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2003

MIMS EPrint: **2008.18**

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ISSN 1749-9097

# Recursive Gramian and Hankel map approximation of large dynamical systems \*

*Younes Chahlaoui and Paul Van Dooren<sup>†</sup>*

## 1 Introduction

In the last twenty years, model reduction of large scale dynamical systems has become very popular. The idea is to construct a “simple” lower order model that approximates well the behavior of the “complex” larger dynamical model. A complex system is essentially a mathematical model which describes a real world physical process. This mathematical model is often characterized by partial differential equations (PDEs). Since improved accuracy (using e.g. a very fine discretization) leads to large and sparse models of high complexity (see e.g. [11]), this may become prohibitive for certain computations (control, optimization, ...). Therefore it is essential to design models of reduced complexity.

Most ideas developed for linear systems are based on the dominant spaces of Gramians [10] (energy functions for in- and outgoing signals), which are the solutions of Lyapunov or Stein equations. A lot of work is still needed to efficiently compute these solutions (or their dominant spaces) when the system matrices are large and

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\*This paper presents research supported by NSF contract CCR-99-12415 and by the Belgian Programme on Inter-university Poles of Attraction, initiated by the Belgian State, Prime Minister's Office for Science, Technology and Culture. The scientific responsibility rests with its author.

The work of the first author has been partially carried out within the framework of a collaboration agreement between CESAME (Université Catholique de Louvain, Belgium) and LINMA of the Faculty of Sciences (Université Chouaib Doukkali, Morocco), funded by the Secretary of the State for Development Cooperation and by the CIUF (Conseil Interuniversitaire de la Communauté Française, Belgium).

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sparse (see e.g.[4]).

In this paper we describe two novel approaches for iterative projection based on low-rank approximations of Gramians and Hankel matrices. We derive bounds for the approximation errors of these methods and illustrate their efficiency on a few numerical examples. These results are presented for linear discrete-time systems, but they extend to linear continuous-time systems as well (see e.g.[4]).

## 2 Time-varying systems

Linear discrete time-varying systems are described by systems of difference equations:

$$\begin{cases} x_{k+1} &= A_k x_k + B_k u_k \\ y_k &= C_k x_k \end{cases} \quad (1)$$

with input  $u_k \in \mathbb{R}^m$ , state  $x_k \in \mathbb{R}^N$  and output  $y_k \in \mathbb{R}^p$ , and  $m, p \ll N$ . The input sequence is assumed to be square-summable, i.e  $u_k \in l_2^{m-1}$ , and we assume that  $\{A_k\}_0^\infty$ ,  $\{B_k\}_0^\infty$ , and  $\{C_k\}_0^\infty$  are bounded<sup>2</sup> sequences of matrices with appropriate dimensions.

Using the recurrence (1) over several time steps, one obtains the state at step  $k$  in function of past inputs over the interval  $[k_i, k)$  :

$$x_k = \Phi(k, k_i) x_{k_i} + \sum_{i=k_i}^{k-1} \Phi(k, i+1) B_i u_i$$

where  $\Phi(k, k_i) \doteq A_{k-1} \dots A_{k_i}$  is the discrete transition matrix over time period  $[k_i, k)$  [7]. We will assume the time-varying system to be asymptotically stable, which means that

$$\forall k \geq k_i \quad \|\Phi(k, k_i)\| \leq m \cdot a^{(k-k_i)}, \text{ with } m > 0, 0 < a < 1.$$

Under such conditions one can define the Gramians over intervals  $[k_i, k)$  and  $[k, k_f]$  as follows :

$$\mathcal{G}_c(k) = \sum_{i=k_i}^{k-1} \Phi(k, i+1) B_i B_i^T \Phi^T(k, i+1), \quad \mathcal{G}_o(k) = \sum_{i=k}^{k_f} \Phi^T(i, k) C_i^T C_i \Phi(i, k).$$

(Notice that the asymptotic stability is needed when  $k_i = -\infty$  or  $k_f = +\infty$ .)

These Gramians can also be obtained from the Stein recurrence formulas:

$$\mathcal{G}_c(k+1) = A_k \mathcal{G}_c(k) A_k^T + B_k B_k^T \quad \text{and} \quad \mathcal{G}_o(k) = A_k^T \mathcal{G}_o(k+1) A_k + C_k^T C_k, \quad (2)$$

with respective initial conditions  $\mathcal{G}_c(k_i) = 0$ ,  $\mathcal{G}_o(k_f+1) = 0$ . These formulas were obtained using

$$\Phi(k_1, k_2) = \Phi(k_1, k_2+1) A_{k_2} \quad \text{and} \quad \Phi(k_1+1, k_2) = A_{k_1} \Phi(k_1, k_2) \quad \text{where } k_1 \geq k_2.$$

<sup>1</sup>This is a Hilbert space with inner product  $\langle x, y \rangle_{l_2^m} \doteq \sum_{k=-\infty}^{\infty} x_k^T y_k$

<sup>2</sup>A sequence of matrices  $\{M_k\}$  is said to be bounded if there exists a constant  $M \in \mathbb{R}$  such that  $\|M_k\| \leq M, \forall k \in \mathbb{Z}$ .

We point out that both recurrences (2) evolve differently with time. The first equation goes “forward” in time, while the second goes “backward” in time.

### 3 Recursive low-rank Gramians approximations

When model reduction is to be used, the Gramians have often rapidly decaying eigenvalues [2], which suggests to approximate the Gramians at each step by a low-rank factorization. We show below how to obtain such approximations and at the same time exploit the sparsity of the model  $\{A_k, B_k, C_k\}$  if there is. Although all material below should be applied to both Gramians  $\mathcal{G}_c(k)$  and  $\mathcal{G}_o(k)$ , we focus on the controllability Gramian only.

It is easy to see that the solution of

$$\mathcal{G}_c(k+1) = A_k \mathcal{G}_c(k) A_k^T + B_k B_k^T. \quad (3)$$

is always symmetric positive semi-definite, so we can substitute it by a symmetric factorization:  $\mathcal{G}_c(k) = \mathcal{C}(k)\mathcal{C}(k)^T$ . The key idea of the low-rank method is to approximate the factor of  $\mathcal{G}_c(k)$ ,  $\mathcal{C}(k)$  by a rank  $n_k$  approximation  $S_c(k)$  at each iteration (typically  $n_k$  is constant). So the proposed algorithm is the following:

**Algorithm 3.1** Let  $S_c(0) \in \mathbb{R}^{N \times n}$  be an initialization satisfying :

$$\mathcal{C}(0)\mathcal{C}(0)^T = S_c(0)S_c(0)^T + E_c(0)E_c(0)^T,^3$$

then the  $k^{th}$  low-rank approximation  $S_c(k)$  is obtained as follows :

$$\begin{aligned} \hat{S}_c(k) &= [ A_{k-1} S_c(k-1) \mid B_{k-1} ] = U_c \Sigma V_c^T \\ S_c(k) &\doteq \hat{S}_c(k) V_c(:, 1:n) \\ E_c(k) &\doteq \hat{S}_c(k) V_c(:, n+1:n+m) \end{aligned} \quad (4)$$

where  $U_c \in \mathbb{R}^{N \times (n+m)}$ ,  $\Sigma \in \mathbb{R}^{(n+m) \times (n+m)}$  and  $V_c \in \mathbb{R}^{(n+m) \times (n+m)}$  are the Short Singular Values Decomposition (SSVD) matrices of  $\hat{S}_c(k)$ .

It is immediate that  $\mathcal{P}_k = S_c(k)S_c(k)^T$  is the best rank  $n$  approximation to  $\hat{S}_c(k)\hat{S}_c(k)^T$ . But we have to compare  $\mathcal{P}_k$  with  $\mathcal{G}_c(k) = \mathcal{C}(k)\mathcal{C}(k)^T$ .

For this, we define  $\mathcal{C}(k) = [ B_{k-1} \mid A_{k-1} B_{k-2} \mid \dots \mid \Phi(k, 0)\mathcal{C}(0) ]$ , and we have the following result <sup>4</sup> :

**Theorem 1.** [4] *At each iteration, there exists an orthogonal matrix  $V_k \in \mathbb{R}^{(n+km) \times (n+km)}$  satisfying :*

$$\mathcal{C}(k)V_k = [ S_c(k) \mid E_c(k) \mid A_{k-1} E_c(k-1) \mid \dots \mid \Phi(k, 0) E_c(0) ]$$

where  $E_c(k)$  is the neglected part at iteration  $k$  (4).

<sup>3</sup>we can always choose, e.g.  $S_c(0)=0$ , which would imply  $E_c(0)=\mathcal{C}(0)$

<sup>4</sup>See [4] for more details and proofs of all theorems of this section

Let  $\eta_c = \max_{0 \leq i \leq \infty} \|E_c(i)\|_2$ . As we suppose that our system is asymptotically stable, we can bound the difference between  $\mathcal{P}_k$  and  $\mathcal{G}_c(k)$  for all  $k$ ,  $\mathcal{E}_c(k) \doteq \mathcal{G}_c(k) - \mathcal{P}_k$ , in terms of the “noise” level  $\eta_c$  as follows :

**Theorem 2.** [4]

$$\|\mathcal{E}_c(k)\|_2 \leq \eta_c^2 m^2 \sum_{i=0}^k a^{2i} \leq \frac{\eta_c^2 m^2}{1 - a^2}, \quad \text{with } m > 0, 0 < a < 1.$$

### The time-invariant case

For linear time-invariant systems  $\{A, B, C\}$ , this difference  $\mathcal{E}_c(k)$  remains bounded for large  $k$ .

**Theorem 3.** [4] Let  $\mathcal{P}$  be the solution of  $\mathcal{P} = A\mathcal{P}A^T + I$  then

$$\|\mathcal{E}_c(k)\|_2 \leq \eta_c^2 \|\mathcal{P}\|_2 \leq \eta_c^2 \frac{\kappa(A)^2}{1 - \rho(A)^2} \quad (5)$$

### Remark 3.1.

- Our bounds are very similar to those obtained in [1] for the time-invariant case; In (5),  $\kappa(A)^2/(1 - \rho(A)^2)$  is constant and it is very small when  $\rho(A) \ll 1$  and  $\kappa(A)$  is reasonable;
- $\eta_c$  can be taken equal to the maximum of  $\|E_c(i)\|_2$  for  $k_i \leq i \leq \infty$ , since we can interpret theorems 2 and 3 as starting with step  $k_i$  rather than 0. This is particularly useful if after step  $k_i$  the errors have converged to their minimal value, i.e. the convergence threshold  $\epsilon_m$ . If  $E_c(i)$  becomes very small, one can expect that the algorithm has a fixed point. This appears very clearly in the numerical tests [4]. Theoretical work on this point is in progress.

### The periodic case

Using the connection between the periodic time-varying system and the time-invariant system [8], and the fact that for a periodic system there exists a periodic controllability Gramian [9], we can extend the previous result to a  $K$ -periodic system, as follows :

**Theorem 4.** [4] Let  $\mathcal{P}$  be the solution of  $\mathcal{P} = \hat{A}\mathcal{P}\hat{A}^T + I_{KN}$  where

$$\hat{A} = \begin{pmatrix} 0 & \dots & 0 & A_0 \\ A_1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ 0 & \dots & A_{K-1} & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{P} \doteq \text{diag}(\mathcal{P}_1, \dots, \mathcal{P}_{K-1}, \mathcal{P}_0)$$

then

$$\|\mathcal{E}_c(k)\|_2 \leq \eta_c^2 \|\mathcal{P}\|_2 \leq \eta_c^2 \frac{\kappa(\hat{A})^2}{1 - \rho(\Phi(K, 0))^2} \quad (6)$$

**Remark 3.2.** Using the connection between periodic system and time-invariant system, and from the convergence we can conclude that we have also a periodicity for the dominant subspaces defined by the  $S_c(i)$ , which implies that the reduced order model will be also periodic.

## 4 Recursive low-rank Hankel approximation

In this section we present the second low-rank approach. The key idea of this approach is to use the underlying recurrences defining the so-called time-varying ‘‘Hankel maps’’. These matrices have a similar structure to the Hankel matrix of the time-invariant case. Let us consider a time window  $[k_i, k_f] = [k - \tau, k + \tau]$  of width  $2\tau$  and centered around  $k$ . If we restrict the inputs to be non-zero only in the interval  $[k_i, k]$  (i.e. the ‘‘past’’), then the outputs in the interval  $[k, k_f]$  (i.e. the ‘‘future’’) are given by the convolution with a finite dimensional ‘‘Hankel map’’  $\mathcal{H}(k, k_i, k_f)$ :

$$\mathcal{H}(k, k_i, k_f) = \begin{bmatrix} C_k B_{k-1} & C_k A_{k-1} B_{k-2} & \dots & C_k \Phi(k, k_i + 1) B_{k_i} \\ C_{k+1} A_k B_{k-1} & C_{k+1} A_k A_{k-1} B_{k-2} & \dots & C_{k+1} \Phi(k + 1, k_i + 1) B_{k_i} \\ \vdots & \vdots & \ddots & \vdots \\ C_{k_f} \Phi(k_f, k) B_{k-1} & C_{k_f} \Phi(k_f, k - 1) B_{k-2} & \dots & C_{k_f} \Phi(k_f, k_i + 1) B_{k_i} \end{bmatrix}$$

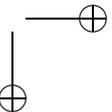
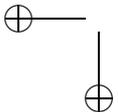
This matrix has a factorization :

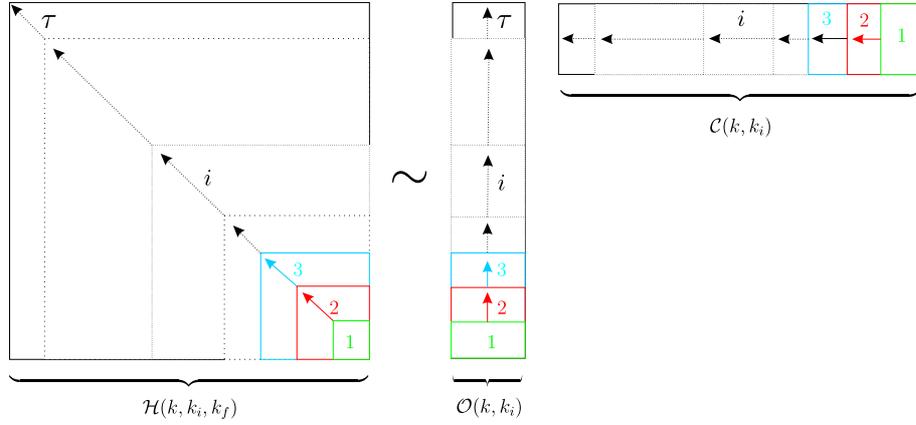
$$\mathcal{H}(k, k_i, k_f) = \underbrace{\begin{bmatrix} C_k \\ C_{k+1} A_k \\ \vdots \\ C_{k_f} \Phi(k_f, k) \end{bmatrix}}_{\mathcal{O}(k, k_f)} \underbrace{\begin{bmatrix} B_{k-1} & A_{k-1} B_{k-2} & \dots & \Phi(k, k_i + 1) B_{k_i} \end{bmatrix}}_{\mathcal{C}(k, k_i)}$$

where  $\mathcal{O}_k \doteq \mathcal{O}(k, k_f)$  and  $\mathcal{C}_k \doteq \mathcal{C}(k, k_i)$  are respectively the observability and the reachability matrices at instant  $k$  related to the finite windows  $[k, k_f]$  and  $[k_i, k]$ . It also follows from the factorization that the submatrices of the factors satisfy the following recurrences :

$$\mathcal{O}_j = \begin{bmatrix} C_j \\ \mathcal{O}_{j+1} A_j \end{bmatrix}, \quad k \leq j < k_f \quad \mathcal{C}_{j+1} = [ B_j \quad A_j \mathcal{C}_j ], \quad k_i \leq j < k.$$

These recurrences construct the controllability matrix forward from  $k_i$  to  $k$  and the observability matrix backward from  $k_f$  to  $k$ . The idea of our Recursive Low-Rank Hankel approximation method (RLRH) is now to compute these recurrences using low-rank approximations at each time step, according to the following recursive scheme :





**Algorithm 4.1** Let the initializing matrices  $S_c(k_i), S_o(k_f) \in \mathbb{R}^{N \times n}$  satisfy

$$\begin{aligned} \mathcal{C}(k_i, k_i)\mathcal{C}^T(k_i, k_i) &= S_c(k_i)S_c^T(k_i) + E_c(k_i)E_c^T(k_i), \\ \mathcal{O}^T(k_f, k_f)\mathcal{O}(k_f, k_f) &= S_o(k_f)S_o^T(k_f) + E_o(k_f)E_o^T(k_f), \end{aligned} \quad (5)$$

then the  $i^{\text{th}}$  ( $i = 1, \dots, \tau$ ) low-rank approximations  $S_c(k_i + i)$  and  $S_o(k_f - i)$  are obtained as follows :

$$[S_c(k_i + i) \mid E_c(k_i + i)] = [B_{k_i + i - 1} \mid A_{k_i + i - 1}S_c(k_i + i - 1)] [V_h^{(1)}(i) \mid V_h^{(2)}(i)] \quad (7)$$

$$\begin{bmatrix} S_o^T(k_f - i) \\ E_o^T(k_f - i) \end{bmatrix} = \begin{bmatrix} U_h^{(1)T}(i) \\ U_h^{(2)T}(i) \end{bmatrix} \begin{bmatrix} C_{k_f - i} \\ S_o^T(k_f - i + 1)A_{k_f - i} \end{bmatrix} \quad (8)$$

where  $U_h^{(1)}(i) \in \mathbb{R}^{(p+n) \times n}$  and  $V_h^{(1)}(i) \in \mathbb{R}^{(m+n) \times n}$  come from

$$\begin{bmatrix} C_{k_f - i} \\ S_o^T(k_f - i + 1)A_{k_f - i} \end{bmatrix} [B_{k_i + i - 1} \mid A_{k_i + i - 1}S_c(k_i + i - 1)] = \underbrace{U_h(i)\Sigma(i)V_h(i)^T}_{\text{SVD}} \quad (9)$$

**Remark 4.1.** It follows from (7) and (8) that at each iteration  $i = 1, \dots, \tau$  we have

$$\begin{bmatrix} S_o^T(k_f - i) \\ E_o^T(k_f - i) \end{bmatrix} [S_c(k_i + i) \mid E_c(k_i + i)] = \begin{bmatrix} \Sigma_1(i) & 0 \\ 0 & \Sigma_2(i) \end{bmatrix} \quad (10)$$

Now in order to make the link between the whole controllability and observability matrices  $\mathcal{C}(\cdot, \cdot)$ ,  $\mathcal{O}(\cdot, \cdot)$  and their low-rank approximations  $S_c(\cdot)$  and  $S_o(\cdot)$ , we have the following theorem<sup>6</sup>:

<sup>5</sup>we can e.g. choose  $S_c(k_i) = 0$  and  $S_o(k_f) = 0$ , which would imply  $E_c(k_i) = \mathcal{C}(k_i, k_i)$  and  $E_o(k_f) = \mathcal{O}^T(k_f, k_f)$

<sup>6</sup>See [5] for more details and proofs of all theorems of this section

**Theorem 5.** [5] At each iteration, there exist orthogonal matrices  $V_i \in \mathbb{R}^{(n+im) \times (n+im)}$  and  $U_i \in \mathbb{R}^{(n+ip) \times (n+ip)}$  satisfying :

$$\mathcal{C}(k_i + i, k_i)V_i = [ S_c(k_i + i) \mid E_c(k_i + i) \mid A_{k_i+i-1}\mathcal{C}_e(k_i + i, k_i) ]$$

$$\mathcal{O}^T(k_f - i, k_f)U_i = [ S_o(k_f - i) \mid E_o(k_f - i) \mid A_{k_f-i}^T\mathcal{O}_e(k_f - i + 1, k_f) ]$$

where  $E_c(k_i + i)$  and  $E_o(k_f - i)$  are the neglected parts at iteration  $i$  (7) and

$$\mathcal{C}_e(j, k_i) \doteq [ E_c(j-1) \mid \dots \mid \Phi(j-1, k_i)E_c(k_i) ]$$

$$\mathcal{O}_e(j, k_f)^T \doteq [ E_o(j) \mid \dots \mid \Phi(k_f, j)^T E_o(k_f) ].$$

As a consequence of this theorem and from remark (4.1), we have the following result which give us an approximation of the original Hankel map  $\mathcal{H}(k, k_i, k_f)$ :

**Theorem 6.** [5] There exist orthogonal matrices  $V_k \in \mathbb{R}^{(n+km) \times (n+km)}$  and  $U_k \in \mathbb{R}^{(n+kp) \times (n+kp)}$  such that :

$$U_k^T \mathcal{H}(k, k_i, k_f)V_k = \begin{bmatrix} S_o^T(k)S_c(k) & 0 & S_o^T(k)A_{k-1}\mathcal{C}_e(k, k_i) \\ 0 & E_o^T(k)E_c(k) & E_o^T(k)A_{k-1}\mathcal{C}_e(k, k_i) \\ \mathcal{O}_e(k+1, k_f)A_k S_c(k) & \mathcal{O}_e(k+1, k_f)A_k E_c(k) & \mathcal{O}_e(k+1, k_f)A_k A_{k-1}\mathcal{C}_e(k, k_i) \end{bmatrix}. \quad (11)$$

This result will enable us to evaluate the quality of our approximations by using the Hankel operator without having to pass by Gramians, which can be very suitable in some cases.

### The time-invariant case

Let us analyze the quality of our approximation for the time-invariant case. Suppose that we have run the above procedure and that we have obtained two matrices  $S_c$  and  $S_o$  of full rank  $n$ . Using those matrices we can approximate the Gramians of the original model by  $S_c S_c^T$  and  $S_o S_o^T$ . The difference between the approximate low-rank Gramians and the exact Gramians

$$\mathcal{E}_c(k) \doteq \mathcal{G}_c(k) - \mathcal{P}_k, \quad \mathcal{E}_o(k) \doteq \mathcal{G}_o(k) - \mathcal{Q}_k$$

remains bounded for large  $k$ , as indicated in the following theorem.

**Theorem 7.** [5] Let  $\mathcal{P}$  and  $\mathcal{Q}$  be respectively the solutions of  $\mathcal{P} = A\mathcal{P}A^T + I$  and  $\mathcal{Q} = A^T\mathcal{Q}A + I$ , then

$$\|\mathcal{E}_c(k)\|_2 \leq \eta_c^2 \|\mathcal{P}\|_2 \leq \eta_c^2 \frac{\kappa(A)^2}{1 - \rho(A)^2}, \quad \text{and} \quad \|\mathcal{E}_o(k)\|_2 \leq \eta_o^2 \|\mathcal{Q}\|_2 \leq \eta_o^2 \frac{\kappa(A)^2}{1 - \rho(A)^2}$$

where  $\eta_c \doteq \max_k \|E_c(k)\|_2$  and  $\eta_o \doteq \max_k \|E_o(k)\|_2$ .

**Theorem 8.** [5] Using the first  $n$  columns  $U_k^{(1)}$  of  $U_k$  and  $V_k^{(1)}$  of  $V_k$ , we obtain a rank  $n$  approximation of the Hankel map :

$$\mathcal{H}(k) - U_k^{(1)} S_o^T(k) \cdot S_c(k) V_k^{(1)T} = \mathcal{E}_h(k),$$

for which we have the error bound :

$$\|\mathcal{E}_h(k)\|_2 \leq \frac{\kappa(A)}{\sqrt{1 - \rho(A)^2}} \max\{\eta_c \|S_o^T A\|_2, \eta_o \|AS_c\|_2\} + \frac{\kappa(A)^2}{1 - \rho(A)^2} \eta_o \eta_c.$$

**Remark 4.2.** In the time-invariant case all matrices  $A$ ,  $B$  and  $C$  are constant. As a consequence all Hankel maps of width  $\tau$  are equal as well and only the interval width plays a role in the obtained decomposition. As a consequence one obtains an approximate rank factorization of a Hankel map with  $i$  block columns and rows at each instant  $i$ . The bounds obtained in Theorem 7 and 8 are moreover independent of  $k$ . As  $i$  grows larger one can expect that reasonable approximations of  $\eta_c$  and  $\eta_o$  are in fact given by the last terms, i.e.  $\eta_c \approx \|E_c(k)\|_2$  and  $\eta_o \approx \|E_o(k)\|_2$  which will give much tighter bounds in these theorems.

## 5 RLRG versus RLRH

Now, let us compare between those two algorithms for model reduction of large scale systems. First, investigating the amount of work involved by both algorithms give:

- For both methods one needs to form products of the type  $A_j S_c(j)$  and  $S_o^T(l+1)A_l$ . If we assume the matrices  $A_k$  to be sparse, then the amount of work needed for this is  $\Theta(\alpha N n)$  where  $\alpha$  is the number of non-zero elements per row or column of  $A_k$  [6];
- For the RLRG method, we need to compute and apply at each step the transformation  $U_c, U_o$ . This requires  $\Theta(N(n+m)^2)$  flops and  $\Theta(N(n+p)^2)$  flops, respectively [6];
- For the RLRH method, the construction of the left hand side of (9) requires an additional  $2N(n+m)(n+p)$  flops and the application of the transformations  $U_h$  and  $V_h$  requires  $\Theta((p+n)(m+n)(2n+p+m))$  flops.

The two methods have thus a comparable complexity  $\Theta(N(n+m)(n+p))$  when the matrices  $A_k$  are sparse. But as RLRH works on the Hankel map rather than on the individual Gramians, it should suffer less from a bad balancing of the original system. This is illustrated in the examples of the next section.

## 6 Numerical examples

In this section we apply our algorithms to three different dynamical systems: a CD Player model, a Building model and an International Space Station model (see [3])

for a brief description of these models). Contrary to the RLRH method, the RLRG, as it works independently on both Gramians, has problems if the original model is badly balanced. To avoid this, one can apply the RLRG algorithm for e.g.  $2n$  instead of only  $n$  and apply balanced truncation to reduce it further to degree  $n$ . It can be seen from the following table that this operation works very well.

The table shows the order of the systems  $N$ , the number of inputs  $m$  and outputs  $p$ , the order of reduced system  $n$ , the condition number of the balancing transformation  $T$ , the spectral radius and condition number of the matrix  $A$ , the  $\mathcal{H}_\infty$  norms of the original system and of the error systems. In the figure 1, the  $\sigma_{max}$ -plot of the full order and the corresponding error systems are shown. It can be seen from the figure and the table that we obtain, with the recursive low-rank Hankel approximation, results which are close to those obtained using Balanced Truncation.

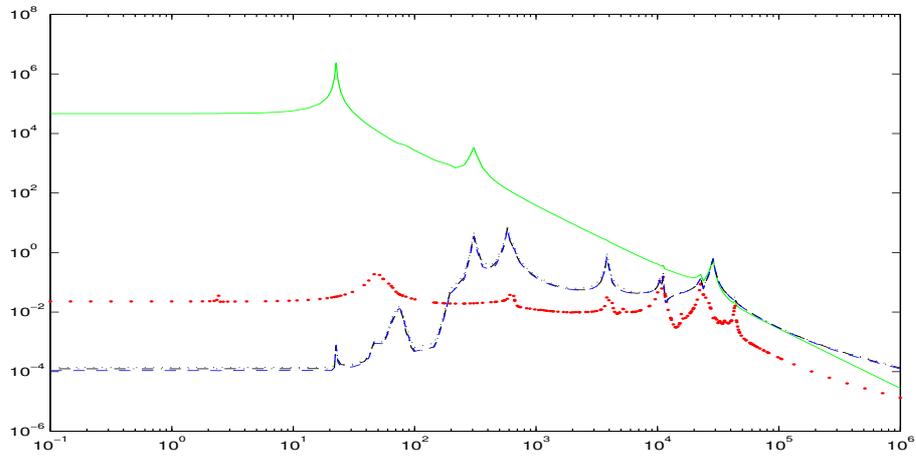
			CD-player model			Building model			ISS model		
N	m=p	n	120	2	24	48	1	10	270	3	32
$cond(T)$			40.7341			347.078			740178		
$\rho(A)$			1			0.998886			0.999837		
$cond(A)$			1.00705			5.8264			5.82405		
$\ \mathcal{S}\ _{\mathcal{H}_\infty}$			2.3198e+006			0.0053			0.1159		
$\ \mathcal{S} - \mathcal{S}_{BT}\ _{\mathcal{H}_\infty}$			0.2040			6.0251e-004			2.3630e-004		
$\ \mathcal{S} - \mathcal{S}_{RLRG(n)}\ _{\mathcal{H}_\infty}$			6.2938			0.0055			0.5637		
$\ \mathcal{S} - \mathcal{S}_{RLRG(2n)}\ _{\mathcal{H}_\infty}$			6.1916			7.0792e-004			0.0010		
$\ \mathcal{S} - \mathcal{S}_{RLRH}\ _{\mathcal{H}_\infty}$			6.1890			6.7317e-004			0.0011		

It is seen here that when the original system is well balanced RLRG( $n$ ) and RLRH( $n$ ) give essentially the same result. The decomposition (4) is then close to the “square root” of decomposition (9) and both approaches should be very close. If the original system is poorly balanced (as e.g. the ISS model) we can see that RLRH( $n$ ) becomes much better. This can be understood from the fact that the poor balancing of the system model vanishes in the Hankel map and hence also in the matrix decomposition in (9). We see in the example that RLRG( $2n$ ) followed by BT manages to compensate this, but at a higher cost.

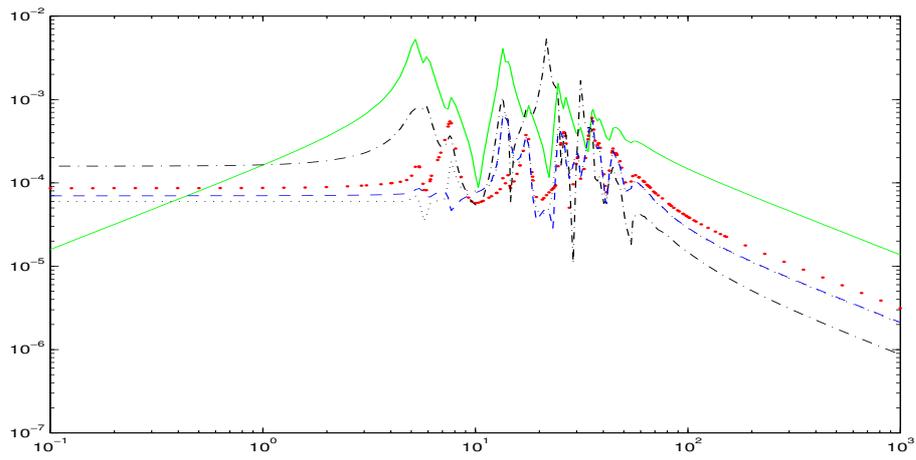
## 7 Conclusion

We describe two novel approaches for iterative projection using recursive low-rank methods. The two algorithms can be used in model reduction of time-varying large-scale dynamical systems. The second algorithm (RLRH) is more attractive as the method yields better approximation for poorly balanced system, and no more operations are needed to provide a good reduced model.

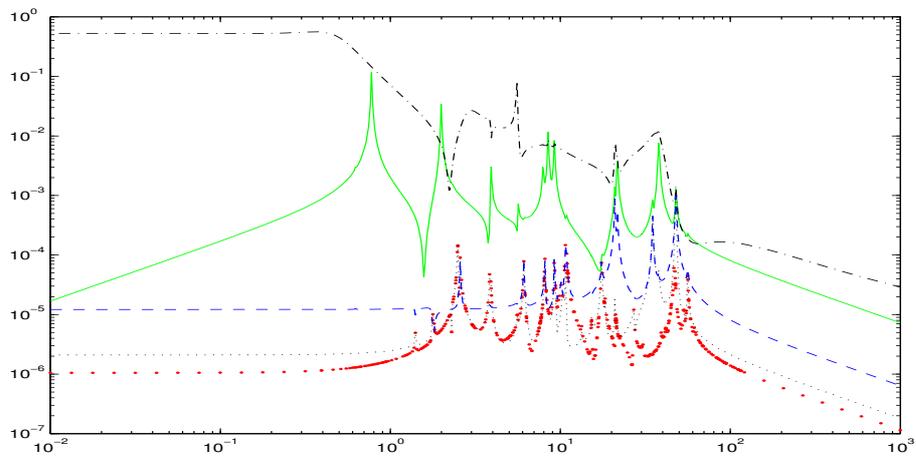
LEGEND : — full model, ···· BT error system, - - - RLRG( $n$ ) error system,  
···· RLRG( $2n$ ) error system, - - - RLRH error system.



(a) CD-player model



(b) Building model



(c) International space station model

**Figure 1.**  $\sigma_{max}$ -plot of the frequency responses.

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