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# Toric Topology of Stasheff Polytopes

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## Abstract

The Stasheff polytopes  $K_n$ ,  $n > 2$ , appeared in the Stasheff paper “Homotopy associativity of H-spaces” (1963) as the spaces of homotopy parameters for maps determining associativity conditions for a product  $a_1 \dots a_n$ ,  $n > 2$ .

Stasheff polytopes are in the limelight of several research areas. Nowadays they have become well-known due to applications of operad theory in physics.

We will describe geometry and combinatorics of Stasheff polytopes using several different constructions of these polytopes and the methods of toric topology.

We will show that the two-parameter generating function  $U(t, x)$ , enumerating the number of  $k$ -dimensional faces of the  $n$ -th Stasheff polytope, satisfies the famous Burgers-Hopf equation  $U_t = UU_x$ .

We will discuss some applications of this result including an interpretation of the Dehn–Sommerville relations in terms of the Cauchy problem and the Cayley formula in terms of conservation laws.

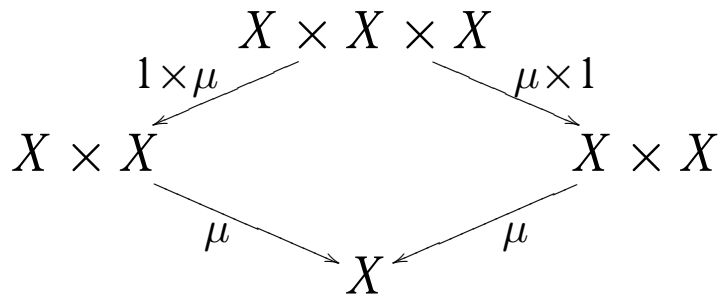
## References

- [1] V. M. Buchstaber, T. E. Panov, *Torus actions and their applications in topology and combinatorics.*, AMS, University Lecture Series, v. 24, Providence, RI, 2002.
- [2] V. M. Buchstaber, T. E. Panov and N. Ray, *Spaces of polytopes and cobordism of quasitoric manifolds.* Moscow Math. J. **7 (2)**, 2007, 219-242.
- [3] V. M. Buchstaber, E. V. Koritskaya, *The Quasi-Linear Burgers-Hopf Equation and the Stasheff Polytopes.*, Funct. Anal. Appl., 41:3, 2007, 196–207.

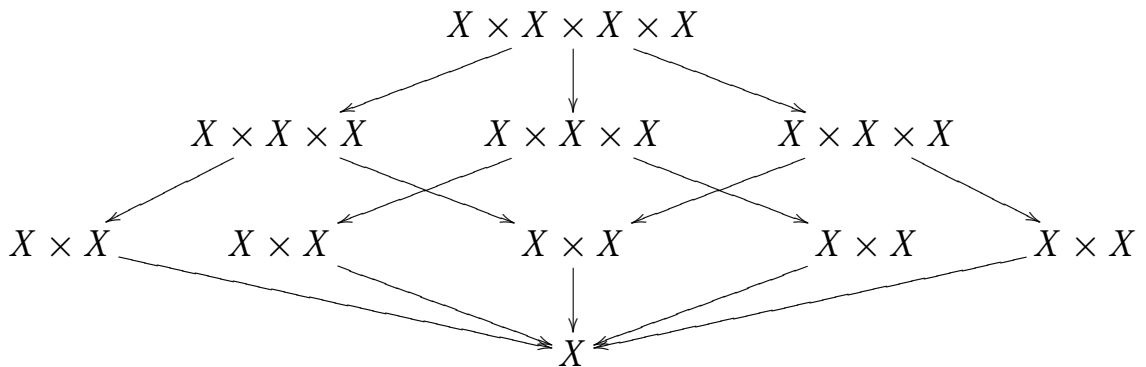
## Homotopy associativity ( $A_n$ -structure)

Take a  $H$ -space  $X$  with a multiplication  $\mu = \mu_2 : X \times X \rightarrow X$ . Then:

$$\mu_3 : K_3 \times X^3 \rightarrow X$$



$$\mu_4 : K_4 \times X^4 \rightarrow X$$



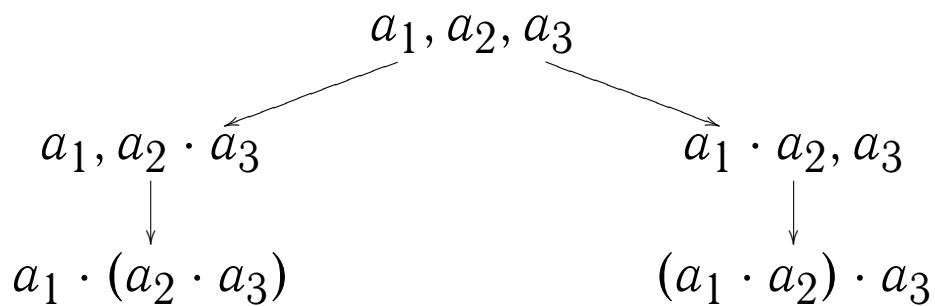
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$$\mu_n : K_n \times X^n \rightarrow X.$$

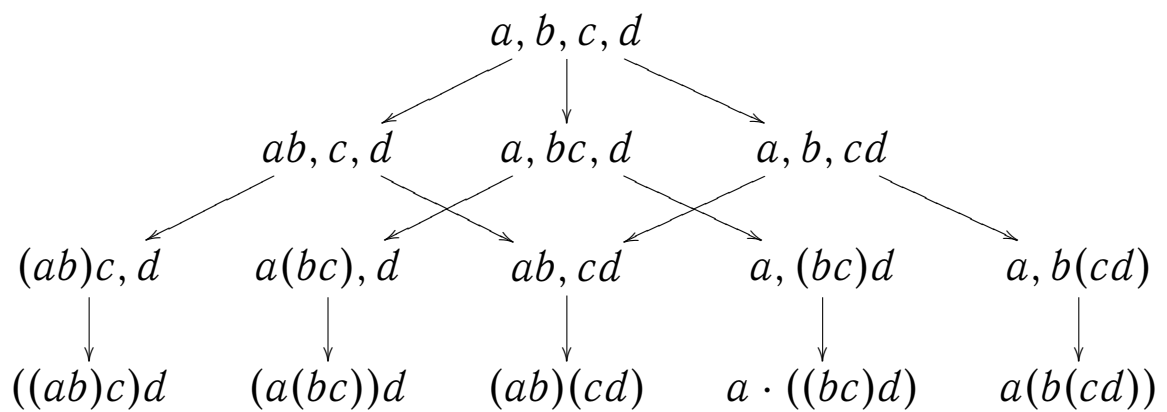
## In how many ways

can a product of  $n$  factors be interpreted  
in a non-commutative and non-associative algebra?

$n = 3$



$n = 4$



We will use four equivalent ways to describe the Stasheff polytope  $K_n$ :

bracketing, polygon dissection, plane trees and intervals.

### The language of brackets

**Definition.** The set  $\Gamma_i$ ,  $0 \leq i < n - 2$ , of  $i$ -dimensional faces of the Stasheff polytope  $K_n$  of dimension  $n - 2$  is the set of correct bracketings of the monomial  $a_1 \cdot \dots \cdot a_n$  with  $n - 2 - i$  pairs of brackets. The outer pair of brackets  $(a_1 \cdot \dots \cdot a_n)$  is not taken into account.

The incidence relation is defined as follows.

Let  $\gamma \in \Gamma_k$  and  $\delta \in \Gamma_l$ , where  $k > l$ . The cell  $\delta$  lies at the boundary of the cell  $\gamma$  (i.e.,  $\delta \subset \partial\gamma$  geometrically) if  $\gamma \subset \delta$  (as bracketings).

The set of 0-dimensional faces of the polytope  $K_n$ , i.e., the set of its vertices, is the set of correct bracketings of the monomial  $a_1, \dots, a_n$  with  $n - 2$  pairs of brackets.

The number of such bracketings is equal to the Catalan number  $C_{n-1} = \frac{1}{n} \binom{2n-2}{n-1}$ .

This is one possible definition of the Catalan numbers.

For example:  $C_2 = 2$ ,  $C_3 = 5$ ,  $C_4 = 14$ .

Two vertices in  $K_n$  are joined by an edge if and only if the bracketing corresponding to one vertex can be obtained from the bracketing corresponding to the other vertex by deleting a pair of brackets and inserting, in a unique way, another pair of brackets different from the deleted one.

For example, in the case  $K_3$ :

$$\begin{array}{c} (a_1 a_2) a_3 \\ \downarrow \\ a_1 (a_2 a_3) \end{array}$$



## The language of diagonals

**Definition.** Consider a convex  $(n + 1)$ -gon  $G_n$ . The set  $\Gamma_i$ ,  $0 \leq i < n - 2$ , of  $i$ -dimensional faces of the Stasheff polytope  $K_n$  of dimension  $n - 2$  is the set of all distinct sets of  $n - i - 2$  disjoint diagonals of  $G_n$ . (That is, each face of  $K_n$  is associated with a set of disjoint diagonals of  $G_n$ , and vice versa.)

The incidence relation is defined in the same way as in the preceding definition. Let  $\gamma \in \Gamma_k$  and  $\delta \in \Gamma_l$ , where  $k > l$ . The cell  $\delta$  lies at the boundary of  $\gamma$  (i.e.,  $\delta \subset \partial\gamma$  geometrically) if  $\gamma \subset \delta$  (as sets of diagonals).

**Corollary.** The dihedral group  $D_{n+1}$  of symmetries of a regular  $(n + 1)$ -gon  $G_n$  is the transformation group of the Stasheff polytope  $K_n$ .

Thus, the number of triangulation of  $(n + 1)$ -gon  $G_n$  is equal to the Catalan number  $C_{n-1}$ .

The problem to find the number of triangulation of  $(n + 1)$ -gon  $G_n$  is known as

*Euler's polygon division problem.*

Euler proposed it to C.Goldbach in 1751.

In 1758 Segner gave the solution of this problem by recurrence formula:

$$E_n = E_2E_{n-1} + E_3E_{n-2} + \cdots + E_{n-1}E_2, \quad n > 3,$$

with  $E_1 = E_2 = E_3 = 1$  and  $C_{n-2} = E_n$ .

The sequence  $\{C_n\}$  is named in honour of E. Catalan, who discovered the connection to bracketings of monomials in 1844.

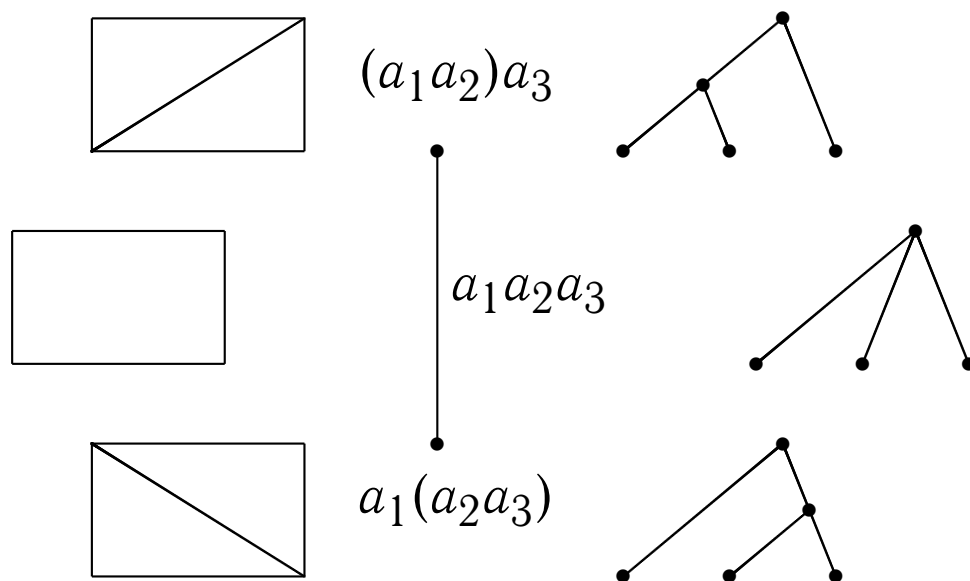
The number of diagonals

$$\frac{(n - 2)(n + 1)}{2} = \binom{n + 1}{2} - (n + 1)$$

of  $G_n$  is equal to the number of  $(n - 3)$ -dimensional faces (facets) of  $K_n$ .

The connection between bracketing and plane trees was known to A. Cayley (see [\*])

### The Stasheff polytope $K_3$

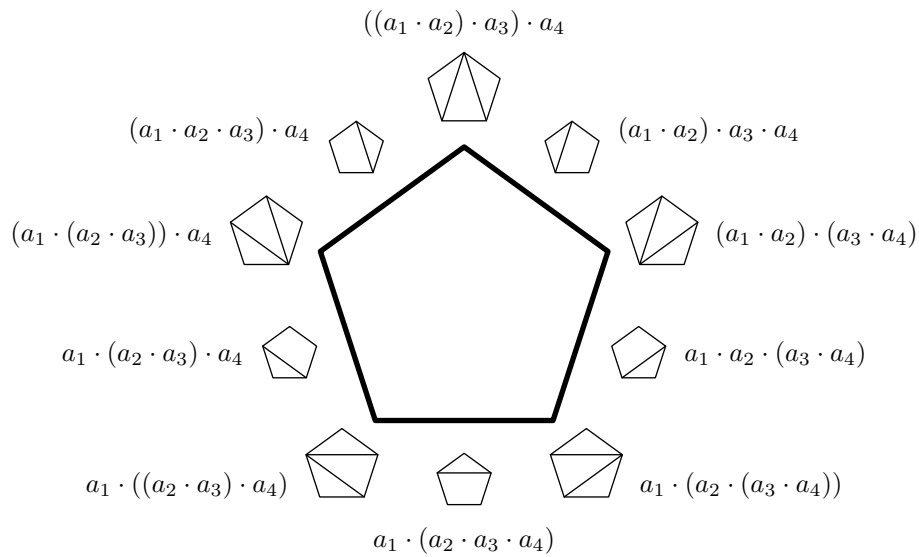


The languages: diagonals, brackets and plane trees.

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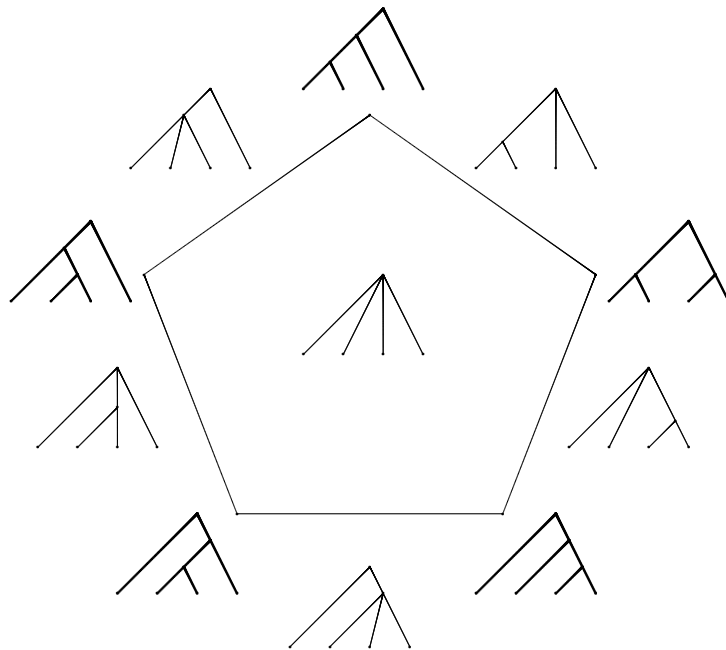
\*A.Cayley, On the analytical form called trees, Part II, Philos. Mag. (4) 18,1859,374–378.

## The Stasheff polytope $K_4$ .



The languages: correct bracketings and disjoint diagonals.

**The Stasheff polytope  $K_4$ .**



The language of plane trees.

## The language of intervals.

To each pair of brackets of the form

$$a_1 \cdots a_i (a_{i+1} \cdots a_{i+l+1}) a_{i+l+2} \cdots a_{n+1}$$

we assign the interval  $I_{i,l} = [i+1, \cdots, i+l] \subset [1, \cdots, n]$ , where  $0 \leq i \leq n-l$  and  $1 \leq l \leq n-1$ .

For example:

$K_3$

$$(a_1 \cdot a_2) \cdot a_3 \longrightarrow I_{0,1}, \quad a_1 \cdot (a_2 \cdot a_3) \longrightarrow I_{1,1}.$$

$K_4$

$$\begin{aligned} (a_1 \cdot a_2 \cdot a_3) \cdot a_4 &\longrightarrow I_{0,2}, & a_1 \cdot (a_2 \cdot a_3) \cdot a_4 &\longrightarrow I_{1,1}, \\ a_1 \cdot (a_2 \cdot a_3 \cdot a_4) &\longrightarrow I_{1,2}, & a_1 \cdot a_2 \cdot (a_3 \cdot a_4) &\longrightarrow I_{2,1}, \\ (a_1 \cdot a_2) \cdot a_3 \cdot a_4 &\longrightarrow I_{0,1}. \end{aligned}$$

**A realization of the Stasheff polytope  $K_{n+1}$   
as a simple polytope in  $\mathbb{R}^n$   
with integer vertices lying in a hyperplane**

Consider the formal monomial  $a_1 \cdot \dots \cdot a_{n+1}$ .

Let us label all multiplication signs “.” in this monomial from left to right with the numbers  $1, 2, \dots, n$ , so that the  $i$ -th multiplication sign,  $1 \leq i \leq n$ , is between  $a_i$  and  $a_{i+1}$ , i.e.

$$a_1 \overset{1}{\cdot} a_2 \cdots a_i \overset{i}{\cdot} a_{i+1} \cdots a_n \overset{n}{\cdot} a_{n+1}.$$

To each correct bracketing of this monomial with  $n - 1$  pairs of brackets, we assign the  $n$ -dimensional vector  $M = (m_1, \dots, m_n)$  whose coordinates  $m_i$  are defined as follows:

each multiplication sign stands for the multiplication of two smaller monomials. Set  $m_i = l_i r_i$ , where  $l_i$  and  $r_i$  are the lengths of the right and left monomials corresponding to the  $i$ -th multiplication sign.

For example, in the case  $n = 3$ , the bracketing

$$a_1 \overset{1}{\cdot} ((a_2 \overset{2}{\cdot} a_3) \overset{3}{\cdot} a_4)$$

gives rise to the vector  $(3, 1, 2)$ ,

because  $m_1 = 1 \cdot 3$ ,  $m_2 = 1 \cdot 1$ , and  $m_3 = 2 \cdot 1$ ;

and the bracketing  $(a_1 \cdot a_2) \cdot (a_3 \cdot a_4)$  gives rise to the vector  $(1, 4, 1)$ .

This defines a mapping of the set of vertices of the  $(n - 1)$ -dimensional Stasheff polytope  $K_{n+1}$  into  $\mathbb{R}^n$ . Extending it by linearity, we obtain a mapping

$$M: K_{n+1} \rightarrow \mathbb{R}^n.$$

For example,

$$M: K_3 \rightarrow \mathbb{R}^2.$$

$$M((a_1 \cdot a_2) \cdot a_3) = (1, 2), \quad M(a_1 \cdot (a_2 \cdot a_3)) = (2, 1).$$



Let  $0 \leq i \leq n - l$ ,  $1 \leq l \leq n - 1$ .

Take the linear function  $p_{i,l} : \mathbb{R}^n \rightarrow \mathbb{R}$ , where

$$p_{0,l}(\mathbf{x}) = \frac{1}{l}(x_1 + \cdots + x_l) - \frac{1}{n-l}(x_{l+1} + \cdots + x_n)$$

$$p_{l,n-l}(\mathbf{x}) = -p_{0,l}(\mathbf{x}), \quad 1 \leq l \leq n - 1,$$

and for  $0 < i < n - l$ ,  $1 \leq l \leq n - 2$

$$p_{i,l}(\mathbf{x}) = \frac{1}{l}(x_{i+1} + \cdots + x_{i+l}) - \frac{1}{n-l}(x_1 + \cdots + x_i + x_{i+l+1} + \cdots + x_n).$$

Set

$$L_{i,l} = \{\mathbf{x} \in \mathbb{R}^n : p_{i,l}(\mathbf{x}) + \frac{1}{2}n \geq 0\}.$$

**Theorem 1.** The mapping  $M : K_{n+1} \rightarrow \mathbb{R}^n$  is an embedding. Its image is the intersection of the hyperplane

$$H = \left\{ \mathbf{x} \in \mathbb{R}^n : \frac{1}{n}(x_1 + \cdots + x_n) = \frac{n+1}{2} \right\}$$

with the half-spaces  $L_{i,l}$ ,  $0 \leq i \leq n-1$ ,  $1 \leq l \leq n-1$ .

For each vertex of  $K_{n+1}$ , its image lies in the intersection of the  $n-1$  half-spaces  $L_{i,l}$  determined by the pairs of brackets occurring in the correct bracketing corresponding to this vertex.

This result is a some improvement of the main result of J.-L. Loday (see [\*]), who used the language of plane binary trees.

Set

$$B = \left\{ \mathbf{x} \in \mathbb{R}^n : -\frac{n}{2} \leq p_{0,l}(\mathbf{x}) \leq \frac{n}{2}, l = 1, \dots, n-1 \right\}.$$

**Corollary.**

The image of  $K_{n+1}$  in  $\mathbb{R}^n$  is the intersection of the  $(n-1)$ -dimensional *cube*  $H \cap B$  with the half spaces  $L_{i,l}$ , where  $0 < i < n-1$ ,  $0 < l < n-1$ .

Thus,  $K_{n+1}$  is a truncated  $(n-1)$ -dimensional cube with  $\binom{n-1}{2}$  truncations.

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\*J.-L. Loday, Realization of the Stasheff polytope., Arch. Math. v. 83, Issue 3, 2004, 267–278.

## The Stasheff polytope $K_3$

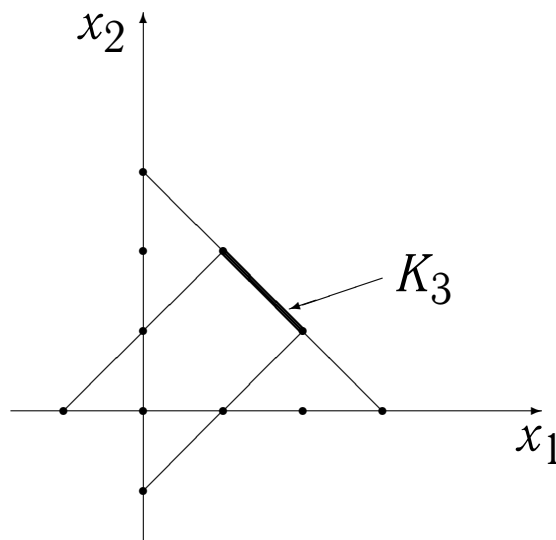
$$(a_1 \cdot a_2) \cdot a_3 \longrightarrow I_{0,1}, \quad a_1 \cdot (a_2 \cdot a_3) \longrightarrow I_{1,1}$$

$$p_{0,1}(\mathbf{x}) = x_1 - x_2 = -p_{1,1}(\mathbf{x})$$

$$B = \{\mathbf{x} \in \mathbb{R}^2 : -1 \leq p_{0,1}(\mathbf{x}) \leq 1\}$$

$$H = \{\mathbf{x} \in \mathbb{R}^2 : x_1 + x_2 = 3\}$$

$$K_3 \simeq H \cap B$$



## The Stasheff polytope $K_4$

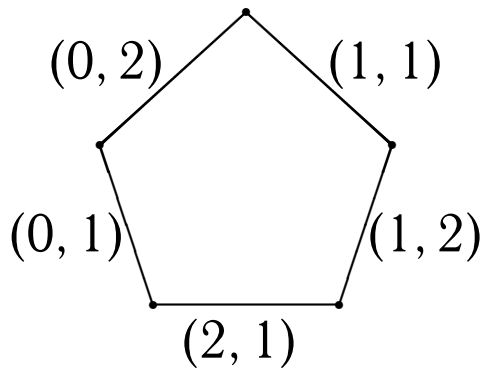
$$p_{0,1}(\mathbf{x}) = x_1 - \frac{1}{2}(x_2 + x_3) = -p_{1,2}(\mathbf{x})$$

$$p_{0,2}(\mathbf{x}) = \frac{1}{2}(x_1 + x_2) - x_3 = -p_{2,1}(\mathbf{x})$$

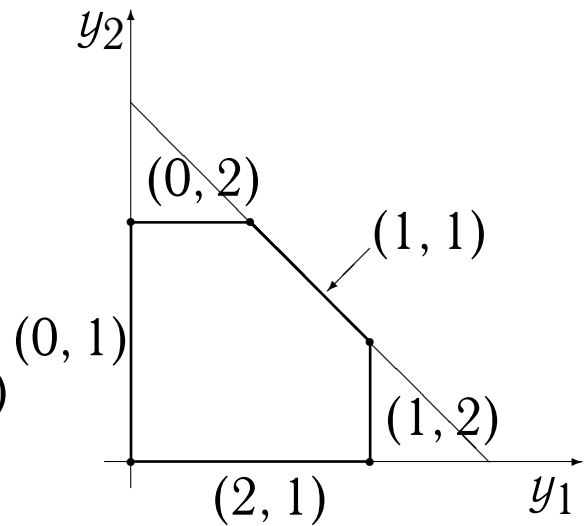
$$p_{1,1}(\mathbf{x}) = x_2 - \frac{1}{2}(x_1 + x_3)$$

$$L_{i,l} = \{\mathbf{x} \in \mathbb{R}^3 : p_{i,l}(\mathbf{x}) + \frac{3}{2} \geq 0\}$$

$$H = \{\mathbf{x} \in \mathbb{R}^3 : \frac{1}{2}(x_1 + x_2 + x_3) = 2\}$$



$$y_1 = 2p_{0,1}(\mathbf{x}) + 3$$



$$y_2 = 2p_{2,1}(\mathbf{x}) + 3$$

Any vertex  $v_q \in K_{n+1}$ ,  $q = 1, \dots, \frac{1}{n+1} \binom{2n}{n} = C_n$  gives a set  $\{I_{i,l}, (i,l) \in s(q)\}$  of intervals determined by the pairs of brackets occurring in the bracketing corresponding to this vertex  $v_q$ . Take

$$\mathcal{L}_{i,l} = \{y \in \mathbb{R}^{n-1} : (i+l)y_{i+l} - iy_i + il \geq 0\}$$

for  $0 \leq i \leq n-l$ ,  $1 \leq l \leq n-1$ , and

$$I^{n-1} = \{y \in \mathbb{R}^{n-1} : 0 \leq y_l \leq n-l, 1 \leq l \leq n-1\}.$$

**Theorem.** There is an embedding

$$\mathcal{M} : K_{n+1} \longrightarrow \mathbb{R}^{n-1}$$

with the image

$$\{y \in I^{n-1} : y \in \mathcal{L}_{i,l}, 0 < i < n-l, 0 < l < n-1\}.$$

For each vertex  $v_q \in K_{n+1}$  we have

$$\mathcal{M}(v_q) = \bigcap_{(i,l) \in s(q)} \mathcal{L}_{i,l}.$$

For example,  $K_3 \simeq I^1$ ,  $K_4 \simeq I^2 \cap \mathcal{L}_{1,1}$ .

## The Stasheff polytope $K_5$

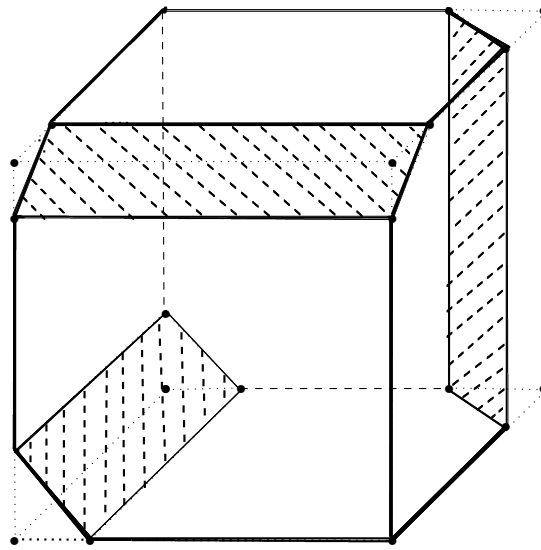
$$\underline{a_1 \cdot a_2 \cdot a_3 \cdot a_4 \cdot a_5}$$

We have 9 pairs of brackets

$$(a_1 \cdot a_2) \rightarrow I_{0,1}; (a_2 \cdot a_3) \rightarrow I_{1,1}; (a_3 \cdot a_4) \rightarrow I_{2,1}; (a_4 \cdot a_5) \rightarrow I_{3,1};$$

$$(a_1 \cdot a_2 \cdot a_3) \rightarrow I_{0,2}; (a_2 \cdot a_3 \cdot a_4) \rightarrow I_{1,2}; (a_3 \cdot a_4 \cdot a_5) \rightarrow I_{2,2};$$

$$(a_1 \cdot a_2 \cdot a_3 \cdot a_4) \rightarrow I_{0,3}; (a_2 \cdot a_3 \cdot a_4 \cdot a_5) \rightarrow I_{1,3}.$$



$$K_5 \simeq I_3 \cap \mathcal{L}_{1,1} \cap \mathcal{L}_{1,2} \cap \mathcal{L}_{2,1}.$$

**Definition.** A polytope  $P$  of dimension  $n$  is said to be *simple* if every vertex of  $P$  is the intersection of exactly  $n$  facets, i.e. faces of dimension  $n - 1$ .

**Proposition.** The Stasheff polytopes are simple.

Theorem 1 provides an explicit description of the  $(n - 1)$  facets whose intersection is a given vertex of the  $(n - 1)$ -dimensional polytope  $K_{n+1}$ .

**Definition.** An  $n$ -dimensional polytope  $P^*$  is said to be *dual to  $P$*  if for each  $i$ ,  $0 \leq i \leq n - 1$ , there exists an one-to-one correspondence between the  $i$ -dimensional faces  $\gamma_i$  of  $P$  and the  $(n - i - 1)$ -dimensional faces  $\gamma_{n-i-1}^*$  of  $P^*$  such that the embedding  $\gamma_{n-j-1}^* \subset \gamma_{n-i-1}^*$  corresponds to the embedding  $\gamma_i \subset \gamma_j$ .

Each pair of brackets in the monomial  $a_1 \cdot \dots \cdot a_{n+1}$  determines a facet of the polytope  $K_{n+1}$ . Thus, in terms of the dual polytope, it corresponds to a vertex of the polytope  $K_{n+1}^*$ .

The number of vertex of  $K_{n+1}^*$  is  $\frac{(n-2)(n+1)}{2}$ .

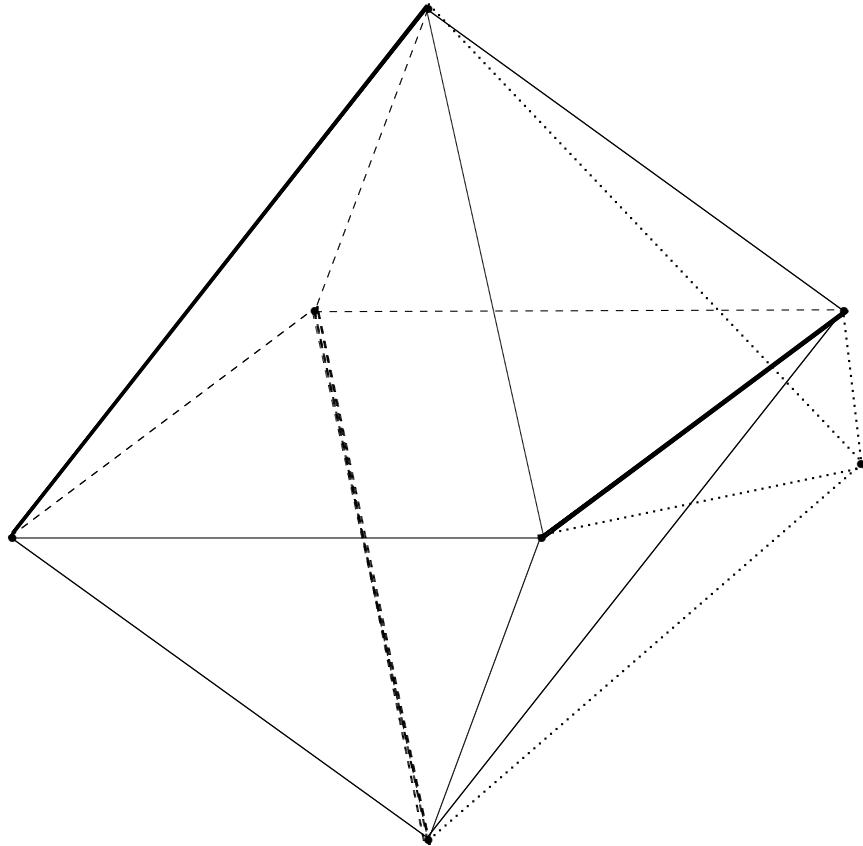
**Definition.** A polytope  $S$  is said to be *simplicial* if every face of  $S$  is a simplex.

The dual  $P^*$  of a simple polytope  $P$  is simplicial and vice versa.

**Proposition.** The dual  $K_n^*$  of a Stasheff polytope  $K_n$  is a simplicial polytope.



## The dual polytope $K_5^*$



Octahedron  $(I^3)^*$  is dual to cube  $I^3$ .

The fragment of construction  $K_5^*$  via stellar subdivision.

**Definition.** A polytope  $P$  is called a *flag polytope* if each set of vertices of  $P$  pairwise joined by edges forms a simplex.

**Proposition.** The dual  $K_n^*$  of a Stasheff polytope  $K_n$  is a flag polytope.

**Proof.** To each set of  $k$  vertices of  $K_n^*$  pairwise joined by edges, there corresponds a set of  $k$  diagonals of a convex  $(n + 1)$ -gon  $G_n$ . Since these vertices are pairwise joined by edges, it follows that the corresponding diagonals are disjoint.

By definition, this collection of diagonals determines a face of  $K_n$  and hence a face of  $K_n^*$ .

Since  $K_n^*$  is a simplicial polytope, it follows that this face is a simplex. □

**Proposition.** The boundary of the polytope  $K_n^*$  dual to  $K_n$  is a triangulation of the  $(n-3)$ -dimensional sphere.

## Stanly–Reisner ring of Stasheff polytopes

Let  $P$  be a simple polytope with  $m$  facets  $F_1, \dots, F_m$ . Fix commutative ring  $k$  with unit. Let  $k[v_1, \dots, v_m]$  be a polynomial graded  $k$ -algebra,  $\deg v_i = 2$ .

**Definition.** The *face ring*  $k(P)$  (or the *Stanly–Reisner ring*) of a simple polytope  $P$  is the quotient ring

$$k(P) = k[v_1, \dots, v_m]/J_P,$$

where  $J_P$  is the ideal, generated by all square-free monomials  $v_{i_1} \cdot v_{i_2} \cdots v_{i_s}$  such that  $F_{i_1} \cap \cdots \cap F_{i_s} = \emptyset$  in  $P$ ,  $i_1 < \cdots < i_s$ .

**Corollary.**  $k(K_n) = k[v_1, \dots, v_m]/J_{K_n}$  where the set  $\{v_1, \dots, v_m\}$  corresponds to the set of diagonals  $\{d_1, \dots, d_m\}$ ,  $m = \frac{(n-2)(n+1)}{2}$  of a convex  $(n+1)$ -gon  $G_n$  and  $J_{K_n}$  is the ideal generated by all monomials  $v_i v_j$ ,  $i \neq j$ , such that  $d_i \cap d_j \neq \emptyset$  in  $G_n$ .

**Example.**  $k(K_3) = k[v_1, v_2]/(v_1 v_2)$ .

**Corollary.** A generator in  $J_{K_n}$  corresponds to a set of four vertices in a convex  $(n + 1)$ -gon  $G_n$ , that is the number of generators  $\delta$  in  $J_{K_n}$ ,  $n \geq 3$ , is  $\binom{n+1}{4}$ .

For example:

$$n = 4 : m = 5, \delta = 5$$

$$n = 5 : m = 9, \delta = 15$$

Let  $k[D_{n+1}]$  be the group ring over  $k$  of the dihedral group  $D_{n+1}$ .

**Corollary.** The face ring  $k(K_n)$  is a  $k[D_{n+1}]$ -module.

**Examples.**

$$k(K_3) = k[v, \tau v]/(v \cdot \tau v),$$

where  $\tau$  is a generator in  $\mathbb{Z}_2$ .

$$k(K_4) = k[\tau^i v, i = 0, \dots, 4]/(\tau^i w, i = 0, \dots, 4),$$

where  $\tau$  is a generator in  $\mathbb{Z}_5$  and  $w = v \cdot v$ .

$$k(K_5) = k[\tau^i v_1, i = 0, \dots, 5, \tau^i v_2, i = 0, 1, 2]/J_{K_5},$$

where  $\tau$  is a generator in  $\mathbb{Z}_6$ ,

$$J_{K_5} = (\tau^i w_j, i = 0, \dots, 5, j = 1, 2, \tau^i w_3, i = 0, 1, 2),$$

and  $w_1 = v_1 \cdot \tau v_1$ ,  $w_2 = v_1 \cdot v_2$ ,  $w_3 = v_2 \cdot \tau v_2$ .

**Definition.** Let  $S$  be a simplicial  $n$ -dimensional polytope. Let  $f_i$  be the number of its  $i$ -dimensional faces. The integer vector  $f(S) = (f_0, \dots, f_{n-1})$  is called the  $f$ -vector of  $S$ . For convenience, we set  $f_{-1} = 1$ .

The  $f$ -vector of a simple polytope  $P$  is, by definition, the  $f$ -vector of the dual simplicial polytope  $P^*$ .

We denote the number  $f_{k-1}(K_n^*)$  of  $(n-k-2)$ -dimensional faces of the Stasheff polytope  $K_n$  by  $u_{k,n}$ .

Set  $U(t, x) = \sum_{n,k} u_{k,n} t^k x^n$ ,  $0 \leq k \leq n - 2$ , where  $u_{0,n} = 1$  for  $n \geq 2$  and  $u_{0,n} = 0$  for  $n = 0, 1$ .

Note that  $u_{n-2,n} = C_{n-1}$ .

**Theorem.** The function  $U$  is a solution of the Burgers–Hopf equation

$$U_t = UU_x$$

with the initial condition  $U(0, x) = \frac{x^2}{1-x}$ .

Recursion formula for the numbers  $u_{k,n}$  are a crucial part of the proof of this theorem.

**Lemma.** The numbers  $u_{k,n}$  satisfy the recursion formula

$$ku_{k,n} = \sum_{i+j=k-1} \sum_{p+q=n+1} pu_{i,p}u_{j,q}, \quad 0 \leq k \leq n-2,$$

where  $u_{0,n} = 1$  for  $n \geq 2$  and  $u_{0,n} = 0$  for  $n = 0, 1$ .

**Proof.** Assume that we have drawn  $k$  nonintersecting diagonals in an  $(n+1)$ -gon. Each of these diagonals dissects the  $(n+1)$ -gon into two smaller ones, in which a total of another  $k-1$  diagonals are drawn.

The number of ways to dissect an  $(n+1)$ -gon into two smaller  $(p+1)$ - and  $(q+1)$ -gons is equal to  $n+1$  in all cases except when  $p = q = (n+1)/2$ , where it is equal to  $(n+1)/2$ .

Note that each way to draw  $k$  diagonals is counted exactly  $k$  times. □

The transformation  $(\tilde{t}, \tilde{x}) = (\alpha t + \beta, \alpha x + \gamma)$  preserves the form of the Burgers–Hopf equation.

This transformation takes the solution  $U(t, x)$  with the initial condition  $U(0, x) = \varphi(x)$  to the solution  $\tilde{U}(t, x)$  with the initial condition  $\tilde{U}(0, x) = \tilde{\varphi}(x) = U(\beta, \alpha x + \gamma)$ .

**Proposition.** The relation  $U(0, x) = U(-1, -x)$  follows from the formula for the Euler characteristic of the sphere.

**Proof.** Since  $U(0, x) = x^2/(1 - x)$ , it follows that the condition  $U(0, x) = U(-1, -x)$  is equivalent to the condition

$$1 = (-1)^n \sum_{k=0}^{n-2} (-1)^k u_{k,n} \quad \text{for each } n \geq 2.$$

We have  $u_{k,n} = \hat{f}_{k-1}(K_n)$  with  $\hat{f}_{-1}(K_n) = 1$ ; hence

$$(-1)^n = \sum_{k=0}^{n-2} (-1)^k \hat{f}_{k-1}(K_n) = 1 - \chi(\partial K_n).$$

Therefore,  $\chi(\partial K_n) = 1 + (-1)^{n-3}$ . □

In the theory of quasilinear equations, there is an analog of the existence and uniqueness theorem for the Cauchy problem for the case in which the initial condition is not characteristic. This is the case in our setting.

Using Proposition, we see that the solution of the Burgers–Hopf equation with the given initial condition has the symmetry

$$U(t, x) = U(-(t + 1), -x).$$

For simple polytopes, the formula for the Euler characteristic admits a generalization in the form of Dehn–Sommerville relations. In terms of the  $f$ -vector of an  $n$ -dimensional polytope  $P$ , they can be written as follows:

$$f_{k-1} = \sum_{j=k}^n (-1)^{n-j} \binom{j}{k} f_{j-1}, \quad k = 0, 1, \dots, n.$$



The Dehn–Sommerville relations have the form

$$h_i = h_{n-i}, \quad i = 0, 1, \dots, n,$$

where  $h_i$  are the coordinates of the  $h$ -vector defined in the following way.

**Definition.** Let  $f_0, \dots, f_{n-1}$  be the coordinates of the  $f$ -vector of an  $n$ -dimensional polytope  $P$ .

Then the integer vector  $h(P) = (h_0, \dots, h_n)$ , where  $h_i$  are determined by the equation

$$h_0 t^n + \dots + h_{n-1} t + h_n = (t-1)^n + f_0 (t-1)^{n-1} + \dots + f_{n-1},$$

is called the  $h$ -vector of  $P$ .

**Proposition.** The Dehn–Sommerville relations are equivalent to the symmetry  $U(t, x) = U(-(t+1), -x)$  of the solution of the Burgers–Hopf equation.

**Proof.** Set  $h(t) = \sum h_i t^{n-i}$ . Then the Dehn–Sommerville relations can be rewritten in the form

$$h(t) = t^n h(1/t).$$

Returning to the  $f$ -vectors, we obtain

$$\begin{aligned} \sum_0^n f_{i-1} (t-1)^{n-i} &= t^n \sum_0^n f_{i-1} \left(\frac{1}{t} - 1\right)^{n-i} \\ &\iff \sum f_{i-1} (t-1)^{n-i} = \sum f_{i-1} t^i (1-t)^{n-i} \\ &\iff \sum f_{i-1} \left(\frac{1}{t-1}\right)^i = (-1)^n \sum f_{i-1} \left(\frac{t}{1-t}\right)^i. \end{aligned}$$

Since  $1/(t-1) = -(1+t/(1-t))$ , we see, by setting  $\tau = 1/(t-1)$ , that the last equation is equivalent to

$$\sum_k u_{k,n} \tau^k = (-1)^n \sum_k u_{k,n} (-(\tau+1))^k,$$

as desired. □

## Quasilinear Burgers–Hopf Equation

The Hopf equation (Eberhard F.Hopf, 1902–1983) is the equation

$$U_t + f(U)U_x = 0.$$

The Hopf equation with  $f(U) = U$  is a limit case of the following equations:

$$U_t + UU_x = \mu U_{xx} \quad (\text{the Burgers equation}),$$

$$U_t + UU_x = \varepsilon U_{xxx} \quad (\text{the Korteweg–de Vries equation}).$$

The Burgers equation (Johannes M.Burgers, 1895–1981) occurs in various areas of applied mathematics (fluid and gas dynamics, acoustics, traffic flow). It is used for describing of wave processes with velocity  $u$  and viscosity coefficient  $\mu$ . The case  $\mu = 0$  is a prototype of equations whose solution can develop discontinuities (shock waves).

K-d-V equation (Diederik J.Korteweg, 1848–1941 and Hugo M. de Vries, 1848–1935) was introduced as equation for the long waves over water (in 1895). It appears also in plasma physics. Today K-d-V equation is a most famous equation in soliton theory.

It follows from the theory of partial differential equations that the quasilinear Hopf equation

$$U_t + f(U)U_x = 0$$

with the initial condition  $U(0, x) = \varphi(x)$  has the solution  $U = \varphi(\xi)$ , where  $\xi = \xi(t, x)$  is determined by the relation  $x = \xi + f(\varphi(\xi))t$ .

We consider only the case  $f(U) = -U$  and refer to the corresponding equation  $U_t = UU_x$  as the Burgers–Hopf equation. The transformation  $t \rightarrow -t$  takes this equation to the equation  $U_t + UU_x = 0$ .

For the initial condition  $\varphi(x) = x^2/(1 - x)$ , the function  $\xi(t, x)$  is given by the quadratic equation

$$(t + 1)\xi^2 - (1 + x)\xi + x = 0.$$

By solving this equation, we obtain a closed-form expression for the solution of the Cauchy problem in a neighborhood of the point  $(0, 0)$ :

$$U(t, x) = \frac{\xi^2}{1 - \xi}, \quad \text{where} \quad \xi = \frac{2x}{x + 1 + \sqrt{(x + 1)^2 - 4(t + 1)x}}.$$

For a general initial condition, the relation  $x = \xi - \varphi(\xi)t$  implies that

$$\varphi(\xi) = \frac{1}{t}(\xi - x).$$

Thus,  $\xi(t, x) = tU(t, x) + x$ ; i.e., we can eliminate the function  $\xi(t, x)$  from the equation  $U = \varphi(\xi)$ .

**Corollary.** The solution of the equation  $U_t = UU_x$  with  $U(0, x) = \varphi(x)$  is a solution of the *functional equation* (equation on the characteristics)

$$U = \varphi(x + tU).$$

In particular, if  $\varphi(x)$  is a rational function, then  $U(t, x)$  satisfies an algebraic functional equation of the form

$$\sum_{k=0}^n a_k(t, x)U^k = 0,$$

where  $a_k(t, x)$  are polynomials in  $t$  and  $x$ .

In our case,  $\varphi(x) = x^2/(1 - x)$ , and the function  $U(t, x)$  satisfies the equation

$$t(1 + t)U^2 + (2xt + x - 1)U + x^2 = 0.$$

It can readily be seen from this equation that  $U$  has the symmetry

$$U(t, x) = U(-(t + 1), -x).$$

Let us treat  $\xi(t, x)$  as a function of  $x$  with parameter  $t$ . Then it is the inverse of the function  $x - \varphi(x)t$ . Hence we can apply the classical Lagrange formula for computing the inverse function:

$$\begin{aligned} \xi(t, x) &= \frac{1}{2\pi i} \int_{|z|=\varepsilon} -\ln \left( 1 - \frac{x}{z} \left( 1 - \frac{\varphi(z)}{z} t \right)^{-1} \right) dz = \\ &= \sum \frac{x^n}{n} \left[ \left( 1 - \frac{\varphi(z)}{z} t \right)^{-n} \right]_{n-1}, \end{aligned}$$

where  $[\gamma(z)]_k$  is the coefficient of  $z^k$  in the series  $\gamma(z)$ .

By substituting the initial condition  $\varphi(x) = x^2/(1 - x)$  into this formula, we obtain

$$\xi(t, x) = \sum_{n \geq 1} \frac{x^n}{n} \left[ \left( 1 + \frac{tz}{1 - (t+1)z} \right)^n \right]_{n-1}.$$

Hence

$$U(t, x) = \sum_{n \geq 2} V_n(t) x^n,$$

where

$$V_n(t) = \frac{1}{n} \sum_{l=0}^{n-2} \binom{n}{l+1} \binom{n-2}{l} t^l (1+t)^{n-2-l}.$$

Note that this formula readily implies the identity

$$U(t, x) = U(-(t+1), -x).$$

Moreover, if we use the identity

$$\sum_{l=0}^k \binom{n}{l+1} \binom{k}{l} = \binom{n+k}{k+1}, \quad 0 \leq k \leq n-2,$$

then this formula for  $V_n(t)$  implies the classical result

$$f_{k-1}(K_n) = \frac{1}{n} \binom{n-2}{k} \binom{n+k}{k+1}, \quad 0 \leq k \leq n-2.$$

Here  $f_{k-1}(K_n)$  is the number of  $(n-k-2)$ -dimensional faces of the Stasheff polytope  $K_n$ .



Another way to obtain the solution is to consider *conservation laws*.

Let  $U(t, x)$  be the solution of the Cauchy problem for the Burgers–Hopf equation

$$U_t = UU_x, \quad U(0, x) = \varphi(x).$$

This equation has the conservation laws

$$\left(\frac{U^{k+1}}{k+1}\right)_x = \left(\frac{U^k}{k}\right)_t, \quad k = 1, 2, \dots$$

Hence for any  $k$  and  $l$ ,  $1 \leq k \leq l$ ,  $l = 1, 2, \dots$ , we have

$$\frac{d^k}{dx^k} \left(\frac{U^{l+1}}{l+1}\right) = \frac{d^{k-1}}{dx^{k-1}} \left(\frac{U^l}{l}\right)_t = \frac{d^k}{dt^k} \left(\frac{U^{l-k+1}}{l-k+1}\right).$$

Let us define  $U_k(x)$  as the coefficient of  $t^k$  in the expansion

$$U(t, x) = \sum_n \sum_k u_{k,n} t^k x^n = \sum U_k(x) t^k.$$

Then

$$\left. \frac{d^k}{dt^k} U \right|_{t=0} = k! U_k(x) = \frac{d^k}{dx^k} \left( \frac{U_0^{k+1}(x)}{k+1} \right)$$

for  $l = k$ . Therefore,

$$U_k(x) = \frac{1}{(k+1)!} \frac{d^k}{dx^k} \varphi^{k+1}(x).$$

By using the binomial expansion

$$(1-x)^{-(k+1)} = 1 + (k+1)x + \dots + \frac{(k+1) \cdots (k+1)}{l!} x^l + \dots,$$

we obtain

$$u_{k,n} = \frac{1}{n} \binom{n-2}{k} \binom{n+k}{k+1} = f_{k-1}(K_n).$$

Thus, we have computed the number

$$f_{k-1}(K_n), \quad n \geq 3, \quad 1 \leq k < n - 2$$

with the help of conservation laws for the Burgers–Hopf equation.

The first computation of this number we can find in the Cayley’s paper (1891), where he also used the function  $U_k(x)$ .

Note that Cayley (Arthur Cayley, 1821–1895) obtained the above form of  $U_k(x)$  by using the recursion formula

$$f(k, n) = \frac{n}{2k} \sum_{l+m=n+2} \sum_{i+j=k-1} f(i, l)f(j, m),$$

where  $f(k, n) = u_{k,n-1} = f_{k-1}(K_{n-1})$ .