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Conjugacy problem in HNN-extensions: regular elements and black holes

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Abstract. We discuss the complexity of conjugacy problem in HNN-extensions of groups. We stratify the groups in question and show that for "almost all", in some explicit sense, elements, the conjugacy search problem is decidable.

Contents

1. Introduction

The present paper continues the development of a new approach to algorithmic problems in groups initiated in [**1**]; see that paper for a detailed introduction into the subject. Following the key idea of [**1**], we stratify a given HNN-extension G into two parts with respect to the "hardness" of the conjugacy problem:

- a *Regular Part* RP, consisting of so-called *regular elements* for which the conjugacy problem is decidable by standard algorithms. We show that the regular part RP satisfies all the necessary conditions from [**1**]:
  - the standard algorithms are very fast on regular elements;

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  – if an element is a conjugate of a given regular element then the algorithms quickly provide a conjugator, so the Search Conjugacy problem is also decidable for regular elements;
  – the set RP is *generic* in G, i.e., it is very "big" in some particular sense explained in the previous paper [**2**];
- the *Black Hole* BH (the complement of the set of regular elements) which consists of elements in G for which either the standard algorithms do not work at all, or they require a considerable modification, or it is not clear yet whether these algorithms work or not.

This paper concentrates on the case of non-degenerate HNN-extensions. The conjugacy problem for the so-called degenerated HNN-extensions ($H = A = B$) is considered in our next paper [**3**] devoted to the analysis of the class of groups constructed by Miller [**6**].

Results of Section 4 are parallel to similar results of [**1**] for amalgamated products, with proofs which can be obtained by obvious transformation of proofs from [**1**] and therefore can be omitted.

2. HNN-extensions

2.1. Preliminaries. We introduce in brief some terminology and formulate several known results on HNN-extensions of groups. We refer to the books [**5, 6**] and one of the original papers [**4**] for more detail.

Let $H = \langle X \mid \mathcal{R}\rangle$ be a group given by generators and relators, $A = \langle U_i \mid i \in I\rangle$ and $B = \langle V_i \mid i \in I\rangle$ two isomorphic subgroups of H with an isomorphism $\phi : A \to B$ given by $\phi : U_i \to V_i$, $i \in I$. Then the group G defined by the presentation

$$G = \left\langle X, t \mid \mathcal{R},\, t^{-1}U_i t = V_i,\, i \in I\right\rangle$$

is called an *HNN-extension* of the *base* group H with the *stable* letter t and *associated* (via the isomorphism ϕ) subgroups A and B. We sometimes write G as

$$G = \left\langle H, t \mid t^{-1}At = B,\, \phi\right\rangle$$

An HNN-extension G is called *degenerate* if $H = A = B$.

A modification of the above definition is that of *multiple HNN-extension.* The data consist of a group H and a set of isomorphisms $\phi_i : A_i \to B_i$ between subgroups of H. Then similar to the above we define a multiple HNN-extension of H as

$$G = \left\langle H, t_i \mid t_i^{-1}At_i = B, \phi_i,\, i \in I\right\rangle.$$

2.2. Reduced and normal forms. In this section following [**5**] we discuss *reduced* and *normal forms* of elements in HNN-extensions of groups. The main focus is on algorithms for computing them. We consider only HNN-extensions with one stable letter, but one can easily extend the results to arbitrary multiple HNN-extensions.

Let $G = \left\langle H, t \mid t^{-1}At = B,\, \phi\right\rangle$ be an HNN-extension of a group H with the stable letter t and associated subgroups A, B. Every element g of G can be written in the form

$$g = w_0 t^{\epsilon_1} w_1 \cdots t^{\epsilon_n} w_n, \tag{1}$$

where $\epsilon_i = \pm 1$ and w_i is a (possibly empty) word in the generating set X. The following result is well known (see, for example, [**5**]).

THEOREM 2.1 ([**5**]). *Let* $G = \langle H, t \mid tAt^{-1} = B, \phi \rangle$, *and let* $g = w_0 t^{\epsilon_1} w_1 \cdots t^{\epsilon_n} w_n$. *If* g *represents the identity element of* G *then either*

(a) $n = 0$ *and* w_0 *represents the identity element of* H*; or*
(b) g *contains either* $t^{-1} w_i t$ *where* $w_i \in A$ *or* $t w_i t^{-1}$ *where* $w_i \in B$ (*words of this type are called* pinches).

Theorem 2.1 immediately gives a decision algorithm for the Word Problem in G provided one can solve effectively in the group H the Word and Membership Problems $w_0 = 1$, $w_i \in A$, $w_i \in B$ from (a) and (b) above. We will have to say more on the time complexity of the Word Problem in G in the sequel.

We say that (1) is a *reduced form* of $g \in G$ if no pinches occur in it. It can be shown that the number of occurrences of t_i in a reduced form of g does not depend on the choice of reduced form; we shall call it *length* of g and denote it by $l(g)$.

We say that an element g with $l(g) > 0$ is *cyclically reduced* if $l(g^2) = 2l(g)$. In addition, we impose extra conditions in case $l(g) = 0$ (which is equivalent to saying that $g \in H$): namely, we say that g is cyclically reduced if either $g \in A \cup B$ or g is not conjugate in H to any element from $A \cup B$.

We warn that our definition of cyclically reduced elements differs from that of [**5**]; elements reduced in our sense are reduced in the sense of [**5**] but not vice-versa.

Reduced forms of elements in G are not unique. To define unique *normal forms* of elements in G one needs to fix systems of right coset representatives of A and B in G.

Let S_A and S_B be systems of right representatives (transversals) of the subgroups A and B in H. A reduced form

$$g = h_0 t^{\epsilon_1} s_1 \cdots t^{\epsilon_n} s_n \tag{2}$$

of an element $g \in G$ is said to be a *normal form* of g if the following conditions hold:

- $h_0 \in H$;
- if $\epsilon_i = -1$ then $s_i \in S_A$;
- if $\epsilon_i = 1$ then $s_i \in S_B$;

Normal forms of elements of G are unique; see, for example, [**5**]. It is convenient sometimes to write down the normal form (2) of g as

$$g = h_0 p_1 \cdots p_k \tag{3}$$

where $p_i = t^{\epsilon_i} s_i$ and $s_i \in S_A$ if $\epsilon_i = -1$, $s_i \in S_B$ if $\epsilon_i = 1$. Observe that this decomposition corresponds to the standard decomposition of elements of G when G is viewed as the universal Stallings group $U(P)$ associated with the pregroup

$$P = \{H, tH, t^{-1}H\},$$

(see a more detailed description of pregroups in [**7**]).

Now the definition of cyclically reduced elements can be formulated as follows. A reduced form

$$g = h t^{\epsilon_1} s_1 \cdots t^{\epsilon_n} s_n$$

of element g is *cyclically reduced* if and only if

- If $n = 0$ then either $h \in A \cup B$ or h is not conjugate in G to any element in $A \cup B$.
- if $n > 0$ then either $\epsilon_1 = \epsilon_n$, or $s_n h$ does not belong to A provided $\epsilon_n = -1$, or $s_n h$ does not belong to B provided $\epsilon_n = 1$.

2.3. Algorithm 0 for computing reduced forms. This algorithm takes as an input a word of the form

$$g = w_0 t^{\epsilon_1} w_1 \cdots t^{\epsilon_n} w_n.$$

If the word contains no pinches then it is reduced. Otherwise find the first pinch We look at the first subword of the form $t^{\epsilon_i} w_i t^{\epsilon_{i+1}}$ and transform the subword according to one of the rules

- If $w_i \in A$ and $\epsilon_i = -1$ then replace $t^{-1} w_i t$ by $\phi(w_i)$
- If $w_i \in B$ and $\epsilon_i = 1$ then replace $t w_i t^{-1}$ by $\phi^{-1}(w_i)$.

After that we multiply the elements $w_{i-1}\phi(w_i)w_{i+1}$ (or, correspondingly, $w_{i-1}\phi^{-1}(w_i)w_{i+1}$), thus decreasing the length $l(g)$ of the word by 2.

Therefore we can formulate the following result (similar to the one for amalgamated products [**1**]).

Proposition 2.2. *Let $G = \langle H, t \mid t^{-1}At = B\rangle$ be an HNN-extension of a group H with associated subgroups A and B. If the Membership Subgroup Problem is decidable for subgroups A and B in H then Algorithm 0 finds the reduced form for every given $g \in G$.*

2.4. Algorithm I for computing normal forms. Assume now that the Coset Representative Search Problem (**CRSP**), as defined in [**1**], is decidable for the subgroups A and B in H, i.e., there exist recursive sets S and T of representatives of A and B in H and two algorithms which for a given word $w \in F(X)$ find, correspondingly, a representative for Aw in S and for Bw in T.

Now we describe the standard Algorithm I for computing normal forms of elements in G.

Algorithm I can be viewed as a sequence of applications of rewriting rules of the type

- $t^{-1}h \to \phi(c)t^{-1}s$, where $h = cs$, $c \in A$, $s \in S_A$;
- $th \to \phi^{-1}(c)ts$, where $h = cs$, $c \in B$, $s \in S_B$;
- $t^{\epsilon}t^{-\epsilon} \to 1$

to a given element $g \in G$ presented as a word in the standard generators of G. Since the problem **CRSP** is decidable for A and B in H the rewriting rules above are effective (i.e., given the left side of the rule one can effectively find the right side of the rule). The rewriting process is organized “from the right to the left”, i.e, the algorithm always rewrites *the rightmost occurrence* of the left side of a rule above.

It is not hard to see that the Algorithm I halts on every input $g \in G$ in finitely many steps and provides with a normal form of g.

We summarize the discussion above in the following well-known theorem.

Theorem 2.3. *Let $G = \big\langle H, t \mid t^{-1}At = B \big\rangle$ be an HNN-extension of a group H with associate subgroups A and B. If the Coset Representative Search Problem* **CRSP** *is decidable for subgroups A and B in H (with respect to fixed transversals S_A and S_B) then Algorithm I finds the normal form for every given $g \in G$.*

2.5. Algorithm II for computing cyclically reduced normal forms. Now we want to briefly outline an algorithm which, given an element $g \in G$ in reduced form, computes its cyclically reduced normal form. We work under the assumption that the Coset Representative Search Problem (**CRSP**) and the Conjugacy Membership Search Problem (**CMSP**) are decidable for subgroups A and B in H.

ALGORITHM II: COMPUTING CYCLICALLY REDUCED NORMAL FORMS.

INPUT: a word in the reduced form

$$g = h_0 t^{\epsilon_1} h_1 \cdots h_{k-1} t^{\epsilon_k} h_k,$$

STEP 0 Find the normal form of g using Algorithm I:

$$g = hp_1 \cdots p_k$$

STEP 1

- If $l(g) = 0$ then $g \in H$.
  - If $g \in C$, where $C = A \cup B$, or if g is not conjugate to an element in C, then g is already in cyclically reduced form.
  - If $g^x \in C$ for some $x \in H$ then use a decision algorithm for **CMSP** to find a particular such x and replace g by g^x.
- If $l(g) = 1$, then g is already in cyclically reduced form.
- If $l(g) \geqslant 2$ and $\epsilon_1 = \epsilon_k$ then g is already in cyclically reduced form.

STEP 2

IF $l(g) \geqslant 2$ and $\epsilon_1 = -\epsilon_k$ and $s_k h \notin A$ (when $\epsilon_k = -1$) or $t_k h \notin B$ (when $\epsilon_k = 1$) then g is in cyclically reduced form.

OTHERWISE, if $s_k h \in A$ then set

$$g^* = t^{-\epsilon_1} h^{-1} g h t^{\epsilon_1};$$

obviously, we have $l(g^*) = l(g) - 2$, and we can apply the algorithm to g^*.

The case $t_k h \in B$ is treated similarly.

3. Transfer machine for free constructions

In this section we study a construction (*transfer machine*) which provides embeddings of HNN-extensions into free products with amalgamation. We show that the transfer machine allows one to reduce algorithmic problems in one class of groups into the other one with linear time overhead.

3.1. Transferring HNN-extensions into amalgamated free profucts. In this section we describe a machine that transfers HNN-extensions into amalgamated free products. This yields an algorithm for computing normal forms in HNN-extensions via computing the corresponding normal forms in suitable free products with amalgamation with linear overhead for the time complexity. Thus one can obtain time complexity estimates for HNN-extensions via the estimates for corresponding free products with amalgamation. [**1**].

3.1.1. *Direct transfer.* Let $G = \left\langle H, t \mid t^{-1}At = B, \phi \right\rangle$ be an HNN-extension. Set

$$P = H * \langle x \rangle\,, \quad \bar{P} = \bar{H} * \langle \bar{x} \rangle$$

and

$$C = H * x^{-1}Ax, \quad \bar{C} = \bar{H} * \bar{x}^{-1}\bar{B}\bar{x}$$

where $\bar{H}$ is an isomorphic copy of H via an isomorphism ϕ_H, $\bar{B}$ is the corresponding copy of B in $\bar{H}$ (via the restriction of ϕ_H onto B). Then C is isomorphic to $\bar{C}$ via an isomorphism $\phi^* : C \to \bar{C}$ such that $\phi^* = \phi_H$ on H and $\phi^*(x^{-1}ax) = \bar{x}^{-1}a^{\phi}\bar{x}$ for $a \in A$.

Then G can be canonically embedded in the amalgamated free product

$$G^* = \langle P, \bar{P} \mid C = \bar{C}, \phi^* \rangle = P *_{C=\bar{C}} \bar{P}.$$

Setting $t = x\bar{x}^{-1}$ and discarding $\bar{x}$ and $\bar{H}$ yields isomorphism

$$G^* = G * \langle x \rangle$$

and the canonical embedding

$$\lambda : G \to G^*.$$

We refer to the triple (G, G^*, λ) as to *HNN-machine*.

As we noticed earlier every HNN-extension has a unique normal form for its elements once transversal for C in A and C in B have been chosen. In a similar way, an amalgamated free product has a unique representation for its elements once transversals for P and Q have been chosen. What we shall do here is to describe the relationship between these transversals and normal forms for G and G^*.

Let U be a transversal for C in G^*. We call U *minimal* if every element of U is of minimal length in its coset class, relative to the normal form for G^* regarded as the free product $G * \langle x \rangle$.

Lemma 3.1.

(i) *Let S be a transversal for A in H. Let S^* be the subset of $P = H * \langle x \rangle$ consisting of the identity element together with all elements of the form*

$$\left\{ x^{i_1} h_1 x^{i_2} \cdots h_{m-1} x^{i_m} \mid i_j \in \mathbb{Z},\, j = 1, \ldots, m,\, i_1 \neq 0 \right\}$$

where either $i_1 \neq -1$ or $i_1 = -1$ and $h_1 \in S$. Then S^* is a minimal transversal for C in P.

(ii) *The dual statement for B and H.*

Proof. The proof is straightforward. □

Lemma 3.2.

(i) *Let U be a minimal transversal for C in P. Define $S(U)$ to be the set of all $h_1 \in H$ such that there exists $u \in U$ with*

$$u = x^{-1} h_1 x^{i_2} h_2 \cdots x^{i_m} h_m,$$

together with the trivial element. Then $S(U)$ is a transversal for A in H.

(ii) *The dual statement for $\bar{C}$ and $\bar{P}$.*

Our aim is to prove a result which can be informally formulated as follows.

Theorem 3.3 (Joint work with D. J. Collins). *Let $G = \left\langle H, t \mid t^{-1}At = B \right\rangle$ be an HNN-extension and S, T transversals of A, B in H. Then for any $g \in G$ given in the normal form relative to S, T one can find in linear time the normal form of the element $\lambda(g)$ in the free amalgamated product*

$$G^* = P *_{C=\bar{C}} \bar{P}$$

with respect to the transversals S^*, T^*.

A more formal statement of this result is contained in Theorems 3.6 and 3.7 below.

3.1.2. *Rewriting procedures.* Let G^* be given in the form

$$G * \langle x \rangle = \left\langle H, t \mid t^{-1}At = B,\, \phi \right\rangle * \langle x \rangle$$

and let us assume that we have fixed transversals S and T for A and B in H so that we have normal forms for elements of G. Then a normal form for an element of G^* is any expression of the form $g_0 x^{i_1} g_1 \cdots x^{i_m} g_m$ where $g_0, g_1, \ldots, g_m$ are normal forms for elements of G and $i_1, \ldots, i_m$ are non-zero integers—possibly $m = 0$, while if $m \geqslant 1$ then g_0 and g_m may be trivial.

Our objective is to transform such an expression into a normal form expression with respect to G^* viewed as the amalgamated product $P *_C \bar{P}$. The procedure has two steps: the first consists of replacing each occurrence of t lying in a normal form for an element of G and then freely reducing the result. At this point we have an expression in the generators of $P *_C \bar{P}$ and we can clearly bracket this so that we alternately have either a word in the generators of P or a word in the generators of $\bar{P}$ (the latter will actually just be a power of y). The second step is to transform this expression into a normal form corresponding to the transversals S^* and T^* that we have defined above. The time complexity of this latter process therefore depends on the time complexity of the process for writing arbitrary elements of P and $\bar{P}$ in the form cs^* or ct^* where $c \in C$, $s^* \in S^*$, $t^* \in T^*$, plus the time complexity of transforming an element of C expressed as an element of P, into an element of $B\bar{P}$ and vice versa. The actual procedure for writing an element p of P in the form cs^* is as follows.

INPUT: an expression of the form $h_0 x^{i_1} \cdots x^{i_r} h_r$ where $h_0, h_1, \ldots, h_r$ are representatives of elements of H.

STEP 1. Write $c_0 = h_0$. If $r = 0$, stop with $a = c = c_0$. Otherwise proceed to Step 1.

STEP 2. Check if $i_1 = -1$. If not, write $c = c_0$,

$$s^* = x^{i_1} h_1 \cdots x^{i_r} h_r$$

and stop with $a = cs^*$.

STEP 3. We have $a = c_0 x^{-1} h_1 \cdots x^{i_r} h_r$. Write $h_1 = as$ where $a \in A$ and $s \in S$. If $s \neq 1$, write $c_1 = x^{-1}ax$, $c = c_0 c_1$ and $s^* = x^{-1} s x^{i_2} h_2 \cdots x^{i_r} h_r$ and stop with $a = cs^*$. Otherwise go to Step 4.

STEP 4. We have $p = h_0 x^{-1} a x^{i_2} \cdots x^{i_r} h_r$. If $i_2 \neq 1$, write $c_1 = x^{-1} p x$, $c = c_0 c_1$ and $s^* = x^{i_2 - 1} h_2 \cdots x^{i_r} h_r$ and stop with $p = cs^*$. Otherwise store c_0, c_1 and iterate the above steps, starting with Step 1 applied to $h_2 \cdots x^{i_r} h_r$ until either a halt is reached.

The procedure is similar for an arbitrary element of $\bar{P}$, although here of course we are working with the copy $\bar{H}$ of H rather than H itself and so we use the transversal $\bar{T}$ rather than T. This easily leads to the following lemma.

LEMMA 3.4.

(i) *The time complexity of expressing an element of P in the form cs^* where $c \in C$ and $s^* \in S^*$ is linear in the time complexity of expressing an element of H in the form as where $a \in A$ and $s \in S$.*

(ii) *The dual statement for A and B.*

It remains therefore to deal with the issue of switching between representations of elements of C according to as we wish to view them as lying in P or lying in $\bar{P}$.

The actual procedure is very simple to describe. Typically an element of C has the form $h_0x^{-1}a_1xh_1\cdots x^{-1}a_rxh_r$ when viewed as an element of P. The transformation required is the replacement of each a_j by $\bar{b}_j$ where b_j is the image of pa_j under the isomorphism $\phi : A \to B$. The reverse process, moving from $\bar{P}$ to P, consists of replacing $\bar{b}_j$ by the corresponding pa_j and removing the bar from each $\bar{h}_j$. This can be summarized as the following lemma.

Lemma 3.5.

(i) *The complexity of expressing an element of C, viewed as an element of P, into an element of $\bar{P}$, is linear in the time complexity of mapping an element $a \in A$ into its image $b = \phi(a) \in B$ in H.*

(ii) *The dual statement for $\bar{P}$.*

Now we are in position to give a more precise form of Theorem 3.3.

Theorem 3.6. *The time complexity of obtaining a normal form for an element of G^* viewed as the amalgamated free product with respect to transversals S^* and T^* for C in P and $\bar{P}$ that come from transversal S and T for A and B in H is linear in terms of the time complexity of obtaining a normal form of an element in G^* viewed as the free product of G and $\langle x\rangle$ where normal forms in G are computed using S and T.*

See [**1**] for a detailed discussion of corresponding algorithms for amalgamated products.

3.1.3. *Inverse transfer.* We now have to return to the rewriting procedure in the opposite direction. Here our starting point is G^* viewed as $P *_C \bar{P}$ so that when we are give an element in normal form we are given an expression $cu_1u_2\cdots u_m$ where $c \in C$ and $u_1, u_2, \ldots, u_m$ lie alternatively in transversals for C in P and for C in $\bar{P}$. To rewrite such an expression in a normal form for G^* viewed as $G^* = G * \langle x\rangle$, we proceed by replacing occurrences of y by $t^{-1}x$ and also by deleting bars over any terms from $\bar{H}$. Upon freely reducing the result we then have an expression $g_0x^{i_1}g_1\cdots x^{i_r}g_r$ where $g_0, g_1, \ldots, g_r$ are normal forms for elements of G and $x^{i_1}, \ldots, x^{i_r}$ are non-trivial powers of x. The latter are easy to recognise but to obtain the former we need to have transversals for A and B in H. For our aims it is naturally to assume that the transversals U and V are both minimal relative to the free product decompositions of P and $\bar{P}$. Then by Lemma 3.2 we have a canonical way to obtain transversals S and T for A and B in H. This amounts, of course, to assuming that U and V have been constructed from S and T in the first place.

The procedure is now quite straightforward. We have an expression

$$g_0x^{i_1}g_1\cdots x^{i_r}g_r.$$

Using the transversals S and T, we can put each element of G into normal form and we are finished, except possibly when some g_i turns out to represent the identity. Then we have to consolidate the adjacent powers of x and repeat the procedure. From an algorithmic standpoint, in practice one would work systematically from g_0 towards g_r.

Theorem 3.7. *The time complexity of obtaining a normal form for an element of G^* viewed as the free product of G and $\langle x\rangle$ is linear in terms of the time complexity of obtaining a normal form for an element of G^* viewed as the amalgamated* *free product provided the transversals U and V for C in P and $\bar{P}$ are minimal and hence yield, in linear time, the transversals S and T that are used to compute the normal forms in G that are used in the normal forms for G^*.*

Theorems 3.3, 3.6 and 3.7 allow us to transfer from amalgamated products to HNN-extensions the results about the complexity of conversion of reduced forms to normal reduced forms [**1**].

In particular, if A, B, C be finitely generated free groups and let C be an infinite index subgroup in both A and B. Then, under modest assumptions about C, the regular part of Algorithm 0 has a linear time complexity on a generic subset [**2**].

Theorem 3.8. *Algorithm 0 and Algorithm 1 have exponential worst case time complexity in respect to the length of input words.*

This theorem is an immediate corollary of [**1**, Theorem 3.7].

Notice that if A, B, C be finitely generated free groups and let C be an infinite index subgroup in both A and B. Then, under modest assumptions about C, the regular part of Algorithm 0 has a linear time complexity on a generic subset; see [**2**] for details.

3.2. Conjugacy criterion. In this section we formulate, in a slightly modified form, the well known conjugacy criterion for HNN-extensions, due to Collins [**4**].

Recall that the *i-cyclical permutation* of a cyclically reduced element $g = h_0 t^{\epsilon_1} \cdots h_{r-1} t^{\epsilon_r}$ is the element

$$g_i = h_i t^{\epsilon_{i+1}} \cdots t^{\epsilon_r} h_0 t^{\epsilon_1} \cdots h_{i-1} t^{\epsilon_i},$$

rewritten in normal form.

Theorem 3.9. *Let $G = \langle H, t \mid t^{-1}At = B \rangle$ be an HNN-extension of the base group H with associated subgroups A and B. Let*

$$g = h_0 t^{\epsilon_1} \cdots h_{r-1} t^{\epsilon_r}, \quad g' = h'_0 t^{\eta_1} \cdots h'_{s-1} t^{\eta_s}$$

be conjugate cyclically reduced elements of G. Then one of the following is true:

- *Both g and g' lie in the base group H. If $g \notin A \cup B$ then $g' \notin A \cup B$ and g and g' are conjugate in H.*
- *If $g \in A \cup B$ then $g' \in A \cup B$ and there exists a finite sequence of elements $c_1, \ldots, c_l \in A \cup B$, such that $c_0 = g$, $c_l = g'$ and c_i is conjugated to c_{i+1} by an element of the form ht^ϵ, $h \in H$, $\epsilon = \pm 1$.*
- *Neither of g, g' lies in the base group H, in which case $r = s$ and g' can be obtained from g by i-cyclically permuting it $(i = 1, \ldots, r)$ and then conjugating it by an element z from A, if $\epsilon_i = -1$, or from B, if $\epsilon_i = +1$.*

4. Conjugacy search problem for regular elements

In this section we introduce and study *regular* elements. Let $C = A \cup B$ and

$$N^*_G(C) = \{g \mid C^g \cap C \neq 1\}$$

be the generalised normaliser of the set C. We say that $(c, g) \in C \times G$ is a *bad* pair if $c \neq 1$, $g \notin C$, and $gcg^{-1} \in C$.

Notice that if (c, g) is a bad pair then $g \in N^*_G(C) \setminus C$ and $c \in Z_g(C)$, where

$$Z_g(C) = \{\, c \in C \mid c^g \in C \,\} = C^{g^{-1}} \cap C.$$

The following lemma gives a more detailed description of bad pairs.

Lemma 4.1. *Let $c \in C \smallsetminus \{1\}$, $g \in G \smallsetminus C$, and $g = hp_1 \cdots p_k$ is the normal form of g. Then (c, g) is a bad pair if and only if he following system of equations has solutions $c_1, \ldots, c_{k+1} \in C$.*

$$\begin{array}{rcl} p_k c p_k^{-1} & = & c_1 \\ p_{k-1} c_1 p_{k-1}^{-1} & = & c_2 \\ & \vdots & \\ p_1 c_{k-1} p_1^{-1} & = & c_k \\ h c_k h^{-1} & = & c_{k+1} \end{array}$$

Moreover, in this case $p_i, h \in N_G^*(C)$.

Proof. This lemma is a special case of Lemma 4.3 below. □

We denote the system of equations in Lemma 4.1 by $B_{c,g}$. Observe that the consistency of the system $B_{c,g}$ does not depend on the particular choice of representatives of A and B in H. Sometimes we shall treat c as a variable, in which case the system will be denoted B_g.

4.1. Black hole. The set

$$BH = N_G^*(C)$$

will be called a *black hole*. Elements from BH are called *singular*, and elements from $R = G \smallsetminus BH$ *regular*. The following description of the black hole is an immediate corollary of Lemma 4.1.

Corollary 4.2. *Let $G = \langle H, t \mid t^{-1}At = B \rangle$. Then an element $g \in G \smallsetminus C$ is singular if and only if the system B_g has a nontrivial solution $c, c_1, \ldots, c_{k+1} \in C$.*

Now we want to study slightly more general equations of the type $gc = c'g'$ and their solutions $c, c' \in C$.

Lemma 4.3. *Let $G = \langle H, t \mid t^{-1}At = B \rangle$. Let $g, g' \in G$ be elements given by their canonical forms*

$$g = hp_1 \cdots p_k, \qquad g' = h'p'_1 \cdots p'_k \tag{4}$$

Then the equation $gc = c'g'$ has a solution $c, c' \in C$ if and only if the following system $S_{g,g'}$ of equations in variables $c, c_1, \ldots, c_k$ has a solution in G.

$$\begin{array}{rcl} p_k c & = & c_1 p'_k \\ p_{k-1} c_1 & = & c_2 p'_{k-1} \\ & \vdots & \\ p_1 c_{k-1} & = & c_k p'_1 \\ h c_k & = & c' h' \end{array}$$

The proof of Lemma 4.3 is a word-by-word reproduction of the proof of Lemma 4.5 in [**1**].

The first k equations of the system $S_{g,g'}$ form what we call the *principal system of equations*, we denote it by $PS_{g,g'}$. In what follows we consider $PS_{g,g'}$ as a system in variables $c, c_1, \ldots, c_k, c'$ which take values in C, the elements $p_1, \ldots, p_k, p'_1, \ldots, p'_k$ are constants.

Let M be a subset of a group G. If $u, v \in G$, we call the set uMv a *G-shift* of M. For a collection $\mathcal{M}$ of subsets in G, we denote by $\Phi(\mathcal{M}, G)$ the least set of subsets of G which contains $\mathcal{M}$ and is closed under G-shifts and intersections.

LEMMA 4.4. *Let G be a group and $C = A \cup B$ be the union of two subgroups A and B of G. If $D \in \Phi(C, G)$ and $D \neq \emptyset$ then D is the union of finitely many sets of the form*

$$D = (A^{g_1} \cap \cdots \cap A^{g_m} \cap B^{g'_1} \cap \cdots \cap B^{g'_n})h$$

for some elements $g_1, \ldots, g_m, g'_1, \ldots, g'_n, h \in G$.

The proof of this lemma repeats the proof of Lemma 4.7 of [**1**].

LEMMA 4.5. *Let $G = \langle H, t \mid t^{-1}At = B \rangle$. Then for any two elements g and g' with canonical forms*

$$g = hp_1 \cdots p_k, \quad g' = h'p'_1 \cdots p'_k \quad (k \geqslant 1)$$

the set $E_{g,g'}$ of all elements c from C for which the system $PS(g, g')$ has a solution $c, c_1, \ldots, c_k$, is equal to

$$E_{g,g'} = C \cap p_k^{-1} C p'_k \cap \cdots \cap p_k^{-1} \cdots p_1^{-1} C p'_1 \cdots p'_k.$$

In particular, if $E_{g,g'} \neq \emptyset$ then it is the union of at most 2^{k+1} cosets with respect to subgroups in A and B of the form described in the previous lemma.

The proof of this lemma is essentially the same as that of Lemma 4.8 in [**1**].

Denote by $Sub(C)$ the set of all subgroups of C. By Lemma 4.4, non-empty sets from $\Phi(Sub(C), H)$ are finite unions of cosets of subgroups from H.

COROLLARY 4.6. *Let $G = \langle H, t \mid t^{-1}At = B \rangle$. If the Cardinality Search Problem is decidable in $\Phi(Sub(C), H)$, then, given g, g' as above, one can effectively find the set $E_{g,g'}$. In particular, one can effectively check whether $E_{g,g'}$ is empty, singleton, or infinite.*

The proof repeats the proof of Corollary 4.9 in [**1**].

LEMMA 4.7. *Let $G = \langle H, t \mid t^{-1}At = B \rangle$ and $g, g' \in G$. If $l(g) = l(g') \geqslant 1$ and the system $PS(g, g')$ has more than one solution in C then the elements g, g' are singular.*

The proof repeats the proof of Lemma 4.10 in [**1**].

LEMMA 4.8. *Let $G = \langle H, t \mid t^{-1}At = B \rangle$ be an HNN-extension of a finitely presented group H with finitely generated associated subgroups A and B. Set $C = A \cup B$. Assume also that H allows algorithms for solving the following problems:*

- *The Coset Representative Search Problem for subgroups A and B in H.*
- *Cardinality Search Problem for $\Phi(Sub(C), H)$ in H.*
- *Malnormality problem for C in H.*

Then there exists an algorithm for deciding whether a given element in G is regular or not.

PROOF. The proof repeats the proof of Lemma 4.11 from [**1**]. □

Corollary 4.9. *Let $G = \langle H, t \mid t^{-1}At = B \rangle$ be an HNN-extension of a free group H with finitely generated associated subgroups A and B. Then the set of regular elements in G is recursive.*

Denote by CR the set of all elements in G which have at least one regular cyclically reduced canonical form, that is, CR is the set of elements in G which are conjugates of cyclically reduced regular elements. The set CR plays an important part in our analysis of the conjugacy search problem in G.

Lemma 4.10. *Let $G = \langle H, t \mid t^{-1}At = B \rangle$. Set $C = A \cup B$. Assume also that H allows algorithms for solving the following problems*

- *The Coset Representative Search Problem for subgroups A and B in H.*
- *The Cardinality Search Problem for $\Phi(Sub(C), H)$ in H.*
- *The Malnormality Problem for C in H.*

Then there exists an algorithm to determine whether a given element in G is in CR or not.

Proof. Proof follows from Lemma 4.8 and Algorithm II from Section 2.5 of this paper. □

4.2. Conjugacy search problem and regular elements. The aim of this section is to study the Conjugacy Search Problem for regular elements in HNN-extensions. We show that the conjugacy search problem for regular elements is solvable under some very natural restrictions on the group H. We start with the following particular case of the Conjugacy Search Problem.

The Conjugacy Search Problem for a fixed element g: this the Conjugacy Search Problem for the set of pairs

$$\Phi_g = \{(g, u) \mid u \in G\}.$$

Theorem 4.11. *Let $G = \langle H, t \mid t^{-1}At = B \rangle$ be an HNN-extension of finitely presented group H with associated finitely generated subgroups A and B. Assume also that H allows algorithms for solving the following problems:*

- *The Coset Representative Search Problem for subgroups A and B in H.*
- *The Cardinality Search Problem for $\Phi(Sub(C), H)$ in H.*

Then the Conjugacy Search Problem in G is decidable for cyclically reduced regular elements g of length $l(g) \geqslant 1$.

The proof of this theorem follows the proof of Theorem 4.15 from [**1**], if we replace the conjugacy criterion for amalgamated products by the conjugacy criterion for HNN-extensions.

Now we study the Conjugacy Search Problem for regular elements of length 0.

Lemma 4.12. *Let $G = \langle H, t \mid t^{-1}At = B \rangle$ and g be a cyclically reduced regular element of G with $l(g) = 0$. If the Coset Representative Search Problem for subgroups A and B in H and the Conjugacy Search Problem for C in H are decidable then the Conjugacy Search Problem for g in G is decidable.*

The proof follows from the conjugacy criterion.

We are ready to formulate a general conjugacy search problem for regular elements.

The Conjugacy Search Problem for CR is the Conjugacy Search Problem for the set of pairs

$$\Phi_{CR} = \{(g, u) \mid g \in CR, u \in G\}.$$

THEOREM 4.13. *Let* $G = \langle H, t \mid t^{-1}At = B \rangle$ *be an HNN-extension of a finitely presented group* H *with associated finitely generated subgroups* A *and* B. *Assume also that* H *allows algorithms for solving the following problems:*

- *The Coset Representative Search Problem for subgroups* A *and* B *in* H.
- *The Cardinality Search Problem for* $\Phi(Sub(C), H)$ *in* H.
- *The Conjugacy Search Problem in* H.
- *The Conjugacy Membership Search Problems for* A *and* B *in* H

Then the Conjugacy Search Problem in G *is decidable for elements from* CR.

COROLLARY 4.14. *Let* $G = \langle H, t \mid t^{-1}At = B \rangle$ *be an HNN-extension of a free* H *with associated finitely generated subgroups* A *and* B.

Then the Conjugacy Search Problem in G *is decidable for elements from* CR.

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