Metric matroids and metric buildings

Borovik, Alexandre

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Alexandre V. Borovik

Abstract. These notes represent fragments of an old joint work with I. M. Gelfand; I put in them some material which did not find its way in our book with Neil White *Coxeter Matroids* [**1**], but, I believe, deserves to be documented.

In particular, I use the old term *W-matroid* for what later became a *Coxeter matroid for Coxeter group W*.

We discuss realizations of W-matroids on Coxeter groups by retractions of buildings and their convexity properties. We define, in terms of metric properties of retractions, some class of geometric objects which includes, as partial cases, buildings and $\mathbb{R}$-trees. For some of these new geometric structures 'abstract' convexity properties of retractions have a natural interpretation in terms of usual Euclidean convexity.

Definitions and notation used in the paper are mostly standard and can be found in [**1**].

1. Convexity properties of W-matroids

Let W be a Coxeter group with the set of distinguished generators R. A *W-matroid* (called *Coxeter matroid* in [**1**]) is a map $\mu : W \longrightarrow W$ satisfying the inequality

$$w^{-1}\mu(u) \leq w^{-1}\mu(w)$$

for all $u, w \in W$ (here $\leq$ is the Bruhat ordering on W).

It will be convenient for us to identify W with its Coxeter complex which we shall denote by the same letter W. A subset $X \subseteq W$ is *convex* if it contains, with every two chambers $u, v \in X$, all geodesic galleries from u to v.

Theorem 1 (A. Borovik and S. Roberts, see [**1**]). *If $\mu : W \longrightarrow W$ is a W-matroid, then*

- **(1):** *The fibers $\mu^{-1}[w]$, $w \in W$ are convex subsets of W.*
- **(2):** *If two fibers $\mu^{-1}[u]$ and $\mu^{-1}[v]$ of μ are adjacent (i.e. have a panel in common), then their images u, v are symmetric with respect to some wall σ of the Coxeter complex W. Moreover, all common panels of $\mu^{-1}[u]$ and $\mu^{-1}[v]$ lie on σ.*
- **(3):** *If W is infinite, then the nonempty fibers of the map μ are also infinite.*

This result provides a combinatorial version of properties of convex polyhedra associated with W-matroids for finite Weyl groups [**2**].

What is more interesting, some other properties of W-matroids can also be interpreted in terms of appropriately defined 'convexity'. The present work is devoted to a brief discussion of related results and constructions.

2. Bruhat convexity

2.1. Convexity. We start with a very general definition of convexity. Let E be a set and $\mathcal{P} \subseteq 2^E$ is a set of subsets of E closed under finite unions and intersections and containing all finite subsets of E. A *convex hull operator* on $\mathcal{P}$ is a closure operator $\tau : \mathcal{P} \longrightarrow \mathcal{P}$ (i.e. monotone, increasing, idempotent map from $\mathcal{P}$ to $\mathcal{P}$) satisfying, in addition, the **Anti-Exchange Principle**:

$$\textit{if } x, y \notin \tau(A),\ x \neq y \textit{ and } y \in \tau(A \cup \{x\}),\textit{ then } x \notin \tau(A \cup \{y\}).$$

Obviously, if E is an Euclidean space, then the usual convex hull

$$conv(X) = \{a_1x_1 + \cdots + a_nx_n \mid x_i \in X, a_i \geq 0, i = 1, \ldots, n, a_1 + \cdots + a_n = 1\},$$

is a convex hull operator on the set of all subsets of E.

The following (rather easy) proposition produces a wide class of convex hull operators.

Proposition 2 (A. Borovik and I. M. Gelfand [**BG2**], Sections 3 and 5). *Assume that a group W acts on an ordered set $(E, \leq)$. We define a family of orderings on E by setting*

$$x \leq^w y \Longleftrightarrow w^{-1}x \leq w^{-1}y.$$

Define for $A \subseteq E$

$$\tau_w(A) = \{x \in E \mid x \leq^w y \textit{ for some } y \in E\}.$$

and

$$\tau_W(A) = \bigcap_{w \in W} \tau_w(A).$$

Then τ_W is a convex hull operator on 2^E.

2.2. Extreme points. For a convex hull operator $\tau : 2^E \longrightarrow 2^E$, a point $x \in A$ is called an *extreme point* of $A \subseteq E$ if

$$x \notin \tau(A \setminus \{x\}).$$

The set of all extreme points of A is denoted by $ex(A)$.

2.3. Convex maps. Let W be a group acting on an ordered set $(E, \leq)$. A map $\mu : W \longrightarrow E$ is called *convex*, if

$$\mu(u) \leq^w \mu(w)$$

for all $u, w \in W$.

Notice that if W is a Coxeter group acting on itself by left multiplication and $\leq$ is the Bruhat ordering on W, then convex maps from W to W are nothing else but W-matroids.

Proposition 3 ([**BG2**], Theorem 3.1). *If $\mu : W \longrightarrow E$ is a convex map from a group W to an ordered set $(E, \leq)$, then*

$$\mu[W] = ex(\tau_W(\mu[W])).$$

It is easy to see that in a finite Coxeter group W the *Bruhat convex hull operator* τ_W associated with the Bruhat ordering $\leq$ on W is trivial: $\tau_W(A) = A$ for every set $A \subseteq W$. But in infinite Coxeter groups the Bruhat convexity is a non-trivial and, from our point of view, very interesting structure.

3. Metricaly convex maps

3.1. Pseudometric. Let W be a set. A *pseudometric* on W is a map

$$r : W \times W \longrightarrow \mathbb{R}^{\geq 0}$$

satisfying the following properties.

Axiom M1: $r(u, u) = 0$,
Axiom M2: $r(u, v) = r(v, u)$,
Axiom M3: $r(u, v) + r(v, w) \geq r(u, w)$ for all $u, v, w \in W$.

Notice that we do not demand that $r(u, v) = 0$ implies $u = v$.

It is easy to see that, given a pseudometric r, the relation $x \sim y \iff r(x, y) = 0$ is an equivalence relation. We will call the classes of the equivalence relation *points* of W. This name is suggested by the following important example of a set with a pseudometric. Let W be the group of all isometries of an affine Euclidean space E. Fix the origin $0 \in E$ and for $u, v \in W$ define $r(u, v) = |u(0) - v(0)|$, where $|\ |$ is the Euclidean norm on E. Obviously r is a pseudometric on W.

Returning to the general case we denote by E the set of all points of W and by $\pi : W \longrightarrow E$ the corresponding factor map. We define the distance $r(w, a)$ of an element $w \in W$ from a point $a \in E$ as $r(w, u)$ for some (or arbitrary) $u \in a$. The inequality of triangle (Axiom M3) warrants that the definition does not depend on choice of a representative $u \in a$. In the same way we can define a pseudometric r on the set E of points. Notice that on E the pseudometric r becomes a metric: $r(x, y) = 0$ for $x, y \in E$ implies $x = y$.

3.2. Metricaly convex maps. We shall call a map $\mu : W \longrightarrow E$ *metricaly convex*, if it satisfies

$$r(w, \mu(u)) \leq r(w, \mu(w))$$

for all $u, w \in W$.

The *ball of radius R centered at $a \in E$* in the metric space E is the set

$$B(a, R) = \{x \in E \mid r(a, x) \leq R\};$$

its *surface* $S(a, R)$ is defined as

$$S(a, R) = \{x \in E \mid r(a, x) = R\}.$$

If X is any bounded subset of E, we define the *hull* $\tau(X)$ as the intersection of all balls containing X and the set $\sigma(X)$ of *surface points* of $\tau(X)$ as

$$\sigma(X) = \{x \in \tau(X) \mid x \text{ lies on the surface of one of the balls containing } X\}.$$

In many interesting cases (for example, in finite-dimensional Euclidean spaces) τ is a convex hull operator on the set of bounded subsets in E.

The following result is obvious from definitions and justifies use of the term 'metricaly convex map'.

PROPOSITION 4. *Let W be a set with a pseudometric r and $E = \pi(W)$ its set of points. If $\mu : W \longrightarrow E$ is a metricaly convex map, then its image $\mu[W]$ lies on the surface of its hull,*

$$\mu[W] \subseteq \sigma(\tau(\mu[W])).$$

3.3. Euclidean convexity. An important class of metricaly convex maps is related to the group W of all isometries of a finite-dimensional affine Euclidean space E. In this case we can identify $\pi(W)$ with E. It is easy to see that for a bounded set $X \subset E$ its hull $\tau(X)$ is actually equal to the topological closure $\overline{conv(X)}$ of the usual convex hull $conv(X)$ and $\sigma(\tau(X))$ lies on the boundary $\delta(\tau(X))$ of $\tau(X)$ and can be interpreted as the set of 'non-flat' points of the boundary. So in this case we have, for a metricaly convex map $\mu : W \longrightarrow E$,

$$\mu[W] \subseteq \delta(\tau(\mu[W])).$$

3.4. Convexity on Coxeter groups. If now W is Coxeter group and $r(u, v) = l(u^{-1}v)$, where l is the usual length function on W, then r is a metric on W and $W = \pi(W)$ coincides with its set of points. Since $u \leq v$ implies $l(u) \leq l(v)$, it is immediate from the definition of a W-matroid $\mu : W \longrightarrow W$ that μ is a metricaly convex map with respect to the (pseudo)metric r on W.

So we have two different notions of convexity for a Coxeter group W. In the traditional meaning of this term, a set $X \subseteq W$ is convex if it contains with every pair of its points $u, v \in X$ all geodesic galleries stretched from u to v. But we also have the Bruhat convex hull operator introduced in the previous section (we have denoted it τ_W). We shall call a subset $X \subseteq W$ *Bruhat convex* if $X = \tau_W(X)$. Theorem 1 shows that there are intimate and intricate relations between these two kinds of convexity on W.

The notion of a metricaly convex map on W seems to be weaker than that of a convex map. But we still do not know whether exist metricaly convex, but not convex, maps on Coxeter groups or not; it is quite possible that there are easy examples. Convex maps (or W-matroids) arise naturally in the geometry of Grassmann and flag varieties (see next Section 4 and also more detailed discussions in [**1**] and [**2**]). On the other hand, it is metricaly convex maps which arise in certain generalizations of buildings. We will discuss this concept in Section 5.

4. W-matroids and buildings

In this section we explain a motivation behind our desire to link the notion of metricaly convex maps with the theory of buildings.

Let W be a Coxeter group, Δ a building of type W and A an apartment in Δ. For a chamber $c \in A$ we denote by $\rho_{c,A}$ the retraction of Δ onto A with the center c. Abusing the notation, we will identify the apartment A with W, $A = W$.

The following surprisingly simple theorem is the starting point of all our considerations. It dramatically simplifies and clarifies results of [**2**] on stratifications of Grassmann and flag varieties.

THEOREM 5 (A. V. Borovik and I. M. Gelfand [**1**]). *Fix a chamber x in Δ and consider the map $\mu : W \longrightarrow W$ defined by $\mu(w) = \rho_{w,W}(x)$. Then μ is a W-matroid. In other words, μ is a convex (and also metricaly convex) map.*

We say in this situation that a W-matroid μ has a geometric representation in the building Δ. This generalizes the classic notion of representability of matroids—this analogy is fully developed in the book [**1**].

Since retractions of buildings play the central role in the theory of W-matroids, we will try in the next section to define buildings in terms of metric properties of their retractions. This more general approach shed some light on convexity properties of W-matroids. Moreover, it was a pleasant surprise for us when we realized that our generalization of buildings includes, as partial cases, $\mathbb{R}$-trees and and some other interesting classes of geometric objects.

5. Metric buildings

Let W be a group. A pseudometric

$$r : W \times W \longrightarrow \mathbb{R}^{\geq 0}$$

is *left-invariant* if in addition to Axioms M1 – M3 it satisfies $r(wu, wv) = r(u, v)$ for all $u, v, w \in W$.

5.1. W-metric. Our approach to generalization of buildings emulates the well-known definition (due to J. Tits [**Ti1**]) of buildings in terms of a 'metric' with values in a Coxeter group.

> **Axiom B1:** *W-metric* on a set A is a map $\delta : A \times A \longrightarrow W$ satisfying $\delta(a, b) = \delta(b, a)^{-1}$ for all $a, b \in A$ and such that $\delta(a, b) = 1$ if and only if $a = b$.

Notice that the group W itself has a natural W-metric $\delta(u, v) = u^{-1}v$. If A and B are two sets with a W-metric, an *W-isometry* from A to B is a map preserving the W-metric. Notice that the left multiplication by an element of W is a W-isometry of W.

If a set A admits a W-metric and W has a left-invariant pseudometric r then, abusing notation, we define a pseudometric r on A by setting

$$r(a, b) = r(1, \delta(a, b)).$$

5.2. Metric buildings of type W. Let W be a group with a a left-invariant pseudometric r. A *metric building $\mathcal{B}$ of type* W is a set B with a W-metric δ together with a family $\mathcal{A}$ of isometries from W to B satisfying the axioms B2 – B5 below.

> **Axiom B2:** If $\alpha \in \mathcal{A}$ and $w \in W$, then their composition $\alpha \circ w \in \mathcal{A}$.
>
> **Axiom B3:** For any a, b in B there is an isometry $\alpha \in \mathcal{A}$ such that both a and b lie in the image of α.

The sets of the form $\alpha[W]$ for $\alpha \in \mathcal{A}$ are called *apartments* of the W-complex B and the group W is called the *Weyl group* of the complex B. We shall refer to elements of B as *repers*.

The name 'reper' in this context is suggested by an analogy with Euclidean geometry: if W is the group of all isometries of n-dimensional Euclidean space E, then W acts simply transitively on the set A of all repers in in E, i.e. objects consisting of a point from E with n pairwise orthogonal vectors of norm 1 attached. If $a, b \in A$ are two repers, then we can define W-distance $\delta(a, b)$ being the only element of W which sends a to b. Obviously A is a W-metric space isometric to W.

Let now A be an apartment and c a reper in A. It is obvious that there is a a unique W-isometry $\alpha \in \mathcal{A}$ onto A which sends 1 (the identity element of W) to c. We define the *retraction* $\rho_{c,A}$ of the W-complex B onto the apartment A by

$$\rho_{c,A}(b) = \alpha(\delta(c,b)),$$

where b runs through B.

Now we state the most important axiom for metric buildings of type W.

Axiom B4: For any apartment $A \subseteq B$ and reper $c \in A$ the retraction $\rho_{c,A}$ satisfies

$$r(\rho_{c,A}(a), \rho_{c,A}(b)) \leq r(a,b)$$

for all $a, b \in B$.

The next, and the last, axiom can be probably deduced from Axioms B1 – B4. When preparing this research report we did not have time to check carefully possible dependences between the axioms.

Axiom B5: Apartments of B are convex in the following sense. If A is an apartment and repers $a, b \in A$ and $c \in B$ satisfy $r(a,c) + r(c,b) = r(a,b)$ then $\pi(c) \in \pi(A)$.

Because of special role of the pseudometric r on W in our definition of metric buildings of type W it is probably more appropriate to use the term 'metric buildings of type (W, r)', specifying the pseudometric r on W. But in the present paper it will be always clear what particular pseudometric on W we are working with.

Lemma 6. *The function r is a pseudometric on the set B of repers of the metric building $\mathcal{B}$.*

Proof: We need to check only that $r(a,c) \leq r(a,b) + r(b,c)$ for any three repers $a, b, c \in B$. Let A be an apartment containing a and c and $\rho = \rho_{c,A}$. The restriction of r on A is a pseudometric on A. So we have by Axiom B4 $r(a,c) = r(\rho(a), \rho(c)) \leq r(\rho(a), \rho(b)) + r(\rho(b), \rho(c)) \leq r(a,b) + r(b,c)$. □

5.3. Classic case. First of all we have to check that our generalization of buildings includes buildings in the classic meaning of the term.

The following proposition is almost obvious.

Proposition 7. *Let W be a Coxeter group with the set of distinguished generators S and $r(u,v) = l(u^{-1}v)$ is the usual word metric on W. Let $\mathcal{B} = (B, \mathcal{A})$ be a metric building of type W. We shall refer to repers of B as chambers. For $s \in S$ we say, as usual, that two chambers $a, b \in B$ are s-adjacent if $\delta(a,b) = s$, thus introducing on B the structure of a chamber complex. Then if $w = s_1 s_2 \cdots s_n$ is a reduced word in W and $a, b \in B$,*

$$\delta(a,b) = w \iff a \text{ and } b \text{ are connected by a geodesic gallery of type } s_1 s_2 \cdots s_n.$$

In particular, B is a building in the sense of the traditional definition. Vice versa, every building is a metric building.

5.4. Convexity properties of retractions. Now we identify W with an apartment in B and consider the map μ from W to its set of points $E = \pi(W)$ defined by

$$\mu(w) = \pi(\rho_{w,W}(a))$$

for a fixed reper $a \in B$. The following theorem follows almost immediately from Axiom B4.

Theorem 8. *μ is a metricaly convex map from W to E.*

Proof: We need to prove that $r(w, \mu(u)) \leq r(w, \mu(w))$ for all $u, w \in W$. For $u \in W$ denote for brevity $\rho_u = \rho_{u,W}$. By definition of the function $r : W \times E \longrightarrow \mathbb{R}$ we have $r(w, \mu(u)) = r(w, \pi(\rho_u(a))) = r(w, \rho_u(a))$. So the desired inequality takes the form

$$r(w, \rho_u(a)) \leq r(w, \rho_w(a)).$$

Notice that $\rho_u(w) = w$ for all $w \in W$. Therefore $r(w, \rho_u(a)) = r(\rho_u(w), \rho_u(a)) \leq r(w, a)$ (here we use Axiom B4). Notice also that it follows from the definition of a retraction that $\delta(w, a) = \delta(w, \rho_w(a))$. Hence $r(w, a) = r(1, \delta(w, a)) = r(1, \delta(w, \rho_w(a))) = r(1, w^{-1}\rho_w(a)) = r(w, \rho_w(a))$. After combining all these inequalities together we obtain the desired result. □

6. $\mathbb{R}$-trees

Дале деревья теряют свои очертанья и глазу
Кажутся то треугольником, то полукругом –
Это уже выражение чистых понятий,
Дерево сфера царствует здесь над другими.
Дерево сфера – это значок беспредельного дерева,
Это итог числовых операций.
Ум, не ищи ты его посредине деревьев:
Он посредине, и с боку, и здесь, и повсюду.

Н. Заболоцкий

6.1. Continuous Euclidean buildings. Let now W be a subgroup of the group of isometries of the affine Euclidean space E of dimension n. Assume that W contains the subgroup of all affine translations of E and endowed with the pseudometric $r(u, v) = |u(0) - v(0)|$. In this case we shall call a metric building B of type W a *continuous Euclidean building.* We will show in this section that $\mathbb{R}$-trees provide natural examples of continuous Euclidean buildings.

6.2. $\mathbb{R}$-trees. Recall that a $\mathbb{R}$-*tree* is a metric space T with the metric d such that

(a): any two points $x, y \in T$ with $r(x, y) = d$ are connected by an isometric image of the real segment $[0, d]$

and

(b): any two points of T can be connected by a unique curve, i.e. an image of a real segment under a one-to-one continuous map from a real segment to T.

A $\mathbb{R}$-tree T has *infinite branches* if every two points of T lie in an isometric image of $\mathbb{R}$. Isometric images of $\mathbb{R}$ are called *apartments* of T.

Let now T be a $\mathbb{R}$-tree with infinite branches and $\mathcal{A}'$ be the set of all isometries of $\mathbb{R}$ into T. We say that two isometries $\alpha, \beta : \mathbb{R} \longrightarrow T$ are equivalent, if $\alpha(x) = \beta(x)$ for all x in some segment $[0, \epsilon] \subset \mathbb{R}$ for $\epsilon > 0$. We shall call the equivalence classes of isometries *repers* and denote the set of them by B. If $a \in B$ and $\alpha \in a$, denote $\theta(a) = \alpha(0)$; we shall say that $\theta(a)$ is the *point* of a.

Let A be an apartment in T. We say that a reper a belongs to A, if $\alpha([0,\epsilon]) \subset A$ for some $\epsilon > 0$ and some $\alpha \in a$ (in particular, the center $\theta(a)$ of a lies on A). Now it is obvious that the set of all repers of a given apartment in T can be identified with the group W of all isometries of $\mathbb{R}$. It is easy to see that any two repers a, b in B belong to some common apartment A in T. We can find $\alpha \in a$ and $\beta \in b$ such that $\alpha(\mathbb{R}) = \alpha(\mathbb{R}) = A$ and define $\delta(a,b)$ as

$$\delta(a,b) = \alpha^{-1}\beta.$$

If we now define a pseudometric on W by $r(u,v) = |u(0) - v(0)|$ and take for $\mathcal{A}$ the set of all W-isometries from W to B then it can be easily checked that these definitions turn $\mathcal{B} = (B, \mathcal{A})$ into a metric building of type W. Moreover, if $a, b \in B$, then $r(a,b) = d(\theta(a), \theta(b))$. It defines a natural one-two-one correspondence between the points $\pi(a)$ and centers $\theta(a)$ of repers in B preserving metrics r on $\pi(B)$ and d on T.

Summarizing the construction, we can state the following theorem.

THEOREM 9. *Let W be the group of all isometries of $\mathbb{R}$ with the left-invariant pseudometric $r(u,v) = |u(0) - v(0)|$, $u, v \in W$. If T is a $\mathbb{R}$-tree with infinite branches, then there exists a metric building B of type W such that the metric space $\pi(B)$ of points of B is isometric to T.*

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School of Mathematics, The University of Manchester, Oxford Street, Manchester M13 9PL, United Kingdom

E-mail address: alexandre.borovik@manchester.ac.uk