are just the usual differential forms. Note that an invariant function with a generic critical point at a pseudo-reflexion hyperplane is non-singular on the quotient space.

3) This follows from the proofs of Theorems 4.1 and 5.1 of Wall in [30], using $\Omega^m/df\wedge\Omega^{m-1}$ rather than $\mathcal{O}_{\mathbf{C}^m}/Jf$, but first we must reduce to the case of a transversal to the stratum.

This reduction proceeds as follows. Clearly, for a stratified critical point to be generic, it is necessary that its restriction to the stratum be a non-degenerate critical point. One can then apply the equivariant splitting lemma to write the function locally as a sum of a non-degenerate quadratic form on the stratum and a generic function on a transversal to the fixed point set invariant under the action of the isotropy subgroup. The multiplicity is then the multiplicity of the restriction to a transversal.

Following Wall, let $f(x,y)=x^a+y^b$. Then $\Omega^m/df\wedge\Omega^{m-1}$ is the sum of a trivial representation and a free $\mathbf{C}G$-module. On deforming f two types of critical point emerge from the origin: those with trivial isotropy and those on the reflecting hyperplanes. By (2), the critical points on the reflecting hyperplanes do not contribute to the multiplicity, so, as in Wall's proof, the effect of the deformation is to reduce $\Omega^m/df\wedge\Omega^{m-1}$ by a number of free $\mathbf{C}G$-modules. Thus, for generic f we have, in Wall's notation,

$$\dim(\underline{\Omega}_X^m/df\wedge\underline{\Omega}_X^{m-1})=1+\nu^G(f).$$

Furthermore, Wall shows (using Koushnirenko's formula for Newton diagrams) that if G is cyclic, then $\nu^G(f)=e-3$. □

If the $\mathbf{C}^*$-action is free outside $\mathcal{B}$ then the multiplicity we have defined gives complete information on the decomposition of a degenerate critical point under a generic perturbation. If, on the other hand, the action is not free outside $\mathcal{B}$ then it is also necessary to be able to compute the number of critical points lying in any given fixed point subspace. By the principal of symmetric criticality (see Section 1) it is enough to repeat the multiplicity computation for the restriction of f to each fixed point space V. However, in the real case there is an easier method, namely factoring out $\mathcal{M}(f)$ by the ideal $I(V)$ of functions vanishing on V. Summing up in the real case, we have the following result.

Corollary 4.3 *Let f be a function invariant under a real action of $\mathbf{C}^*$ with an isolated critical point at 0, and let f_t be a generic invariant deformation of f. Then the number of critical points of f_t emanating from 0 is equal to*

$$\dim_{\mathbf{C}}\mathcal{M}(f)=\dim_{\mathbf{C}}[\mathcal{O}_X/Jf]_0,$$

where Jf is the jacobian ideal of f, and the subscript 0 means the invariant part. Moreover, the number of critical points of f_t with isotropy group H is equal to

$$\dim_{\mathbf{C}}\left[\frac{\mathcal{O}_X}{Jf+I}\right]_0,$$

where I is the ideal of functions vanishing on $\mathrm{Fix}(H;\mathbf{C}^{n+1})$.

PROOF The first observation is that generic functions have non-degenerate critical points. By the proposition above, these have multiplicity 1, and so $\dim \mathcal{M}(f)$ does indeed count the number of critical points. Now, for a real action, the isomorphism $\mathcal{O}_X \to \Omega_X^{n+1}$, $h \mapsto h\omega$, with $\omega = dx_1 \wedge \ldots \wedge dx_a \wedge dy_1 \wedge \ldots \wedge dy_a$, is equivariant. Moreover, this isomorphism maps Jf to $df \wedge \Omega_X^n$. The first part follows.

The final part is proved using the Principle of Symmetric Criticality, as stated in Lemma 1.4. For the multiplicity of the restriction $f_{|V}$ is given by $\dim[\mathcal{O}_V/Jf_{|V}]_0$, but

$$\left[\frac{\mathcal{O}_V}{Jf_{|V}}\right]_0 \cong \left[\frac{\mathcal{O}_X}{Jf_{|V} + I(V)}\right]_0 = \left[\frac{\mathcal{O}_X}{Jf + I(V)}\right]_0 .$$

□

Finite extensions of $\mathbf{C}^*$

We consider briefly the effect of a finite extension of $\mathbf{C}^*$ acting on $X = \mathbf{C}^{n+1}$. Let G be such an extension, so

$$1 \to \mathbf{C}^* \to G \to \Gamma \to 1,$$

with Γ a finite group.

As before, let Y denote the quotient by $\mathbf{C}^*$, and $\underline{\Omega}_X^p$ denote the $\mathbf{C}^*$-invariant p-forms. Then Γ acts on Y, and on the $\underline{\Omega}_X^p$, the local cohomology groups computed in Section 2, and so on. We denote the full quotient space by Y/Γ, the G-invariant forms by Ω_G^p, and the G-invariant basic forms by $\Omega_{Y/\Gamma}^p$.

Let f be a G-invariant function on $\mathbf{C}^{n+1}$ having an isolated critical point on Y (or, what is the same, on Y/Γ). Then Γ acts on the cohomology groups $H^i(\underline{\Omega}_X^{\cdot}, df\wedge)$ and $H^i(\Omega_Y^{\cdot}, df\wedge)$. For any G-invariant function f denote by $C(f)$ the set of critical points of f in Y. Now, Γ acts on $C(f)$ by permutations, and we denote the associated permutation representation by $[C(f)]$.

Proposition 4.4 *Let G act on $\mathbf{C}^{n+1}$ as a real representation, and let f_t be a generic invariant deformation of f with f having isolated critical points on Y. Then there is an isomorphism of representations of Γ:*

$$[C(f_t)] \cong \mathcal{M}(f).$$

The action of Γ on the 1-dimansional groups $H^k(\Omega_Y^{\cdot}, df\wedge)$ for $k = 3,5,7,\ldots,2b-1$ depend on its action on the vector field ϑ generating the $\mathbf{C}^*$-action, since H_Y^{2k+1} is generated by $df \wedge \omega^k$, and ω is defined by $df = \iota_\vartheta \omega$. Furthermore, by the results of Section 6, the action of Γ on the cohomology groups of the Milnor fibre of f in the quotient space are isomorphic to its actions on the H_Y^k.

Similarly, other results of Wall [29] and Roberts [21] generalize to this setting.

5 Equivariant vector fields and critical points

Liftable vector fields

The action of $\mathbf{C}^*$ on X induces an action on Θ_X, the $\mathcal{O}_X$-module of analytic vector fields on X. The vector fields v satisfying $\mathcal{L}_\vartheta v = 0$, those fixed by the action, are called *equivariant* vector fields; they form an $\mathcal{O}_Y$-module denoted $\underline{\Theta}_X$. Clearly,

$$\underline{\Theta}_X = \left(\bigoplus_{i=1}^{a}[\mathcal{O}_X]_{\lambda_i}\frac{\partial}{\partial x_i}\right) \bigoplus \left(\bigoplus_{j=1}^{b}[\mathcal{O}_X]_{\mu_j}\frac{\partial}{\partial y_j}\right) \bigoplus \left(\bigoplus_{k=1}^{c}\mathcal{O}_Y\frac{\partial}{\partial z_k}\right). \tag{5.19}$$

Any equivariant vector field on X defines a derivation of $\mathcal{O}_Y$, and so passes down to a vector field on Y. It is well-known (and not hard to show) that such a vector field on Y is tangent to the stratification by orbit type (see Section 1). This suggests defining the $\mathcal{O}_Y$-module of all vector fields on Y tangent to the stratification by orbit type, which we denote Θ_Y. It should be emphasized that this Θ_Y does not coincide with the usual module of vector fields tangent to a variety, unless the stratification of Y as a quotient space is the same as its logarithmic stratification.

There is a homomorphism $p : \underline{\Theta}_X \to \Theta_Y$, whose kernel consists of equivariant vector fields tangent to the orbits. Thus $\ker p = \mathcal{O}_Y\vartheta$. The question of whether p is surjective is a 'lifting problem', which, for reductive group actions, has been studied by G. Schwarz [25]. We begin this section by giving a more precise result in the case of $\mathbf{C}^*$-actions. We say that a $\mathbf{C}^*$-action has the *lifting property* if p is surjective.

Theorem 5.1 *A $\mathbf{C}^*$-action has the lifting property if and only if one of the following conditions holds:*

1. $b > 1$,
2. $a = b = 1$,
3. *For $a > b = 1$, there are no non-negative integer solutions r_j, s to the equation,*

$$\lambda_i = \sum_{j\neq i} r_j\lambda_j + s(-\mu),$$

with $s > 0$ and at least one of the $r_j > 0$. In particular, this condition holds if the sum of the weights is zero.

PROOF It is enough to prove this for $F = 0$ since the general case is just a product of this case with a smooth space.

1) Consider the exact sequence of sheaves on Y

$$0 \to \mathcal{O}_Y\vartheta \to \underline{\Theta}_X \xrightarrow{p} \Theta_Y \to \mathcal{N} \to 0, \tag{5.20}$$

which defines the cokernel $\mathcal{N}$ consisting of non-liftable vector fields. We wish to find criteria which ensure $\mathcal{N} = 0$.

The first observation is that $\operatorname{supp}\mathcal{N} \subset F$ since outside of F the isotropy is finite, and by [3] and [25], we have that p is surjective off F. If $\mathcal{N} \neq 0$ then it follows that $\operatorname{depth}\mathcal{N} = 0$, where by depth we mean m_Y-depth.

By (5.19) we have that

$$\operatorname{depth}\underline{\Theta}_X = \min\{\operatorname{depth}[\mathcal{O}_X]_{\lambda_i}, \operatorname{depth}[\mathcal{O}_X]_{\mu_j}\}. \tag{5.21}$$

It now follows from Theorem 2.6 that $\operatorname{depth}\underline{\Theta}_X \geq b$. Clearly, $\operatorname{depth}\Theta_Y > 0$ since Θ_Y is torsion free. Finally, $\operatorname{depth}\mathcal{O}_Y = \operatorname{codim}_Y(F) = a+b-1$. Taking the Čech resolution of (5.20) for the subset F of Y gives the following fact (Proposition A.2): if $\operatorname{depth}\mathcal{O}_Y > 2$, $\operatorname{depth}\underline{\Theta}_X > 1$ and $\operatorname{depth}\Theta_Y > 0$ then $\operatorname{depth}\mathcal{N} > 0$. Thus, if $b > 1$ then all these conditions are satisfied, so indeed $\mathcal{N} = 0$.

2) If $a = b = 1$ then this follows from [25, Proposition 7.2] (or by direct calculation as for case (3)).

3) The third case is proved in the same way that Wall proves it for $a = 2$, $b = 1$, $c = 0$ in [30, Example 2.3]. (In fact Wall makes an error as he does not allow for the possibility that the stratifications of $Y = V/G$ and $Y_1 = W/H$ differ at the origin.) At the end of Section 1, we note that if $b = 1$, the quotient Y is isomorphic to the quotient of X_1 by $\mathbf{Z}/\mu\mathbf{Z}$, and their stratifications differ at most at the origin. Now, since $\mathbf{Z}/\mu\mathbf{Z}$ is finite, it follows that every vector field on the quotient Y_1 tangent to the stratification is liftable. Thus we can represent Θ_{Y_1} by equivariant vector fields on X_1. Thus,

$$\Theta_{Y_1} = \bigoplus_{i=1}^{a}\{x^r\tfrac{\partial}{\partial x_i} \mid (r,\lambda) - \lambda_i \equiv 0(\mu)\}.$$

To obtain the vector fields Θ_Y on Y tangent to the stratification, we can use Θ_{Y_1}, but we must ensure that the vector fields vanish at 0, thus

$$\Theta_Y = \{v \in \Theta_{Y_1} \mid r \neq 0\}$$

On the other hand,

$$\underline{\Theta}_X = \bigoplus_{i=1}^{a}\{x^r y^s\tfrac{\partial}{\partial x_i} \mid (r,\lambda) - \lambda_i + s\mu = 0\}\bigoplus\{x^r y^s\tfrac{\partial}{\partial y} \mid (r,\lambda) + (s-1)\mu = 0\}.$$

Using the same argument as Wall, we can ignore the last summand (because of the 1-dimensional kernel of p). Then using $x \mapsto (x,1)$ to identify X_1 with a subset of X (as in Section 1), we see that $v = x^r\frac{\partial}{\partial x_i} \in \Theta_Y$ lifts if and only if there is an $s \geq 0$ such that $x^r y^s\frac{\partial}{\partial x_i} \in \underline{\Theta}_X$, that is, $(r,\lambda) + s\mu = \lambda_i$. Thus, it fails to be liftable precisely when the congruence $(r,\lambda) - \lambda_i \equiv 0(\mu)$ is satisfied by $s < 0$. That is, non-liftable vector fields correspond to multiindices satisfying

$$(r,\lambda) + (-s)\mu = \lambda_i$$

with $r \neq 0$ and $-s > 0$, as was required. □

Multiplicity after Bruce & Roberts

In [7, Section 8], Bruce & Roberts consider the multiplicity of critical points of analytic functions on quotient varieties. Their approach is to work directly on the stratified quotient space Y; they show that critical points of the function f correspond to intersections of graph(df) and $LC^-(Y)$, the logarithmic characteristic variety of Y, and that the intersection multiplicity is given by $\dim_{\mathbf{C}}(\mathcal{O}_Y/\Theta_Y(f))$. In the case that the group is finite, they prove that $LC^-(Y)$ is a Cohen-Macaulay space, and so deduce that intersection multiplicites are preserved under deformations. On the other hand they point out that it is easy to find examples of reductive group actions for which $LC^-(Y)$ is not Cohen-Macaulay. Such an example is provided by $\mathbf{C}^*$ acting on $\mathbf{C}^{a+1}$ ($a > 1$) with weights $(1,1,\dots,1,-1)$. The quotient space is then smooth with orbit type strata $Y_{\mathrm{reg}} = \mathbf{C}^a \setminus \{0\}$ and $\{0\}$. Thus $LC^-(Y)$ consists of two transverse a-dimensional subspaces of $\mathbf{C}^{2a}$ and is therefore not Cohen-Macaulay. Bruce and Roberts suggest that $LC^-(Y)$ is Cohen-Macaulay for any real action of a reductive group.

It turns out that there are many instances of $\mathbf{C}^*$-actions for which $LC^-(Y)$ is indeed Cohen-Macaulay, and not just the real actions conjectured by Bruce and Roberts.

We begin with an obvious result. Recall from Section 4 that by definition, $\mathcal{M}(f) := \underline{\Omega}_X^{n+1}/df\wedge\underline{\Omega}_X^n$.

Proposition 5.2 *Suppose that $\mathbf{C}^*$ acts on $\mathbf{C}^{n+1}$ and the sum of the weights is zero. Then the two modules $\mathcal{O}_Y/\Theta_Y(f)$ and $\mathcal{M}(f)$ are isomorphic.*

PROOF There is always an isomorphism of $\mathcal{O}_X$-modules $\mathcal{O}_X \to \Omega_X^{n+1}$ given by multiplication by the $(n+1)$-form $\omega = dx_1\wedge\ldots\wedge dz_c$. Furthermore, for any function f, one has, $\omega J(f) = df\wedge\Omega_X^n$, where $J(f) = \Theta_X(f)$ is the jacobian ideal generated by the partial derivatives of f.

Suppose now that the sum of the weights is zero, so that ω is $\mathbf{C}^*$-invariant. Then the isomorphism is also an isomorphism of $\mathbf{C}^*$-modules. Taking invariant parts, it follows that $\omega.\mathcal{O}_Y = \underline{\Omega}_X^{n+1}$, and $\omega.\underline{\Theta}_X(f) = df\wedge\underline{\Omega}_X^n$ for an invariant function f. Thus, $\mathcal{M}(f) \cong \mathcal{O}_Y/\underline{\Theta}_X(f)$.

Finally, it follows from Theorem 5.1, that if the sum of the weights is zero then $\underline{\Theta}_X(f) = \Theta_Y(f)$. □

Corollary 5.3 *If the sum of the weights is zero, and f has an isolated critical point at 0, then $\mathcal{O}_Y/\Theta_Y(f)$ deforms flatly under any invariant deformation of f.* □

Corollary 5.4 *If the sum of the weights is zero then the logarithmic characteristic variety $LC^-(Y)$ of the quotient space is Cohen-Macaulay.*

PROOF First let f be an invariant function with an isolated critical point in Y, and let $\hat{f}$ be an extension to an open set in $\mathbf{C}^l$ (the ambient space of Y). Consider the family of functions parametrized by $a\in(\mathbf{C}^l)^*$ defined by $\hat{f}_a(u) = \hat{f}(u) - au$. Now define a map

$$\begin{aligned}\Phi : T^*\mathbf{C}^l &\to (\mathbf{C}^l)^*\\ (u,a) &\to d\hat{f}_a(u) = d\hat{f}(u) - a.\end{aligned}$$

For each a, the intersection $\Phi^{-1}(a) \cap LC^{-}(Y)$ is finite (by the preparation theorem, since f has an isolated critical point at 0). By Corollary 5.3 above, the restriction of Φ to $LC^{-}(Y)$ is flat. Consequently, $LC^{-}(Y)$ is Cohen-Macaulay. □

Another case where $LC^{-}(Y)$ is Cohen-Macaulay occurs when Y is isomorphic as a stratified variety to a finite quotient, see the examples in Section 1. In general, we do not have necessary and sufficient conditions for $LC^{-}(Y)$ to be Cohen-Macaulay. Indeed, the only general negative result we have is the following.

Proposition 5.5 *Consider the action with weights* $(1,\ldots,1,-1,\ldots,-1)$. *Then* $LC^{-}(Y)$ *is Cohen-Macaulay if and only if* $a = b$.

PROOF Write $M = M_{a,b}$ for the space of $a \times b$ matrices. The quotient space Y is isomorphic to the subvariety of M of matrices of rank at most 1, which has an isolated singularity at 0. Thus $LC^{-}(Y) \subset T^*M$ has two components, one is T_0^*M, the fibre over the zero matrix, and the other is the closure of the conormal bundle over the smooth part. The conormal space over the matrix $Q \in Y$ consists of matrices $P \in M$ for which $P^tQ = QP^t = 0$. These two components are of dimension $ab = \dim M$, and each is Cohen-Macaulay. Their intersection is the subset of $T_0^*M = M$ of matrices P of rank at most $b-1$, which has codimension $a-b+1$ in each of the components. At a generic point of the intersection, the variety is just a union of two smooth subspaces intersecting along a subspace of codimension $a-b+1$. At such a point, the variety cannot be Cohen-Macaulay unless they intersect in a hypersurface, i.e. unless $a = b$. □

Remark 5.6 There are examples of Y for which $LC^{-}(Y)$ is Cohen-Macaulay which are not accounted for by the results above. For example, we found using the computer package Macaulay [2] that for the action with weights $(1,1,-1,-2)$ $LC^{-}(Y)$ is Cohen-Macaulay, while for the action with weights $(1,1,-1,-3)$ it is not.

Remark 5.7 In the situation of Proposition 5.5 with $a = 2, b = 1$, one sees that $LC^{-}(Y)$ is the union of two transverse 2-planes in $\mathbf{C}^4$ which is not Cohen-Macaulay. However, it is the quotient by $\mathbf{C}^*$ of a Cohen-Macaulay space of dimension 3 in $\mathbf{C}^5$, given by equations,

$$l_1x_1 + l_2x_2 = l_1y = l_2y = 0,$$

where $\mathbf{C}^*$ acts on (x_1,x_2,y,l_1,l_2)-space with weights $(1,1,-1,0,0)$. (This space is in fact the appropriate Z defined in [7, Section 8].)

6 The quotient Milnor fibre

Let $f:(Y,0)\to(\mathbf{C},0)$ be an invariant function germ with isolated singularity, and let $f: U_\varepsilon \to S$ be a representative (with $U_\varepsilon = Y\cap B_\varepsilon$, the intersection of Y with the ε-ball in the ambient space of Y, and f non-singular on $U\setminus\{0\}$). For any $t\in S$ one can define the fibre $Y_t = f^{-1}(t)\subset U$. We call Y_t the quotient Milnor fibre as it is the quotient of $f^{-1}(t)\cap\pi^{-1}(U_\varepsilon)\subset X$ by $\mathbf{C}^*$. (We do not assume $F=0$ in this section.)

Theorem 6.1 *ε and $\eta\in\mathbf{R}_+$ can be chosen sufficiently small so that for any $t\in D_\eta$ (the disk in $\mathbf{C}$ centre 0 and radius η), and for each $i\geq 1$,*

$$\dim_{\mathbf{C}} H^i(Y_t,\mathbf{C}) = \dim_{\mathbf{C}} H^{i+1}(\Omega^{\cdot}_Y, df\wedge).$$

This agrees with the classical case where Y is smooth. However, in the smooth case $H^{i+1}(\Omega^{\cdot}_Y, df\wedge)=0$ for $i+1<\dim Y$.

In the classical case of an isolated singularity on a smooth space, one knows that the Milnor fibre is homotopic to a wedge of spheres of middle dimension. In the present case this is clearly not so, though it seems likely that Y_t is homotopic to a wedge of spheres of middle dimension and the generic hyperplane section:

$$Y_t \sim \vee_{i=1}^{\mu} S^{n-1}\vee L_t,$$

where L_t is the Milnor fibre of a generic linear function L on Y. We conjecture that this is the case at least if Y has an isolated singularity, and that following Funar [11] the integer cohomology of the Milnor fibre is torsion free.

The proof of this theorem follows closely the proofs of Brieskorn [6] and Malgrange [18]. There are also discussions of this theorem in [29] and [26] for the case that the group is finite and the function f on X has an isolated critical point.

We will need a (well-known) Poincaré Lemma for the basic forms.

Lemma 6.2 *The complex of sheaves $(\Omega^{\cdot}_Y, d)$ is a resolution of the constant sheaf $\mathbf{C}_Y$.*

PROOF Away from $F\subset Y$ this follows from the Poincaré lemma for finite groups by the slice theorem.

On F a different argument is needed. Let $z\in F$ and let U be a contractible Stein neighbourhood of z in Y. First observe that $(\Omega^{\cdot}_X(\pi^{-1}(U)), d)$ is acyclic by the usual Poincaré lemma and the fact that $\pi^{-1}(U)$ is contractible Stein. Then by taking invariant parts we deduce the acyclicity of $(\underline{\Omega}^{\cdot}_X(U), d)$.

Consider now the double complex

$$K^{p,q} = \begin{cases} \underline{\Omega}_X^{p-q}(U) & \text{if } p\neq q\\ m_z(U) & \text{if } p=q,\end{cases}$$

where $m_z(U)$ is the ideal of functions on U vanishing at z. The maps on this complex are $d: K^{p,q}\to K^{p+1,q}$ and $\iota_\vartheta: K^{p,q}\to K^{p,q+1}$. Since $(\underline{\Omega}_X(U), d)$ is exact, the homology of the total complex is zero. Now, the spectral sequence commencing with ι_ϑ degenerates at E_2 to give

$$E_\infty^{p,q} = E_2^{p,q} = \begin{cases} H^p(\Omega^\cdot_Y, d) & \text{if } q = 0 \\ 0 & \text{otherwise.} \end{cases}$$

The result now follows. □

As usual, define the sheaf of relative differentials as

$$\Omega^p_{Y/S} := \frac{\Omega^p_Y}{df \wedge \Omega^{p-1}_Y}.$$

We use d to denote both the absolute exterior differential on $\Omega^\cdot_Y$ as well as the relative exterior differential on $\Omega^\cdot_{Y/S}$. As a preliminary result, we need the following.

Lemma 6.3

$$\begin{aligned} H^0(\Omega^\cdot_{Y/S}, d) &= f^{-1}\mathcal{O}_S \\ H^i(\Omega^\cdot_{Y/S}, d)_{|Y\setminus\{0\}} &= 0 \quad \textit{for } i > 0. \end{aligned}$$

PROOF This follows from the Poincaré Lemma 6.2, and the exactness of $(\Omega^\cdot_Y, df\wedge)$ outside 0, Lemma 3.4. □

Proposition 6.4 (Brieskorn [6]) *For* $p \geq 0$,

1) $H^p(f_*\Omega^\cdot_{Y/S}, d)$ *is* $\mathcal{O}_S$*-coherent,*

2) $H^p(f_*\Omega^\cdot_{Y/S}, d)_0 \cong H^p(\Omega^\cdot_{Y/S,0}, d)$,

3) *For* $t \neq 0$, $H^p(f_*\Omega^\cdot_{Y/S}, d)_t \cong H^p(Y_t, \mathbf{C}) \otimes_{\mathbf{C}} \mathcal{O}_{S,t}$.

PROOF It follows from Lemma 6.3 that $(\Omega^\cdot_{Y/S}, d)$ is a concentrated complex in the sense of [27]. The three statements are general properties of concentrated complexes, and as such are proved as Theorem 1 and Propositions 1 and 2 of [27]. □

The coherent sheaf $\mathcal{H}^p := H^p(f_*\Omega^\cdot_{Y/S}, d)$ restricts on $S\setminus\{0\}$ to (the sheaf associated to) the bundle of p-th cohomology groups of the fibres Y_t. Corresponding to parallel transport of cohomology classes there is a connection

$$\nabla_t : \mathcal{H}^p_{|S\setminus\{0\}} \longrightarrow \mathcal{H}^p_{|S\setminus\{0\}},$$

the so-called Gauss-Manin connexion. This connexion does not extend to a connexion over the zero stalk $\mathcal{H}^p_0$ of $\mathcal{H}^p$; it is necessary to map $\mathcal{H}^p_0$ to a slightly larger module.

In order to deal with such a problem, Malgrange [18] introduced the notion of (E,F) connection, where $E \subset F$ are finitely generated $\mathcal{O}_{S,0}$-modules with F/E torsion, and $D : E \to F$ is a $\mathbf{C}$-linear derivation. That is, for $e \in E$ and $h \in \mathcal{O}_{S,0}$

$$D(h.e) = \frac{dh}{dt}.e + h.D(e).$$

There is an index theorem for such a set-up [18, Theorem 2.3] which states that, with $(D;E,F)$ as above, D has finite dimensional kernel and cokernel and

$$\dim_{\mathbf{C}} \ker(D) - \dim_{\mathbf{C}} \operatorname{Coker}(D) = \operatorname{rk}(E) - \dim_{\mathbf{C}}(F/E).$$

We apply these ideas to the following pairs of $\mathcal{O}_{S,0}$-modules and derivations:

$$E_i = H^i(\Omega_{Y/S,0}, d) = \frac{\{\omega \in \Omega_Y^i \mid d\omega \in df \wedge \Omega_Y^{i-1}\}}{d\Omega_Y^{i-1} + df \wedge \Omega_Y^{i-1}},$$

(where Ω_Y now means germs at 0 rather than sheaves) and

$$F_i = \frac{\{\omega \in \Omega_Y^i \mid df \wedge d\omega = 0\}}{d\Omega_Y^{i-1} + df \wedge \Omega_Y^{i-1}}.$$

Clearly $E_i \subset F_i$. The derivation is

$$\begin{array}{rcl} \partial_t : E_i & \to & F_i, \\ \omega & \mapsto & \eta \end{array}$$

where η satisfies $d\omega = df \wedge \eta$.

Proposition 6.5 *1. ∂_t is an isomorphism, and*

2. $F_i/E_i \xrightarrow{d} H^{i+1}(\Omega_Y^\cdot, df\wedge)$ is an isomorphism.

PROOF 1) First we show that ∂_t is well-defined. Let $\omega \in \Omega_Y^i$ represent $[\omega] \in E_i$. Then $d\omega = df \wedge \eta$ for some $\eta \in \Omega_Y^i$. This η is determined up to elements of $\ker[df\wedge : \Omega_Y^i \to \Omega_Y^{i+1}]$, which are zero in F_i, because by Proposition 3.6,

$$d\Omega_Y^{i-1} + df \wedge \Omega_Y^{i-1} = d\Omega_Y^{i-1} + \ker[df\wedge : \Omega_Y^i \to \Omega_Y^{i+1}].$$

Changing the representative of $[\omega]$ by $d\alpha + df \wedge \beta$ changes η by $-d\beta$ which is also zero in F_i. Thus ∂_t is well-defined.

Suppose $\partial_t[\omega] = 0$, that is $d\omega = df \wedge \eta$ where $\eta = d\alpha + df \wedge \beta$ for some α, β. Then $d\omega = df \wedge d\alpha$ for some α, so $\omega = df \wedge \alpha + d\gamma$ by the Poincaré Lemma 6.2. This means that $[\omega] = 0$ in E_i.

Let η represent $[\eta] \in F_i$, so $d(df \wedge \eta) = -df \wedge d\eta = 0$. Using the Poincaré lemma we see that $df \wedge \eta = d\omega$ for some $\omega \in \Omega_Y^i$. This proves the surjectivity of ∂_t.

2) Now,

$$F_i/E_i = \frac{\{\omega \in \Omega_Y^i \mid df \wedge d\omega = 0\}}{\{\omega \in \Omega_Y^i \mid d\omega \in df \wedge \Omega_Y^{i-1}\}},$$

so the statement is obvious. □

PROOF OF THEOREM 6.1. Firstly, Proposition 6.4 states that for $t \neq 0$,

$$\mathrm{rk}(E_i) = \dim_{\mathbf{C}}(H^i(Y_t, \mathbf{C})).$$

Secondly, using Proposition 6.5, Malgrange's index theorem applied to $(\partial_t; E_i, F_i)$ gives

$$\mathrm{rk}(E_i) = \dim_{\mathbf{C}}(H^{i+1}(\Omega^{\cdot}_Y, df\wedge)).$$

□

Remark 6.6 Exactly as in [18], one can show that the E_i and F_i are free. For $i = 2, 4, \ldots$ with $i < n-1$ and $c = 0$ they have rank 1, and representatives of generators of these modules are as follows.

$$\begin{aligned} \varepsilon_{2k} &= \frac{1}{6} f^3 d\left(\frac{\alpha}{f}\right) \wedge (d\alpha)^{k-1} \\ \phi_{2k} &= \frac{1}{2} f^2 d\left(\frac{\alpha}{f}\right) \wedge (d\alpha)^{k-1} \end{aligned}$$

where $\iota_\vartheta(\alpha) = f$ as in (3.10). One checks that

$$\begin{aligned} \varepsilon_{2k} &\in E_{2k} \\ \phi_{2k} &\in F_{2k} \\ \partial_t \varepsilon_{2k} &= \phi_{2k} \\ d\phi_{2k} &= df \wedge (d\alpha)^k. \end{aligned}$$

It follows that ε_{2k} and ϕ_{2k} are generators of E_{2k} and F_{2k} respectively. Furthermore

$$t\partial_t \varepsilon_{2k} = 3\varepsilon_{2k},$$

so ∂_t is regular singular, and the monodromy on the cohomology groups with $i < n-1$ is trivial.

If $F \neq 0$, then these modules have rank $\dim_{\mathbf{C}} \mathcal{M}_F(f)$, and generators are given by a construction similar to that in Remark 3.8, namely by taking the exterior product with ν as ν varies over $\mathcal{M}_F(f)$. In this case the monodromy of the low dimensional cohomology groups will be just the monodromy associated to the restriction f_F of f to F.

Remark 6.7 One can introduce the equivariant version of the Gauss-Manin system as the cohomology of a complex analogous to the one in [20, p. 158] or [24]. This equivariant version is the total complex of a triple complex with terms

$$C^{p,q,r} = \underline{\Omega}_X^{p-q+r},$$

and differentials d, ι_ϑ, and $df\wedge$. To be more precise, we consider $(\underline{\Omega}_X[u,D], \mathbf{d})$, where u and D are commuting symbols and where

$$\mathbf{d}\omega.u^k.D^l = d\omega.u^k.D^l + \iota_\vartheta\omega.u^{k+1}.D^l - df\wedge\omega.u^k.D^{l+1}.$$

Because d, ι_ϑ and $df\wedge$ pairwise anticommute we have $\mathbf{d}^2 = 0$.

On this complex, one has three additional operators u, t and ∂_t:

$$\begin{aligned} u.\omega.u^k.D^l &= \omega.u^{k+1}.D^l \\ t.\omega.u^k.D^l &= f\omega.u^k.D^l - l\omega.u^k.D^{l-1} \\ \partial_t.\omega.u^k.D^l &= \omega.u^k.D^{l+1} \end{aligned}$$

These commute with $\mathbf{d}$, and $\partial_t t - t\partial_t = 1$, whereas u commutes with t and ∂_t. The cohomology $\mathcal{H}$ gets the structure of a $\mathcal{D}[u]$-module, where $\mathcal{D} = \mathbf{C}\{t,\partial_t\}$, and it is not hard to see that in fact $\mathcal{H}$ is a coherent $\mathcal{D}[u]$-module.

There is a natural filtration, called the Hodge filtration, $F^\cdot$ on this (triple) complex with terms

$$F^{p-1} = \sum_{k,l,m\geq 0} \underline{\Omega}_X^{p+k-l+m} u^l.D^m.$$

One has that $\mathbf{d}F^{p-1} \subset F^p$, so $F^\cdot$ induces a filtration on $\mathcal{H}$. It seems that this $F^\cdot$ can be used to define a mixed Hodge structure on $H^\cdot(Y_t,\mathbf{C})$ in a manner completely analogous to [24]. We hope to elaborate on this on another occasion.

Chern class of the quotient map

We end this section with a discussion of various closed 2-forms on the quotient space Y and the Milnor fibre Y_t, and the relationship between them.

With the usual notation, we have $f = \iota_\vartheta\alpha$ and $\omega = d\alpha$; consequently, there is the fundamental "Hamiltonian" relationship

$$df = -\iota_\vartheta\omega. \tag{6.22}$$

Example 6.8 (*Symplectic Reduction*)
In symplectic geometry, if ω is a symplectic form, this equation is used to define the Hamiltonian f of the symplectic vector field ϑ. Note that if the invariant form ω is non-degenerate, then by Darboux' theorem, it can be written in the form $\omega = \sum dx_i \wedge dy_i$, and so the $\mathbf{C}^*$-action must be real since each "monomial form" $dx_i\wedge dy_i$ must be invariant. (In other words, ω defines an equivariant isomorphism of $\mathbf{C}^{n+1}$ with its dual, which implies that the action is real.)

The quotient Milnor fibres Y_t are in this case the reduced spaces for the $\mathbf{C}^*$-action. The restriction of ω to X_t is a basic form on X_t, i.e. $\omega_t := i_t^*\omega \in \Omega^2_{Y_t}$, where $i_t : X_t \hookrightarrow X$ is the inclusion (we also write $i_t : Y_t \hookrightarrow Y$). Thus any statements about quotient Milnor fibres can be viewed as generalizations of statements about reduced spaces in symplectic geometry. A particular result is the following:

Let $\mathbf{C}^*$ act symplectically on the symplectic space (X,ω) with an isolated fixed point at 0, and let f be the Hamiltonian, with $f(0) = 0$. Then for $t \neq 0$, the cohomology of the reduced space Y_t is given by

$$H^i(Y_t,\mathbf{C}) = \begin{cases} \mathbf{C} & \text{if } i \leq n-2 \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

This is clear from Theorem 6.1 and Example 3.7, since the relation $f = \iota_\vartheta \alpha$ implies that f is homogeneous of degree 2, and (6.22) implies that it has an isolated critical point.

Returning to the general (non-symplectic) case, consider the meromorphic 1-form $\alpha' = \alpha/f$. This is an invariant form with poles along $X_0 = \{f = 0\}$. Since it satisfies $\iota_\vartheta \alpha' = 1$ off X_0, it is a connection 1-form for the principle fibration $\pi : X_{\neq 0} \to Y_{\neq 0}$, and $\omega' = d\alpha'$ is a curvature 2-form. It follows that the Chern class ch of this fibration is given by the cohomology class $[\omega'] \in H^2(Y_{\neq 0}, \mathbf{Z}) \subset H^2(Y_{\neq 0}, \mathbf{C})$. Notice that ω' is indeed a basic form on $Y_{\neq 0}$, as

$$\iota_\vartheta \omega' = \iota_\vartheta d\alpha' = -d(\iota_\vartheta \alpha') = 0.$$

It has a pole of order 2 along X_0 (or Y_0).

Now, $i_t^* \omega' = i_t^* d\alpha' = d i_t^*(\alpha/f) = (d i_t^* \alpha)/t = i_t^* \omega/t$. Thus we have the following result on the variation in the cohomology class of ω_t, similar to the theorem of Duistermaat & Heckman [10, Theorem 1.1]:

Theorem 6.9 *Let $t \neq 0$ and suppose $\omega \in \underline{\Omega}^2$ is a closed form satisfying (6.22) where f has an isolated critical point on Y. Then the cohomology class defined by ω_t and the Chern class* ch *of the fibration $X_t \to Y_t$ (which is independent of t) are related by*

$$[\omega_t] = t\,\mathrm{ch}\,.$$

These forms are also related to the generators of E_2 and F_2 given in Remark 6.7:

$$\begin{aligned} [\varepsilon_2] &= \frac{1}{6} f^3\,\mathrm{ch} \\ [\phi_2] &= \frac{1}{2} f^2\,\mathrm{ch}\,. \end{aligned}$$

Remark 6.10 There is no basic form $\eta \in \Omega_Y^2$ (defined on a neighbourhood of $0 \in Y$) with the property that the restriction of η to Y_t is ω_t, for otherwise $df \wedge \eta = df \wedge \omega$, but $df \wedge \omega$ is non-trivial in $H^3(\Omega^{\cdot}_Y, df\wedge)$, by Proposition 3.6. However, $t\omega_t = i_t^*(f\omega)$ is the restriction of the basic form $f^2\omega' = f\omega - df \wedge \alpha$. This is of course consistent with the fact that the cohomology group $H^3(\Omega^{\cdot}_Y, df\wedge)$ is killed by m_Y and $f \in m_Y$.

As a final observation, note that the "reduced form" ω_t can be obtained from the special form $df \wedge \omega$ by taking residues:

$$\omega_t = \mathrm{Res}_{\{f=t\}} \left(\frac{df \wedge \omega}{f - t} \right).$$

A Čech complexes and local cohomology

In this appendix, we describe a complex associated to any R-module M, and any finite set of functions $\Phi = \{\phi_1, \dots, \phi_r\}$ in R. The main property of this complex is that it computes the (algebraic) local cohomology of M along $Z = Z_\Phi = V(I)$, where I is the ideal generated by the ϕ_i. See Remark A.7 for why the algebraic cohomology is sufficient for our purposes.

Let X be a space, $\mathcal{U} = \{U_i\}$ an open cover of X and $\mathcal{F}$ a sheaf on X. Associated to this data there is the complex $C^\cdot(\mathcal{U},\mathcal{F})$ of alternating Čech cochains:

$$C^p(\mathcal{U},\mathcal{F}) = \bigoplus_{i_0<\cdots<i_p} \Gamma(U_{i_0,\dots,i_p},\mathcal{F}) =: \bigoplus_{i_0<\cdots<i_p} \mathcal{F}_{i_0,\dots,i_p},$$

for example, see [4].

In the case that the U_i are 'sufficiently small' this complex can be used to compute $H^i(X,\mathcal{F})$. Exactly what 'sufficiently small' means depends on the context. In the topological case, the U_I would have to be contractible and $\mathcal{F}$ constant; in the analytic category U_I would have to be Stein and $\mathcal{F}$ coherent; in the algebraic category, the U_I would need to be affine and $\mathcal{F}$ quasi-coherent.

Thus, if $X = \operatorname{Spec}(R)$, $\mathcal{F} = \tilde{M}$ where M is an R-module, and $\Phi = \{\phi_1, \dots, \phi_r\}$, then we can form a covering of $X \setminus Z$, where $Z = V(I)$ as above, by the open sets $U_i = \operatorname{Spec}(R_{\phi_i})$, where R_ϕ is the localization of R with resect to the multiplicative set generated by ϕ, that is $R_\phi = R[\phi^{-1}]$. Thus

$$C^p(\mathcal{U},\mathcal{F}) = \oplus M_{\phi_{i_0},\dots,\phi_{i_p}}$$

where $M_\phi = R_\phi \otimes M$. As all the U_i are affine, we have

$$H^p(X \setminus Z,\mathcal{F}) = H^p(C^p(\mathcal{U},\mathcal{F})).$$

In this setting, the local cohomology groups [12] sit in exact sequences:

$$\begin{gathered} 0 \to H^0_Z(M) \to M \to H^0(X\setminus Z, M) \to H^1_Z(M) \to 0, \\ 0 \to H^i(X\setminus Z, M) \xrightarrow{\cong} H^{i+1}_Z(M) \to 0 \qquad \text{for } i > 0. \end{gathered} \tag{A.1}$$

In order to perform calculations easily in local cohomology we have found it convenient to modify the notation as follows. Consider the two-term complexes,

$$K_i = K_i(R,\Phi) := [R \to R_{\phi_i} c_i],$$

with R in degree 0 and R_{ϕ_i} in degree 1. For any R-module M define

$$\check{C}^\cdot(M,\Phi) = \check{C}^\cdot_\Phi M := K_1 \otimes K_2 \otimes \cdots \otimes K_r \otimes M$$

where all tensor products are over R, and M is considered as a complex concentrated in degree 0. The symbols c_i are used to make a distinction between elements of R and their images in R_{ϕ_i}. In other words, it helps keep track of the Čech cover. We let the c_i anticommute, so

$$\check{C}^p(M,\Phi) = \bigoplus_{i_1<\cdots<i_p} M_{i_1,\ldots,i_p} c_I,$$

and the differential is just

$$\mathbf{c} = (c_1+\ldots+c_r)\wedge : \check{C}^p(M,\Phi) \to \check{C}^{p+1}(M,\Phi).$$

Thus, for example, $\check{C}^0 M = M$ and $\check{C}^1 M = \bigoplus_{i=1}^r M[\phi_i^{-1}]c_i$.

The complex $\check{C}^{\cdot}(M,\Phi)$ is isomorphic to the ordinary Čech complex $C^{\cdot}(M,\mathcal{U})$ with index shifted by 1, and augmented by the module M in degree 0. These constructions lead to the following result.

Theorem A.1 *Let R be a commutative ring, M an R-module and $\Phi = \{\phi_1,\ldots,\phi_r\}$ a subset of R. Let $Z = V(I)$, where I is the ideal generated by $\{\phi_1,\ldots,\phi_r\}$. Then*

$$H^i_Z(M) \cong H^i(\check{C}^{\cdot},\mathbf{c}).$$

For more background information, see the book of J. Strooker [28].

Consider a complex $(\mathcal{F}.)$ of sheaves on X:

$$0 \to \mathcal{F}_n \to \mathcal{F}_{n-1} \to \ldots \to \mathcal{F}_1 \to \mathcal{F}_0 \to 0, \tag{A.2}$$

and assume that the homology of this complex is supported on a closed subset Z of X. Associated to such a complex is *the Čech complex over* $(\mathcal{F}.)$ (with respect to Z), which is defined as follows. Let $\Phi = \{\phi_1,\ldots,\phi_r\}$ define Z, and consider the Čech complexes $\check{C}^{\cdot}_{\Phi}\mathcal{F}_i = \check{C}^{\cdot}\mathcal{F}_i$ for each i. These form a double complex, $K^{pq} = \check{C}^q(\mathcal{F}_{n-p})$:

$$\begin{array}{ccccccccccc}
0 & \to & \check{C}^r\mathcal{F}_n & \to & \check{C}^r\mathcal{F}_{n-1} & \to \cdots \to & \check{C}^r\mathcal{F}_1 & \to & \check{C}^r\mathcal{F}_0 & \to & 0\\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \check{C}^{r-1}\mathcal{F}_n & \to & \check{C}^{r-1}\mathcal{F}_{n-1} & \to \cdots \to & \check{C}^{r-1}\mathcal{F}_1 & \to & \check{C}^{r-1}\mathcal{F}_0 & \to & 0\\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \check{C}^1\mathcal{F}_n & \to & \check{C}^1\mathcal{F}_{n-1} & \to \cdots \to & \check{C}^1\mathcal{F}_1 & \to & \check{C}^1\mathcal{F}_0 & \to & 0\\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \mathcal{F}_n & \to & \mathcal{F}_{n-1} & \to \cdots \to & \mathcal{F}_1 & \to & \mathcal{F}_0 & \to & 0
\end{array}$$

Proposition A.2 *Let $(\mathcal{F}.)$ be a complex as in (A.2) with homology supported on Z. Then there is a spectral sequence whose E_1 term is*

$$E_1^{pq} = H^q_Z(\mathcal{F}_{n-p})$$

and which converges to

$$E_\infty^{pq} = \begin{cases} H_{n-p}(\mathcal{F}.) & \text{if } q = 0\\ 0 & \text{if } q > 0.\end{cases}$$

PROOF All the rows of the double complex $K^{\cdot,\cdot}$ except the bottom one are exact because $\mathcal{F}_i$ is exact outside Z. Taking the horizontal homology we get $H_{n-p}(\mathcal{F}_\cdot)$ along the bottom row, and zeros above. Taking first vertical homology we pick up $H_Z^q(\mathcal{F}_{n-p})$ as E_1. □

Corollary A.3 (Acyclicity lemma) *Suppose that the complex (A.2) is exact outside Z. If $\operatorname{depth}_Z \mathcal{F}_i \geq i$ for all i, then the complex is acyclic (that is, $H_i(\mathcal{F}_\cdot) = 0$ for $i > 0$).*

Lemma A.4 *Let $X = X_1 \times X_2$, and let $Z_1 \subset X_1$ and $Z_2 \subset X_2$, with $Z = Z_1 \times Z_2 \subset X$. Suppose $\mathcal{F}_i$ is a sheaf on X_i and let $\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2$ be the tensor product sheaf on X (in the appropriate category), then*

$$H_Z^k(\mathcal{F}) = \bigoplus_{i+j=k} H_{Z_1}^i(\mathcal{F}_1) \otimes H_{Z_2}^j(\mathcal{F}_2).$$

PROOF Consider Čech complexes $\check{C}_{Z_1}^\cdot(\mathcal{F}_1)$ and $\check{C}_{Z_2}^\cdot(\mathcal{F}_2)$. Then $\check{C}_Z^\cdot(\mathcal{F}) := \check{C}_{Z_1}^\cdot(\mathcal{F}_1) \otimes \check{C}_{Z_2}^\cdot(\mathcal{F}_2)$ is a Čech complex for $Z_1 \times Z_2$. The lemma follows from a spectral sequence argument on the double complex $K^{p,q} = \check{C}_{Z_1}^p(\mathcal{F}_1) \otimes \check{C}_{Z_2}^q(\mathcal{F}_2)$, as the higher differentials all vanish. □

Example A.5 We derive the local cohomology groups $H_{\mathbf{C}^s}^i(\mathbf{C}[x_1, \ldots, x_n])$, where $\mathbf{C}^n = \mathbf{C}^r \times \mathbf{C}^s$. First we calculate $H_{\{0\}}^i(\mathbf{C}[x])$ in one variable, and then proceed by induction using Lemma A.4. For any set of coordinates on $\mathbf{C}^n$ say $\{x_1, \ldots, x_r\}$ define

$$A(x_1, \ldots, x_r) = \frac{1}{x_1 \ldots x_r} \mathbf{C}[x_1^{-1}, \ldots, x_r^{-1}].$$

(A is for ‘antiworld’.) To find $H_{\{0\}}(\mathbf{C}[x])$, we use the cover with one open set $U = \mathbf{C} \setminus \{0\}$, and function $\phi = x$. The Čech complex is just

$$0 \to \mathbf{C}[x] \to \mathbf{C}[x, x^{-1}]c \to 0$$

and so $H_{\{0\}}^0(\mathbf{C}[x]) = 0$, and $H_{\{0\}}^1(\mathbf{C}[x]) \cong \mathbf{C}[x, x^{-1}]/\mathbf{C}[x]c \cong A(x)c$.

Then, by induction, using Lemma A.4, we find that $H_{\{0\}}^i(\mathbf{C}[x_1, \ldots, x_r]) = 0$ for $i < r$, and

$$H_{\{0\}}^r(\mathbf{C}[x_1, \ldots, x_r]) \cong A(x_1, \ldots, x_r) c_1 \wedge \ldots \wedge c_r.$$

Finally, by Lemma A.4, we get that

$$H_{\mathbf{C}^s}^i(\mathbf{C}[x_1, \ldots, x_n]) \cong \begin{cases} A(x_1, \ldots, x_r) \otimes_{\mathbf{C}} \mathbf{C}[x_{r+1}, \ldots, x_n] c_1 \wedge \ldots \wedge c_r & \text{if } i = r \\ 0 & \text{otherwise.} \end{cases}$$

Example A.6 Suppose $f : \mathbf{C}^n \to \mathbf{C}$ has an isolated critical point at 0. Then the complex $(\Omega_{\mathbf{C}^n}^\cdot, df\wedge)$ induces a complex $(H_{\{0\}}^n(\Omega_{\mathbf{C}^n}^\cdot), df\wedge)$. The cohomology of this complex is given by

$$H^i(H_{\{0\}}^n(\Omega_{\mathbf{C}^n}^\cdot), df\wedge) = \begin{cases} \Omega_{\mathbf{C}^n}^n / df \wedge \Omega_{\mathbf{C}^n}^{n-1} & \text{if } i = 0 \\ 0 & \text{otherwise.} \end{cases}$$

This is seen by taking the Čech resolution $A^{\cdot,\cdot}$ of the $(\Omega_{\mathbf{C}^n}^\cdot, df\wedge)$ complex. The first horizontal spectral sequence degenerates at E_1 to give

$$H^n(A, D) = \Omega_{\mathbf{C}^n}^n / df \wedge \Omega_{\mathbf{C}^n}^{n-1},$$

and $H^i(A,D) = 0$ for $i \neq n$. On the other hand the first vertical spectral sequence degenerates at E_2 to give the cohomology groups $H^i(H^n_{\{0\}}(\Omega^{\cdot}_{\mathbf{C}^n}), df\wedge)$.

Remark A.7 (*Algebraic local cohomology applied to coherent analytic sheaves*) Our applications of local cohomology are to analytic rather than algebraic sheaves. Nonetheless, the results remain valid as all the sheaves are coherent, and algebraic local cohomology of coherent analytic sheaves is a well-defined functor. For example, for the acyclicity lemma, if a complex of coherent sheaves is exact off a subvariety Z then its cohomology is annihilated by a power of the ideal defining Z, and consequently it is enough to consider algebraic local cohomology.

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