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Tressl, Marcus

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Pseudo completions and completions in stages of o-minimal structures

Marcus Tressl

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Abstract For an o-minimal expansion R of a real closed field and a set $\mathscr{V}$ of $Th(R)$-convex valuation rings, we construct a "pseudo completion" with respect to $\mathscr{V}$. This is an elementary extension S of R generated by all completions of all the residue fields of the $V \in \mathscr{V}$, when these completions are embedded into a big elementary extension of R. It is shown that S does not depend on the various embeddings up to an R-isomorphism. For polynomially bounded R we can iterate the construction of the pseudo completion in order to get a "completion in stages" S of R with respect to $\mathscr{V}$. S is the "smallest" extension of R such that all residue fields of the unique extensions of all $V \in \mathscr{V}$ to S are complete.

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Let R be a real closed field. There is a largest ordered field $\hat{R}$ such that R is dense in $\hat{R}$. $\hat{R}$ is again real closed and $\hat{R}$ is called the completion of R (cf. [7]). If v is a proper real valuation on R, then $\hat{R}$ is also the underlying field of the completion of the valued field (R, v) and $\hat{R}$ is obtained by adjoining limits of Cauchy sequences with respect to v as explained in [8].

We generalize this construction as follows. Let $\mathscr{V}$ be a set of convex valuation rings, possibly containing R itself. We construct a "smallest" real closed

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M. Tressl (✉)
NWF-I Mathematik, Universität Regensburg, 93040 Regensburg, Germany
e-mail: marcus.tressl@mathematik.uni-regensburg.de

field containing R which has a limit for all sequences of R that become Cauchy sequences after passing to the residue field of some $V \in \mathscr{V}$. This can also be done for o-minimal expansions of real closed fields and $Th(R)$-convex valuation rings (see Sect. 3 for the definition of the completion in this case).

Our first result (Theorem 4.1) basically says that we can adjoin the missing limits to R in any order and that the resulting elementary extension R' of R does not depend on the choices, up to an R-isomorphism. We call R' the pseudo completion of R with respect to $\mathscr{V}$. If R is a pure real closed field (more generally, a polynomially bounded o-minimal expansion of a real closed field), then we can compute the value groups and the residue fields of convex valuation rings of R'. Moreover for every valuation ring $V \in \mathscr{V}$ the convex hull V' of V in R' is the unique convex valuation ring of R', lying over V.

It turns out that R' is not "complete in stages" with respect to $\mathscr{V}' := \{V' | V \in \mathscr{V}\}$ in general, i.e. not all residue fields of the V' are complete in general [cf. Example 5.7]. Therefore, in order to get a "smallest" extension of R, which is complete in stages, we have to iterate the construction of the pseudo completion. The iteration stops at an ordinal and the resulting extension S of R is called the completion in stages of R with respect to $\mathscr{V}$. In Theorem 5.10, we compute the value groups and the residue fields of convex valuation rings of S. Moreover in Theorem 5.10 it is shown that every element $s \in S \setminus R$ is of the form $ax + b$ where $a, b \in R$ and $x \in S$ such that for a unique convex valuation ring W of S with $W \cap R \in \mathscr{V}$, $s/\mathfrak{m}_W$ is the limit of a Cauchy sequence of $V/\mathfrak{m}_V$ without limits in $V/\mathfrak{m}_V$; here $\mathfrak{m}_V$, $\mathfrak{m}_W$ denote the maximal ideal of V, W, respectively.

Finally we want to point out a combinatorial tool which we use in our arguments. This is a dimension in o-minimal structures, we call it the realization rank, which is coarser than the ordinary dimension associated to o-minimal structures. For real closed fields $R \subseteq S$, with tr.deg. S/R finite, the realization rank of S over R is the maximal number of elements $s_1, \ldots, s_k \in S$ such that $tp(s_1, \ldots, s_k/R)$ is uniquely determined by the open boxes contained in it [cf. Proposition 1.15]. We first analyze this new dimension.

The explanation of the valuation theoretic notions and facts used for o-minimal expansions of fields can be found in [2]. Readers who are mainly interested in the case of real closed fields may replace "o-minimal structure" by "real closed field", "definable" by "semi-algebraic" and "definable closure" by "real closure". Moreover if $R \subseteq S$ are real closed fields and $B \subseteq S$, then the type $tp(B/R)$ of B over R can be identified with the ordering of $R[t_b | b \in B]$ (where the t_b are indeterminates) induced by the evaluation map $t_b \mapsto b$.

1 The realization rank

We start with a reminder on dependence relations as in van der Waerden's "Algebra" ([10]).

Definition 1.1 *A relation $x \ll A$ between elements x and subsets A of a given set X is called a* dependence relation *if the following conditions are fulfilled:*

(D1) $x \ll \{x\}$.

(D2) *if* $x \ll A$ *and* $A \subseteq B$ *then* $x \ll B$.
(D3) *if* $x \ll A$ *then there is a finite subset* B *of* A, *such that* $x \ll B$.
(D4) *(exchange lemma)*
if A *is finite,* $x \ll A \cup \{y\}$ *and* $x \not\ll A$, *then* $y \ll A \cup \{x\}$.
(D5) *(transitivity)*
if A *is finite,* $x \ll A$ *and* $a \ll B$ *for every* $a \in A$, *then* $x \ll B$.

We rephrase this notion in terms of independent sets:

Definition 1.2 *Let* X *be a set and let* $\mathcal{I}$ *be a nonempty set of finite subsets of* X. $\mathcal{I}$ *is called a* system of independence *if the following two properties hold.*

(I1) *If* $A \subseteq B \in \mathcal{I}$ *and* $B \in \mathcal{I}$, *then* $A \in \mathcal{I}$.
(I2) *If* $A, B \in \mathcal{I}$, $x \in X \setminus B$ *and if* $B \cup \{x\} \in \mathcal{I}$, *then* $A \cup \{x\} \in \mathcal{I}$ *or there is some* $a \in A \setminus B$ *such that* $B \cup \{a\} \in \mathcal{I}$.

Observe that $\emptyset \in \mathcal{I}$ if $\mathcal{I}$ is an independence system. Dependence relations and systems of independence describe the same concept:

Proposition 1.3 *If* $\mathcal{I}$ *is a system of independence of a set* X *then we define a relation between elements and subsets of* X *by*

$$x \ll_{\mathcal{I}} A \;:\Longleftrightarrow\; x \in A \text{ or there is some } A_0 \subseteq A, A_0 \in \mathcal{I} \text{ such that } A_0 \cup \{x\} \notin \mathcal{I}.$$

If $\ll$ *is a dependence relation of* X *then we define*

$$\mathcal{I}(\ll) := \{A | A \text{ is finite and } a \not\ll A \setminus \{a\} \text{ for all } a \in A\}.$$

(i) *If* $\ll$ *is a dependence relation of* X, *then* $\mathcal{I}(\ll)$ *is a system of independence and*

$$\ll_{\mathcal{I}(\ll)} = \ll .$$

(ii) *If* $\mathcal{I}$ *is a system of independence of* X, *then* $\ll_{\mathcal{I}}$ *is a dependence relation and*

$$\mathcal{I}(\ll_{\mathcal{I}}) = \mathcal{I}.$$

Proof This is a folklore fact, we omit the easy proof. □

If $\mathcal{I}$ is a system of independence of X with corresponding dependence relation $\ll$ and $A \subseteq X$, then we write $\mathcal{I} - \mathrm{rk}(A)$ or $\ll -\mathrm{rk}(A)$ respectively, for the cardinality of a basis – i.e. a maximal $\ll$-independent subset–of A.

1.1 The realization rank

We always work with small subsets of a large *o*-minimal structure $\mathfrak{M}$ expanding a dense linear order without endpoints; that means $\mathfrak{M}$ will be λ-big for some large infinite cardinal λ, whereas "small" means "of cardinality λ" (cf. [4], 10.1). $\mathfrak{M}$ is not mentioned always.

Moreover we fix a (small) subset A of $\mathfrak{M}$. A is always assumed to be definably closed. For a set X, $\mathrm{cl}(X)$ denotes the definable closure of X (in $\mathfrak{M}$). If $D \subseteq \mathfrak{M}$ is definably closed, then $D\langle X\rangle$ also denotes $\mathrm{cl}(D \cup X)$.

Lemma 1.4 *If p is a 1-type over A and $A \subseteq B \subseteq \mathfrak{M}$, then the following conditions are equivalent.*

(i) *p has a unique extension to B.*
(ii) *If p is realized in* $\mathrm{cl}(B)$ *then p is realized in A.*

Proof The set A is definably closed. Therefore each formula with parameters in A with one free variable is equivalent to a quantifier free formula of the language $\{<\}$ with parameters in A. Now the lemma follows easily. □

Definition 1.5 *If B is a subset of $\mathfrak{M}$ and if c is an element from $\mathfrak{M}$, we say that c is* dominated *by B over A (or A-dominated by B) and write $c \lhd_A B$, if $tp(c/A)$ is realized in* $\mathrm{cl}A \cup B$*; otherwise c is called A-* indominated *by B.*

Counterexample 1.6 A-dominance is not a dependence relation, since transitivity is violated. To see an example let $\mathfrak{M}$ be a big real closed field containing $\mathbb{R}$, take $A = R_0$ to be the real closure of $\mathbb{Q}$ and let $\mu \in \mathfrak{M}$ be positive and infinitesimal over $\mathbb{R}$. Then

(*a*) $\mu \in R_0(\pi, \pi + \mu)$, thus μ is R_0-dominated by $\{\pi, \pi + \mu\}$.
(b) $\pi + \mu$ is R_0-dominated by $\{\pi\}$.
(c) μ is R_0-indominated by $\{\pi\}$.

In spite of this example, the A-dominance relation leads to a dependence relation. Before introducing this relation we prove that $\lhd_A$ satisfies axioms (D1)–(D4) of a dependence relation. We suppress the index A and write dominated or indominated only. The set A is always fixed and, as mentioned in the beginning, definably closed.

Certainly we have for all $c \in \mathfrak{M}$ and all $B, C \subseteq \mathfrak{M}$:

(D1) c is dominated by $\{c\}$.
(D2) c dominated by B, $B \subseteq C \Rightarrow c$ dominated by C.
(D3) if c is dominated by B, then there is a finite subset B_0 of B, such that c is dominated by B_0.

From Lemma 1.4 we know for any element $c \notin A$ the equivalence of

(i) c is indominated by B.
(ii) $tp(c/A \cup B)$ is the unique extension of $tp(c/A)$ on $A \cup B$.
(iii) If $c' \in \mathfrak{M}$ such that $tp(c/A) = tp(c'/A)$, then $c' \notin \mathrm{cl}A \cup B$.

Exchange Lemma for A-dominance 1.7 *If c is indominated by B and dominated by Bd, then d is dominated by Bc.*

Proof We search for a realization of $tp(d/A)$ in $\mathrm{cl}(ABc)$. Since c is dominated by Bd there is some realization $c' \in \mathrm{cl}(ABd)$ of $tp(c/A)$. Since c is indominated by B, it follows that $c' \notin \mathrm{cl}(AB)$. From the exchange lemma for the definable closure "cl" in o-minimal structures (cf. [6], Theorem 4.1) we get $d \in \mathrm{cl}(ABc')$. Since c is indominated by B and $t(c'/A) = t(c/A)$ it follows from the equivalence preceding our lemma that $tp(c/AB) = tp(c'/AB)$. Let σ be an $(A \cup B)$-automorphism of $\mathfrak{M}$ such that $\sigma(c') = c$. Then $\sigma(d) \in \mathrm{cl}ABc$ is a realization of $tp(d/A)$ as desired. □

The next proposition implies a variant of transitivity for A-dominance, which we will use to define a system of independence.

Proposition 1.8 *Let I be an index set, let $\{b_i \mid i \in I\}$, C and D be sets such that for each $i \in I$, b_i is indominated by $C \cup \{b_j | j \neq i\}$. Suppose b_i is dominated by $C \cup D$ for every $i \in I$. Then $tp((b_i)_{i\in I}/A \cup C)$ is realized in* $\mathrm{cl}\ A \cup C \cup D$. *More precisely: If b_i' is a realization of $tp(b_i/A)$ in* $\mathrm{cl}\ A \cup C \cup D$, *then $(b_i')_{i \in I}$ is a realization of $tp((b_i)_{i\in I}/A \cup C)$.*

Proof We have $b_i \neq b_j$ if $i \neq j$ and it is enough to prove the Proposition for finite I. We do an induction on $n = \mathrm{card}\ I$:

$n = 1$: suppose b is dominated by $C \cup D$, indominated by C and b' realizes $tp(b/A)$ in $\mathrm{cl}\ A \cup C \cup D$. Since b is indominated by C the type $tp(b/A \cup C)$ is realized by b' too.

$n \to n+1$. Suppose $\{b_1, \ldots, b_{n+1}\}$ is indominated by C and b_i is dominated by $C \cup D$. Let $b_1', \ldots, b_{n+1}' \in \mathrm{cl}\ A \cup C \cup D$ be realizations of $tp(b_1/A), \ldots, tp(b_{n+1}/A)$ respectively. By the induction hypothesis we have $tp(b_1, \ldots, b_n/A \cup C) = tp(b_1', \ldots, b_n'/A \cup C)$.

Let σ be an $A \cup C$-automorphism of $\mathfrak{M}$ such that $\sigma(b_i) = b_i'$ $(1 \leq i \leq n)$. Since b_{n+1} is indominated by $C \cup \{b_1, \ldots, b_n\}$, we see that $\sigma(b_{n+1})$ is indominated by $C \cup \{b_1', \ldots, b_n'\}$, that is $tp(f(b_{n+1})/A \cup C \cup \{b_1', \ldots, b_n'\}) = tp(b_{n+1}'/A \cup C \cup \{b_1', \ldots, b_n'\})$. Hence $(b_1', \ldots, b_{n+1}')$ is a realization of $tp(b_1, \ldots, b_{n+1}/A \cup C)$. □

Corollary and Definition 1.9 *Let $A, C \subseteq \mathfrak{M}$ and let A be definably closed. For elements $x \in \mathfrak{M}$ and subsets B of $\mathfrak{M}$ we define $x \triangleleft_{A,C} B :\iff x \triangleleft_A (C \cup B)$. Then $\triangleleft_{A,C}$ satisfies properties (D1)–(D4) of a dependence relation (cf. Definition 1.1).*

Proof Properties (D1)–(D3) are obviously true for $\triangleleft_{A,C}$. (D4) holds by Exchange Lemma for A-dominance 1.7. □

Definition 1.10 *Let $A, C \subseteq \mathfrak{M}$ and let A be definably closed. We define*

$$\mathcal{I}(A, C) := \{B \subseteq \mathfrak{M} | B \textit{ is finite and for all } b \in B \textit{ we have } b \not\triangleleft_{A,C} B \setminus \{b\}\}.$$

Proposition 1.11 *$\mathcal{I}(A, C)$ is a system of independence.*

Proof Certainly, property (I1) of an independence system holds for $\mathcal{I}(A, C)$ and we show that also property (I2) of an independence system holds for $\mathcal{I}(A, C)$.

To see this let $B, D \in \mathcal{I}(A, C)$ and let $x \notin D$. Suppose $B \cup \{x\} \notin \mathcal{I}(A, C)$ and $D \cup \{b\} \notin \mathcal{I}(A, C)$ for all $b \in B$. We have to show $D \cup \{x\} \notin \mathcal{I}(A, C)$. Since $\triangleleft_{A,C}$ satisfies (D1)–(D4) (cf. Corollary and Definition 1.9), this means $x \triangleleft_{A,C} B$ and $b \triangleleft_{A,C} D$ for all $b \in B$. As $B \in \mathcal{I}(A, C)$ we can apply Proposition 1.8:

Let $B = \{b_1, \ldots, b_n\}$ and let F be an $A \cup C$-definable map, such that $F(b_1, \ldots, b_n)$ is a realization of $tp(x/A)$. From Proposition 1.8 we know that $tp(b_1, \ldots, b_n/A \cup C)$ is realized in $\mathrm{cl} A \cup C \cup D$ by some n-tuple $(b'_1, \ldots, b'_n)$. If σ is an $A \cup C$-automorphism of $\mathfrak{M}$ such that $\sigma(b_i) = b'_i$ then $\sigma(F(b_1, \ldots, b_n)) = F(b'_1, \ldots, b'_n)$ is a realization of $tp(x/A)$ in $\mathrm{cl} A \cup C \cup D$.

Hence $x \triangleleft_{A,C} D$ and $D \cup \{x\} \notin \mathcal{I}(A, C)$ as desired. □

Notations 1.12 *The dependence relation corresponding to $\mathcal{I}(A, C)$ as explained in Proposition 1.3 is denoted by $\ll_{A,C}$. The dimension associated with $\ll_{A,C}$ is denoted by $\mathrm{rk}_{A,C}$ and is called the realization rank with respect to A, C.*

If the set C is contained in A we write $\ll_A$ and rk_A instead of $\ll_{A,C}$ and $\mathrm{rk}_{A,C}$. A set B is called $\ll_{A,C}$-independent if every finite subset of B is in $\mathcal{I}(A, C)$.

Proposition 1.13 *We have for every set $B \subseteq \mathfrak{M}$:*

(i) *B is $\ll_{A,C}$-independent if and only if $b \not\triangleleft_{A,C} B \setminus \{b\}$ for all $b \in B$.*
(ii) *For all $x \in \mathfrak{M}$, $x \ll_{A,C} B \iff x \triangleleft_{A,C} B_0$ for some $\ll_{A,C}$-independent subset B_0 of B.*
(iii) $\mathrm{rk}_{A,C}(B) = \min\{$*card* $B_0 \mid B_0 \subseteq B$ *and* $b \triangleleft_{A,C} B_0$ *for all* $b \in B\}$

Proof (i) holds by definition of $\ll_{A,C}$ and since $\triangleleft_{A,C}$ satisfies (D1)–(D4) and (ii) is implied by (i).

(iii) $\geq$ holds, since by (ii), for a $\ll_{A,C}$-basis B_0 of B we have $b \triangleleft_{A,C} B_0$ for all $b \in B$.

Conversely let B' be a $\ll_{A,C}$-basis of B and let $B_0 \subseteq B$, such that each $b \in B$ is A-dominated by $B_0 \cup C$. By Proposition 1.8 the type of B' over $A \cup C$ is realized in $\mathrm{cl}(A \cup C \cup B_0)$. Since $\dim B'/A \cup C = \mathrm{card}\, B'$ it follows that $\dim B_0/A \cup C \geq \mathrm{card}\, B'$. Hence $\mathrm{rk}_{A,C} B = \mathrm{card}\, B' \leq \dim B_0/A \cup C \leq \mathrm{card}\, B_0$. □

A set $B_0 \subseteq B$, which is minimal with the property

$$b \in B \;\Rightarrow\; b \text{ is dominated by } C \cup B_0$$

need not be indominated over C. Look at the following example.

Examples Here are three examples which shows that the ranks $\mathrm{rk}_{A,C}$ do not behave as one might expect. Let R_0 be the real closure of $\mathbb{Q}$ in $\mathbb{R}$ and let μ be some positive infinitesimal. Then we have

(i) $\mathrm{rk}_{R_0}(R_0(\pi + \mu, \pi)/R_0) = 2$. But the set $\{\pi + \mu, \pi\}$ is not an R_0-dominance basis of $R_0(\pi + \mu, \pi)$ over R_0. In particular $\ll_{R_0}$ is different from $\triangleleft_{R_0}$.
(ii) $\mathrm{rk}_{R_0,\mu}(\{\pi + \mu, \pi\}) = 1 = \mathrm{rk}_{R_0}(\{\pi + \mu, \pi\}/R_0)$
and

$\mathrm{rk}_{R_0,\{\pi,\pi+\mu\}}(\mu) = 0 \neq 1 = \mathrm{rk}_{R_0}(\mu)$
That is: the symmetry

$$\mathrm{rk}_{A,D}(B) = \mathrm{rk}_A B \;\Rightarrow\; \mathrm{rk}_{A,B}(D) = \mathrm{rk}_A(D)$$

does not hold in general.

(iii) If A is a subset of $\mathbb{R}$, and B is an arbitrary set, then $\mathrm{rk}_{A,\mathbb{R}}(B) \leq 1$, since $\mathbb{R}$ is Dedekind complete. Hence, if $p = tp(\mu, \pi/R_0)$ and (α, β) is another realization of p, we have

$$\mathrm{rk}_{R_0,\mathbb{R}}(\{\alpha, \beta\}) \leq 1 < 2 = \mathrm{rk}_{R_0}(\mu, \pi).$$

Intuitively speaking this means that p cannot be extended to a type of $\mathbb{R}$ "in an independent way".

The next proposition gives a geometric interpretation of $\mathrm{rk}_{A,C}$.

Definition 1.14 *Let R be o-minimal and let C be subset of an elementary extension of R. Let p be an n-type over R. We say that p is* a box type over C *if p is uniquely determined as an element of $S_n(R \cup C)$ by those formulas from p which define the open boxes $\prod_{i=1}^{n}(a_i, b_i)$, $a_i, b_i \in R$.*

If $C \subseteq R$, we just say p is a box type.

So if p is a box type over C, then p has a unique extension to $S_n(R \cup C)$ and the open R-definable boxes containing p imply this extension.
Note that if $\bar{a} \in R^n$, then $\{tp(\bar{a}/R)\}$ is a neighborhood of $tp(\bar{a}/R)$, which does not contain an open box.

Proposition 1.15 *Let R be o-minimal and let C be a subset of an elementary extension of R. If $p \in S_n(R)$, then the following conditions are equivalent:*

(i) *For some (hence for each) realization $\bar{\alpha}$ of p we have* $\mathrm{rk}_{R,C}(\bar{\alpha}) = n$.
(ii) *p is a box type over C.*
(iii) *If $p_1, \ldots, p_n$ are the projections of p onto the coordinate axis, then each p_i is a cut of R and p is the unique n-type over $R \cup C$ containing each p_i.*

Proof Obviously each of the conditions (i) and (ii) imply $\dim p = n$.

(i)⇒(ii). By induction on n. If $n = 1$, then p is omitted in $R\langle C\rangle$, thus (ii) holds. For the induction step, let $\bar{\alpha}$ to be an $n-1$-tuple and let β be an element, such that p is realized by $\bar{\alpha}\hat{\ }\beta$ with $\mathrm{rk}_{R,C}(\bar{\alpha}\beta) = n$. By the induction hypothesis, $tp(\bar{\alpha}/R)$ is a box type over C. Let X be an $R \cup C$-definable set which contains $tp(\bar{\alpha}, \beta/R \cup C)$. Since $\dim \bar{\alpha}, \beta/R \cup C = n$, we can suppose that X is an open cell $(F, G)_Y$, where F, G and Y are $R \cup C$-definable. We have $F(\bar{\alpha}) < \beta$. As $tp(\beta/R)$ is omitted in $R\langle C\bar{\alpha}\rangle$, there is some $a_1 \in R$ with $F(\bar{\alpha}) \leq a_1 < \beta$. Similar we can find some $a_2 \in R$ with $\beta < a_2 \leq G(\bar{\alpha})$. Since $tp(\bar{\alpha}/R)$ is a box type over C, there is an open R-definable box $Y_0 \subseteq \{\bar{b} \in Y \mid F(\bar{b}) \leq a_1,\ a_2 \leq G(\bar{b})\}$ with $tp(\bar{\alpha}/R) \in Y_0$. Finally $Y \times (a_1, a_2)$ is an open box, which contains $tp(\bar{\alpha}, \beta/R \cup C)$ and which is contained in X.

(ii)⇒(i). We do again an induction on n. If $n = 1$, then (ii) implies that p is omitted in $R\langle C\rangle$, thus $\mathrm{rk}_{R,C}(\alpha) = 1$ for all realizations α of p. Assume $p \in S_n(R)$ is a box type over C and $\bar{\alpha}\hat{}\beta$ is a realization of p. Certainly $tp(\bar{\alpha}/R)$ is a box type over C and by the induction hypothesis $\mathrm{rk}_{R,C}(\bar{\alpha}) = n-1$. We have to show that $tp(\beta/R)$ is omitted in $R\langle C\bar{\alpha}\rangle$: Let F be an $R \cup C$-definable map, say $F(\bar{\alpha}) < \beta$. Let $Y \subseteq R^{n-1}$ be an open box and let $a_1 < a_2 \in R$ with $p \in Y \times (a_1, a_2)$ and $Y \times (a_1, a_2) \subseteq \{(\bar{b}, b') \in R^n \mid F(\bar{b}) < b'\}$. That is $F(\bar{b}) \leq a_1$ for all $\bar{b} \in Y$, hence $F(\bar{\alpha}) \leq a_1 < \beta$.

(ii)⇔(iii) If $p_1, \dots, p_n$ are the projections of p and each p_i is a cut over R, then the intersection of all open boxes containing p in $S_n(R \cup C)$ is the set of all n-types $q \in S_n(R \cup C)$ which contain $p_1, \dots, p_n$. □

The next corollary and the subsequent remark will not be used later on. They relate the notion "box type" to the real spectrum (cf. [1]), for the reader who is aquainted with this point of view. Recall that quantifier elimination for real closed fields says that for every real closed field R, the natural map $S_n(R) \longrightarrow \mathrm{Sper}\, R[t]$, $t = (t_1, \dots, t_n)$ is a bijection. We say that an element $p \in \mathrm{Sper}\, R[t]$ is a box type if the corresponding n-type is a box type.

Corollary 1.16 *If R is a real closed field and $p \in$ Sper $R[t]$, $t = (t_1, \dots, t_n)$ is a box type such that R is archimedean in the real closure of p, then p is minimal and maximal in* Sper $R[t]$.

Proof Since $\dim p = n$, p is minimal in Sper $R[t]$. On the other hand, if $q \in$ Sper $R[t]$ is different from p then there is an open box B containing p and not containing q. Since R is archimedean in the real closure of p, we can find a smaller open box B' containing p with $\overline{B'} \subseteq B$. Hence p does not specialize to q. □

Observe that the converse of Corollary 1.16 fails in general. The reason is that a semi-algebraic homeomorphism $R^n \longrightarrow R^n$ respects the topology of Sper $R[t]$ – hence minimal, maximal points are mapped to minimal, maximal points – but not the property "p is a box type".

In other words, box types cannot be detected with the topology of Sper $R[t]$.

Proposition 1.17 *Let $R \prec \mathfrak{M}$ and let $B, C \subseteq \mathfrak{M}$ such that B is $\ll_{R,C}$-independent. Then $tp(B/R \cup C)$ is the unique extension of $tp(B/R)$. In particular $tp(B/R \cup C)$ is an heir of $tp(B/R)$ (cf.* [4]*, p.* 292*, for the definition of "heir")*

Proof We may assume that $B = \{b_1, \dots, b_n\}$ is finite and we do an induction on n. If $n = 1$, then $tp(b_1/R \cup C)$ is the unique extension of $tp(b_1/R)$, since $b_1 \not\triangleleft_R C$. In the induction step we have: $tp(b_1, \dots, b_n/R \cup C)$ is the unique extension of $tp(b_1, \dots, b_n/R)$ (from the induction hypothesis) and $tp(b_{n+1}/R \cup C \cup \{b_1, \dots, b_n\})$ is the unique extension of $tp(b_{n+1}/R)$ (since B is $\ll_{R,C}$-independent). These two properties are equivalent to the property that tp $(b_1, \dots, b_{n+1}/R \cup C)$ is the unique extension of $tp(b_1, \dots, b_{n+1}/R)$. □

1.2 Behavior of $\ll_R$ under base change

First a reminder on the functional version of the Marker–Steinhorn Theorem. Recall that an elementary extension $R \prec S$ of o-minimal structures is called **tame**, if every $s \in S$, which is R-bounded, is infinitely close to an element of R wit respect to R.

Theorem 1.18 *Let $R \prec S$ be a tame extension of o-minimal expansions of fields. Let V be the convex hull of R in S and let $\lambda : S \longrightarrow R \cup \{\infty\}$ be the place according to V. Furthermore let $X \subseteq S^n$ and $F : X \longrightarrow S$ be definable in (S, V) with parameters from S. For a subset Y of S^n let $H(Y) := \bigcup_{y \in Y} y + \mathfrak{m}_V^n$ denote the set of all points of S^n which are infinitely close to a point of Y with respect to R. Then*

(i) *The composed map*

$$\lambda F : F^{-1}(V) \cap R^n \longrightarrow S^n \xrightarrow{F} S \xrightarrow{\lambda} R \cup \{\infty\}$$

is R-definable. λF is the unique map $F^{-1}(V) \cap R^n \longrightarrow R$ with the property $(\lambda F)(\bar{a}) = \lambda(F(\bar{a}))(\bar{a} \in R \cap F^{-1}(V))$.

(ii) *There is a decomposition $R^n = E \,\dot\cup\, D \,\dot\cup\, D' \,\dot\cup\, C$ of R^n in R-definable sets, such that:*
  (a) *F is positive infinite on $H(D)$.*
  (b) *F is negative infinite on $H(D')$.*
  (c) *$F - (\lambda F)_S$ is infinitesimal on $H(C)$ and*
  (d) $\dim E < n$.

Proof This is [5], Theorem 3.3. □

Proposition 1.19 *Let $R \prec S$ be o-minimal expansions of fields and let B be from an elementary extension of S such that $b \not\triangleleft_R S$ for all $b \in B$.*

(i) *If B is $\ll_S$-independent then B is $\ll_R$-independent.*
(ii) *If R is tame in S, then B is $\ll_S$-independent if and only if B is $\ll_R$-independent.*

Proof (i) Suppose B is $\ll_S$-independent and not $\ll_R$-independent. By induction on n we may assume that there are $b, b_1, \ldots, b_n$ such that $\{b_1, \ldots, b_n\}$ is $\ll_R$-independent and such that $F(b_1, \ldots, b_n)$ and b realize the same cut of R for some R-definable map $F : R^n \longrightarrow R$. By assumption, $F(b_1, \ldots, b_n)$ and b realize the same cut of S, hence $\{b, b_1, \ldots, b_n\}$ is $\ll_S$-dependent, a contradiction.

(ii) Now suppose R is tame in S. Let V be the convex hull of R in S and let $\lambda : S \longrightarrow R \cup \{\infty\}$ be the place according to V. Suppose B is $\ll_R$-independent and not $\ll_S$-independent. Again, by induction we find $b, b_1, \ldots, b_n \in B$ such that $\{b_1, \ldots, b_n\}$ is $\ll_S$-independent, but for some S-definable map $F : S^n \longrightarrow S$, the element $F(b_1, \ldots, b_n)$ induces the same cut over S as b. Let $Z := F^{-1}(V) \cap R^n$

and $\lambda F : Z \longrightarrow R$ as in Theorem 1.18 (i). Let $R^n = E \,\dot\cup\, D \,\dot\cup\, D' \,\dot\cup\, C$ of R^n be a decomposition as in 1.18(ii).

Since $F(b_1, \ldots, b_n) = b$ is R-bounded, there is an S-definable set Z_0 such that F is R-bounded on Z and such that $Z_0 \in tp(b_1, \ldots, b_n)$. By Proposition 1.15 and since $\{b_1, \ldots, b_n\}$ is $\ll_S$-independent, there is an S-definable, open box O such that $\overline{O} \subseteq Z_0 \setminus E$ and such that $O \in tp(b_1, \ldots, b_n)$. Since $b_i \not\lhd_R S$ $(1 \leq i \leq n)$, we may shrink O so that O is R-definable. From Theorem 1.18 (ii) we get that $F - (\lambda F)_S$ has values in $\mathfrak{m}_V$ on $H(\overline{O}) \supseteq \overline{O}_S$. But then also $F(b_1, \ldots, b_n) - \lambda F(b_1, \ldots, b_n)$ is infinitesimal with respect to R. Since the cut of b over R is not definable and $F(b_1, \ldots, b_n)$ realizes this cut, also $\lambda F(b_1, \ldots, b_n)$ realizes this cut. Since λF is R-definable, $\{b, b_1, \ldots, b_n\}$ is $\ll_R$-dependent. □

Lemma 1.20 *Let $R \prec S$ be o-minimal expansions of fields and let b be from an elementary extension of S.*

(i) *If S is dense in $S\langle b\rangle$ and $b \not\lhd_R S$ then R is dense in $R\langle b\rangle$.*
(ii) *If R is dense in $R\langle b\rangle$ and S does not contain infinitesimal elements with respect to R, then S is dense in $S\langle b\rangle$.*

Proof (i) Suppose there are $\alpha, \beta \in R\langle b\rangle$, $\alpha < \beta$ with $(\alpha, \beta) \cap R = \emptyset$. We may assume that $\alpha, \beta \notin R$. Since S is dense in $S\langle b\rangle$ there is some $s \in S$ with $\alpha < s < \beta$, thus $tp(s/R) = tp(\alpha/R)$. Since $\alpha \in R\langle b\rangle \setminus R$ there is an R-definable map $f : S\langle b\rangle \longrightarrow S\langle b\rangle$ such that $f(\alpha) = b$. Hence $f(s)$ realizes $tp(b/R)$, a contradiction.

(ii) Suppose S is not dense in $S\langle b\rangle$ and S does not contain infinitesimal elements with respect to R. Take $\alpha, \beta \in S\langle b\rangle \setminus S$ with $\alpha < \beta$ such that $(\alpha, \beta) \cap S = \emptyset$, in particular $tp(\alpha/S) = tp(\beta/S)$. Let $f : S \longrightarrow S$ be S-definable such that $f(\alpha) = b$. Then f is strictly monotonic on the realizations of $tp(\alpha/S)$, hence $\gamma := f(\beta) \neq b$ is a realization of $tp(b/S)$. Say $b < \gamma$. Since S does not contain infinitesimal elements with respect to R there is some $m \in R$ such that $0 < m < \gamma - b$. Then $b < b + m < \gamma$ and there is no element in R between b and $b + m$. □

Proposition 1.21 *Let $R \prec S$ be an o-minimal expansions of fields and let $B, D \subseteq S$. Let B be $\ll_R$-independent such that R is neither dense nor tame in $R\langle b\rangle$ for all $b \in B$. If D is another $\ll_R$-independent set such that R is dense in $R\langle d\rangle$ for each $d \in D$, then $R\langle B\rangle$ is dense in $R\langle B \cup D\rangle$ and $B \cup D$ is $\ll_R$-independent.*

Proof We may assume that $B = \{b_1, \ldots, b_n\}$ and $D = \{d_1, \ldots, d_k\}$ are finite. First observe that R is archimedean in $R\langle B\rangle$, otherwise by induction, b_n is infinitely close to some $c \in R\langle b_1, \ldots, b_{n-1}\rangle$ and R is archimedean in $R\langle b_1, \ldots, b_{n-1}\rangle$. But then either b_n has a definable type over R or c and b_n have the same type over R, a contradiction to our assumption.

By Lemma 1.20 (ii) applied to $R \prec R\langle B\rangle$ and d_1, $R\langle B\rangle$ is dense in $R\langle B, d_1\rangle$. By Lemma 1.20(ii) applied to $R \prec R\langle B, d_1\rangle$ and d_2, $R\langle B, d_1\rangle$ is dense in $R\langle B, d_1, d_2\rangle$. Continuing in this way we see that $R\langle B\rangle$ is dense in $R\langle B \cup D\rangle$.

Now we prove by induction on n that $B \cup D$ is $\ll_R$-independent. Suppose we know that $\{b_1, \ldots, b_{n-1}\} \cup D$ is $\ll_R$-independent. Suppose $tp(b_n/R)$ is realized in

$R\langle\{b_1,\ldots,b_{n-1}\}\cup D\rangle$. Since B is $\ll_R$-independent, $tp(b_n/R\langle b_1,\ldots,b_{n-1}\rangle)$ is realized in $R\langle\{b_1,\ldots,b_{n-1}\}\cup D\rangle$. Since $R\langle b_1,\ldots,b_{n-1}\rangle$ is dense in $R\langle\{b_1,\ldots,b_{n-1}\}\cup D\rangle$, also $R\langle b_1,\ldots,b_{n-1}\rangle$ is dense in $R\langle B\rangle$. By Lemma 1.20 (i), R is dense in $R\langle b_n\rangle$ a contradiction. □

2 *V*-limits

Let $K\subseteq L$ be ordered fields. In this section we study elements b of $L\setminus K$ which become limits of Cauchy sequences of K after passing to some residue field of a convex valuation ring V of K. It turns out that this property only depends on the cut that b generates over K, these cuts are then called V-limits.

We first recall some notions from [9]. If X is a totally ordered set, then a cut p of X is a tuple $p=(p^L,p^R)$ with $X=p^L\cup p^R$ and $p^L<p^R$. If $Y\subseteq X$ then Y^+ denotes the cut p of X with $p^R=\{x\in X|x>Y\}$. Y^+ is called the upper edge of Y. Similarly the lower edge Y^- of Y is defined.

Definition 2.1 *Let p be a cut of an ordered abelian group K, The convex subgroup*

$$G(p):=\{a\in K|a+p=p\}$$

of K is called the invariance group *of p (here $a+p:=(a+p^L,a+p^R)$).*

If K is an ordered field, then the convex valuation ring

$$V(p):=\{a\in K|a\cdot G(p)\subseteq G(p)\}$$

is called the invariance valuation ring *of p. If $s\notin K$ is from an ordered field extension of K then we write $G(s/K)$ and $V(s/K)$ for the invariance group and the invariance ring of the cut induced by s on K.*

Definition 2.2 *Let K* be a divisible *ordered abelian group and let p be a cut of K. We may define the signature of p as*

$$\operatorname{sign}p:=\begin{cases}1 & \text{if there is a convex subgroup } G \text{ of } K \text{ and some } a\in K \text{ with } p=a+G^+\\ -1 & \text{if there is a convex subgroup } G \text{ of } K \text{ and some } a\in K \text{ with } p=a-G^+\\ 0 & \text{otherwise}\end{cases}$$

Since K is divisible we cannot have $a+G^+=b+H^-$ for $a,b\in K$ and convex subgroups G,H of K. Hence the signature is well defined.

In what follows the units of a ring A will be denoted by A^*.

Definition 2.3 *Let K be an ordered field and let $V\subseteq K$ be a convex valuation ring with maximal ideal $\mathfrak{m}_V$. A cut p of K is called a V*-limit *if* $\operatorname{sign}p=0$ *and if there is some $a\in K^*$ such that $G(p)=a\cdot\mathfrak{m}_V$. Observe that $V(p)=V$ in this case.*

If in addition $G(p)=\mathfrak{m}_V$ and $\mathfrak{m}_V^+\le p\le V^+$, then p is called a proper V-limit. *Observe that $\mathfrak{m}_V^+<p<V^+$ in this case, as* $\operatorname{sign}p=0$.

An element b from an ordered field extension L of K is called a (proper) V-limit if $b \notin K$ and if the cut of b induced on K is a (proper) V-limit.

The next proposition states some reformulations of the notion "proper V-limit". First some notations. If K is an ordered field, then a sequence $(a_\alpha)_{\alpha<\lambda}$ of elements of K is called a Cauchy sequence, if it is a Cauchy sequence with respect to the order topology of K. Observe that for a non-trivial convex valuation ring V of K, a Cauchy sequence with respect to V in the valuation theoretic sense (cf. [8]) is a Cauchy sequence in our sense. Recall, if $(a_\alpha)_{\alpha<\lambda}$ is a Cauchy sequence, then a subsequence of $(a_\alpha)_{\alpha<\lambda}$ is a Cauchy sequence with respect to V in the valuation theoretic sense.

An element b from an ordered field extension of K is the limit of a Cauchy sequence $(a_\alpha)_{\alpha<\lambda}$ of K if

$$\forall \varepsilon \in K,\ \varepsilon > 0\ \exists \alpha_0 < \lambda\ \forall \alpha > \alpha_0\ |b - a_\alpha| < \varepsilon.$$

If T is an o-minimal extension of the theory of real closed fields, then a convex valuation ring V of a model R of T is called T-convex, if V is the convex hull of an elementary substructure of R. In this case, every maximal definably closed subfield $K \subseteq V$ is an elementary substructure of R (cf. [2]).

If T is the theory of real closed fields, then all convex subrings of R are T-convex.

Proposition 2.4 *Let $L \subseteq M$ be an extension of ordered fields, let $W \subseteq M$ be a convex subring and let $V := W \cap L$. Let $K \subseteq W$ be a subfield such that $K/\mathfrak{m}_W = V/\mathfrak{m}_W$. The following are equivalent for every $b \in M$:*

(i) *b is a proper V-limit.*
(ii) *$b \in W^*$ and $b/\mathfrak{m}_W$ is the limit of a Cauchy sequence of $V/\mathfrak{m}_V$ without limits in $V/\mathfrak{m}_V$.*
(iii) *b is the limit of a Cauchy sequence of K without limits in K.*
(iv) *$b \notin K$ and K is dense in the ordered group $K + bK$.*
(v) *$b \notin K$,* $\operatorname{sign}(b/K) = 0$ *and* $G(b/K) = \{0\}$.

Proof We may assume that $b > 0$.

(i)⇒(v) First we prove that $b \notin V + \mathfrak{m}_W$. Suppose $b - a \in \mathfrak{m}_W$ for some $a \in V$, say $a < b$. Since $\operatorname{sign} b/L = 0$ and $G(b/L) = \mathfrak{m}_V$, there is some $c \in V$, $c > \mathfrak{m}_V$ with $a + c < b$. Hence $\mathfrak{m}_V < c < b - a \in \mathfrak{m}_W$ in contradiction to $V = L \cap W$.

This proves $b \notin K$ and for all $c \in K$, $a \in V$ with $c - a \in \mathfrak{m}_W$ we have $c < b$ iff $a < b$.

Let $c \in K$, $c > 0$. We prove that $c \notin G(b/K)$. Let $v \in V$ with $c - v \in \mathfrak{m}_W$, say $v < c$. Then $v > \mathfrak{m}_V$, since $c > 0$. Since $G(b/L) = \mathfrak{m}_V$ there is some $a \in V^{>0}$ such that $a < b < a + v$. Let $c_1 \in K$ with $c_1 - a \in \mathfrak{m}_W$. Then also $a + v - (c + c_1) \in \mathfrak{m}_W$ and by what we have shown, $a < b < a + v$ implies $c_1 < b < c_1 + c$. Thus $c \notin G(b/K)$ as desired.

It remains to show that $\operatorname{sign}(b/K) = 0$, say $\operatorname{sign}(b/K) \geq 0$. Since $G(b/K) = 0$ it is enough to find for every element $c \in K$ with $c < b$ an element $c_1 \in K, c_1 > 0$

with $c + c_1 < b$. Let $a \in V$ with $c - a \in \mathfrak{m}_W$. Then $a < b$ and from $\mathrm{sign}(b/L) = 0$, $G(b/L) = \mathfrak{m}_V$ we get some $v \in V$, $v > \mathfrak{m}_V$ with $a + v < b$. Take $c_1 \in K$ with $c_1 - v \in \mathfrak{m}_W$. Then $c_1 > 0$ and $c + c_1 < b$ since $c + c_1 - (a + v) \in \mathfrak{m}_W$.

(v)⇒(i). First we prove that $b \notin L + \mathfrak{m}_W$. Suppose $b - a \in \mathfrak{m}_W$ for some $a \in L$. Since $\mathrm{sign}(b/K) = 0$, there is some $v \in V$ with $b < v$. But then also $a \in V$. Let $c \in K$ with $a - c \in \mathfrak{m}_W$. Then $b - c \in \mathfrak{m}_W$ and there is no element in K between b and c. This implies that the cut of b over K is definable, a contradiction to $\mathrm{sign}\, b/K = 0$.

Hence $b \notin L + \mathfrak{m}_W \supseteq V + \mathfrak{m}_W = K + \mathfrak{m}_W$. We prove $G(b/L) = \mathfrak{m}_V$. First let $v \in \mathfrak{m}_V$, $v \geq 0$ and suppose there is some $l \in L$ with $b < l < b + v$. Then $l - b \in \mathfrak{m}_W$ in contradiction to $b \notin L + \mathfrak{m}_W$. Hence $\mathfrak{m}_V \subseteq G(b/L)$. Conversely let $a \in V$, $a > \mathfrak{m}_V$ and take $c \in K$ with $a - c \in \mathfrak{m}_W$. Then $c > 0$ and since $G(b/K) = 0$ there is some $c_1 \in K$, $c_1 > 0$ with $c_1 < b < c_1 + c$. Let $a_1 \in V$ with $a_1 - c_1 \in \mathfrak{m}_W$. Then $a_1 < b < a_1 + a$, hence $a \notin G(b/L)$.

Thus we know $G(b/L) = \mathfrak{m}_V$ and it remains to show that $\mathrm{sign}(b/L) = 0$. Suppose there is some $a \in V$ such that the cut η of b over L is $a \pm \mathfrak{m}_V^+$, say $\eta = a + \mathfrak{m}_V^+$. Let $c \in K$ with $a - c \in \mathfrak{m}_W$. Then $c < b$. Since $\mathrm{sign}(b/K) = 0$ there is some $c_1 \in K$, $c_1 > 0$ with $c + c_1 < b$. Let $a_1 \in V$ with $c_1 - a_1 \in \mathfrak{m}_W$. Then $a + a_1 < b$ and $a_1 > \mathfrak{m}_W \supseteq \mathfrak{m}_V$, in contradiction to $\eta = a + \mathfrak{m}_V^+$.

So we know that (i) is equivalent to (v). The equivalences (i)⇔(ii)⇔(iii) and (iv)⇔(v) are easy and left to the reader. □

Remarks Observe that an ordered field K need not be dense in $K(b)$ if b is the limit of a Cauchy sequence of K without limits in K. For example if $K = \mathbb{Q}$, $\varepsilon \neq 0$ is infinitesimal and $b = \sqrt{2 + \varepsilon}$. Also, a field K as in Proposition 2.4 cannot be found inside V in general. For example if $L = \mathbb{Q}(\sqrt{2} + \varepsilon)$, where ε is infinitesimal and V is the convex hull of $\mathbb{Q}$ in L. Then $\mathbb{Q}$ is the unique subfield of V and $V/\mathfrak{m}_V \cong \mathbb{Q}(\sqrt{2})$.

Here is another reformulation of the notion "proper V-limit" in terms of so-called distinguished Cauchy sequences as explained in [8], section D:

If (K, V_0) is a valued field, then a sequence $(a_\alpha)_{\alpha<\lambda}$ is called distinguished Cauchy sequence if $(a_\alpha)_{\alpha<\lambda}$ is a pseudo Cauchy sequence of the valued field (K, V_0), such that $a_\alpha \in V_0$ for all α and such that for some valuation ring V of K with $V_0 \subsetneqq V$ the $(a_\alpha - a_\beta)/V_0^*$ are unbounded in the convex subgroup V^*/V_0^* of K^*/V_0^*.

We call V the valuation ring associated to $(a_\alpha)_{\alpha<\lambda}$.

Corollary 2.5 *In the situation of Proposition 2.4, let $V_0 \subsetneqq V$ be another convex valuation ring. Then b is a proper V-limit if and only if b is the pseudo limit of a distinguished pseudo Cauchy sequences of the valued field (L, V_0) which does not have a limit in L and which has V as associated valuation ring.*

Proof Easily from Proposition 2.4. □

Proposition 2.6 *Let $L \subseteq M$ be an extension of ordered fields, let $W \subseteq M$ be a convex subring and let $V := W \cap L$. Let $K \subseteq W$ be a subfield such that $K/\mathfrak{m}_W = V/\mathfrak{m}_W$ and let $b \in M$. Then*

(i) *b is a V-limit if and only if there are $a_1, a \in L$, $a \neq 0$ such that $a_1 + ab$ is a proper V-limit.*
(ii) *If b is a V-limit and a V'-limit, where V' is a convex valuation ring of L, then $V = V'$.*
(iii) *If L, M are models of an o-minimal extension T of the theory of real closed fields, $K, L \prec M$ and if V is T-convex, then K is dense in $K\langle b\rangle$ if b is a proper V-limit.*

Proof (i) If $a_1, a \in L$, $a \neq 0$, then a straightforward computation shows that $G(a_1 + ab/L) = a\cdot G(b/L)$ and $\text{sign}(b/L) = \text{sign}(a_1 + ab/L)$.

So if $a_1, a \in L$, $a \neq 0$, such that $a_1 + ab$ is a proper V-limit, then $\text{sign}(b/L) = \text{sign}(a_1+ab/L) = 0$ and from $G(a_1+ab/L) = \mathfrak{m}_V$ we get $G(b/L) = \frac{1}{a}G(ab/L) = \frac{1}{a}G(a_1 + ab/L) = \frac{1}{a}\cdot\mathfrak{m}_V$, hence b is a V-limit.

Conversely if b is a V-limit, $a_0 \in L^*$ and $G(b/L) = a_0\cdot\mathfrak{m}_V$, then $G(b\cdot a_0^{-1}/L) = \mathfrak{m}_V$, so there is $a_1 \in L$ with $a_1 < b\cdot a_0^{-1} < a_1 + 1$. Then $a_1 - b\cdot a_0^{-1}$ is a proper V-limit and we may take $a := -a_0^{-1}$.

(ii) If b is a V-limit and a V'-limit, where V' is a convex valuation ring of L, then $G(b/L) = a\cdot\mathfrak{m}_V$ and $G(b/L) = a'\cdot\mathfrak{m}_{V'}$ for some $a, a' \in L^*$. But then $\mathfrak{m}_{V'} = \frac{a}{a'}\mathfrak{m}_V$ and this is only possible if $V = V'$.

(iii) Suppose b is a proper V-limit and K is not dense in $K\langle b\rangle$. Let $a, c \in K\langle b\rangle$ with $a < c$ and $(a, c) \cap K = \emptyset$. We may assume that $a, c \notin K$. There is a K-definable map $f : K \longrightarrow K$ such that $f(a) = b$. As a and c realize the same cut over K, b and $f(c)$ realize the same cut over K. Moreover f is strictly monotonic in $[a, c] \subseteq K\langle b\rangle$, say $b < f(c)$. Since $\text{sign}\, b/K = 0$ by Proposition 2.4(v), there is some $d \in K$ with $0 < d < f(c) - b$. As $(b, f(c)) \cap K = \emptyset$ we get $d \in G(b/K)$, a contradiction to $G(b/K) = 0$ (cf. Proposition 2.4(v)). □

Proposition 2.7 *Let T be an o-minimal expansion of fields in the language $\mathscr{L}$, let $R \models T$ and let $\mathscr{V}$ be a set of T-convex valuation rings of R. For each $V \in \mathscr{V}$ let $K_V \subseteq V$ be a maximal definably closed subfield of V. Let $S \succ R$ and for each $V \in \mathscr{V}$ let $B_V \subseteq S$ be a set of proper V-limits.*

Then $\bigcup_{V\in\mathscr{V}} B_V$ is $\ll_R$-independent if and only if B_V is $\ll_{K_V}$-independent for all $V \in \mathscr{V}$.

Proof We write $B := \bigcup_{V\in\mathscr{V}} B_V$. By Proposition 2.4 each $b \in B_V$ has signature 0 over K_V. Hence $b \not\lhd_{K_V} R$ for all $b \in B_V$. So if B is $\ll_R$-independent, then B_V is $\ll_R$-independent and by Proposition 1.19, B_V is $\ll_{K_V}$-independent for all $V \in \mathscr{V}$.

For the converse we may assume that $\mathscr{V}$ is finite, say $\mathscr{V} = \{V_1, \ldots, V_n\}$ and $V_1 \subsetneqq \cdots \subsetneqq V_n$. Let $B_i \subseteq B_{V_i}$ be finite. It is enough to prove by induction on n that $B = B_1 \cup \cdots \cup B_n$ is $\ll_R$-independent if each B_i is $\ll_{K_{V_i}}$-independent. If $n = 1$, then we know this from Proposition 1.19(ii).

Induction step. Let $L_i \subseteq V_i$ be a maximal definably closed subfield of V_i with $L_1 \subseteq \cdots \subseteq L_{n+1}$. From the case $n = 1$ we know that each B_i is $\ll_R$-independent. By what we have above, B_i is $\ll_{L_i}$-independent. By Proposition 2.4, each $b \in B_i$ is a $V_i \cap L_{n+1}$-limit. By the induction hypothesis, $B_1 \cup \cdots \cup B_n$ is $\ll_{L_{n+1}}$-independent.

By Proposition 2.6(iii), L_{n+1} is dense in $L_{n+1}\langle b\rangle$ for each $b \in B_{n+1}$. On the other hand if $b \in B_1 \cup \cdots \cup B_n$, then L_{n+1} is neither dense nor tame in $L_{n+1}\langle b\rangle$. By Proposition 1.21, $B_1 \cup \cdots \cup B_n \cup B_{n+1}$ is $\ll_{L_{n+1}}$-independent. Again by Proposition 1.19, $B_1 \cup \cdots \cup B_n \cup B_{n+1}$ is $\ll_R$-independent. □

3 The completion of an o-minimal structure

Proposition 3.1 *Let T be an o-minimal extension of the theory of real closed fields. Let $R \prec \mathcal{M}$ be models of T. Then there is a model S of T with $R \prec S \prec \mathcal{M}$, such that:*

(i) *R is dense in S.*
(ii) *If R' is an elementary substructure of $\mathcal{M}$, $R \prec R'$ and if R is dense in R', then there is an elementary embedding $R' \longrightarrow S$ over R.*

The embedding in (ii) *is unique. If $R \prec S' \prec \mathcal{M}$ and S' has properties* (i) *and* (ii)*, then there is a unique R-isomorphism $S \longrightarrow S'$.*

Proof Let $X \subseteq \mathcal{M}$ be the set of all $\alpha \in \mathcal{M}$, such that R is dense in $R\langle\alpha\rangle$. Let B be a $\ll_R$-basis of X over R. We claim, that $S := R\langle B\rangle$ has the required properties. Clearly R is an elementary substructure of S.

By Proposition 1.21, R is dense in S. Let $R' \prec \mathcal{M}$ be an elementary extension of R, such that R is dense in R'. Let B' be a transcendence basis of R' over R. Clearly B' is an $\ll_R$-basis of R'. By the choice of B, the type of every $b' \in B'$ over R is realized in $R\langle B\rangle$. By Proposition 1.8 we know that $tp(B'/R)$ is realized in $R\langle B\rangle = S$. Hence $tp(R'/R)$ is realized in S and there is an elementary R-embedding $R' \longrightarrow S$.

Both additions are obvious. □

Corollary 3.2 *Let T be an o-minimal extension of the theory of real closed fields. Let $R \prec \tilde{R} \prec \mathcal{M}$ be models of T, suppose that R is archimedean in $\tilde{R}$ and $\tilde{R}$ is tame in $\mathcal{M}$. We provide $\tilde{R}$ with the topology induced by the ordering of $\tilde{R}$. Let S be the topological closure $\overline{R}$ in this topology. Then $R \prec S \prec \tilde{R} \prec \mathcal{M}$ and S fulfills the conditions (i) and (ii) of Proposition* 3.1*, both for R and $\tilde{R}$ as well as for R and $\mathcal{M}$. We have*

$$S = \{\alpha \in \tilde{R} \mid R \text{ is dense in } R\langle\alpha\rangle\}$$

Proof Let $R \prec S_1 \prec \mathcal{M}$ as in Proposition 3.1 and let $S_1 \prec \tilde{S}_1 \prec \mathcal{M}$, such that S_1 is archimedean in $\tilde{S}_1$ and $\tilde{S}_1$ is tame in $\mathcal{M}$. Since $\tilde{S}_1$ and $\tilde{R}$ are isomorphic over R we can suppose that $S_1 \subseteq \tilde{R} = \tilde{S}_1$ (T is an expansion of RCF). Since R is archimedean in $\tilde{R}$, S_1 is contained in $\overline{R} = S$. If $\alpha \in \overline{R}$, then R is dense in $R\langle\alpha\rangle$. If $\alpha \in \tilde{R}$, such that R is dense in $R\langle\alpha\rangle$, then by Lemma 1.20, the set S_1 is dense in $S_1\langle\alpha\rangle$. By the choice of S_1 we get therefore $\alpha \in S_1$. This proves $S_1 = S = \{\alpha \in \tilde{R} \mid R \text{ is dense in } R\langle\alpha\rangle\}$. □

Proposition 3.1 applied to a sufficiently large, elementary extension $\mathfrak{M}$ of R yields

Corollary 3.3 *Let T be an o-minimal extension of the theory of real closed fields and let R be a model of T. Then there is a model $S \succ R$ with:*

(i) *R is dense in S.*
(ii) *If R' is an elementary extension of R and R is dense in R', then there is an elementary embedding $R' \longrightarrow S$ over R.*

The embedding in (ii) *is unique. S is uniquely determined up to a unique R-isomorphism by conditions* (i) *and* (ii). □

The model S in Corollary 3.3 is the largest elementary extension of R, such that R is dense in S. S is not dense in any proper elementary extension of S. S is called the **completion** of R and is denoted by $\hat{R}$

We get S by Corollary 3.2 in the following manner: choose $R \prec R_1 \prec \mathfrak{M}$ such that R is archimedean in R_1 (i.e. R_1 is the convex hull of R), R_1 is tame in $\mathfrak{M}$ and $\mathfrak{M}$ is $|R_1|^+$-saturated. Take

$$S = \{\alpha \in R_1 \mid R \text{ is dense in } R\langle\alpha\rangle\}$$

Since R is dense in $R\langle\alpha\rangle$ if and only if R is dense in the field $R(\alpha)$ (by Propositions 2.4 and 2.6(iii)), the underlying field of the completion of R does not depend on the theory T.

If V is a convex valuation ring of R and $\hat{V}$ is the convex hull of V in $\hat{R}$, then the valued field $(\hat{R}, \hat{V})$ is the completion of the valued field (R, V).

4 Definition of the Pseudo Completion

Theorem 4.1 *Let T be an o-minimal expansion of fields in the language $\mathscr{L}$, let $R \models T$ and let $\mathscr{V}$ be a set of T-convex valuation rings of R (the case $R \in \mathscr{V}$ is not excluded). For each $V \in \mathscr{V}$ let $K_V, L_V \subseteq V$ be maximal definably closed subfields of V.*

Let $S \succ R$ so that S contains completions $\hat{K}_V$ of K_V and $\hat{L}_V$ of L_V for all $V \in \mathscr{V}$. Then

(i) *There is an $\mathscr{L}$-isomorphism $\varphi : R\langle\bigcup_{V\in\mathscr{V}} \hat{K}_V\rangle \longrightarrow R\langle\bigcup_{V\in\mathscr{V}} \hat{L}_V\rangle$ over R sending $R\langle\hat{K}_V\rangle$ onto $R\langle\hat{L}_V\rangle$.*
(ii) *If $K_V = L_V$ for each $V \in \mathscr{V}$ and φ_V denotes the unique $\mathscr{L}$-isomorphism $\varphi_V : \hat{K}_V \longrightarrow \hat{L}_V$ over K_V, then there is a unique $\mathscr{L}$-isomorphism $\varphi : R\langle\bigcup_{V\in\mathscr{V}} \hat{K}_V\rangle \longrightarrow R\langle\bigcup_{V\in\mathscr{V}} \hat{L}_V\rangle$ over R extending all the φ_V.*
(iii) *The product map*

$$\bigotimes_{V\in\mathscr{V}}{}_R (R \otimes_{K_V} \hat{K}_V) \longrightarrow S$$

which sends $\left(\sum r_{1i} \otimes b_{1i}\right) \otimes \cdots \otimes \left(\sum r_{ki} \otimes b_{ki}\right)$ to $\left(\sum r_{1i} \cdot b_{1i}\right) \cdot \cdots \cdot \left(\sum r_{ki} \cdot b_{ki}\right)$ is injective.

Proof Let $B_V \subseteq \hat{K}_V$ be a basis of $\hat{K}_V$ over K_V in the sense of T. Since K_V is dense in $\hat{K}_V$, B_V is $\ll_{K_V}$-independent. By Proposition 2.7, $B := \bigcup_{V\in\mathscr{V}} B_V$ is $\ll_R$-independent. Moreover by Proposition 2.6(iii), for $V \in \mathscr{V}$ and $b \in B_V$, the cut of b over R is realized by some $c_b \in \hat{L}_V$ and in the situation of (ii) we must take $c_b := \varphi_V(b)$. Then, by Proposition 1.8, there is an elementary R-embedding $\varphi : R\langle B\rangle \longrightarrow R\langle\bigcup_{V\in\mathscr{V}} \hat{L}_V\rangle$ sending b to c_b for each $b \in B_V,\ V \in \mathscr{V}$.

In order to prove that φ is surjective and that $\varphi(R\langle\hat{K}_V\rangle) = R\langle\hat{L}_V\rangle$ it is enough to show that $C_V := \{c_b | b \in B_V\}$ is a basis of $\hat{L}_V$ over L_V in the sense of T. Clearly C_V is independent over L_V. Let $l \in \hat{L}_V \setminus L_V, l \notin C_V$. Then l is a V-limit, so $tp(l/K_V)$ is realized in $\hat{K}_V$. Then also $tp(l/R)$ is realized in $R\langle B_V\rangle$, hence $tp(l/R)$ is realized in $R\langle C_V\rangle$. This means that $C_V \cup \{l\}$ is $\ll_R$-dependent and by Proposition 1.19, $C_V \cup \{l\}$ is $\ll_{L_V}$-dependent. Since $l \in \hat{L}_V$ and L_V is dense in $\hat{L}_V$ this is only possible if $l \in L_V\langle C_V\rangle$.

This proves (i) and (ii).

(iii). First we show that $R \otimes_{K_V} \hat{K}_V \longrightarrow S$ is injective, i.e. R and $\hat{K}_V$ are linearly disjoint over K_V. Since B_V is $\ll_{K_V}$-independent, B_V is $\ll_R$-independent by Proposition 2.7. Since $tp(b/K_V)$ is omitted in R for all $b \in B_V$ it follows that B_V is $\ll_{K_V,R}$-independent. By Proposition 1.17, $tp(B_V/R)$ is an heir of $tp(B_V/K_V)$. This property implies that every linear equation with coefficients in $\hat{K}_V$ which has a solution in R, also has a solution in K_V. Hence R and $\hat{K}_V$ are linearly disjoint over K_V.

It remains to show that the domains $R \otimes_{K_V} \hat{K}_V$ are linearly disjoint over R. By what we have shown we may identify $R \otimes_{K_V} \hat{K}_V$ with $R[\hat{K}_V] \subseteq S$. Moreover we may assume that $\mathscr{V}$ is finite, say $\mathscr{V} = \{V_1, \ldots, V_n\}$. We write B_i for B_{V_i}. Since $\bigcup_{i=1}^n B_i$ is $\ll_R$-independent, the type $tp(B_n/R \cup B_1 \cup \cdots \cup B_{n-1})$ is an heir over R (c.f. Proposition 1.17). Again it follows that every linear equations with coefficients in $R\langle B_1 \cup \cdots \cup B_{n-1}\rangle$ which has a solution in $R\langle B_n\rangle$, also has a solution in R. By induction on n we get (iii). □

Definition 4.2 *In the situation of Theorem* 4.1 *the model* $R\langle\bigcup_{V\in\mathscr{V}} \hat{K}_V\rangle$ *of* T *is called the pseudo completion of* R *with respect to* $\mathscr{V}$.

By Theorem 4.1 this model of T is up to an R-isomorphism independent of S, K_V and $\hat{K}_V$; it can be constructed in the following way. Let A be the ring

$$A := \bigotimes_{V\in\mathscr{V}} {}_R (R \otimes_{K_V} \hat{K}_V).$$

Then A is an R-algebra without zero divisors and there is an injective R-algebra homomorphism f from A into an elementary extension of R. Then the pseudo completion is the definable closure of $f(A)$.

If T is the theory of real closed fields then the pseudo completion is the real closure of the quotient field of A with respect to any ordering.
The next proposition describes in what sense the pseudo completion is minimal.

Proposition 4.3 *Let T be an o-minimal expansion of fields in the language $\mathscr{L}$, let $R \models T$, let $\mathscr{V}$ be a family of T-convex subrings of R and let R' be the pseudo completion of R with respect to $\mathscr{V}$. Let $S \succ R$ be an elementary extension of R.*

(i) *Suppose each cut of R, which is a V-limit for some $V \in \mathscr{V}$ is realized in S. Then there is an elementary embedding $R' \longrightarrow S$ over R.*
(ii) *For each $V \in \mathscr{V}$, let $W(V)$ be the convex hull of V in S. Let $\mathscr{W}$ be a set of T-convex valuation rings of S with $W(V) \in \mathscr{W}$ for all $V \in \mathscr{V}$ and let S' be the pseudo completion of S with respect to $\mathscr{W}$. Then there is an elementary R-embedding $R' \longrightarrow S'$.*
If $\mathscr{W}$ is precisely the set of all $W(V)$ with $V \in \mathscr{V}$ and for each $V \in \mathscr{V}$, the residue field of V is equal to the residue field of $W(V)$, then we can choose this embedding φ so that S' is the definable closure of $S \cup \varphi(R')$.

Proof For $V \in \mathscr{V}$ let $K_V \subseteq V$ be a maximal definably closed subfield of V.

(i). Let $B_V \subseteq \hat{K}_V \subseteq R'$ be a transcendence basis of $\hat{K}_V$ over K_V ($V \in \mathscr{V}$). Pick some $V \in V$. By assumption and Proposition 2.6(iii), for $b \in B_V$ the cut of b over K_V is realized in S. Since K_V is dense in $\hat{K}_V$, B_V is $\ll_{K_V}$-independent. By Proposition 1.8, $\hat{K}_V$ can be embedded into S over K_V. By Theorem 4.1, R' can be embedded over R into S.

(ii). Let $L_V \subseteq W(V)$ be a maximal definably closed subfield of $W(V)$ containing K_V for every $V \in \mathscr{V}$. Since K_V is archimedean in L_V it follows from Lemma 1.20(ii), that there is a (unique) elementary K_V-embedding $\varphi_V : \hat{K}_V \longrightarrow \hat{L}_V$. By Theorem 4.1 we may assume that S' contains the definable closure of $S[\bigcup_{V\in\mathscr{V}} \hat{L}_V]$. By Theorem 4.1, R' is R-isomorphic to the definable closure of $R[\bigcup_{V\in\mathscr{V}} \varphi(\hat{K}_V)]$ in S'.

Now suppose $\mathscr{W}$ is precisely the set of all $W(V)$ with $V \in \mathscr{V}$ and for each $V \in \mathscr{V}$, the residue field of V is equal to the residue field of $W(V)$. Then $L_V = K_V$ and S' is R-isomorphic to the definable closure of $S[\bigcup_{V\in\mathscr{V}} \hat{K}_V]$. □

By Example 5.11 below, a pseudo completion R' of a pure real closed field R is in general not minimal in the sense that any R-endomorphism of R' is an automorphism. Moreover it is unclear if R' is uniquely determined up to an R-isomorphism by the minimality demand of Proposition 4.3(i); this is the content of the open problem 5.12 at the end of the paper.

5 Completion in stages of polynomially bounded structures

An o-minimal expansion R of a field is called polynomially bounded if every definable function $R \longrightarrow R$ is ultimately bounded by some polynomial. Here all polynomially bounded structures are additionally assumed to have an archimedean prime model. In particular, pure real closed fields are polynomially bounded. If R is polynomially bounded, then every convex subring is $Th(R)$-convex (cf. [2]).

Definition 5.1 *Let K be an ordered field and let $\mathcal{V}$ be a set of convex valuation rings of K. We say that K is complete in stages with respect to $\mathcal{V}$ if all residue fields of elements of $\mathcal{V}$ are complete.*

By Proposition 2.4, K is complete in stages with respect to $\mathcal{V}$ if and only if there are no V-limits in any ordered field extension of K, for all $V \in \mathcal{V}$.

For ordered fields, this definition is more general than the definition of Ribenboim [8]. Let V be a convex valuation ring of an ordered field. Then the valued field (K, V) is complete in stages in the sense of Ribenboim ([8], section D) if and only if K is complete in stages with respect to

$$\{W \subseteq K | W \text{ is a convex valuation ring with } V \subsetneq W\}$$

in our sense. This follows from Corollary 2.5 together with [8], section D, Théorème 3, which says that the valued field (K, V) is complete in stages if and only if every distinguished pseudo Cauchy sequence of (K, V) has a pseudo limit in K.

In this section we construct a completion in stages of R with respect to $\mathcal{V}$ for a polynomially bounded expansion R of a real closed field and a set $\mathcal{V}$ of convex valuation rings of R. This is a smallest elementary extension S which is complete in stages with respect to the set of convex hulls of the $V \in \mathcal{V}$. We get S by iterating the construction of the pseudo completion. Before we can do this, we have to compute the residue fields and the value groups of the pseudo completion.

Proposition 5.2 *Let R be polynomially bounded and let s be an element from an elementary extension of R, $s \notin R$. The following are equivalent.*

- (i) $\mathrm{sign}(s/R) = 0$.
- (ii) *If G is a convex subgroup of $(R, +)$, then G^+ is omitted in $R\langle s\rangle$.*
- (iii) *If W is a convex valuation ring of $R\langle s\rangle$, then the value group of W is equal to the value group of $W \cap R$.*

Proof Clearly (ii) implies (i). Also (ii) implies (iii), since an element $R\langle s\rangle$ which is not in the value group of $W \cap R$ is the edge of a convex subgroup of R.

Conversely suppose $\alpha \in R\langle s\rangle$ realizes G^+ for a convex subgroup G of $(R, +)$. The proposition is proved if we show that $\mathrm{sign}(s/R) \neq 0$ and that $w(\alpha)$ is not in the value group of $V(\alpha/R)$, where w is the valuation of $R\langle s\rangle$ with respect to the convex hull W of $V(\alpha/R)$ in $R\langle s\rangle$.

In order to see this, let $r \in R$ and suppose $\alpha/r \in W^*$, say $\alpha/r > 0$. Then there are $y, z \in V$ with $0 < \alpha/r < y$ and $0 < r/\alpha < z$, thus $0 < r/z < \alpha < y \cdot r$. Hence $r/z \in G$ and $zy \cdot r/z \notin G$ in contradiction to $z \cdot y \in V(\alpha/R)$.

Hence $w(\alpha)$ is not in the value group of $V(\alpha/R)$. By the valuation property ([3]) there must be some $b \in R$ such that $w(s - b)$ is not in the value group of $V(\alpha/R)$. But then $s - b$ realizes the edge of a convex subgroup of R, i.e. $\mathrm{sign}(s/R) \neq 0$. □

Lemma 5.3 *Let R be polynomially bounded and let s be from an elementary extension of R with* $\operatorname{sign}(s/R) = 0$. *If $F : R \longrightarrow R$ is R-definable with $F(s) \notin R$, then there are $a, b \in R$, $a < s < b$ such that F is differentiable in (a, b) and for all $r \in R$ with $a < r < b$ we have*

$$G(F(s)/R) = F'(r) \cdot G(s/R).$$

Proof By C^1-cell decomposition and since the cut of s over R is not definable, we may assume that F is C^1 in an open neighborhood of $[a, b]$ for some $a, b \in R$ with $a < s < b$. We write F' for the derivative of F in $[a, b]$. If F is a linear map in some interval (c, d) with $c, d \in R$, $c < s < d$ the lemma holds since $G(ys + z/R) = yG(s/R)$ for all $y, z \in R, y \neq 0$. Hence we may assume that $F'(s) \notin R$.

Let W be the convex hull of V in $R\langle s\rangle$. Since $\operatorname{sign}(s/R) = 0$, Proposition 5.2 implies that the value group of W is equal to the value group of V. Hence there is some $z \in R$ such that $z \cdot F'(s) \in W^*$. We may replace F by $z \cdot F$, hence we may assume that $F'(s) \in W^*$, say $F'(s) > 0$. Since $F'(s) \notin R$, Proposition 5.2 gives us $c, d \in V$, $\mathfrak{m}_V < c < d$ with $c < F'(s) < d$. By shrinking (a, b) if necessary we may assume that $F|_{[a,b]} : [a, b] \longrightarrow [F(a), F(b)]$ is a strictly increasing homeomorphism with $F'(x) \in (c, d)$ on $[a, b]$. We prove $G(F(s)/R) = G(s/R)$; this also proves the lemma, since $G(s/R) = F'(r) \cdot G(s/R)$ for all $r \in R$, $a < r < b$.

In order to show $G(s/R) \subseteq G(F(s)/R)$ we take $g \in G(s/R), g > 0, r \in R$ with $a < r < s$ and we show that $F(r) + g < F(s)$. Since $F'(x) > c$ in $[a, b]$ we know that $F(x) > F(r) + c \cdot (x - r)$ for $x \in (r, b)$. Since $g \in G(s/R)$ and $c \in V^*$, we know that $r + g/c < s$, hence $F(x) > F(r) + c \cdot (x - r) \geq F(r) + g$ for $x \in (r + g/c, b)$ and $F(s) > F(r) + g$ as desired.

Conversely let $y \in R$ with $y > G(s/R)$. Then also $y/d > G(s/R)$ and there is some $r \in (a, b)$ with $r < s < r + y/d$. Since $F'(x) < d$ in $[a, b]$ we know that $F(x) < F(r) + d \cdot (x - r)$ for $x \in (r, b)$. Hence also $F(x) < F(r) + d \cdot (x - r) < F(r) + y$ for all $x \in R$ with $r < x < \min\{b, r + y/d\}$. Since $r < s < \min\{b, r + y/d\}$ it follows $F(r) < F(s) < F(r) + y$, thus $y \notin G(F(s)/R)$ as desired. □

Lemma 5.4 *Let $R \prec S$ be polynomially bounded, such that* $\operatorname{sign}(s/R) = 0$ *for all $s \in S \setminus R$. Let α be from an elementary extension of S and let $F : S \longrightarrow S$ be S-definable such that $F(\alpha) \notin S$. Suppose* $\operatorname{sign}(\alpha/S) = 0$ *and the cuts of α and $F(\alpha)$ over R are omitted in S. Then there is some $c \in R^*$ such that $G(F(\alpha)/R) = c \cdot G(\alpha/R)$.*

Proof By Lemma 5.3 applied to S and α we get some $s \in S^*$ with $G(F(\alpha)/S) = s \cdot G(\alpha/S)$. Let $V = V(\alpha/R)$ and let W be the convex hull of V in S. By assumption and by Proposition 5.2, there is some $c \in R^*$ such that $c/s \in W^*$. Since the cut of α over R is omitted in S, $G(\alpha/S)$ contains $G(\alpha/R)$. Since S does not realize the upper edge of $G(\alpha/R)$, $G(\alpha/S)$ is the convex hull of $G(\alpha/R)$ in S. This implies that $V(\alpha/S)$ contains $V = V(\alpha/R)$. Again, since S does not realize the upper edge of V, $V(\alpha/S)$ is the convex hull of V in S. Thus c/s is a unit in $V(\alpha/S) = W$ and $s \cdot G(\alpha/S) = s \cdot (c/s) \cdot G(\alpha/S) = c \cdot G(\alpha/S)$.

By Proposition 5.2, $\operatorname{sign} F(\alpha)/S = 0$, hence also $G(F(\alpha)/S)$ is the convex hull of $G(F(\alpha)/R)$ in S. Thus $G(F(\alpha)/S) = c \cdot G(\alpha/S)$ implies $G(F(\alpha)/R) = c \cdot G(\alpha/R)$. □

Corollary 5.5 *Let R be polynomially bounded, let $s_1, \ldots, s_n$ be from an elementary extension of R with $\mathrm{rk}_R(s_1, \ldots, s_n) = n$ and $\operatorname{sign}(s_i/R) = 0$ for all $i \in \{1, \ldots, n\}$. If $F : R^n \longrightarrow R$ is R-definable and $F(s_1, \ldots, s_n) \notin R$, then $\operatorname{sign}(F(s_1, \ldots, s_n)/R) = 0$ and $G(F(s_1, \ldots, s_n)/R) = c \cdot G(s_i/R)$ for some $i \in \{1, \ldots, n\}$ and some $c \in R$.*

Proof For $i \in \{1, \ldots, n\}$ the cut of s_i over R is omitted in $R\langle s_1, \ldots, s_{i-1}\rangle$. As $\operatorname{sign}(s_i/R) = 0$ it follows that $\operatorname{sign}(s_i/R\langle s_1, \ldots, s_{i-1}\rangle) = 0$. Then by induction on n, Proposition 5.2 implies that every $s \in R\langle s_1, \ldots, s_i\rangle \setminus R$ has signature 0.

Let $\alpha := F(s_1, \ldots, s_n)$. Since $\alpha \notin R$ there is some $i \in \{1, \ldots, n\}$ such that $\alpha \cup (\{s_1, \ldots, s_n\} \setminus \{s_i\})$ is a $\ll_R$-basis of $R\langle s_1, \ldots, s_n\rangle$. Say $i = 1$. Let $S := R\langle s_2, \ldots, s_n\rangle$. Since the cuts of α and s_1 over R are omitted in S and $\operatorname{sign}(s_1/S) = 0$, we can apply Lemma 5.4. Hence $G(\alpha/R) = c \cdot G(s_1/R)$ for some $c \in R^*$. □

Now we compute the residue fields and the value groups of convex valuation rings of the pseudo completion of a polynomially bounded structure:

Theorem 5.6 *Let R be polynomially bounded and let S be the pseudo completion of R with respect to a set $\mathcal{V}$ of convex subrings of R.*

(i) *Every $s \in S \setminus R$ is a V-limit for a unique convex valuation ring V of R and this ring is in $\mathcal{V}$.*
(ii) *Let V_0 be any convex valuation ring of R. Then the convex hull W_0 of V_0 in S is the unique convex valuation ring of S lying over V_0. The value group of W_0 is the value group of V_0 and*
   (*a*) *if $V_0 \subsetneqq V$ for all $V \in \mathcal{V}$, then the extension $(R, V_0) \subseteq (S, W_0)$ of valued fields is immediate;*
   (*b*) *if $V \subseteq V_0$ for some $V \in \mathcal{V}$, then $W_0/\mathfrak{m}_{W_0}$ is the pseudo completion of $V_0/\mathfrak{m}_{V_0}$ with respect to $\{V/\mathfrak{m}_{V_0} | V \in \mathcal{V}, V \subseteq V_0\}$.*

Proof (i) follows from Corollary 5.5, since S is the definable closure of a $\ll_R$-independent set of elements, each being a V-limit for some $V \in \mathcal{V}$ (cf. Propositions 2.7 and 2.4). The uniqueness statement holds by Proposition 2.6(ii).

(ii) By (i), every $s \in S \setminus R$ is a V-limit for some $V \in \mathcal{V}$, in particular $\operatorname{sign}(s/R) = 0$. By Proposition 5.2, no edges of convex subgroups of R are realized in S. Consequently W_0 is the unique convex valuation ring of S, lying over V_0 and W_0 must have the same value group as V_0.

In order to see (a) and (b) let $K_V \subseteq V$ be a maximal definably closed subfield for each $V \in \mathcal{V} \cup \{V_0\}$.

(a) Suppose $V_0 \subsetneqq V$ for all $V \in \mathcal{V}$. Let $s \in W_0$. We have to show that the cut p of s over K_{V_0} is definable. Suppose p is not definable. If $G(s/K_{V_0}) = 0$, then s is a V_0-limit by Proposition 2.4. By (i), s is a V-limit for some $V \in \mathcal{V}$. Since $V \neq V_0$ this is impossible (cf. Proposition 2.6(i)). Hence $G(s/K_{V_0}) \neq 0$ and $V(s/K_{V_0})$ is a proper convex valuation ring of K_{V_0}. Since p is omitted in R, $G(s/R)$ is the largest convex subgroup of R with $G(s/R) \cap K_{V_0} = G(s/K_{V_0})$. This implies that $V(s/R)$ is a convex valuation ring, lying over $V(s/K_{V_0})$. As $V(s/K_{V_0})$ is proper it follows $V(s/R) \subseteq V_0$. On the other hand – by (i) – $V(s/R) \in \mathscr{V}$ and this contradicts our assumption on V_0.

(b) By Theorem 4.1 we may assume that $K_{V_0} \subseteq K_V$ for every $V \in \mathscr{V}$ with $V_0 \subseteq V$. For $V \in \mathscr{V}$ with $V \subseteq V_0$, any maximal definably closed subfield of $V \cap K_{V_0}$ is also a maximal definably closed subfield of V (this is so, since such a field L is archimedean in $V \cap K_{V_0}$ and tame in K_{V_0} – as $V \cap K_{V_0}$ is archimedean in V and K_{V_0} is tame in R, also L is archimedean in V and tame in R). So by Theorem 4.1 we may assume that $K_V \subseteq K_{V_0}$ for all $V \in \mathscr{V}$ with $V \subseteq V_0$, too. Let $\mathscr{V}' := \{V \in \mathscr{V} | V \subseteq V_0\}$ and let $R' := R\langle \bigcup_{V \in \mathscr{V}'} \hat{K}_V \rangle$ be the pseudo completion of R with respect to $\mathscr{V}'$.

First we prove (ii) (b) for $\mathscr{V}'$ and $W_0' := W_0 \cap R'$. By Theorem 4.1 it is enough to show that $K_{V_0}\langle \bigcup_{V \in \mathscr{V}'} \hat{K}_V \rangle$ is a maximal definably closed subfield of W_0'. In order to prove this it suffices to take $V_1, \dots, V_n \in \mathscr{V}'$ and finite subsets $B_i \subseteq K_{V_i}$ independent over K_{V_i} $(1 \le i \le n)$ and to show that $K_{V_0}\langle B \rangle$ is a maximal definably closed subfield of $W_0 \cap R\langle B \rangle$. By Proposition 2.7, the B_i are mutually disjoint and their union B is $\ll_{K_{V_0}}$-independent. Hence for each $b \in B$ the cut p of b over K_{V_0} is omitted in $K_{V_0}\langle B \setminus \{b\} \rangle$. Since p is not definable, the unique extension to $K_{V_0}\langle B \setminus \{b\} \rangle$ is not definable as well. This shows that $K_{V_0}\langle B \setminus \{b\} \rangle$ is archimedean in $K_{V_0}\langle B \rangle$. Hence, by induction, K_{V_0} is archimedean in $K_{V_0}\langle B \rangle$ and $K_{V_0}\langle B \rangle$ is a subfield of W_0. Since $R\langle B \rangle$ is generated by B and $\dim R\langle B \rangle / R$ is greater or equal to the dimension of the residue field of $W_0 \cap R\langle B \rangle$ over $V_0/\mathfrak{m}_{V_0}$ (cf. [2]), $K_{V_0}\langle B \rangle$ must be a maximal definably closed subfield of $W_0 \cap R\langle B \rangle$.

Hence we know that the residue field of $W_0' = W_0 \cap R'$ is the pseudo completion of $V_0/\mathfrak{m}_{V_0}$ with respect to $\{V/\mathfrak{m}_{V_0} | V \in \mathscr{V}, V \subseteq V_0\}$ and it remains to show that $(R', W_0') \subseteq (S, W_0)$ is immediate. But this follows from a., since by Theorem 4.1, S is the pseudo completion of R' with respect to the set of convex hulls of all $V \in \mathscr{V}$ with $V_0 \subsetneq V$. □

Example 5.7 The pseudo completion R' of a real closed field R with respect to a set $\mathscr{V}$ of convex valuation rings of R, containing R, is not complete in general. In particular, if $\mathscr{V}'$ denotes the set of convex hulls of elements from $\mathscr{V}$ in R', then R' need not be complete in stages with respect to $\mathscr{V}'$.
To see an example, let K be a real closed field with completion $\hat{K} \neq K$ and let Γ be a divisible subgroup of $(\mathbb{R}, +)$ containing $1 \in \mathbb{R}$. We also assume that Γ is an ordered subgroup of $(K, +)$. In this situation we can equip the generalized power series field $\hat{K}((t^\Gamma))$ with the derivative

$$\left(\sum a_\gamma t^\gamma\right)' = \sum a_\gamma \cdot \gamma \cdot t^{\gamma - 1}.$$

Let R be the real closure of $K(t^\gamma | \gamma \in \Gamma)$ in $\hat{K}((t^\Gamma))$ and let V be the convex hull of K in R. The completion $\hat{R}$ of R is

$$\hat{R} = \left\{ \sum_{n=0}^{\infty} a_n t^{\gamma_n} | a_n \in K,\ \gamma_n \in \Gamma \text{ and } \gamma_n \to \infty\ (n \to \infty) \right\}.$$

Hence the pseudo completion of R with respect to $\{V, R\}$ is $\hat{R}\langle\hat{K}\rangle$. We claim that for $x \in \hat{K} \setminus K$, the element

$$\exp(x\cdot t) := \sum_{i=0}^{\infty} \frac{x^i}{i!} t^i$$

is not in $\hat{R}\langle\hat{K}\rangle$. Since $\exp(x\cdot t)$ is in the completion of $\hat{K}(t^\gamma|\gamma \in \Gamma) \subseteq \hat{R}\langle\hat{K}\rangle$, this will show the incompleteness of $\hat{R}\langle\hat{K}\rangle$. We use a differential algebraic argument:

Lemma 5.8 *Let $K \subseteq L$ be ordinary differential fields of characteristic 0, let $y, x \in L$, $y \neq 0$ such that x is transcendental over K. Suppose $g, h \in K[x]$, with $y' = g\cdot y$ and $x' = h$. If $g \notin K$ and $\deg g \geq \deg h$, then y and x are algebraically independent over K. Here the degree is the degree with respect to x.*

Proof Suppose y is algebraic over $K(x)$. Let $f_{d-1}, \ldots, f_0 \in K(x)$ be rational functions, such that

$$\mu(T) := T^d + f_{d-1}T^{d-1} + \cdots + f_0$$

is the minimal polynomial of y over $K(x)$. Then

$$\begin{aligned} 0 = \mu(y)' &= d\cdot y^{d-1}\cdot y' + f'_{d-1}y^{d-1} + f_{d-1}\cdot(d-1)\cdot y^{d-2}y' + \cdots + f'_1 y + f_1 y' + f'_0 \\ &= d\cdot g\cdot y^d + (f'_{d-1} + f_{d-1}\cdot(d-1)\cdot g)\cdot y^{d-1} + \cdots + (f'_1 + f_1 g)y + f'_0 =: \eta(y). \end{aligned}$$

Since $x' \in K(x)$, $K(x)$ is a differential subfield of L and $\eta(y) = 0$ is an algebraic relation of y over $K(x)$ of degree d. Hence $\eta(y) = d\cdot g\cdot\mu(y)$ and a comparison of the constant coefficients with respect to y implies $f'_0 = d\cdot g\cdot f_0$. Let $P, Q \in K[T]$ with $f_0 = P(x)/Q(x)$, $Q(x) \neq 0$. Since $y \neq 0$, $P(x) \neq 0$. From $f'_0 = d\cdot g\cdot f_0$ we get

$$Q(x)\cdot P(x)' - P(x)\cdot Q(x)' = d\cdot g\cdot P(x)\cdot Q(x).$$

Since $x' = h$ and $\deg g \geq \max\{1, \deg h\}$, $P(x)'$ is a polynomial in x of degree $< \deg P + \deg g$. Also $\deg Q(x)' < \deg Q + \deg g$, hence $\deg(Q(x)\cdot P(x)' - P(x)\cdot Q(x)') < \deg P + \deg Q + \deg g = \deg(d\cdot g\cdot P(x)\cdot Q(x))$, a contradiction. □

Now we prove $\exp(x\cdot t) \notin \hat{R}\langle\hat{K}\rangle$. Let $B \subseteq \hat{K}$ be a transcendence basis of $\hat{K}$ over K containing x and let C be a transcendence basis of $\hat{R}$ over R. By Proposition 2.7, $B \cup C$ is a transcendence basis of $\hat{R}\langle\hat{K}\rangle$ over R and $B \cap C = \emptyset$. Let $L := \hat{R}\langle B \setminus \{x\}\rangle$. The field $\hat{R}$ is a differential subfield of $\hat{K}((t^\Gamma))$, equipped with the derivative introduced above (thus $(t^\gamma)' = \gamma\cdot t^{\gamma-1}$ for $\gamma \in \Gamma$). Since L is obtained from $\hat{R}$ by adjoining constants to $\hat{R}$ and then taking the real closure,

L is also a differential subfield of $\hat{K}((t^{\Gamma}))$. Moreover x is transcendental over L. Since $\exp(x{\cdot}t)' = x{\cdot}\exp(x{\cdot}t)$ and $x' = 0$, Lemma 5.8 implies that $\exp(x{\cdot}t)$ and x are algebraically independent over L. Hence $\exp(x\cdot t) \notin L\langle x\rangle = \hat{R}\langle\hat{K}\rangle$ as desired.

□

In the example above, the pseudo completion of $S := \hat{R}\langle\hat{K}\rangle$ with respect to $\{S, W\}$, where W is the convex hull of V, is the completion of $\hat{K}(t^{\gamma}|\gamma \in \Gamma)$.

More generally, if R is polynomially bounded, $\mathscr{V}$ is a set of convex valuation rings of R, let R' be the pseudo completion of R with respect to $\mathscr{V}$ and let $\mathscr{V}'$ be the set of convex hulls of elements from $\mathscr{V}$ in R'. We write $(R, \mathscr{V})'$ for $(R', \mathscr{V}')$. We define for each ordinal α the pair $(R^{(\alpha)}, \mathscr{V}^{(\alpha)})$ by $(R^{(0)}, \mathscr{V}^{(0)}) := (R, \mathscr{V})$, $(R^{(\alpha+1)}, \mathscr{V}^{(\alpha+1)}) := (R^{(\alpha)}, \mathscr{V}^{(\alpha)})'$ and for a limit ordinal α we take $R^{(\alpha)} = \bigcup_{\beta<\alpha} R^{(\beta)}$ and $\mathscr{V}^{(\alpha)} := \{\bigcup_{\beta<\alpha} V^{(\beta)} | V \in \mathscr{V}\}$.

Let $V_0 \subseteq \bigcap_{V\in\mathscr{V}} V$ be a convex subring and let $V_0^{(\alpha)}$ be the convex hull of V_0 in $R^{(\alpha)}$.

Claim The extension $(R^{(1)}, V_0^{(1)}) \subseteq (R^{(\alpha)}, V_0^{(\alpha)})$ is immediate for all $\alpha \geq 1$.

Proof By induction on α, where the limit step is obvious. Suppose we know that $(R^{(1)}, V_0^{(1)}) \subseteq (R^{(\alpha)}, V_0^{(\alpha)})$ is immediate for some $\alpha \geq 1$. We show that $(R^{(\alpha)}, V_0^{(\alpha)}) \subseteq (R^{(\alpha+1)}, V_0^{(\alpha+1)})$ is immediate. If $V_0 \notin \mathscr{V}$, then $V_0^{(\alpha)} \notin \mathscr{V}^{(\alpha)}$ and we can apply Theorem 5.6(ii)(a).

Hence we may assume that V_0 is the least element in V. Then also $V_0^{(\alpha)}$ is the least element of $\mathscr{V}^{(\alpha)}$. By Theorem 5.6(ii)(b), $V_0^{(1)}$ has a complete residue field. By induction, $V_0^{(\alpha)}$ has a complete residue field, too. Hence $R^{(\alpha+1)}$ is the pseudo completion of $R^{(\alpha+1)}$ with respect to $\mathscr{V}^{(\alpha)} \setminus \{V_0^{(\alpha)}\}$. But then again by Theorem 5.6(ii)(a) , $(R^{(\alpha)}, V_0^{(\alpha)}) \subseteq (R^{(\alpha+1)}, V_0^{(\alpha+1)})$ is immediate. □

From the claim it follows that $R^{(\alpha)}$ can be embedded as a field into the maximal immediate extension of the valued field $(R^{(1)}, V_0^{(1)})$. Consequently there must be some ordinal α with $R^{(\alpha)} = R^{(\alpha+1)}$.

Definition 5.9 *The* completion in stages *of R with respect to $\mathscr{V}$ is defined to be the elementary extension $R^{(\alpha)}$ for an ordinal α with $R^{(\alpha)} = R^{(\alpha+1)}$.*

By construction, the completion in stages is complete in stages with respect to the family of convex hulls of the rings from $\mathscr{V}$. Moreover the properties of the pseudo completion from Theorem 5.6 are inherited by the completion in stages:

Theorem 5.10 *Let R be polynomially bounded and let S be the completion in stages of R with respect to a set $\mathscr{V}$ of convex subrings of R.*

(i) *Every $s \in S \setminus R$ is a V-limit for a unique convex valuation ring V of R and this ring is in $\mathscr{V}$.*
(ii) *Let V_0 be any convex valuation ring of R. Then the convex hull W_0 of V_0 in S is the unique convex valuation ring of S lying over V_0. The value group of W_0 is the value group of V_0 and*

(*a*) *if* $V_0 \subsetneqq V$ *for all* $V \in \mathscr{V}$, *then the extension* $(R, V_0) \subseteq (S, W_0)$ *of valued fields is immediate;*

(*b*) *if* $V \subseteq V_0$ *for some* $V \in \mathscr{V}$, *then* $W_0/\mathfrak{m}_{W_0}$ *is the completion in stages of* $V_0/\mathfrak{m}_{V_0}$ *with respect to* $\{V/\mathfrak{m}_{V_0} | V \in \mathscr{V}, V \subseteq V_0\}$.

(iii) *Let* S' *be an elementary extension of* R *and for each* $V \in \mathscr{V}$ *let* W'_V *be the convex hull of* V *in* S'. *If* S' *is complete in stages with respect to* $\{W'_V | V \in \mathscr{V}\}$, *then there is an elementary embedding* $\varphi : S \longrightarrow S'$ *over* R.

Proof For an ordinal α, let $R^{(\alpha)}$ and $\mathscr{V}^{(\alpha)}$ be as in the construction of S above. First we prove (ii). Let V_0 be any convex valuation ring of R and let $V_0^{(\alpha)}$ be the convex hull of V_0 in $R^{(\alpha)}$. By induction on α we get from Theorem 5.6 that $V_0^{(\alpha)}$ is the unique convex valuation ring of $R^{(\alpha)}$, lying over V_0 and the value group of $V_0^{(\alpha)}$ is the value group of V_0. Moreover item (ii)(a) follows immediately from Theorem 5.6(ii)(a) by induction on α.

(ii)(b) By Theorem 5.6(ii)(b) for every ordinal α, $V_0^{(\alpha+1)}/\mathfrak{m}_{V_0^{(\alpha+1)}}$ is the pseudo completion of $V_0^{(\alpha)}/\mathfrak{m}_{V_0^{(\alpha)}}$ with respect to $\{V^{(\alpha)}/\mathfrak{m}_{V_0^{(\alpha)}} | V \in \mathscr{V},\ V \subseteq V_0\}$. By induction on α we get that $V_0^{(\alpha)}/\mathfrak{m}_{V_0^{(\alpha)}}$ is the α-fold iterated pseudo completion of $V_0/\mathfrak{m}_{V_0}$ with respect to $\{V/\mathfrak{m}_{V_0} | V \in \mathscr{V}, V \subseteq V_0\}$. This easily implies (ii)(b)

(i) The uniqueness statement is obviously true.

By induction on α we prove that every $x \in R^{(\alpha)} \setminus R$ is a V-limit for some $V \in \mathscr{V}$. For $\alpha = 1$ we know this from Theorem 5.6(i). For limit ordinals there is nothing to do. Now suppose $x \in R^{(\alpha+1)}$. If the cut of x over R is realized in $R^{(\alpha)}$, then by the induction hypothesis, x is a V-limit for some $V \in \mathscr{V}$. Hence we may assume that the cut of x over R is omitted in $R^{(\alpha)}$. Since $R^{(\alpha+1)}$ is the pseudo completion of $R^{(\alpha)}$ with respect to $\mathscr{V}^{(\alpha)}$, Theorem 5.6(i) gives us some $V \in \mathscr{V}$ such that x is a $V^{(\alpha)}$-limit. Thus $\mathrm{sign}(x/R^{(\alpha)}) = 0$ and for some $a \in R^{(\alpha)}$, $G(x/R^{(\alpha)}) = a \cdot \mathfrak{m}_{V^{(\alpha)}}$. Since the cut of x over R is omitted in $R^{(\alpha)}$, we have $\mathrm{sign}(x/R) = 0$.

Since the value group of $V^{(\alpha)}$ is the value group of V, there is some $r \in R$ such that $r/a \in (V^{(\alpha)})^*$. Hence $a \cdot \mathfrak{m}_{V^{(\alpha)}} = r \cdot \mathfrak{m}_{V^{(\alpha)}}$. Since the cut of x over R is omitted in $R^{(\alpha)}$ and $G(x/R)^+$ is omitted in $R^{(\alpha)}$, $G(x/R^{(\alpha)}) = r \cdot \mathfrak{m}_{V^{(\alpha)}}$ is the convex hull of $G(x/R)$. Since $\mathfrak{m}_{V^{(\alpha)}}$ is the convex hull of $\mathfrak{m}_V$, it follows that $G(x/R) = r \cdot \mathfrak{m}_V$. Together with $\mathrm{sign}(x/R) = 0$, this means that x is a V-limit.

(iii) Since W'_V is the convex hull of V and the residue field of W'_V is complete, for every maximal definably closed subfield K of V there is a completion of K inside W'_V. By Theorem 4.1, there is an elementary embedding of $R^{(1)}$ into S' over R. By an obvious induction this can be iterated until we reach the completion in stages. □

If $\mathscr{V}$ is finite of size n, then $R^{(n)}$ is complete in stages with respect to $\mathscr{V}^{(n)}$. This follows from Theorem 5.6 by induction on n: if $\mathscr{V} = \{V_1, \ldots, V_n\}$ with $V_1 \subsetneqq \cdots \subsetneqq V_n$, then by Theorem 5.6(ii) (b), $V_1^{(1)}$ has a complete residue field. Thus $R^{(2)}$ is the pseudo completion of $R^{(1)}$ with respect to $\{V_2^{(1)}, \ldots, V_n^{(1)}\}$.

Moreover $V_1^{(1)} \subseteq V_1^{(2)}$ is immediate by Theorem 5.6(ii)(a) Hence by induction, $R^{(n)}$ is complete in stages with respect to $\mathscr{V}^{(n)}$.

Example 5.11 One might ask if the pseudo completion or the completion in stages S of a real closed field R with respect to a set of convex valuation rings is minimal in the sense that every R-embedding $S \longrightarrow S$ is surjective. This is not true in general. Look at the following example.

Let $R = R_0\langle\mu\rangle$ be the real closure of $\mathbb{Q}(\mu)$, where μ is infinitesimal and let S be the pseudo completion of R with respect to the valuation ring $V :=$ the convex hull of $\mathbb{Q}$ in R. Then S is $\mathbb{R}\langle\mu\rangle$, which is the completion in stages of R with respect to the valuation ring V, too. We now construct a proper real closed subfield R of $\mathbb{R}\langle\mu\rangle$, which contains μ and which is isomorphic over $R_0\langle\mu\rangle$ to S. In particular R realizes every cut of R_0.

Let $T \subseteq \mathbb{R}$ be a transcendence basis over R_0 and let $B = \{b_1, b_2, \ldots\}$ be a countable subset of T. Let

$$R := R_0\langle(T \setminus B) \cup \{\mu, b_1 + \mu b_2, b_2 + \mu b_3, \ldots\}\rangle.$$

Then $b_1 \notin R$, otherwise there is some $n \in \mathbb{N}$ such that $b_1 \in R_1 := R_0\langle(T \setminus B) \cup \{\mu, b_1 + \mu b_2, \ldots, b_n + \mu b_{n+1}\}\rangle$. But then $b_1, \ldots, b_{n+1}, \mu \in R_1$, hence R_1 has transcendence degree $\geq n+2$ over $R_0\langle T \setminus B\rangle$, which is not possible.

R is isomorphic to S over $R_0\langle(T \setminus B) \cup \{\mu\}\rangle$, the isomorphism is given by sending b_i to $b_i + \mu \cdot b_{i+1}$ (observe that $T \cup \{\mu\}$ is $\ll_{R_0}$-independent and b_i and $b_i + \mu b_{i+1}$ realize the same cut over R_0. Then use Proposition 1.8).

Open Problem 5.12 Let S be a real closed field containing $\mathbb{R}$, of transcendence degree 1 over $\mathbb{R}$. Let S_0 be a real closed subfield of S which realizes every cut of $\mathbb{Q}$. Is S_0 isomorphic to S?

More general, let S be the pseudo completion of a real closed field and let $\varphi : S \longrightarrow S$ be an R-algebra homomorphism. Let S_0 be a real closed field with $\varphi(S) \subseteq S_0 \subseteq S$. Is S_0 isomorphic to S over R? In the example above, R is the real closure of $\mathbb{Q}(\mu)$, where μ is infinitesimal and S is the pseudo completion of R with respect to the valuation ring $V :=$ the convex hull of $\mathbb{Q}$ in R. Then $S = \mathbb{R}\langle\mu\rangle$ also is the completion in stages of R with respect to $\{V\}$.

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