Metamathematics of Elementary Mathematics

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2008

MIMS EPrint: **2008.36**

ISSN 1749-9097

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Why is mathematics so difficult? My talk will be devoted to hidden structures and concepts of elementary mathematics which frequently remain unnoticed but seriously influence students' perception of mathematics.

In this abstract, I will develop one example.

1. "Named" numbers. When, as a child, I was told by my teacher that I had to be careful with *"named"* numbers and not to *add* apples and people, I remember asking her why in that case we can *divide* apples by people:

$$10\,\text{apples} \,:\, 5\,\text{people} = 2\,\text{apples}. \tag{1}$$

Even worse: when we distribute 10 apples giving 2 apples to a person, we have

$$10\,\text{apples} \,:\, 2\,\text{apples} = 5\,\text{people} \tag{2}$$

Where do "people" on the right hand side of the equation come from? Why does "people" appear and not, say, "kids"? There were no "people" on the left hand side of the operation! How do numbers on the left hand side know the name of the number on the right hand side?

I did not get a satisfactory answer from my teacher and only much later did I realize that the correct naming of the numbers should be

$$10\,\text{apples} \,:\, 5\,\text{people} = 2\,\frac{\text{apples}}{\text{people}}, \qquad 10\,\text{apples} \,:\, 2\,\frac{\text{apples}}{\text{people}} = 5\,\text{people}. \tag{3}$$

It is a commonplace wisdom that the development of mathematical skills in a student goes alongside the gradual expansion of the realm of numbers with which he or she works, from natural numbers to integers, then to rational, real, complex numbers:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

What is missing from this natural hierarchy is that already at the level of elementary school arithmetic children are working in a much more sophisticated structure, a graded ring

$$\mathbb{Q}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}].$$

of Laurent polynomials in n variables over $\mathbb{Q}$, where symbols $x_1, \ldots, x_n$ stand for the names of objects involved in the calculation: apples, persons, etc. This explains why educational psychologists confidently claim that the operations (1) and (2) have little in common [1]—indeed, operation (2) involves operands of much more complex nature.

Usually, only Laurent monomials are interpreted as having physical (or real life) meaning. But the addition of heterogeneous quantities still makes sense and is done componentwise: if you have a lunch bag with (2 apples + 1 orange), and another bag, with (1 apple + 1 orange), together they make

$$(2\text{ apples } + 1\text{ orange}) + (1\text{ apple } + 1\text{ orange}) = (3\text{ apples } + 2\text{ oranges}).$$

[1] P. Bryant and S. Squire, The influence of sharing on children's initial concept of division, J. Experimental Child Psychology 81 no. 1 (January 2002) 1–43.

Notice that this gives a very intuitive and straightforward approach to vectors.[2]

Of course, there is no need to teach Laurent polynomials to kids; but it would not harm to teach them to teachers. I have an ally in François Viéte who in 1591 wrote in his *Introduction to the Analytic Art* that

> *If one magnitude is divided by another, [the quotient] is heterogeneous to the former ... Much of the fogginess and obscurity of the old analysts is due to their not paying attention to these [rules].*

It pays to be attentive to the dimensions of quantities involved in a physical formula: the balance of names of units (*dimensions*) on the left and right hand sides may suggest the shape of the formula. Such *dimensional analysis* quickly leads to immensely deep results, like, for example, Kolmogorov's celebrated "5/3 Law" for the energy spectrum of turbulence.

2. And much more ... I can continue the list of my case studies; many of them can be found in my book [3]. For example,

- **The minus sign** "−". It is used to stand for a binary operation of subtraction and also for a unary operation of taking the opposite, creating confusion.
- **"Semantic symmetry"** in problems and proofs (like in Pappus' proof of *Pons Asinorum*) and why it is so hard to grasp without using the concept of a formal proof.
- **Carry**, a digit that is transferred from one column of digits to another column of more significant digits during addition of decimals. Perhaps it holds a record of being the most advanced concept of primary school arithmetic: it is a 2-cocycle responsible for the extension of additive groups

$$0 \longrightarrow 10\mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z}/10\mathbb{Z} \longrightarrow 0.$$

The last example is especially deep and multilayered.

3. Conclusions. All my examples will lead to the same conclusions:

- We should not underestimate the immense richness of basic elementary mathematics.
- Glossing over difficulties presented by hidden structures may seriously imperil students' progress.
- The teacher has to be aware about the hidden structures and be able to guide pupils around dangerous spots—perhaps without needlessly alerting them every time.
- Sensitivity to the presence of "hidden" structures is an important component of mathematical ability in so-called "mathematically gifted" children.

[2] By the way, this "lunch bag" approach to vectors allows a natural introduction of duality and tensors: the total cost of a purchase of amounts g_1, g_2, g_3 of some goods at prices p^1, p^2, p^3 is a "scalar product"-type expression $\sum g_i p^i$. We see that the quantities g_i and p_i could be of completely different nature. The standard treatment of scalar (dot) product in undergraduate linear algebra usually conceals the fact that dot product is a manifestation of duality of vector spaces, creating immense difficulties in the subsequent study of tensor algebra.

[3] A. V. Borovik, Mathematics under the Microscope, submitted; available for free download from http://www.maths.manchester.ac.uk/∼avb/micromathematics/downloads.

