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Scattering of sound waves by an infinite grating composed of rigid plates

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Abstract

A plane sound wave is incident at an angle θ upon an infinite array of rigid plates, equally spaced and lying along the y-axis, where (x, y) are two-dimensional Cartesian coordinates. The boundary value problem is formulated into a matrix Wiener–Hopf equation whose kernel is, when the plates and interstices are of equal length, decomposable into two factors which commute and have algebraic behaviour at infinity. A closed form analytical solution is then obtained following the usual Wiener–Hopf procedure and numerical results are given for various angles of incidence, as well as different spacings.

Keywords: Wiener–Hopf technique; Matrix Wiener–Hopf equations; Diffraction grating; Acoustics

1. Introductory remarks and background

There are numerous interesting physical problems in the fields of acoustics, electromagnetism, elasticity, etc., which, when modelled mathematically, are exactly solvable by the Wiener–Hopf technique [1,6,12,25]. Since its invention in 1931 [24], the method has been used to tackle problems which have semi-infinite or infinite geometries. For simple geometries the Wiener–Hopf technique leads to a scalar equation and, apart from computational difficulties, this equation always has an exact solution [18]. However, for complex boundary value problems, the procedure often leads to a matrix Wiener–Hopf equation. The crucial step in the solution of such an equation is to decompose the so-called Wiener–Hopf kernel into a product of two factors with certain analyticity properties. Although it is possible to decompose scalar kernels with the help of Cauchy-type integrals (see [18, pp. 11–16]), no procedure has yet been devised to exactly factorise general matrix kernels. However, there is a special class of problems which, although reducable to *matrix* Wiener–Hopf equation form, have kernels which nevertheless allow a commutative decomposition. Khrapkov [14] was the first to offer an elegant explicit factorization scheme for such 2 by 2 Wiener–Hopf matrices. A generalization to com-

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mutative factorization of N by N matrices ($N > 2$) was offered by Jones [13] and very recently, for matrices of relevance to diffraction theory, by Veitch and Abrahams [22].

This paper examines scattering of plane sound waves by an infinite grating composed of equally spaced rigid barriers, of length $2\bar{a}$ separated by gaps of length $2\bar{b}$. The model is a classical problem in diffraction theory; it was first investigated by Baldwin and Heins [9] who considered only normal incidence of electromagnetic waves (with the electric vector polarized parallel to the edges of the strips) and interstice length equal to strip width (i.e., $\bar{b} = \bar{a}$). Weinstein [23], in an independent work, permitted the discrete incident wave angles $\pi n/(2\bar{a}k)$ where k is the incident wave angle and n is an integer. These special incident angles allowed both sets of authors to obtain solutions via scalar Wiener–Hopf equations. In 1971, Lüneburg and Westpfahl [16] examined the model for arbitrary incidence angle and equal strip and gap lengths; their interesting approach was function theoretic, reducing the problem to a singular integral of the first kind on a complicated contour. The latter, perhaps rather surprisingly, was found to be exactly solvable after further conversion to two uncoupled Riemann Hilbert Problems. Lüneburg and Westpfahl described their work as an '*extension of Sommerfeld's heuristic method for the half-plane*' and it does indeed seem rather complex and detailed, lacking the straightforward approach of the Wiener–Hopf technique. The paper was contemperaneous with that by Khrapkov [14], and so clearly the authors were unaware of its potential application to the present problem.[1] Some years later, in an interesting article, Achenbach and Li [7] reduced the problem to a singular integral equation and then used Chebyshev polynomials to obtain a set of algebraic equations, from which they obtained the solution. Scarpetta and Sumbatyan [21] also reduced the problem to an integral equation of the first kind, and then approximated the kernel to derive explicit results for the reflection and transmission coefficients. Other authors who have investigated the planar grating problem are Dalyrmple and Martin [10] who again considered only normal incidence and Porter and Evans [19] who allowed arbitrary oblique incidence angles and unequal spacing. Both sets of authors employed eigenfunction expansion techniques to solve the problem.

The present article seeks to tackle the model problem described above by application of Fourier transforms and then by reduction to a system of four Wiener–Hopf equations. The resultant matrix kernel, $\mathbf{K}(\alpha)$ say, belongs to a meromorphic class possessing the special property $\mathbf{K}(\alpha) = \mathbf{Q}(\alpha)/\Delta(\alpha)$ with $\mathbf{Q}^2(\alpha) = \Delta^2(\alpha)\mathbf{I}$, where $\mathbf{I}$ is the identity matrix, $\mathbf{Q}(\alpha)$ has entire elements and $\Delta(\alpha)$ is its determinant. The authors have found a number of other physical problems which reduce to the same, or similar, Wiener–Hopf form, and so a factorization scheme for matrices of this type is likely to be of broad use. Here we report on an exact factorization in the case when the strips and gaps are of identical length. The case of dissimilar a and b can be found in the thesis by the first author [11].

The paper is organised as follows. In Section 2 we shall pose the boundary value problem of diffraction of sound waves obliquely incident on an infinite plane grating composed of rigid plates with spacing dissimilar to the interstices. Then, in Section 3 the problem will be reduced to a *matrix* Wiener–Hopf equation with its kernel, $\mathbf{K}(\alpha)$ say, having the above mentioned property (in the general case $a \neq b$). The factorisation of the matrix kernel is discussed in the following Section 4, for the case of a grating composed of equal length gaps and plates. The approach used here makes use of the Khrapkov-type matrices (see [14,3]) and decomposes the kernel into two factors which are commutative and have appropriate algebraic behaviour in respective domains of the complex-plane. After the exact decomposition is achieved, an *analytical* solution of the full boundary value problem is determined in Section 5. In the final section we present numerical evaluations of the reflection and transmission coefficients for various angles of incidence and varying gap spacing, and compare these with data obtained by Porter and Evans [19].

2. The boundary value problem

It is required to deduce the two-dimensional scattering of plane sound waves by a diffraction grating composed of an infinite array of rigid strips. Introducing Cartesian coordinates $(\bar{x}, \bar{y})$ (overbar here and henceforth denoting dimensional quantities), the strips of length $2\bar{a}$ are located on $\bar{x} = 0$ between $-\bar{a} + 2n(\bar{a} + \bar{b}) \leqslant$

[1] It may be interesting for the reader to compare Lüneburg and Westpfahl's approach with the *ad hoc* approach by Rawlins [20] on the generalized Sommerfeld half-plane problem (i.e., different conditions on upper and lower faces), which can also be tackled by Khrapkov's factorization method [2].

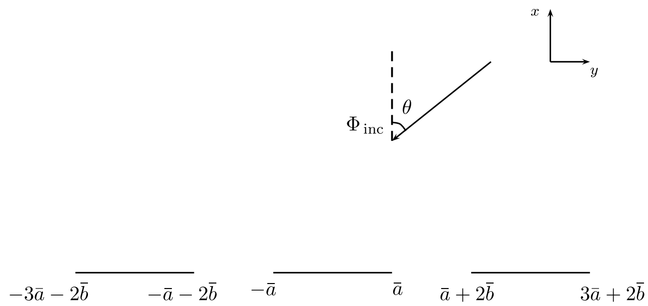

Fig. 1. A wave is incident at angle θ. The diffraction grating composed of finite length rigid plates of length $2\bar{a}$ lying on $y = 0$.

$\bar{y} \leqslant \bar{a} + 2n(\bar{a}+\bar{b})$, $n = -\infty, \ldots, 0, 1, \ldots, \infty$. Then, the sound can transmit through the gaps between the plates, $\bar{x} = 0$, $\bar{a} + 2n(\bar{a}+\bar{b}) \leqslant \bar{y} \leqslant -\bar{a} + 2(n+1)(\bar{a}+\bar{b})$ (see Fig. 1). For a compressible, inviscid fluid the acoustic disturbances can be represented by a velocity potential $\Phi(\bar{x}, \bar{y}; t)$, where t is time; that is, the velocity field is $\nabla\Phi(\bar{x}, \bar{y}; t)$ and pressure fluctuations $p(\bar{x}, \bar{y}; t) = -\rho_0 \frac{\partial\Phi}{\partial t}(\bar{x}, \bar{y}; t)$ where ρ_0 is the mean fluid density. The potential $\Phi(\bar{x}, \bar{y}; t)$ satisfies the two-dimensional wave equation

$$\nabla^2\Phi = \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}, \tag{1}$$

in which c is the speed of propagation of waves in the fluid, and for simplicity the forcing is chosen as a plane time-harmonic wave incident from $x = +\infty$ inclined at an angle θ to the vertical:

$$\Phi^{\rm inc}(\bar{x}, \bar{y}; t) = {\rm Re}\{A{\rm e}^{-{\rm i}k\bar{x}\cos\theta - {\rm i}k\bar{y}\sin\theta}{\rm e}^{-{\rm i}\omega t}\}, \quad 0 \leqslant \theta < \pi/2. \tag{2}$$

Note that, because of the symmetry of the grating, there is no loss of generality in taking θ to be positive and to lie in the indicated range.

The scattered field, if the transient motions are neglected, may be written as the difference between the total field and the incident potential in the form

$${\rm Re}\{\Phi^{\rm s}(\bar{x}, \bar{y}){\rm e}^{-{\rm i}\omega t}\} = \Phi(\bar{x}, \bar{y}; t) - \Phi^{\rm inc}(\bar{x}, \bar{y}; t). \tag{3}$$

Thus, the scattered potential $\Phi^{\rm s}(\bar{x}, \bar{y})$ satisfies the reduced wave equation

$$\frac{\partial^2\Phi^{\rm s}}{\partial\bar{x}^2} + \frac{\partial^2\Phi^{s}}{\partial\bar{y}^2} + k^2\Phi^{\rm s} = 0, \tag{4}$$

in which $k = \omega/c$ is the wavenumber, and on the rigid screens along $\bar{x} = 0$, for which $\partial\Phi/\partial\bar{x} = 0$, the derivative of the potential $\Phi^{\rm s}$ takes the value

$$\frac{\partial\Phi^{\rm s}}{\partial\bar{x}}(0, \bar{y}) = A{\rm i}k\cos\theta{\rm e}^{-{\rm i}k\bar{y}\sin\theta}, \quad -\bar{a} + 2n(\bar{a}+\bar{b}) \leqslant \bar{y} \leqslant \bar{a} + 2n(\bar{a}+\bar{b}), \quad n \in \mathbb{N}. \tag{5}$$

Between the rigid scatterers of the grating, the pressure and velocity must be continuous. Hence we require

$$\frac{\partial\Phi^{\rm s}}{\partial\bar{x}}(0+, \bar{y}) = \frac{\partial\Phi^{\rm s}}{\partial\bar{x}}(0-, \bar{y}); \quad \Phi^{\rm s}(0+, \bar{y}) = \Phi^{\rm s}(0-, \bar{y}) \tag{6}$$

on $\bar{a} + 2n(\bar{a}+\bar{b}) \leqslant \bar{y} \leqslant -\bar{a} + 2(n+1)(\bar{a}+\bar{b})$, in which $0\pm$ means $\bar{x} = \pm\epsilon$, $\epsilon \downarrow 0$. Further physical requirements are finite pressure everywhere in the fluid (and in particular finite energy density at the plate edges) and purely outgoing behaviour as $|\bar{x}| \to \infty$ for the scattered field. With the above conditions, uniqueness of the scattered potentials, and hence of the total sound field, is assumed in the usual manner.

To solve the boundary value problem it is helpful to exploit the periodicity of the geometry. Defining non-dimensional Cartesian coordinates within each strip region $\bar{a} + 2(n-1)(\bar{a}+\bar{b}) \leqslant \bar{y} \leqslant \bar{a} + 2n(\bar{a}+\bar{b})$, as

$$x = k\bar{x}; \quad y_n = k[\bar{y} - 2(\bar{a}+\bar{b})n], \quad n = -\infty, \ldots, -1, 0, 1, \ldots, \infty, \tag{7}$$

then in the nth strip the non-dimensional scattered potential may be written as

$$\Phi_n(x, y_n) = \frac{e^{2(a+b)in\sin\theta}}{A}\Phi^s(\bar{x}, k^{-1}y_n + 2(\bar{a}+\bar{b})n), \quad -a-2b \leqslant y_n \leqslant a, \quad \forall n, \tag{8}$$

where

$$a = k\bar{a}, \quad b = k\bar{b}. \tag{9}$$

Hence $\Phi_n(x, y_n)$ satisfies the reduced wave equation

$$\frac{\partial^2\Phi_n}{\partial x^2} + \frac{\partial^2\Phi_n}{\partial y_n^2} + \Phi_n = 0, \quad -a-2b \leqslant y_n \leqslant a, \tag{10}$$

the boundary condition

$$\frac{\partial\Phi_n}{\partial x}(0, y_n) = i\cos\theta e^{-iy_n\sin\theta}, \quad -a \leqslant y_n \leqslant a, \tag{11}$$

and the continuity requirements, (6),

$$\frac{\partial\Phi_n}{\partial x}(0+, y_n) = \frac{\partial\Phi_n}{\partial x}(0-, y_n), \quad \Phi_n(0+, y_n) = \Phi_n(0-, y_n), \quad -a-2b \leqslant y_n \leqslant -a. \tag{12}$$

Further, the scattered potential (and its derivatives) must, of course, be continuous across the fictitious strip boundaries, which condition leads to a discontinuous boundary condition in Φ_n owing to the exponential factor on the right-hand side of (8). This yields, across the boundaries $y_n = \pm a$, the *quasi-periodicity* requirement

$$\Phi_{n+1}(x, -a-2b) = e^{2(a+b)i\sin\theta}\Phi_n(x, +a), \quad \forall n \tag{13}$$

and

$$\frac{\partial\Phi_{n+1}}{\partial y_n}(x, -a-2b) = e^{2(a+b)i\sin\theta}\frac{\partial\Phi_n}{\partial y_n}(x, +a), \quad \forall n. \tag{14}$$

It is clear that the model problem for $\Phi_n(x, y_n)$ is invarient under translations in y, i.e., $y_n \to y_{n+m}$, for any integers n and m, and hence we can write

$$\Phi_n(x, y_n) \equiv \Phi(x, y), \quad \forall n. \tag{15}$$

Thus, the suffices can be dropped from the Φ_n and y_n variables appearing in Eqs. (8)–(12), and (13), (14) become

$$\Phi(x, -a-2b) = e^{2(a+b)i\sin\theta}\Phi(x, +a), \tag{16}$$

$$\frac{\partial\Phi}{\partial y}(x, -a-2b) = e^{2(a+b)i\sin\theta}\frac{\partial\Phi}{\partial y}(x, +a). \tag{17}$$

A further consideration of the geometry and forcing indicates that the potential is odd in x[2], and so the condition on the gap between the plates (12) is reduced simply to

$$\Phi(0, y) = 0, \quad -a-2b \leqslant y \leqslant -a. \tag{18}$$

Hence, the boundary value problem is to be solved over a semi-infinite strip ($x \geqslant 0$, $-a-2b \leqslant y \leqslant a$) and consists of the governing equation (10), the boundary conditions (11), (18) and the quasi-periodicity, or 'wrap-around', conditions (16), (17). These, together with the requirements of finite pressure (finite Φ) at the plate edges ($x = 0$, $y = \pm a$), and outgoing waves as $x \to +\infty$, ensure that a unique solution is obtainable. In the following section the boundary value problem is reduced to a matrix Wiener–Hopf equation by means of transform methods.

[2] The odd x-behaviour of the scattered potential is easily seen by changing x to $-x$ in the boundary conditions and governing equation, noting that it remains the same except for a change in the sign of the forcing.

3. Reduction to a matrix Wiener–Hopf equation

The physical problem discussed in the previous section was expressed as a boundary value problem in a semi-infinite strip region, $x \geqslant 0$, $-a-2b \leqslant y \leqslant a$. The boundary conditions on the end face, (11) and (18), suggest the use of Fourier cosine and sine transforms in $-a \leqslant y \leqslant a$, $-a-2b \leqslant y \leqslant -a$, respectively. Defining the sine transform as

$$\psi_s(\alpha, y) = 2\mathrm{i}\int_0^\infty \Phi(x, y)\sin \alpha x \,\mathrm{d}x, \quad -2b-a \leqslant y \leqslant -a, \tag{19}$$

then the governing equation (10) reduces to

$$\frac{\mathrm{d}^2\psi_s}{\mathrm{d}y^2} + (1-\alpha^2)\psi_s = 0 \tag{20}$$

due to the vanishing of the potential on $x = 0$ in this y region (and assuming suitable convergence of the integral at infinity, which is easily verified once the solution has been obtained). Note that the Fourier sine transform, in the form defined herein, may be expressed as

$$\psi_s(\alpha, y) = \int_0^\infty \Phi(x, y)\mathrm{e}^{\mathrm{i}\alpha x}\mathrm{d}x - \int_0^\infty \Phi(x, y)\mathrm{e}^{-\mathrm{i}\alpha x}\,\mathrm{d}x. \tag{21}$$

The first integral on the right-hand side is a half-range Fourier transform, i.e., a transform with integrand zero over negative x-values, and so must have no singularities in the upper-half of the α-plane (Noble, 1988). That is, this integral function of α is analytic on and above the inverse contour path, $\mathscr{D}^+$ say, and hence is written

$$\int_0^\infty \Phi(x, y)\mathrm{e}^{\mathrm{i}\alpha x}\,\mathrm{d}x = \psi^+(\alpha, y), \tag{22}$$

where the $+$ denotes a function devoid of singularities in the upper half-plane and, as will be shown, of at worst algebraic growth as $|\alpha| \to \infty$ in this region. Similarly, the second term on the right-hand side of (21) is easily shown to be (noting the odd x behaviour of Φ)

$$\psi^-(\alpha, y) = +\int_{-\infty}^0 \Phi(+x, y)\mathrm{e}^{\mathrm{i}\alpha x}\,\mathrm{d}x = -\psi^+(-\alpha, y) \tag{23}$$

which is singularity free, of algebraic growth at worse, in the lower half of the complex α-plane, $\mathscr{D}^-$ say. Note that the inverse transform

$$\Phi(x, y) = \frac{1}{2\pi}\int_{\mathscr{C}} \{\psi^+(\alpha, y) - \psi^+(-\alpha, y)\}\mathrm{e}^{-\mathrm{i}\alpha x}\,\mathrm{d}\alpha, \quad -b-2a \leqslant y \leqslant -a, \tag{24}$$

requires that $\mathscr{C}$ runs from $\alpha = -\infty$ to $+\infty$ in a strip $\mathscr{D}$ which is defined as $\mathscr{D}^+ \cap \mathscr{D}^-$; so clearly, there must be a common strip of analyticity between $\mathscr{D}^+$ and $\mathscr{D}^-$ otherwise the path $\mathscr{C}$ is not defined. The precise choice of $\mathscr{C}$ will be specified later (see Fig. 2).

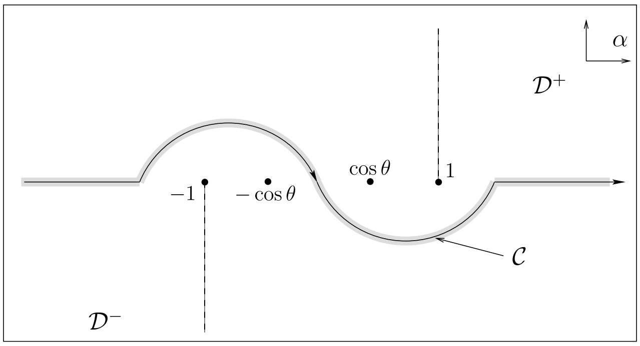

Fig. 2. The shaded area is the common strip of analyticity. Strip of analyticity for the integration path.

In the region $-a \leqslant y \leqslant a$ the boundary condition (11) suggests the application of the Fourier cosine transform

$$\psi_c(\alpha, y) = 2\int_0^\infty \Phi(x, y)\cos\alpha x \, dx = \psi^+(\alpha, y) + \psi^+(-\alpha, y) \tag{25}$$

which turns the governing equation into

$$\frac{d^2\psi_c}{dy^2} + (1-\alpha^2)\psi_c = 2i\cos\theta e^{-iy\sin\theta}, \quad -a \leqslant y \leqslant a. \tag{26}$$

Note that the inverse transform is

$$\Phi(x, y) = \frac{1}{2\pi}\int_{\mathscr{C}} \{\psi^+(\alpha, y) + \psi^+(-\alpha, y)\} e^{-i\alpha x} \, d\alpha, \quad -a \leqslant y \leqslant a, \tag{27}$$

where $\mathscr{C}$ is the same contour as that to be chosen for the integral in (24). Now (20) and (26) are trivially solved to yield

$$\psi^+(\alpha, y) - \psi^+(-\alpha, y) = A(\alpha)e^{\gamma(\alpha)y} + B(\alpha)e^{-\gamma(\alpha)y}, \quad -2b - a \leqslant y \leqslant -a, \tag{28}$$

and

$$\psi^+(\alpha, y) + \psi^+(-\alpha, y) = C(\alpha)e^{\gamma(\alpha)y} + D(\alpha)e^{-\gamma(\alpha)y} + \frac{2i\cos\theta}{(\cos^2\theta - \alpha^2)} e^{-iy\sin\theta}, \quad -a \leqslant y \leqslant a, \tag{29}$$

respectively, where $\gamma(\alpha) = (1-\alpha^2)^{1/2}$ is the ubiquitous square root function found in diffraction problems. In fact, it will be revealed that the solution will not contain branch points at $\alpha = \pm 1$, that is, only powers of $\gamma^2(\alpha)$ occur (not unsurprisingly because this is essentially a *ducted* boundary value problem) and so there is no need to specify the Riemann surface or location of the branch cuts etc. What is required is to specify that $\mathscr{C}$ passes below any singularity at $\alpha = 1$, or indeed at any point on the positive real axis (including that at $\alpha = \cos\theta$), and above singularities occurring on the negative real line (including at $\alpha = -\cos\theta$). This choice of $\mathscr{C}$ is sufficient to guarantee only outgoing or decaying waves at infinity.

Satisfaction of the governing equations and boundary conditions on the face $x = 0$ is ensured by (28), (29), and so all that now remains is to impose continuity of potential (pressure) and its normal derivative (velocity) across the line $y = -a$ plus the quasi-periodicity conditions (16), (17). Writing

$$\psi^+(\alpha, -a) = t^+(\alpha), \quad \frac{d\psi^+}{dy}(\alpha, -a) = s^+(\alpha), \tag{30}$$

gives the continuity conditions on $y = -a$:

$$t^+(\alpha) - t^+(-\alpha) = A(\alpha)e^{-\gamma(\alpha)a} + B(\alpha)e^{+\gamma(\alpha)a}, \tag{31}$$

$$t^+(\alpha) + t^+(-\alpha) = C(\alpha)e^{-\gamma(\alpha)a} + D(\alpha)e^{+\gamma(\alpha)a} + \frac{2i\cos\theta}{(\cos^2\theta - \alpha^2)} e^{ia\sin\theta}, \tag{32}$$

$$s^+(\alpha) - s^+(-\alpha) = \gamma(\alpha)A(\alpha)e^{-\gamma(\alpha)a} - \gamma(\alpha)B(\alpha)e^{+\gamma(\alpha)a}, \tag{33}$$

$$s^+(\alpha) + s^+(-\alpha) = \gamma(\alpha)C(\alpha)e^{-\gamma(\alpha)a} - \gamma(\alpha)D(\alpha)e^{+\gamma(\alpha)a} + \frac{2\sin\theta\cos\theta}{(\cos^2\theta - \alpha^2)} e^{ia\sin\theta}. \tag{34}$$

Similarly, on $y = a$ the transformed potential and its derivative are called

$$\psi^+(\alpha, a) = v^+(\alpha), \quad \frac{d\psi^+}{dy}(\alpha, a) = u^+(\alpha), \tag{35}$$

so that

$$v^+(\alpha) + v^+(-\alpha) = C(\alpha)e^{+\gamma(\alpha)a} + D(\alpha)e^{-\gamma(\alpha)a} + \frac{2i\cos\theta}{(\cos^2\theta - \alpha^2)}e^{-ia\sin\theta}, \tag{36}$$

$$u^+(\alpha) + u^+(-\alpha) = \gamma(\alpha)C(\alpha)e^{+\gamma(\alpha)a} - \gamma(\alpha)D(\alpha)e^{-\gamma(\alpha)a} + \frac{2\sin\theta\cos\theta}{(\cos^2\theta - \alpha^2)}e^{-ia\sin\theta}. \tag{37}$$

The quantities on $y = -2b - a$ are related to those on $y = a$ via the quasi-periodicity conditions (16), (17):

$$\begin{aligned}\psi^+(\alpha, -2b-a) - \psi^+(-\alpha, -2b-a) &= e^{2(a+b)i\sin\theta}\{v^+(\alpha) - v^+(-\alpha)\} \\ &= A(\alpha)e^{-\gamma(\alpha)(2b+a)} + B(\alpha)e^{+\gamma(\alpha)(2b+a)},\end{aligned} \tag{38}$$

$$\begin{aligned}\frac{d\psi^+}{dy}(\alpha, -2b-a) - \frac{d\psi^+}{dy}(-\alpha, -2b-a) &= e^{2(a+b)i\sin\theta}\{u^+(\alpha) - u^+(-\alpha)\} \\ &= \gamma(\alpha)A(\alpha)e^{-\gamma(\alpha)(2b+a)} - \gamma(\alpha)B(\alpha)e^{+\gamma(\alpha)(2b+a)}.\end{aligned} \tag{39}$$

Eqs. (31)–(34), (36)–(39) may be reduced to four by eliminating the unknown functions $A(\alpha)$, $B(\alpha)$, $C(\alpha)$, $D(\alpha)$. These remaining equations can then be arranged into a single 4×4 matrix Wiener–Hopf equation after substantial manipulation. The details are given in Appendix A, and here the most convenient form is offered:

$$\mathbf{M}(\alpha)\mathbf{t}^+(\alpha) = \mathbf{J}\mathbf{M}(\alpha)\mathbf{t}^+(-\alpha) + \mathbf{F}(\alpha), \tag{40}$$

in which the unknown column vector is

$$\mathbf{t}^+(\alpha) = (t^+(\alpha), s^+(\alpha), v^+(\alpha), u^+(\alpha))^{\mathrm{T}}, \tag{41}$$

the square matrix is

$$\mathbf{M}(\alpha) = \begin{pmatrix} \cosh\gamma(\alpha)a & \frac{1}{\gamma(\alpha)}\sinh\gamma(\alpha)a & -\cosh\gamma(\alpha)a & \frac{1}{\gamma(\alpha)}\sinh\gamma(\alpha)a \\ \gamma(\alpha)\sinh\gamma(\alpha)a & \cosh\gamma(\alpha)a & \gamma(\alpha)\sinh\gamma(\alpha)a & -\cosh\gamma(\alpha)a \\ \cosh\gamma(\alpha)b & -\frac{1}{\gamma(\alpha)}\sinh\gamma(\alpha)b & -e^{i\zeta}\cosh\gamma(\alpha)b & -e^{i\zeta}\frac{1}{\gamma(\alpha)}\sinh\gamma(\alpha)b \\ \gamma(\alpha)\sinh\gamma(\alpha)b & -\cosh\gamma(\alpha)b & e^{i\zeta}\gamma(\alpha)\sinh\gamma(\alpha)b & e^{i\zeta}\cosh\gamma(\alpha)b \end{pmatrix} \tag{42}$$

with

$$\zeta = 2(a+b)\sin\theta, \tag{43}$$

$$\mathbf{J} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{44}$$

and the forcing term is

$$\mathbf{F}(\alpha) = \frac{2i\cos\theta}{(\cos^2\theta - \alpha^2)} \times \begin{pmatrix} e^{ia\sin\theta}\left(\cosh\gamma(\alpha)a - \frac{i\sin\theta}{\gamma(\alpha)}\sinh\gamma(\alpha)a\right) - e^{-ia\sin\theta}\left(\cosh\gamma(\alpha)a + \frac{i\sin\theta}{\gamma(\alpha)}\sinh\gamma(\alpha)a\right) \\ e^{ia\sin\theta}(\gamma(\alpha)\sinh\gamma(\alpha)a - i\sin\theta\cosh\gamma(\alpha)a) + e^{-ia\sin\theta}(\gamma(\alpha)\sinh\gamma(\alpha)a + i\sin\theta\cosh\gamma(\alpha)a) \\ 0 \\ 0 \end{pmatrix}. \tag{45}$$

Note that $\mathbf{F}(\alpha)$ is entire, $\mathbf{M}(\alpha)$ contains only zeros in its elements and not branch cuts, and it is insisted that $\mathbf{t}^+(\pm\alpha)$ is analytic, and of at worst algebraic qrowth, in $\mathscr{D}^\pm$. The Wiener–Hopf kernel is

$$\mathbf{K}(\alpha) = \frac{1}{\Delta(\alpha)}\mathbf{Q}(\alpha) = \mathbf{M}^{-1}(\alpha)\mathbf{J}\mathbf{M}(\alpha), \tag{46}$$

where $\Delta(\alpha)$ is the determinant of $\mathbf{M}(\alpha)$ and takes the value

$$\Delta(\alpha) = 2e^{i\zeta}\cosh[2(a+b)\gamma(\alpha)] - e^{2i\zeta} - 1, \tag{47}$$

and the entire matrix $\mathbf{Q}(\alpha)$ is

$$\mathbf{Q}(\alpha) = 2e^{i\zeta} \times \begin{pmatrix} \frac{e^{i\zeta}-e^{-i\zeta}}{2} & -\frac{\sinh[2(a+b)\gamma]}{\gamma} & (\cosh[2b\gamma] - e^{i\zeta}\cosh[2a\gamma]) & \frac{(e^{i\zeta}\sinh[2a\gamma]+\sinh[2b\gamma])}{\gamma} \\ -\gamma\sinh[2(a+b)\gamma] & \frac{e^{i\zeta}-e^{-i\zeta}}{2} & \gamma(e^{i\zeta}\sinh[2a\gamma] + \sinh[2b\gamma]) & (\cosh[2b\gamma] - e^{i\zeta}\cosh[2a\gamma]) \\ \cosh[2b\gamma] - e^{-i\zeta}\cosh[2a\gamma] & -\frac{(e^{-i\zeta}\sinh[2a\gamma]+\sinh[2b\gamma])}{\gamma} & \frac{e^{-i\zeta}-e^{i\zeta}}{2} & \frac{\sinh[2(a+b)\gamma]}{\gamma} \\ -\gamma(e^{-i\zeta}\sinh[2a\gamma] + \sinh[2b\gamma]) & \cosh[2b\gamma] - e^{-i\zeta}\cosh[2a\gamma] & \gamma\sinh[2(a+b)\gamma] & \frac{e^{-i\zeta}-e^{i\zeta}}{2} \end{pmatrix}. \tag{48}$$

Note that $\mathbf{K}(\alpha)$ is its own inverse owing to the fact that $\mathbf{J}^2(\alpha)$ is the identity matrix, $\mathbf{I}$, and hence

$$\mathbf{Q}^2(\alpha) = \Delta^2(\alpha)\mathbf{I}. \tag{49}$$

It will prove useful later to work with Eq. (40) in the form

$$\mathbf{K}(\alpha)[\mathbf{t}^+(\alpha) - \mathbf{F}_1(\alpha)] = \mathbf{t}^+(-\alpha) - \mathbf{F}_1(\alpha), \tag{50}$$

where

$$\mathbf{F}_1(\alpha) = \frac{ie^{ia\sin\theta}\cos\theta}{(\cos^2\theta - \alpha^2)}(1, -i\sin\theta, e^{-2ia\sin\theta}, -i\sin\theta e^{-2ia\sin\theta})^{\mathrm{T}}. \tag{51}$$

The solution of the Wiener–Hopf equation (50) is dependent on the factorization of $\mathbf{K}(\alpha)$ into a product of two matrices, one analytic (with analytic inverse) and of suitable behaviour at infinity in the half-plane $\mathscr{D}^+$, and the other of similar behaviour in the overlapping lower half-plane $\mathscr{D}^-$. This decomposition is the focus of the following section.

4. Kernel decomposition for case $a = b$

The boundary value problem defined in Section 2 was reduced to a 4×4 matrix Wiener–Hopf equation (50) valid in the $\mathscr{S}$-shaped strip of analyticity $\mathscr{D}$. In this section the problem will be considered for the case of equal spacing, i.e. $a = b$. For this special case setting $a = b$ in the matrix kernel $\mathbf{K}(\alpha)$ given by (46) we can rewrite the kernel as

$$\mathbf{K}_0(\alpha) = \frac{1}{\Delta_0(\alpha)}\mathbf{Q}_0(\alpha), \qquad \Delta_0(\alpha) = 2e^{i\xi}(\cosh[4a\gamma(\alpha)] - \cos\xi) \tag{52}$$

where the subscript zero, here and henceforth, signifies the particular case $a = b$, and we also denote ζ in this case as

$$\xi = 4a\sin\theta. \tag{53}$$

To solve the matrix Eq. (50) for the unknown vector functions $\mathbf{t}(\alpha)$ and $\mathbf{t}(-\alpha)$, which are analytic in $\mathscr{D}^+$, $\mathscr{D}^-$, respectively, it is necessary to factorise $\mathbf{K}_0(\alpha)$ into the product form

$$\mathbf{K}_0(\alpha)\mathbf{U}_0(\alpha) = \mathbf{L}_0(\alpha). \tag{54}$$

Here $\mathbf{U}_0(\alpha)$ and $\mathbf{L}_0(\alpha)$ denote matrix functions analytic, and with analytic inverses, in the upper and lower half-planes $\mathscr{D}^+$, $\mathscr{D}^-$, respectively. To proceed it is useful to pre-multiply Eq. (54) by the matrix

$$\boldsymbol{\tau} = \begin{pmatrix} \tau & 0 & 1 & 0 \\ 0 & -\tau & 0 & 1 \\ -\tau & 0 & 1 & 0 \\ 0 & \tau & 0 & 1 \end{pmatrix}, \tag{55}$$

where

$$\tau = \mathrm{e}^{-\mathrm{i}\xi/2}. \tag{56}$$

Performing this multiplication we get

$$\left(\frac{2\mathrm{e}^{\mathrm{i}\xi}}{\Delta_0}\right)^{-1}(\tau l_1 + l_3) = f_-(\alpha)(\tau u_1 - u_3) - \frac{1}{\gamma(\alpha)} g_+(\alpha)(\tau u_2 - u_4), \tag{57}$$

$$-\left(\frac{2\mathrm{e}^{\mathrm{i}\xi}}{\Delta_0}\right)^{-1}(\tau l_2 - l_4) = \gamma(\alpha) g_-(\alpha)(\tau u_1 + u_3) - f_+(\alpha)(\tau u_2 + u_4), \tag{58}$$

$$-\left(\frac{2\mathrm{e}^{\mathrm{i}\xi}}{\Delta_0}\right)^{-1}(\tau l_1 - l_3) = -f_+(\alpha)(\tau u_1 + u_3) + \frac{1}{\gamma(\alpha)} g_-(\alpha)(\tau u_2 + u_4), \tag{59}$$

$$\left(\frac{2\mathrm{e}^{\mathrm{i}\xi}}{\Delta_0}\right)^{-1}(\tau l_2 + l_4) = -\gamma(\alpha) g_+(\alpha)(\tau u_1 - u_3) + f_-(\alpha)(\tau u_2 - u_4), \tag{60}$$

where, for brevity, we work only with the ith column of each side of (54), namely

$$\mathbf{U}_{0i}(\alpha) = \begin{pmatrix} u_1(\alpha) \\ u_2(\alpha) \\ u_3(\alpha) \\ u_4(\alpha) \end{pmatrix}_i; \quad \mathbf{L}_{0i}(\alpha) = \begin{pmatrix} l_1(\alpha) \\ l_2(\alpha) \\ l_3(\alpha) \\ l_4(\alpha) \end{pmatrix}_i, \quad i = 1,2,3,4 \tag{61}$$

and

$$f_\pm(\alpha) = \frac{\mathrm{e}^{\mathrm{i}\xi} - \mathrm{e}^{-\mathrm{i}\xi}}{2} \pm \mathrm{e}^{-\mathrm{i}\xi/2}(1 - \mathrm{e}^{\mathrm{i}\xi})\cosh 2\gamma(\alpha)a, \tag{62}$$

$$g_\pm(\alpha) = \sinh 4\gamma(\alpha)a \pm \mathrm{e}^{-\mathrm{i}\xi/2}(1 + \mathrm{e}^{\mathrm{i}\xi})\sinh 2\gamma(\alpha)a. \tag{63}$$

Note that the subscripts $\pm$ on $f_\pm(\alpha)$ and $g_\pm(\alpha)$ do not imply anything about their respective analyticities; in fact they are all entire functions of α.

The system of Eqs. (57)–(60) has uncoupled somewhat, so that we can form a matrix equation from the first and last equations in the form

$$\begin{pmatrix} \tau l_1 + l_3 \\ \tau l_2 + l_4 \end{pmatrix} = \mathbf{K}_1(\alpha)\begin{pmatrix} \tau u_1 - u_3 \\ \tau u_2 - u_4 \end{pmatrix}, \tag{64}$$

where

$$\mathbf{K}_1(\alpha) = \frac{2\mathrm{e}^{\mathrm{i}\xi}}{\Delta_0}\left[f_-(\alpha)\mathbf{I} - \frac{g_+(\alpha)}{\gamma(\alpha)}\bar{\mathbf{J}}(\alpha)\right], \tag{65}$$

$\mathbf{I}$ is the 2×2 identity matrix, and

$$\bar{\mathbf{J}}(\alpha) = \begin{pmatrix} 0 & 1 \\ \gamma^2(\alpha) & 0 \end{pmatrix}. \tag{66}$$

It can easily be seen that the matrix $\mathbf{K}_1(\alpha)$ is of Khrapkov type, a class that has been studied extensively by various authors (see, for example, [1,14]). It can be factorized commutatively as

$$\mathbf{K}_1(\alpha) = \mathbf{K}_1^+(\alpha)\mathbf{K}_1^-(\alpha) = \mathbf{K}_1^-(\alpha)\mathbf{K}_1^+(\alpha), \tag{67}$$

where

$$\mathbf{K}_1^\pm(\alpha) = a_\pm(\alpha)\left(\cosh(\gamma\theta_\pm)\mathbf{I} + \frac{1}{\gamma(\alpha)}\sinh(\gamma\theta_\pm)\bar{\mathbf{J}}(\alpha)\right), \tag{68}$$

in which $a_\pm(\alpha)$, $\theta_\pm(\alpha)$ are scalar functions with the indicated analyticity properties, $\bar{\mathbf{J}}(\alpha)$ is the entire matrix given in Eq. (66) with the further property

$$\bar{\mathbf{J}}^2(\alpha) = \gamma^2(\alpha)\mathbf{I}. \tag{69}$$

The aforementioned constraint on $\det[\mathbf{K}_1^{\pm}(\alpha)] = (a^{\pm}(\alpha))^2$ implies further that $a^{\pm}(\alpha)$ are zero free as well as singularity free in $\mathscr{D}^{\pm}$. Multiplying $\mathbf{K}_1^{+}(\alpha)$ with $\mathbf{K}_1^{-}(\alpha)$ and equating with (65) gives

$$a^{+}(\alpha)a^{-}(\alpha)\cosh[\gamma(\alpha)(\theta_{+}(\alpha)+\theta_{-}(\alpha))] = 2\mathrm{e}^{\mathrm{i}\xi}\frac{f_{-}(\alpha)}{\Delta_0}, \tag{70}$$

and

$$a^{+}(\alpha)a^{-}(\alpha)\sinh[\gamma(\alpha)(\theta_{+}(\alpha)+\theta_{-}(\alpha))] = -2\mathrm{e}^{\mathrm{i}\xi}\frac{g_{+}(\alpha)}{\Delta_0}. \tag{71}$$

By rearrangement, the required factorizations, namely, the product split

$$[a^{+}(\alpha)a^{-}(\alpha)]^2 = 4\mathrm{e}^{2\mathrm{i}\xi}\frac{(f_{-}^2(\alpha)-g_{+}^2(\alpha))}{\Delta_0^2}, \tag{72}$$

and the sum split

$$\tanh[\gamma(\alpha)(\theta_{+}(\alpha)+\theta_{-}(\alpha))] = -\frac{g_{+}(\alpha)}{f_{-}(\alpha)}, \tag{73}$$

or

$$\theta_{+}(\alpha)+\theta_{-}(\alpha) = \frac{1}{\gamma(\alpha)}\tanh^{-1}\left[-\frac{g_{+}(\alpha)}{f_{-}(\alpha)}\right]. \tag{74}$$

can be achieved. The product split is trivial and can be written, applying the Cauchy's theorem, as

$$a_{\pm}(\alpha) = \mathrm{e}^{\mathrm{i}\pi/4}\exp\left\{\pm\frac{1}{4\pi\mathrm{i}}\int_{\mathscr{C}}\frac{\ln[4\mathrm{e}^{2\mathrm{i}\xi}(g_{+}^2(\zeta)-f_{-}^2(\zeta))/\Delta_0^2]}{\zeta-\alpha}\mathrm{d}\zeta\right\}, \tag{75}$$

and the second is a standard sum factorization (see Noble [18])

$$\theta_{\pm}(\alpha) = \pm\frac{1}{2\pi\mathrm{i}}\int_{\mathscr{C}}\frac{\tanh^{-1}[\mathrm{i}\sinh(2\gamma(\zeta)a)/\sin(\xi/2)]}{\gamma(\zeta)(\zeta-\alpha)}\mathrm{d}\zeta. \tag{76}$$

The contour $\mathscr{C}$ in both of these integrals passes around the singularities in the ζ-plane as indicated in Fig. 2, and the point α lies above (below) $\mathscr{C}$ for $a_{+}(\alpha)$, $\theta_{+}(\alpha)$ ($a_{-}(\alpha)$, $\theta_{-}(\alpha)$).

The same procedure may be applied to Eqs. (58) and (59) to get another matrix equation

$$\begin{pmatrix}\tau l_1 - l_3\\ \tau l_2 - l_4\end{pmatrix} = \mathbf{K}_2(\alpha)\begin{pmatrix}\tau u_1 + u_3\\ \tau u_2 + u_4\end{pmatrix}, \tag{77}$$

with

$$\mathbf{K}_2(\alpha) = \frac{2\mathrm{e}^{\mathrm{i}\xi}}{\Delta_0}\left[f_{+}(\alpha)\mathbf{I} - \frac{g_{-}(\alpha)}{\gamma(\alpha)}\bar{\mathbf{J}}(\alpha)\right], \tag{78}$$

and $\bar{\mathbf{J}}(\alpha)$ defined as in (66). It is apparent that $\mathbf{K}_2(\alpha)$ is also of Khrapkov type and can be decomposed immediately into commuting product factors

$$\mathbf{K}_2^{\pm}(\alpha) = b_{\pm}(\alpha)\left(\cosh(\gamma\beta_{\pm})\mathbf{I} + \frac{1}{\gamma(\alpha)}\sinh(\gamma\beta_{\pm})\bar{\mathbf{J}}(\alpha)\right), \tag{79}$$

where $b_{\pm}(\alpha)$, $\beta_{\pm}(\alpha)$ are regular functions in $\mathscr{D}^{\pm}$. The Khrapkov procedure may again be perfomed to determine the scalar functions $b_{\pm}(\alpha)$, $\beta_{\pm}(\alpha)$; however, a little work will show that $\mathbf{K}_2(\alpha)$ is in fact the inverse of $\mathbf{K}_1(\alpha)$. Therefore, we can immediately conclude that we can choose $\beta_{\pm}(\alpha) = -\theta_{\pm}(\alpha)$ and $b_{\pm}(\alpha) = 1/a_{\pm}(\alpha)$. Thus, henceforth we need employ only the functions $\theta_{\pm}(\alpha)$ and $a_{\pm}(\alpha)$.

Turning back to Eqs. (64) and (77), on using the standard Wiener–Hopf procedure, we get

$$(\mathbf{K}_1^-)^{-1}(\alpha)\begin{pmatrix}\tau l_1 + l_3\\ \tau l_2 + l_4\end{pmatrix} = \mathbf{K}_1^+(\alpha)\begin{pmatrix}\tau u_1 - u_3\\ \tau u_2 - u_4\end{pmatrix} \equiv \begin{pmatrix}C_1\\ C_2\end{pmatrix}, \tag{80}$$

$$(\mathbf{K}_2^-)^{-1}(\alpha)\begin{pmatrix}\tau l_1 - l_3\\ \tau l_2 - l_4\end{pmatrix} = \mathbf{K}_2^+(\alpha)\begin{pmatrix}\tau u_1 + u_3\\ \tau u_2 + u_4\end{pmatrix} \equiv \begin{pmatrix}C_3\\ C_4\end{pmatrix}, \tag{81}$$

where C_1, C_2, C_3, C_4 must be, by the usual analytic continuation argument, arbitrary entire functions. As we are constructing a factorization there is no loss in generality in taking these as constants. We can now use only one side of each equation to determine the values of $u_1,\dots,u_4$ and $l_1,\dots,l_4$. To this end let us take the left-hand side of both equations giving

$$\begin{pmatrix}\tau l_1 + l_3\\ \tau l_2 + l_4\end{pmatrix} = \mathbf{K}_1^-(\alpha)\begin{pmatrix}C_1\\ C_2\end{pmatrix} \tag{82}$$

and

$$\begin{pmatrix}\tau l_1 - l_3\\ \tau l_2 - l_4\end{pmatrix} = \mathbf{K}_2^-(\alpha)\begin{pmatrix}C_3\\ C_4\end{pmatrix}. \tag{83}$$

Addition and subtraction of (82), (83) yields

$$\begin{pmatrix}l_1\\ l_2\end{pmatrix} = \frac{1}{2\tau}\left[\mathbf{K}_1^-(\alpha)\begin{pmatrix}C_1\\ C_2\end{pmatrix} + \mathbf{K}_2^-(\alpha)\begin{pmatrix}C_3\\ C_4\end{pmatrix}\right], \tag{84}$$

$$\begin{pmatrix}l_3\\ l_4\end{pmatrix} = \frac{1}{2}\left[\mathbf{K}_1^-(\alpha)\begin{pmatrix}C_1\\ C_2\end{pmatrix} - \mathbf{K}_2^-(\alpha)\begin{pmatrix}C_3\\ C_4\end{pmatrix}\right]. \tag{85}$$

So with four different independent choices of C_1–C_4 we can get four 4×1 column vectors. Therefore, $\mathbf{L}_0(\alpha)$ can be constructed as a 4×4 matrix in the following form:

$$\mathbf{L}_0(\alpha) = \frac{1}{2\tau}\times\begin{pmatrix} a_-\cosh\gamma\theta_- & \frac{1}{\gamma}a_-\sinh\gamma\theta_- & a_-^{-1}\cosh\gamma\theta_- & -\frac{1}{\gamma}a_-^{-1}\sinh\gamma\theta_- \\ \gamma a_-\sinh\gamma\theta_- & a_-\cosh\gamma\theta_- & -\gamma a_-^{-1}\sinh\gamma\theta_- & a_-^{-1}\cosh\gamma\theta_- \\ \tau a_-\cosh\gamma\theta_- & \frac{1}{\gamma}\tau a_-\sinh\gamma\theta_- & -\tau a_-^{-1}\cosh\gamma\theta_- & \frac{\tau}{\gamma}a_-^{-1}\sinh\gamma\theta_- \\ \tau\gamma a_-\sinh\gamma\theta_- & \tau a_-\cosh\gamma\theta_- & \tau\gamma a_-^{-1}\sinh\gamma\theta_- & -\tau a_-^{-1}\cosh\gamma\theta_- \end{pmatrix} \tag{86}$$

and $\mathbf{U}_0(\alpha)$ and $\mathbf{L}_0(\alpha)$ are related through the equation

$$\mathbf{U}_0(\alpha) = \mathbf{L}_0(-\alpha)\mathbf{X}, \tag{87}$$

where

$$\mathbf{X} = \begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}, \qquad \mathbf{X}^2 = \mathbf{I}. \tag{88}$$

This concludes the exact factorization of $\mathbf{K}_0(\alpha)$ and the solution of the Wiener–Hopf equation will be the main subject of the next section.

5. Solution of the Wiener–Hopf equation for *a* = *b*

In the previous section an exact factorization of the Wiener–Hopf kernel into two matrices, $\mathbf{U}_0(\alpha)$ and $\mathbf{L}_0(\alpha)$, regular in $\mathscr{D}^-$ and $\mathscr{D}^+$ was obtained. These can be employed in the Wiener–Hopf equation (50), which may be rewritten as

$$\mathbf{U}_0^{-1}(\alpha)\mathbf{t}^+(\alpha) - \frac{\mathrm{i}e^{\mathrm{i}\xi/4}}{2(\cos\theta-\alpha)}\{\mathbf{U}_0^{-1}(\alpha) - \mathbf{U}_0^{-1}(\cos\theta)\}\mathbf{F}_p - \frac{\mathrm{i}e^{\mathrm{i}\xi/4}}{2(\cos\theta+\alpha)}\{\mathbf{U}_0^{-1}(\alpha) - \mathbf{L}_0^{-1}(-\cos\theta)\}\mathbf{F}_p$$
$$\equiv \mathbf{E}(\alpha) \equiv \mathbf{L}_0^{-1}(\alpha)\mathbf{t}^+(-\alpha) - \frac{\mathrm{i}e^{\mathrm{i}\xi/4}}{2(\cos\theta-\alpha)}\{\mathbf{L}_0^{-1}(\alpha) - \mathbf{U}_0^{-1}(\cos\theta)\}\mathbf{F}_p$$
$$- \frac{\mathrm{i}e^{\mathrm{i}\xi/4}}{2(\cos\theta+\alpha)}\{\mathbf{L}_0^{-1}(\alpha) - \mathbf{L}_0^{-1}(-\cos\theta)\}\mathbf{F}_p, \tag{89}$$

for $\alpha \in \mathscr{D}$ where

$$\mathbf{F}_p = \begin{pmatrix}1, & -\mathrm{i}\sin\theta, & e^{-\mathrm{i}\xi/2}, & -\mathrm{i}\sin\theta e^{-\mathrm{i}\xi/2}\end{pmatrix}^{\mathrm{T}}, \tag{90}$$

and $\mathbf{U}_0^{-1}(\alpha)$ is given by

$$\mathbf{U}_0^{-1}(\alpha) = \begin{pmatrix} \tau a_+\cosh\gamma\theta_+ & \frac{\tau}{\gamma}a_+\sinh\gamma\theta_+ & -a_+\cosh\gamma\theta_+ & -\frac{1}{\gamma}a_+\sinh\gamma\theta_+ \\ \tau\gamma a_+\sinh\gamma\theta_+ & \tau a_+\cosh\gamma\theta_+ & -\gamma a_+\sinh\gamma\theta_+ & -a_+\cosh\gamma\theta_+ \\ \tau a_+^{-1}\cosh\gamma\theta_+ & -\frac{\tau}{\gamma}a_+^{-1}\sinh\gamma\theta_+ & a_+^{-1}\cosh\gamma\theta_+ & -\frac{1}{\gamma}a_+^{-1}\sinh\gamma\theta_+ \\ -\tau\gamma a_+^{-1}\sinh\gamma\theta_+ & \tau a_+^{-1}\cosh\gamma\theta_+ & -\gamma a_+^{-1}\sinh\gamma\theta_+ & a_+^{-1}\cosh\gamma\theta_+ \end{pmatrix}. \tag{91}$$

Note that $\mathbf{L}_0^{-1}(\alpha)$ can easily be found from Eq. (87). The terms with the arguments $\pm\cos\theta$ are included to remove the poles at $\alpha = \pm\cos\theta$ which were defined in Section 3 to lie in $\mathscr{D}^{\pm}$, respectively. Thus, the left hand side of (89) is analytic in $\mathscr{D}^+$, and the right hand side is analytic in $\mathscr{D}^-$, and so by analytic continuation argument they define an entire vector function in the whole complex α-plane $\mathscr{D}^+ \cup \mathscr{D}^-$, $\mathbf{E}(\alpha)$, say. The precise form of this vector is determined by examining the behaviour of both sides of (89) as $|\alpha| \to \infty$ in their respective half-planes of analyticity. First, by standard techniques, it can be proved that

$$\theta_{\pm}(\alpha) = \mp\mathrm{sgn}(\xi)\frac{\log(\alpha)}{2\alpha} + \mathscr{O}(\alpha^{-1}), \quad |\alpha| \to \infty. \tag{92}$$

where sgn denotes the sign of its argument; however, in Section 2 we placed a restriction on the incident angle θ to lie in the first quadrant. This implies that $\xi \geqslant 0$ and hence we can dispense henceforth with $\mathrm{sgn}(\xi)$. Using asymptotic analysis, it can be found from (92) that

$$\cosh[\gamma(\alpha)\theta_{\pm}(\alpha)] \sim \tfrac{1}{2}\alpha^{1/2}, \quad \gamma\sinh[\gamma(\alpha)\theta_{\pm}(\alpha)] \sim \mp\tfrac{1}{2}\alpha^{3/2}, \tag{93}$$

as $|\alpha| \to \infty$ in $\mathscr{D}^{\pm}$. Recalling the definition of the functions $a_{\pm}(\alpha)$ given by (75), it may easily be seen that

$$a_{\pm}(\alpha) \sim e^{\pi\mathrm{i}/4}, \qquad |\alpha| \to \infty\text{in } \mathscr{D}^{\pm}. \tag{94}$$

Substituting these expansions into (91), we can write

$$\mathbf{U}_0^{-1}(\alpha) \sim \frac{e^{-\pi\mathrm{i}/4}}{2}\begin{pmatrix} \mathrm{i}\tau\alpha^{1/2} & -\mathrm{i}\tau\alpha^{-1/2} & -\mathrm{i}\alpha^{1/2} & \mathrm{i}\alpha^{-1/2} \\ -\mathrm{i}\tau\alpha^{3/2} & \mathrm{i}\tau\alpha^{1/2} & \mathrm{i}\alpha^{3/2} & -\mathrm{i}\alpha^{1/2} \\ \tau\alpha^{1/2} & \tau\alpha^{-1/2} & \alpha^{1/2} & \alpha^{-1/2} \\ \tau\alpha^{3/2} & \tau\alpha^{1/2} & \alpha^{3/2} & \alpha^{1/2} \end{pmatrix}, \tag{95}$$

and the asymptotic behaviour of $\mathbf{L}_0^{-1}(\alpha)$ is given directly from Eq. (87).

The next element is to estimate the sizes of $\mathbf{t}^+(\alpha)$, $\mathbf{t}^+(-\alpha)$ for large $|\alpha|$. A local analysis around the plate edges, $x = 0$, $y = \pm a$, reveals that

$$\Phi(x,-a) \sim c_1|x|^{1/2}, \tag{96}$$
$$\Phi_y(x,-a) \sim c_2|x|^{-1/2}, \tag{97}$$
$$\Phi(x,a) \sim c_3|x|^{1/2}, \tag{98}$$
$$\Phi_y(x,a) \sim c_4|x|^{-1/2}, \tag{99}$$

as $|x| \to 0$ for finite energy density at the edge of the barrier, where c_1, c_2, c_3, c_4 are some unknown constants. There is a direct relationship between the small x expansion of a function and the large α behaviour of its half-range Fourier transform. The reader is referred to the the Abelian theorem quoted in Noble [18], and omitting the details, we find

$$\mathbf{t}^{\pm}(\alpha) \sim \frac{\sqrt{\pi}}{2}\begin{pmatrix} c_1 e^{\pm 3i\pi/4}\alpha^{-3/2} \\ 2c_2 e^{\pm i\pi/4}\alpha^{-1/2} \\ c_3 e^{\pm 3i\pi/4}\alpha^{-3/2} \\ 2c_4 e^{\pm i\pi/4}\alpha^{-1/2} \end{pmatrix}, \tag{100}$$

for α in $\mathscr{D}^{\pm}$, respectively. Both sides of Eq. (89) can now be estimated for large $|\alpha|$; from (91) and (100) we find that they both behave as

$$\mathscr{O}(\alpha^{-1/2}, 1, \alpha^{-1/2}, 1)^{\mathrm{T}}, \quad |\alpha| \to \infty, \alpha \in \mathscr{D}^{\pm}. \tag{101}$$

Therefore, by Liouville's theorem both sides must be equal to the constant vector

$$\mathbf{E}(\alpha) \equiv (0, C_5, 0, C_6)^{\mathrm{T}}, \tag{102}$$

where C_5 and C_6 are constants. This gives the column vector $\mathbf{t}^{+}(\alpha)$ as

$$\begin{aligned}\mathbf{t}^{+}(\alpha) = \mathbf{U}_0(\alpha)\left((0, C_5, 0, C_6)^{\mathrm{T}} - \frac{ie^{i\xi/4}\mathbf{U}_0^{-1}(\cos\theta)}{2(\cos\theta - \alpha)}\mathbf{F}_p - \frac{ie^{i\xi/4}\mathbf{L}_0^{-1}(-\cos\theta)}{2(\cos\theta + \alpha)}\mathbf{F}_p\right) \\ + \frac{ie^{i\xi/4}}{2}\mathbf{F}_p\left(\frac{1}{\cos\theta - \alpha} + \frac{1}{\cos\theta + \alpha}\right)\end{aligned} \tag{103}$$

which, by inspection, will have growth

$$\mathscr{O}(\alpha^{-1/2}, \alpha^{1/2}, \alpha^{-1/2}, \alpha^{1/2})^{\mathrm{T}} \tag{104}$$

as $|\alpha| \to \infty$, a contradiction to the energy requirement (100), unless C and D are chosen appropriately. It can be shown, without difficulty, that to enforce this growth behaviour C_5 and C_6 should be equal and must be chosen as

$$C_5 = C_6 = \frac{ie^{i\xi/4}}{2}(-1, 0, 1, 0)\mathbf{U}_0^{-1}(\cos\theta)\mathbf{F}_p. \tag{105}$$

Thus, the solution is

$$\begin{aligned}\mathbf{t}^{+}(-\alpha) = {} & \frac{ie^{i\xi/4}}{2}\mathbf{F}_p\left(\frac{1}{\alpha + \cos\theta} - \frac{1}{\alpha - \cos\theta}\right) - \frac{ie^{i\xi/4}}{2(\alpha + \cos\theta)}\mathbf{K}_0(\alpha)\mathbf{U}_0(\alpha)\mathbf{L}_0^{-1}(-\cos\theta)\mathbf{F}_p \\ & + \frac{ie^{i\xi/4}}{2(\alpha - \cos\theta)}\mathbf{K}_0(\alpha)\mathbf{U}_0(\alpha)\mathbf{U}_0^{-1}(\cos\theta)\mathbf{F}_p \\ & + \frac{ie^{i\xi/4}}{2}\mathbf{K}_0(\alpha)\mathbf{U}_0(\alpha)\begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}\mathbf{U}_0^{-1}(\cos\theta)\mathbf{F}_p.\end{aligned} \tag{106}$$

The scattered potential may now be constructed from this column vector via Eqs. (28)–(34). We may thus write the solution to the boundary value problem as

$$\begin{aligned}\Phi(x, y) = \frac{1}{2\pi}\int_{\mathscr{C}} \Big\{ & (t^{+}(\alpha) - t^{+}(-\alpha))\cosh[\gamma(a + y)]. \\ & + \frac{1}{\gamma}(s^{+}(\alpha) - s^{+}(-\alpha))\sinh[\gamma(a + y)]\Big\} e^{-i\alpha x}\, d\alpha, \quad \forall y,\end{aligned} \tag{107}$$

where $\mathscr{C}$ lies in the strip $\mathscr{D}$ as indicated on Fig. 2 and $t^+(\pm\alpha)$ and $s^+(\pm\alpha)$ are derived from Eq. (106) by multiplying by row vectors $(1,0,0,0)$, $(0,1,0,0)$, respectively, and using the relation

$$\mathbf{K}_0(\alpha)\mathbf{U}_0(-\alpha) = \mathbf{L}_0(-\alpha) \tag{108}$$

where appropriate.

The full form of the solution can now be written down by deforming the contour $\mathscr{C}$ in the upper half-plane for $x < 0$ or lower half-plane for $x > 0$ and evaluating by using the residue calculus. It can be shown after substantial algebra, omitted here for brevity, that the *total* wave field (including the incident wave (3)) is

$$\Phi^{\text{tot}}(x,y) = \begin{cases} (1-P_0)\mathrm{e}^{-\mathrm{i}x\cos\theta-\mathrm{i}y\sin\theta} - S(x,y), & x<0, \\ \mathrm{e}^{-\mathrm{i}x\cos\theta-\mathrm{i}y\sin\theta} + P_0\mathrm{e}^{\mathrm{i}x\cos\theta-\mathrm{i}y\sin\theta} + S(x,y), & x>0, \end{cases} \tag{109}$$

where

$$S(x,y) = \sum_{n=-\infty,n\neq 0}^{\infty} P_n \mathrm{e}^{-\mathrm{i}n\pi/2}\mathrm{e}^{\gamma_n y}\mathrm{e}^{\mathrm{i}\lambda_n|x|}, \tag{110}$$

$$P_0 = \frac{1}{2} + \frac{\mathbf{G}_0\mathbf{U}_0(\cos\theta)\mathbf{L}_0^{-1}(-\cos\theta)}{16a\cos^2(\theta)}\mathbf{F}_p - \frac{\mathbf{G}_0\mathbf{U}_0(\cos\theta)}{8a\cos\theta}(0,1,0,1)^{\mathrm{T}}q_1^*, \tag{111}$$

$$P_n = \frac{\mathbf{G}_n}{8a\lambda_n}\left\{\mathbf{U}_0(\lambda_n)\left(\frac{\mathbf{L}_0^{-1}(-\cos\theta)}{\cos\theta+\lambda_n} + \frac{\mathbf{U}_0^{-1}(\cos\theta)}{\cos\theta-\lambda_n}\right)\mathbf{F}_p - \mathbf{U}_0(\lambda_n)(0,1,0,1)^{\mathrm{T}}q_1^*\right\}, \tag{112}$$

$$q_1^* = (-1,0,1,0)\mathbf{U}_0^{-1}(\cos\theta)\mathbf{F}_p, \tag{113}$$

in which

$$\lambda_n = \left(1-\left(\frac{\xi+2\pi n}{4a}\right)^2\right)^{1/2}, \qquad \gamma_n = -\mathrm{i}\left(\frac{\xi+2\pi n}{4a}\right), \tag{114}$$

and

$$\mathbf{G}_n = (-\gamma_n, -1, (-1)^n\mathrm{e}^{\mathrm{i}\xi/2}\gamma_n, (-1)^n\mathrm{e}^{\mathrm{i}\xi/2}). \tag{115}$$

An avid reader may spot the fact that P_0 does not contain a contribution arising from the penultimate term of $\mathbf{t}^+(-\alpha)$ of (106) at $\alpha = \cos\theta$. This is because the residue of this expression turns out, after Taylor series expansion, to be zero. Note that the scattered part of the potential in (109) is, as presumed at the outset, odd in x, but the transmitted field is composed of the scattered waves plus the incident wave. Further, for each a and θ there is a finite number of propagating (non-evanescent) wave terms out of the infinite sum which are defined by λ_n taking real values; the condition for these are defined by

$$-M \leqslant n \leqslant N, \tag{116}$$

where M and N are the largest integers less than or equal to $2a(1+\sin\theta)/\pi$ and $2a(1-\sin\theta)/\pi$, respectively, i.e.

$$M = \text{Floor}\left[\frac{2a(1+\sin\theta)}{\pi}\right], \qquad N = \text{Floor}\left[\frac{2a(1-\sin\theta)}{\pi}\right]. \tag{117}$$

Our particular interest is in the energy transmitted and reflected from the barriers. The total energy flux for the transmitted and reflected waves are easily shown to be given by (see, p. 65 of [11], also see, [9], pp.115–116)

$$\mathscr{E}_{\mathrm{T}} = |1-P_0|^2 + \sum_{m=-M,n\neq 0}^{N} \frac{\lambda_m}{\lambda_0}|P_m|^2, \tag{118}$$

$$\mathscr{E}_{\mathrm{R}} = \sum_{m=-M}^{N} \frac{\lambda_m}{\lambda_0}|P_m|^2. \tag{119}$$

For ease of exposition of the numerical results in the following Section, it is useful to define the reflection and transmission coefficients, respectively, as

$$|T_0| = |1 - P_0|, \qquad |T_m| = |P_m|, n \neq 0, \tag{120}$$
$$|R_m| = |P_m|. \tag{121}$$

6. Discussion and concluding remarks

In Section 5 we obtained a closed form analytic solution to the grating problem for the special case $a = b$. We have derived explicit expressions for the reflection and transmission coefficients given by Eqs. (120) and (121). It should be noted, however, that although the solution is explicit, determination of P_n requires the evaluation of the expressions (75) and (76). Such integrals are frequently encountered in the Wiener–Hopf method and their evaluation is straightforward using a mathematical software package such as *Mathematica*. Taking the integration contour $\mathscr{C}$ of Fig. 2, numerical results for $a_{\pm}(\alpha)$ and $\theta_{\pm}(\alpha)$ can be obtained easily to an accuracy of the order of $10^{-12} - 10^{-15}$. Note that the accuracy of all of the numerical results to follow can be checked

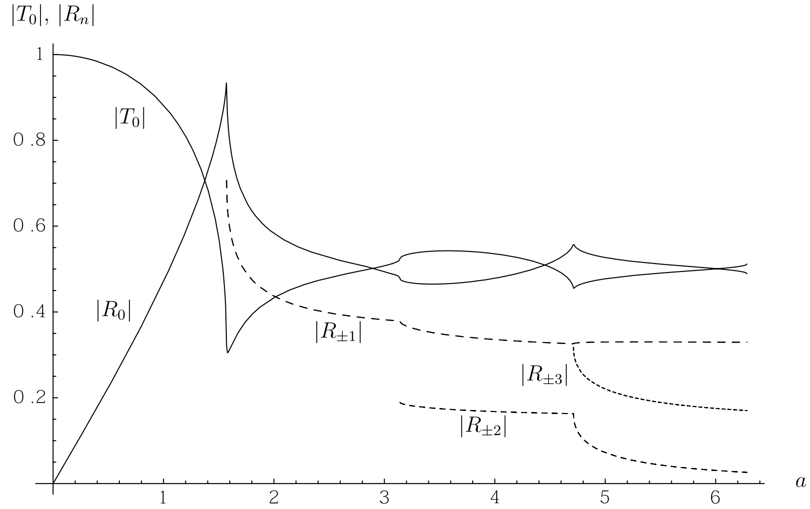

Fig. 3. $|R_n|$, $|T_0|$ against non-dimensional wavenumber a; $\theta = 0$.

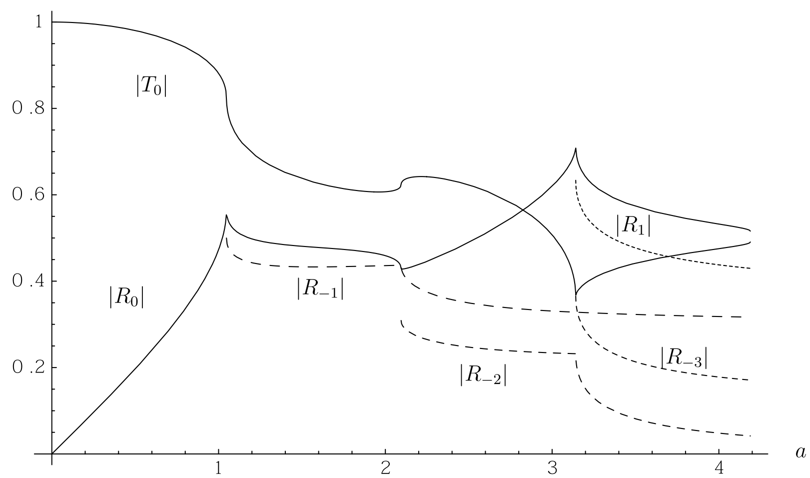

Fig. 4. $|R_n|$, $|T_0|$ against non-dimensional wavenumber a; $\theta = \pi/6$.

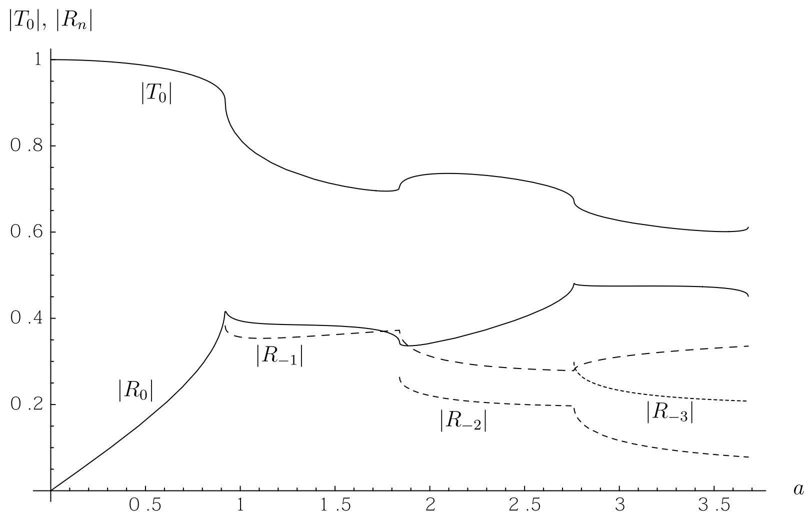

Fig. 5. $|R_n|$, $|T_0|$ against non-dimensional wavenumber a; $\theta = \pi/4$.

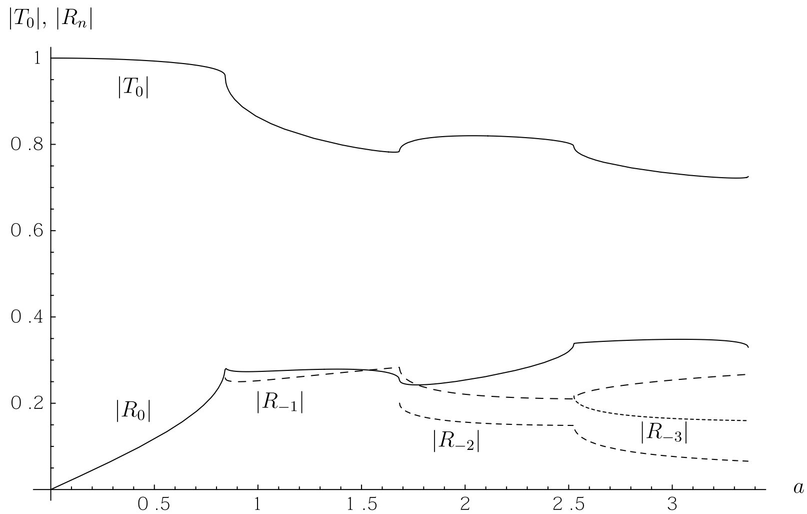

Fig. 6. $|R_n|$, $|T_0|$ against non-dimensional wavenumber a; $\theta = \pi/3$.

through the percentage error of the energy balance, i.e. $\mathscr{E}_R + \mathscr{E}_T = 1$ ((118) and (119)); however, since our solution is a closed form analytic expression, it is not surprising that we find the error to be, at most, of order 10^{-8}%! We can therefore be confident that our numerical evaluations act a benchmark to test the accuracy of alternative numerical schemes. We will now present our results for different values of θ and nondimensionalised wavelength a.

Fig. 3 shows the behaviour of the reflection coefficients, $|R_n|$ ($=|P_n|$), as the wavenumber a varies for the periodic array of rigid plates, where the incident angle is normal to the screens. The results agree closely with those of Porter and Evans [19] and Dalrymple and Martin [10]. The first of these authors used eigenfunction expansions and employed Galerkin's method for their solution, where they reached three significant figure accuracy for their numerical results. The latter authors also used matched eigenfunction expansions; however, they did not solve the problem for higher reflected and transmitted modes. Note that we do not show curves for $|T_n|$ for $n \neq 0$ because, due to symmetry, we have $|R_n| = |T_n|$!

In Fig. 4 we see that increasing the incident wave angle, θ, to $\theta = \pi/6$ increases the value of the zeroth transmission mode. The zeroth reflection and transmission coefficients intersect in the middle of the cut-on region

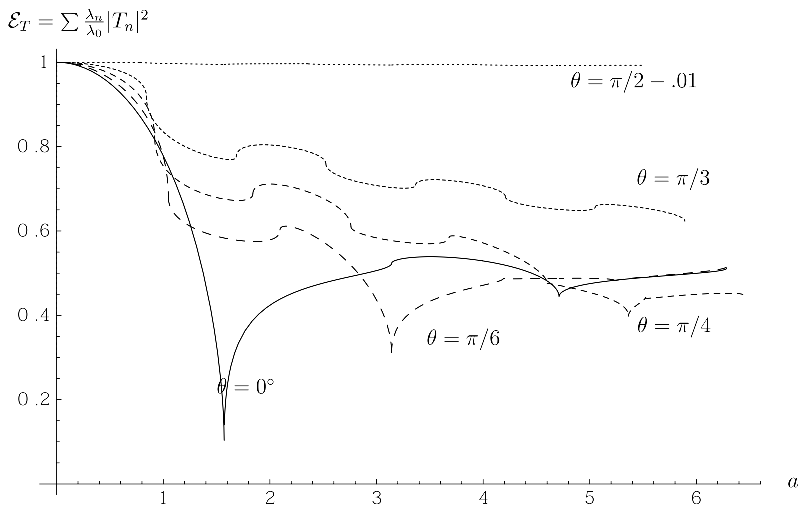

Fig. 7. Proportion of the incident wave energy transmitted through the barrier for distinct θ values vs. a.

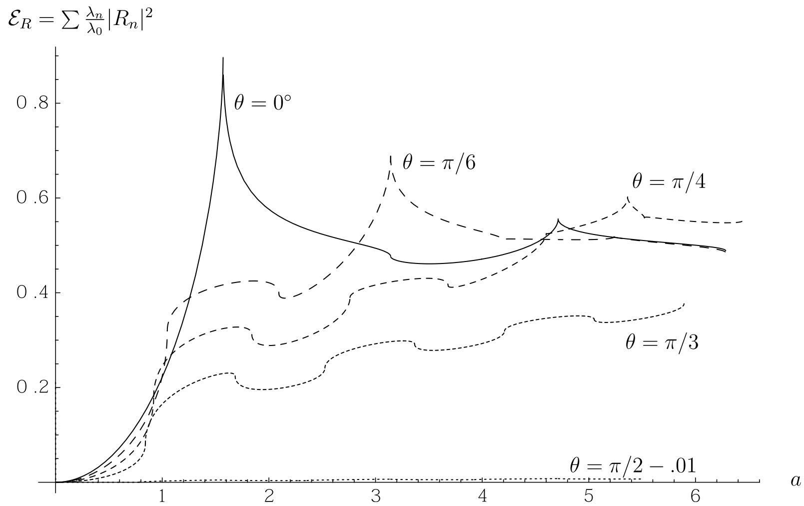

Fig. 8. Proportion of the incident wave energy reflected by the barrier for distinct θ values vs. a.

of the second negative mode. Plotting the transmission coefficients for increasing θ, Figs. 5 and 6, it becomes clear that the magnitudes of $|T_0|$ and $|R_n|$ drift away from each other. Considering the flux of energy and referring to the energy Eqs. (118) and (119), we see, Figs. 7 and 8, that the greatest reflection of wave energy occurs for normal wave incidence, except possibly near the cut-off frequencies.

It may be helpful for the reader to offer a little discussion of the cut-on values of the reflection coefficients illustrated in the figures. The curve of the reflected energy $\mathscr{E}_{\mathrm{R}}$ (shown as the unbroken line emanating from 0 at $a = 0$ in Fig. 8) is determined by adding all the propagating modes appearing in the solution (119). As mentioned, the number of such propagating wave modes in the reflected and transmitted wave fields varies according to the integer values n which satisfy:

$$\frac{-2a(1+\sin\theta)}{\pi} \leqslant n \leqslant \frac{2a(1-\sin\theta)}{\pi}. \tag{122}$$

The variations in the cut-on values of the higher modes in the reflection coefficient, for different incident angles are due to this equation. When $\theta = 0°$, the negative and positive values of index n have the same number;

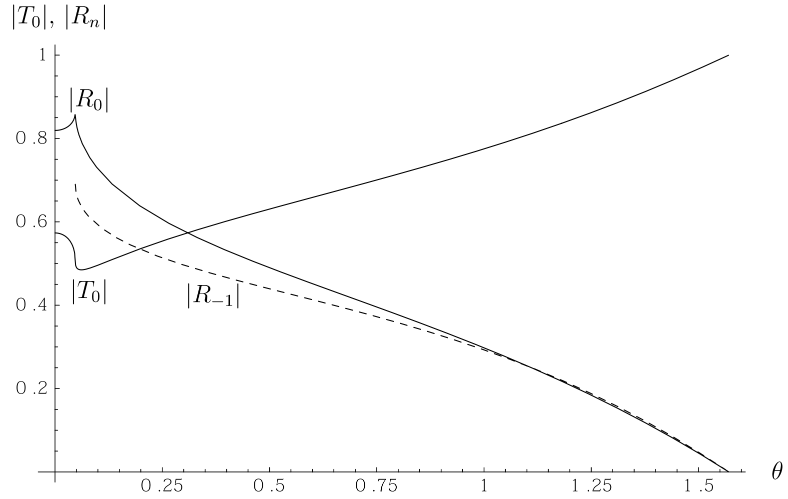

Fig. 9. $|T_0|$, $|R_n|$ against θ when $a = 1.5$.

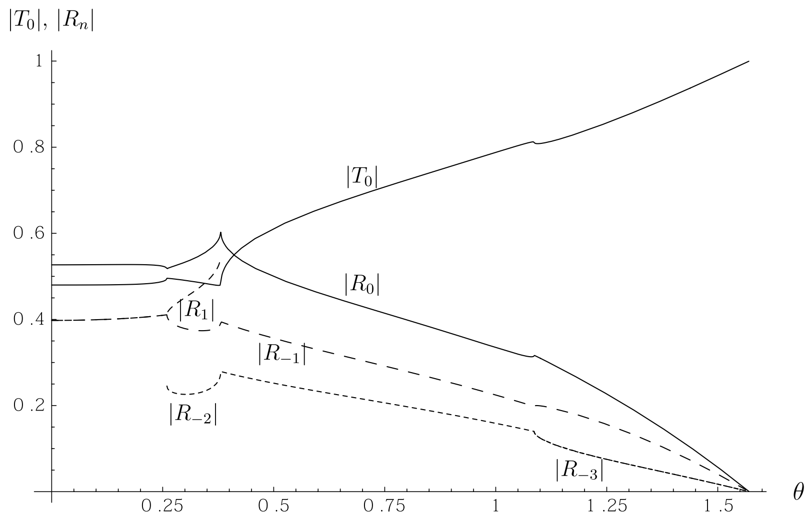

Fig. 10. $|T_0|$, $|R_n|$ against θ when $a = 2.5$.

however as θ approaches $\pi/2$ the negative n terms remain but there are only contributions from the positive n index as a becomes very large. At grazing incidence $\theta = \pi/2$ there are no $|R_n|$, $n > 0$, contributions.

The particular focus of this article is the energy flux through the periodic array. As already mentioned, although the reflection coefficient curves for non-normal incidence have the same general form as those for the $\theta = 0°$ case, the largest energy reflection typically occurs for normal incidence. Increasing the angle of incidence has the general effect of increasing the total transmitted energy, a result that has been observed in many barrier problems (see, for example, Porter and Evans [19] and references therein). Figs. 7 and 8 show graphs of the transmitted and reflected wave fields against incidence angle, θ. Recalling numerical results by Erbaş ([11], Chapter 4) and Baldwin and Heins ([9]) for the duct problem with equal length barriers and gaps, one interesting result is that, for very large wavenumber a, or very short wavelengths, the total transmitted energy approaches one half. This is not difficult to explain. At high frequencies the

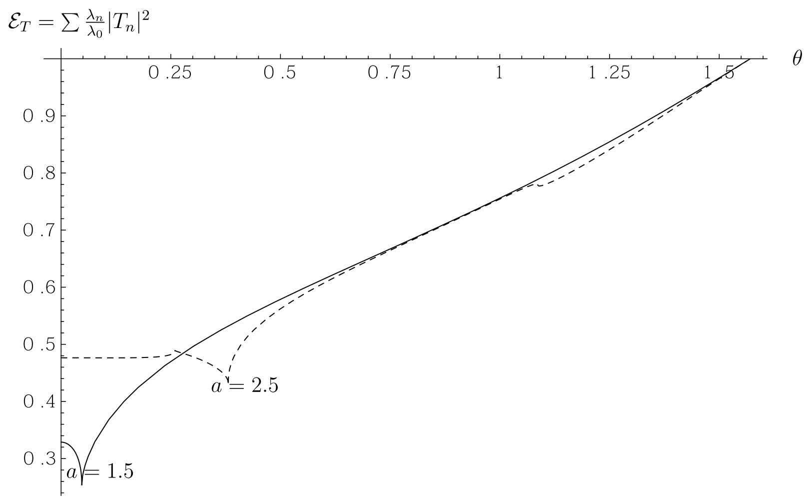

Fig. 11. Proportion of the incident wave energy transmitted through the barrier for fixed a against θ.

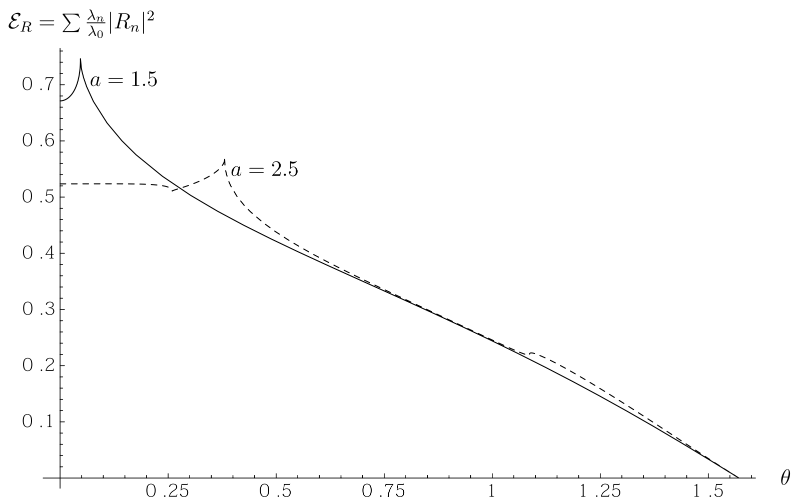

Fig. 12. Proportion of the incident wave energy reflected from the barrier for fixed a against θ.

field scattered by the strip edges is small. Thus, the portion of the waves that are incident on the rigid barriers will be blocked effectively by the array and hence reflected, whereas the waves incident on the gaps will not feel the presence of the grating, and transmit through the gap without any loss of energy. This yields the relation $\mathscr{E}_{\mathrm{T}} \sim b/2a = 1/2$, which is the total transmitted energy of the incident wave. Another interesting phenomenon occurs as the incident angle increases and approaches $\pi/2$. In this case the wave field transmits, at near grazing angle, through the infinite grating without any significant loss of energy. This fact is observed from Figs. 5–7.

We now plot the transmission coefficients for distinct values of wavenumber a and varying incidence angle θ. In Figs. 9 and 10 two values for a, $a = 1.5$ and $a = 2.5$, respectively, are chosen. As can be seen from the graphs, the larger values of θ result in a larger transmission coefficient and virtually no reflection of the wave field. The penultimate figures, Figs. 11 and 12, reveal that the maximum amount of energy transmitted through the grating occurs at $\theta = \pi/2$, which is independent of the gap size. This fact corroborates the discussion above. Our last figure, Fig. 13, displays the comparison of the numerical results of Porter & Evans [19] to that found in this article. The incident wave angle is taken as $\theta = 30°$, and only the first two modes are considered. The agreement for both small and large wavenumbers is excellent.

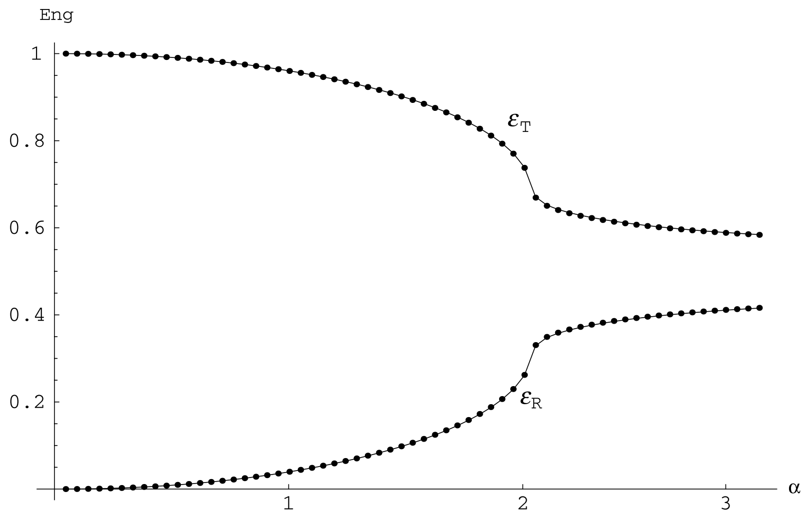

Fig. 13. Comparison of results for transmitted and reflected wave fields from Porter & Evans [19] (dotted lines) and from the exact result using the Wiener–Hopf technique (continuous lines) for $\theta = 30°$. Comparison of results to those of Porter and Evans.

Appendix A. Derivation of the matrix kernel

In Section 3, we obtained a matrix kernel for the grating problem omitting the details. Here we shall show in some detail how the matrix kernel can be obtained. Let us start by eliminating the unknown coefficients A, B, C, D of Eqs. (31)–(34). Multiplying (31) by γ (we use γ instead of $\gamma(\alpha)$ where there is no risk of confusion) and adding and subtracting the resulting equation with (33), respectively, A and B are found as

$$A = \frac{\gamma^{-1}e^{\gamma a}}{2}\{\gamma(t^+(\alpha) - t^+(-\alpha)) + (s^+(\alpha) - s^+(-\alpha))\}, \tag{A.1}$$

$$B = \frac{\gamma^{-1}e^{-\gamma a}}{2}\{\gamma(t^+(\alpha) - t^+(-\alpha)) - (s^+(\alpha) - s^+(-\alpha))\}. \tag{A.2}$$

Equations for the unknowns C and D are found similarly on multiplying (32) by γ and adding and subtracting (32) and (34) to give

$$C = \frac{\gamma^{-1}e^{\gamma a}}{2}\left\{\gamma(t^+(\alpha) + t^+(-\alpha)) + (s^+(\alpha) + s^+(-\alpha)) - \frac{2\cos\theta e^{ia\sin\theta}}{(\cos^2\theta - \alpha^2)}(\gamma i + \sin\theta)\right\}, \tag{A.3}$$

$$D = \frac{\gamma^{-1}e^{-\gamma a}}{2}\left\{\gamma(t^+(\alpha) + t^+(-\alpha)) - (s^+(\alpha) + s^+(-\alpha)) - \frac{2\cos\theta e^{ia\sin\theta}}{(\cos^2\theta - \alpha^2)}(\gamma i - \sin\theta)\right\}. \tag{A.4}$$

Eqs. (38) and (39) can be combined, after a little modification, to give the following relations

$$\gamma(38) + (39) = e^{-i\zeta-2\gamma b}\{\gamma(t^+(\alpha) - t^+(-\alpha)) + (s^+(\alpha) - s^+(-\alpha))\}, \tag{A.5}$$

$$\gamma(38) - (39) = e^{-i\zeta+2\gamma b}\{\gamma(t^+(\alpha) - t^+(-\alpha)) - (s^+(\alpha) - s^+(-\alpha))\}. \tag{A.6}$$

Now, if we multiply (A.5) and (A.6) by $e^{\gamma b}$ and $e^{-\gamma b}$, respectively, adding and subtracting the resulting equations, we get:

$$\gamma(38)\cosh\gamma b + (39)\sinh\gamma b = e^{-i\zeta}\{\gamma\cosh\gamma b(t^+(\alpha) - t^+(-\alpha)) - \sinh\gamma b(s^+(\alpha) - s^+(-\alpha))\}, \tag{A.7}$$

$$\gamma(38)\sinh\gamma b + (39)\cosh\gamma b = e^{-i\zeta}\{-\gamma\sinh\gamma b(t^+(\alpha) - t^+(-\alpha)) + \cosh\gamma b(s^+(\alpha) - s^+(-\alpha))\}. \tag{A.8}$$

In a similar fashion, (36) and (37) can be combined to give:

$$\begin{aligned}\gamma(36)+(37)=&\mathrm{e}^{2\gamma a}\{\gamma(t^+(\alpha)+t^+(-\alpha))+(s^+(\alpha)+s^+(-\alpha))\}\\&-\frac{2\cos\theta\mathrm{e}^{\mathrm{i}a\sin\theta+2\gamma a}}{(\cos^2-\alpha^2)}(\gamma\mathrm{i}+\sin\theta)+\frac{2\cos\theta\mathrm{e}^{-\mathrm{i}a\sin\theta}}{(\cos^2-\alpha^2)}(\gamma\mathrm{i}+\sin\theta),\end{aligned}\tag{A.9}$$

$$\begin{aligned}\gamma(36)-(37)=&\mathrm{e}^{-2\gamma a}\{\gamma(t^+(\alpha)-t^+(-\alpha))-(s^+(\alpha)+s^+(-\alpha))\}\\&-\frac{2\cos\theta\mathrm{e}^{\mathrm{i}a\sin\theta-2\gamma a}}{(\cos^2-\alpha^2)}(\gamma\mathrm{i}-\sin\theta)+\frac{2\cos\theta\mathrm{e}^{-\mathrm{i}a\sin\theta}}{(\cos^2-\alpha^2)}(\gamma\mathrm{i}-\sin\theta).\end{aligned}\tag{A.10}$$

Now, let us multiply (A.9) and (A.10) by $e^{-\gamma a}$ and $e^{\gamma a}$ respectively to get

$$\begin{aligned}\gamma(36)\cosh\gamma a-(37)\sinh\gamma a=&\ \gamma\cosh\gamma a(t^+(\alpha)+t^+(-\alpha))\\&+\sinh\gamma a(s^+(\alpha)+s^+(-\alpha))-\frac{2\cos\theta}{(\cos^2-\alpha^2)}[\mathrm{e}^{\mathrm{i}a\sin\theta}\{\mathrm{i}\gamma\cosh(\gamma a)\\&+\sin\theta\sinh(\gamma a)\}-\mathrm{e}^{-\mathrm{i}a\sin\theta}\{\mathrm{i}\gamma\cosh(\gamma a)-\sin\theta\sinh(\gamma a)\}],\end{aligned}\tag{A.11}$$

$$\begin{aligned}\gamma(36)\sinh\gamma a-(37)\cosh\gamma a=&-\gamma\sinh\gamma a(t^+(\alpha)+t^+(-\alpha))-\cosh\gamma a(s^+(\alpha)\\&+s^+(-\alpha))+\frac{2\cos\theta}{(\cos^2-\alpha^2)}[\mathrm{e}^{\mathrm{i}a\sin\theta}\{\mathrm{i}\gamma\sinh(\gamma a)\\&+\sin\theta\cosh(\gamma a)\}+\mathrm{e}^{-\mathrm{i}a\sin\theta}\{\mathrm{i}\gamma\sinh(\gamma a)-\sin\theta\cosh(\gamma a)\}].\end{aligned}\tag{A.12}$$

Eqs. (A.7), (A.8) and (A.11), (A.12) are the main equations from which the matrix kernel will be obtained, recalling that $v^+(\pm\alpha)$ and $u^+(\pm\alpha)$ are given by (36) and (37). To achieve this, the best practice is to collect the '+' and '−' functions on each sides of these equations in their explicit form. Performing this operation gives the following four equations:

$$\begin{aligned}&\mathrm{e}^{-\mathrm{i}\zeta}\cosh(\gamma b)t^+(\alpha)-\mathrm{e}^{-\mathrm{i}\zeta}\gamma^{-1}\sinh(\gamma b)s^+(\alpha)-\gamma^{-1}\sinh(\gamma b)u^+(\alpha)-\cosh(\gamma b)v^+(\alpha)\\&\quad=\mathrm{e}^{-\mathrm{i}\zeta}\cosh(\gamma b)t^+(-\alpha)-\mathrm{e}^{-\mathrm{i}\xi}\gamma^{-1}\sinh(\gamma b)s^+(-\alpha)-\gamma^{-1}\sinh(\gamma b)u^+(-\alpha)-\cosh(\gamma b)v^+(-\alpha),\end{aligned}\tag{A.13}$$

$$\begin{aligned}&\mathrm{e}^{-\mathrm{i}\zeta}\sinh(\gamma b)t^+(\alpha)-\mathrm{e}^{-\mathrm{i}\zeta}\gamma^{-1}\cosh(\gamma b)s^+(\alpha)+\gamma^{-1}\cosh(\gamma b)u^+(\alpha)+\sinh(\gamma b)v^+(\alpha)\\&\quad=\mathrm{e}^{-\mathrm{i}\zeta}\sinh(\gamma b)t^+(-\alpha)-\mathrm{e}^{-\mathrm{i}\zeta}\gamma^{-1}\cosh(\gamma b)s^+(-\alpha)+\gamma^{-1}\cosh(\gamma b)u^+(-\alpha)+\sinh(\gamma b)v^+(-\alpha),\end{aligned}\tag{A.14}$$

$$\begin{aligned}&\cosh(\gamma a)t^+(\alpha)+\gamma^{-1}\sinh(\gamma a)s^+(\alpha)+\gamma^{-1}\sinh(\gamma a)u^+(\alpha)-\cosh(\gamma a)v^+(\alpha)\\&\quad=-\cosh(\gamma a)t^+(-\alpha)-\gamma^{-1}\sinh(\gamma a)s^+(-\alpha)+\gamma^{-1}\sinh(\gamma a)u^+(-\alpha)\\&\qquad+\cosh(\gamma a)v^+(-\alpha)+\frac{2\mathrm{i}\cos\theta}{(\cos^2\theta-\alpha^2)}\{\mathrm{e}^{\mathrm{i}a\sin\theta}(\cosh\gamma a-\mathrm{i}\gamma^{-1}\sin\theta\sinh\gamma a)\\&\qquad-\mathrm{e}^{-\mathrm{i}a\sin\theta}(\cosh\gamma a+\mathrm{i}\gamma^{-1}\sin\theta\sinh\gamma a)\},\end{aligned}\tag{A.15}$$

$$\begin{aligned}&\sinh(\gamma a)t^+(\alpha)+\gamma^{-1}\cosh(\gamma a)s^+(\alpha)-\gamma^{-1}\cosh(\gamma a)u^+(\alpha)+\sinh(\gamma a)v^+(\alpha)\\&\quad=-\sinh(\gamma a)t^+(-\alpha)-\gamma^{-1}\cosh(\gamma a)s^+(-\alpha)+\gamma^{-1}\cosh(\gamma a)u^+(-\alpha)\\&\qquad-\sinh(\gamma a)v^+(-\alpha)+\frac{2\mathrm{i}\cos\theta}{(\cos^2\theta-\alpha^2)}\{\mathrm{e}^{\mathrm{i}a\sin\theta}(\sinh\gamma a-\mathrm{i}\gamma^{-1}\sin\theta\cosh\gamma a)\\&\qquad+\mathrm{e}^{-\mathrm{i}a\sin\theta}(\sinh\gamma a+\mathrm{i}\gamma^{-1}\sin\theta\cosh\gamma a)\}.\end{aligned}\tag{A.16}$$

Multiplying (A.13) by $\mathrm{e}^{\mathrm{i}\zeta}$, (A.14) by $\gamma(\alpha)\mathrm{e}^{\mathrm{i}\zeta}$, (A.16) by $\gamma(\alpha)$, we can finally combine the resulting equations into the matrix Wiener–Hopf equation of Section 3:

$$\mathbf{M}(\alpha)\mathbf{t}^+(\alpha)=\mathbf{J}\mathbf{M}(\alpha)\mathbf{t}^+(-\alpha)+\mathbf{F}(\alpha),\tag{A.17}$$

where $\mathbf{t}^+(\alpha)$, $\mathbf{M}(\alpha)$, and $\mathbf{F}(\alpha)$ are given by Eqs. (41), (42), and (45) respectively. The matrix Wiener–Hopf kernel, $\mathbf{K}(\alpha)$, is then given by

$$\mathbf{K}(\alpha)=\mathbf{M}^{-1}(\alpha)\mathbf{J}\mathbf{M}(\alpha).\tag{A.18}$$

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