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2005

MIMS EPrint: **2007.188**

ISSN 1749-9097

Available online at www.sciencedirect.com
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ELSEVIER
Advances in Mathematics 198 (2005) 222–274
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

Rational Cherednik algebras and Hilbert schemes

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Received 11 August 2004; received in revised form 22 November 2004; accepted 6 December 2004

Communicated by Johan De Jong
Available online 18 April 2005

Dedicated to Mike Artin on the occasion of his 70th birthday

Abstract

Let H_c be the rational Cherednik algebra of type A_{n-1} with spherical subalgebra $U_c = eH_ce$. Then U_c is filtered by order of differential operators, with associated graded ring $\operatorname{gr} U_c = \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W$ where W is the nth symmetric group. We construct a filtered $\mathbb{Z}$-algebra B such that, under mild conditions on c:

- the category B-qgr of graded noetherian B-modules modulo torsion is equivalent to U_c-mod;
- the associated graded $\mathbb{Z}$-algebra $\operatorname{gr} B$ has $\operatorname{gr} B$-$\mathsf{lqgr} \simeq \mathsf{coh}\ \mathsf{Hilb}(n)$, the category of coherent sheaves on the Hilbert scheme of points in the plane.

This can be regarded as saying that U_c simultaneously gives a non-commutative deformation of $\mathfrak{h} \oplus \mathfrak{h}^*/W$ and of its resolution of singularities $\operatorname{Hilb}(n) \to \mathfrak{h} \oplus \mathfrak{h}^*/W$. As we show elsewhere, this result is a powerful tool for studying the representation theory of H_c and its relationship to $\operatorname{Hilb}(n)$.

MSC: 14C05; 32S45; 16S80; 16D90

Keywords: Cherednik algebra; Hilbert scheme; Equivalence of categories; Resolution of quotient singularities

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[1]Supported by Nuffield Foundation Grant NAL/00625/A and the Leverhulme trust.
[2]Supported in part by the NSF through Grants DMS-9801148 and DMS-0245320.

doi:10.1016/j.aim.2004.12.005

Contents

1. Introduction

1.1. This is the first of two closely related papers on rational Cherednik algebras.

In their short history, Cherednik algebras have been influential in a surprising range of subjects: for example they have been used to answer questions in integrable systems, combinatorics, and symplectic quotient singularities (see [BEG1,Go2,BFG,GK]). In this paper, we strengthen the connections between Cherednik algebras and geometry by showing that they can be regarded as non-commutative deformations of Hilbert schemes of points in the plane. In the sequel [GS] this will be used to show the close relationship between modules over the Cherednik algebra and sheaves on the Hilbert scheme as well as to answer various open problems about these modules.

1.2. Fix $c \in \mathbb{C}$. We assume throughout the paper that $c \notin \frac{1}{2} + \mathbb{Z}$ and, for simplicity, we will also assume that $c \notin \mathbb{R}_{\leqslant 0}$ in this introduction; see (3.13) and (3.14) for the more general case.

Let $H_c = H_{1,c}$ be the rational Cherednik algebra of type A_{n-1} with spherical subalgebra $U_c = eH_ce$. The formal definition of H_c is given in (2.1) but one may regard it as a deformation of the twisted group ring $D(\mathfrak{h}) * W$, where $D(\mathfrak{h})$ is the ring of differential operators on $\mathfrak{h} \cong \mathbb{C}^{n-1}$ with the natural action of the symmetric group $W = \mathfrak{S}_n$. The algebra U_c is then the corresponding deformation of the fixed ring $D(\mathfrak{h})^W$. The algebras U_c and H_c have a natural filtration by order of differential operators with associated graded rings $\operatorname{gr} U_c \cong \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W$ and $\operatorname{gr} H_c \cong \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] * W$. Thus we may also regard U_c as a deformation of $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W$. In this introduction we will mostly be concerned with U_c, but since U_c and H_c are Morita equivalent (Corollary 3.13) the results we prove for U_c also apply to H_c.

It is well-known that $\mathfrak{h} \oplus \mathfrak{h}^*/W$ has a crepant resolution $\operatorname{Hilb}(n) \to \mathfrak{h} \oplus \mathfrak{h}^*/W$, where $\operatorname{Hilb}(n)$ is a variant on the Hilbert scheme of n points in the plane (see (4.9) for the formal definition). The ring U_c has finite global homological dimension (see Corollary 3.15) and so one should expect that it has the properties of a smooth deformation of $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W$; in other words its properties should be more closely related to those of $\mathrm{Hilb}(n)$ than to $\mathfrak{h} \oplus \mathfrak{h}^*/W$. Hints of this have been reported in [Go2,BEG2]: finite-dimensional H_c-modules deform the sections of some remarkable sheaves on $\mathrm{Hilb}(n)$. The main aim of this paper is to formalise this idea by showing that there is a second way of passing to associated graded objects that maps $U_c\text{-}\mathsf{mod}$ precisely to $\mathsf{Coh}(\mathrm{Hilb}(n))$.

1.3. We take our cue from the theory of semisimple Lie algebras. When $n = 2$, U_c is isomorphic to a factor of $U(\mathfrak{sl}_2)$ [EG, Section 8] and, for all n, the properties of U_c are similar to those of $U(\mathfrak{g})/P$, where P is a minimal primitive ideal in the enveloping algebra of a complex semisimple Lie algebra $\mathfrak{g}$ (see, for example, [Gi,GGOR,Gu]). The intuition from the last paragraph not only holds for enveloping algebras but can also be formalised through the Beilinson–Bernstein equivalences of categories. This gives a diagram

$$\begin{tikzcd} D_{\mathcal{B}} \arrow[d, "\mathrm{gr}"'] & U(\mathfrak{g})/P \arrow[l, "\sim"'] \arrow[d, "\mathrm{gr}"] \\ \mathcal{O}_{T^*\mathcal{B}} & \mathcal{O}(\mathcal{N}), \arrow[l, "\tau"'] \end{tikzcd}$$

where $\mathcal{B} = G/B$ is the flag variety, the primitive ideal P has trivial central character and $\tau : T^*\mathcal{B} \to \mathcal{N}$ is the Springer resolution of the nullcone $\mathcal{N}$. The Morita equivalence from the sheaf of differential operators $D_{\mathcal{B}}$ to $U(\mathfrak{g})/P$ is obtained by taking global sections under the identification $U(\mathfrak{g})/P \cong D(\mathcal{B})$.

Ginzburg has raised the question of whether a similar phenomenon holds for Cherednik algebras (see [GK, Conjecture 1.6] for a variant on this conjecture). In other words, can one complete the following diagram?

$$\begin{tikzcd} ? \arrow[d, "\mathrm{gr}"'] & U_c \arrow[l, "\sim"'] \arrow[d, "\mathrm{gr}"] \\ \mathcal{O}_{\mathrm{Hilb}(n)} & \mathcal{O}(\mathfrak{h} \oplus \mathfrak{h}^*/W). \arrow[l, "\tau"'] \end{tikzcd}$$

The main result of the paper gives a positive answer to this question. Given a graded ring R, we write $R\text{-}\mathsf{qgr}$ for the quotient category of noetherian graded R-modules modulo those of finite length.

1.4. Main Theorem. *There exists a graded ring B, filtered by order of differential operators, such that*

(1) *there is an equivalence of categories* $U_c\text{-}\mathsf{mod} \simeq B\text{-}\mathsf{qgr}$;
(2) *there is an equivalence of categories* $\mathrm{gr}\, B\text{-}\mathsf{qgr} \simeq \mathsf{Coh}(\mathrm{Hilb}(n))$.

1.5. The construction of B needs some explanation. For $n > 2$, it can be shown that the Hilbert scheme $\mathrm{Hilb}(n)$ is not a cotangent bundle, so we cannot use sheaves of differential operators as a non-commutative model. Instead, we take as our starting point Haiman's description of $\mathrm{Hilb}(n)$ as a blow-up of $\mathfrak{h} \oplus \mathfrak{h}^*/W$ and deform this to a non-commutative setting. Set $A^0 = \mathcal{O}(\mathfrak{h} \oplus \mathfrak{h}^*/W)$ with ideal $I = A^1\delta$, where δ is the discriminant and $A^1 = \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^{\varepsilon}$ is the module of anti-invariants. Then [Ha1, Proposition 2.6] proves that $\mathrm{Hilb}(n) = \mathsf{Proj}\, A$ where $A = A^0[tI]$ is the Rees ring of I (see Section 4 for the details).

Unfortunately one cannot construct B as an analogous Rees ring over U_c, since U_c is a simple ring for generic values of c. We circumvent this problem by using *$\mathbb{Z}$-algebras* (see Section 5). Specifically, the ring B from Theorem 1.4 is an algebra $B = \bigoplus_{i \geqslant j \geqslant 0} B_{ij}$ whose multiplication is defined in matrix fashion: $B_{ij}B_{jk} \subseteq B_{ik}$ but $B_{ij}B_{\ell k} = 0$ when $j \neq \ell$. The diagonal terms are just $B_{ii} = U_{c+i}$ while the off-diagonal terms B_{ij} are given as the appropriate tensor products of the (U_{d+1}, U_d)-bimodules $Q_d^{d+1} = eH_{d+1}\delta e$. The shift functors $S_d : U_d\text{-}\mathsf{mod} \to U_{d+1}\text{-}\mathsf{mod}$ given by tensoring with Q_d^{d+1} are important operators in the theory of Cherednik algebras and have already played a crucial role in combinatorics and representation theory; see, for example, [BEG1,BEG2,Go2]. A good way to think of the functor S_d is as the analogue of the translation functor [BG] from Lie theory.

In order to have control over B we need to know that the Q_d^{d+1} are progenerators for all $d \in c + \mathbb{N}$; equivalently that the S_d are Morita equivalences. This is a conjecture from [GGOR, Remark 5.17] which we answer with

1.6. Theorem (*Corollary* 3.13). *The shift functor S_d is a Morita equivalence for all $d \in c + \mathbb{N}$.*

The significance of this result is that B now has rather pleasant properties; in particular Theorem 1.4(1) is an easy consequence. For the second assertion of Theorem 1.4, we note that it is easy to obtain a $\mathbb{Z}$-algebra $\widehat{A} = \bigoplus_{i \geqslant j \geqslant 0} A_{ij}$ from the graded algebra $A = \bigoplus_{k \geqslant 0} I^k$ for which $A\text{-}\mathsf{qgr} \simeq \widehat{A}\text{-}\mathsf{qgr}$. One simply takes $A_{ij} = I^{i-j}$ for each i, j. Thus the main step in the proof of Theorem 1.4 is given by

1.7. Proposition (*Theorem* 6.4). *Under the filtration induced from the order filtration of differential operators*, $\mathrm{gr}\, B_{ij} \cong A_{i-j} = I^{i-j}$ *and so* $\mathrm{gr}\, B \cong \widehat{A}$ *as $\mathbb{Z}$-algebras.*

In this result the inclusion $I^{i-j} \subseteq \mathrm{gr}\, B_{ij}$ is straightforward. The opposite inclusion is much more subtle as it is difficult to keep close control of the filtration on B_{ij}. Our proof leans heavily on the work of Haiman [Ha3,Ha4] surrounding the $n!$ and polygraph theorems and the strategy is outlined in more detail in (6.6).

1.8. Applications. Theorem 1.4 gives a powerful technique for relating H_c- or U_c-modules to sheaves on $\mathrm{Hilb}(n)$: given a U_c-module M with a good filtration Λ we obtain a filtered object $(\widetilde{M}, \Lambda) \in B\text{-}\mathsf{qgr}$ by tensoring with B and then a coherent sheaf $\Phi_\Lambda(M) \in \mathsf{Coh}(\mathrm{Hilb}(n))$ by taking the associated graded module.

This process is studied in [GS] where we show there that the subtle combinatorics and geometry of $\mathrm{Hilb}(n)$ is reflected in the representation theory of U_c and H_c. Let

$\Delta_c(\mu)$ be the *standard H_c-module* corresponding to $\mu \in \mathsf{Irrep}(W)$ (this is the analogue of a Verma module) with unique simple factor $L_c(\mu)$. These modules have a natural good filtration Λ and we mention a couple of illustrative results from [GS].

- Suppose that $c = 1/n + k$ for $k \in \mathbb{N}$, so that $L_c(\mathsf{triv})$ is the unique finite-dimensional simple H_c-module. Then $\Phi_\Lambda(eL_c(\mathsf{triv})) \cong \mathcal{O}_{Z_n} \otimes \mathcal{L}^k$, where $Z_n = \tau^{-1}(0)$ is the *punctual Hilbert scheme* and $\mathcal{L} = \mathcal{O}_{\mathrm{Hilb}(n)}(1)$ is the Serre twisting sheaf.
- For any c, the characteristic cycle of $\Phi_\Lambda(e\Delta_c(\mu))$ equals $\sum_\lambda K_{\mu\lambda}[Z_\lambda]$, where $K_{\mu\lambda}$ are Kostka numbers and the Z_λ are particular irreducible components of $\tau^{-1}(\mathfrak{h} \oplus \{\mathbf{0}\}/W)$.

The first of these results is used in [GS] to show that the natural bigraded structure on $\mathrm{gr}_\Lambda(eL_{1/n+k})$ coincides with that on $H^0(Z_n, \mathcal{L}^k)$, thus confirming a conjecture of Berest et al. [BEG2]. The second of these results illustrates the subtlety of Φ: if one passes directly from U_c to $\mathrm{gr}\, U_c \cong \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W$ then $\mathrm{gr}_\Lambda(e\Delta_c(\mu)) \cong \mathbb{C}[\mathfrak{h}] \otimes \mu$ for any μ and c. Thus the support variety of $\mathrm{gr}_\Lambda\, e\Delta_c(\mu)$ is independent of μ.

We prove one such correspondence in this paper. Let $\mathcal{P}$ denote the Procesi bundle on $\mathrm{Hilb}(n)$, the vector bundle of rank $n!$ coming from Haiman's $n!$ theorem, see (4.10). Then Corollary 6.22 proves:

1.9. Corollary. *If eH_c is given the order filtration Λ, then $\Phi_\Lambda(eH_c) = \mathcal{P}$.*

1.10. One reason why Theorem 1.4 provides such a strong bridge between Hilbert schemes and Cherednik algebras is that the construction of B carries within it key elements of both theories. For instance, we have already mentioned that the shift functor S_c is an analogue of the translation functor from Lie theory. It is also the analogue of the shift functor in $\mathsf{Coh}(\mathrm{Hilb}(n))$ given by tensoring with $\mathcal{O}_{\mathrm{Hilb}(n)}(1)$. Indeed, given a U_c-module M with a good filtration Λ, it is easy to show that $\Phi_\Gamma(Q_c^{c+1} \otimes M) = \mathcal{O}_{\mathrm{Hilb}(n)}(1) \otimes \Phi_\Lambda(M)$, for the appropriate filtration Γ (see [GS]).

Similarly, Corollary 1.9 can be interpreted as saying that H_c is a non-commutative analogue of the isospectral scheme X_n, as defined in (4.10) (see (6.23) for further details).

1.11. The $\mathbb{Z}$-algebra has the virtue that it exists whenever one has an analogue of the translation principle; that is, one has algebras R_i and progenerative (R_{i+1}, R_i)-bimodules $Q_{i,i+1}$ (these algebras can also be indexed by more general lattices than $\mathbb{Z}$). One can then construct a $\mathbb{Z}$-algebra as we have done and Theorem 1.4(1) will still hold. It is not clear when Theorem 1.4(2) will hold and, even when it is true, it will undoubtedly be rather subtle.

Hilbert schemes realise crepant resolutions for the symplectic quotient singularity $(\mathbb{C}^2)^n/G$ whenever G is the wreath product of a finite subgroup of $SL_2(\mathbb{C})$ with the symmetric group W, see [Wa, Theorem 4.2]. We believe that our methods will generalise to the symplectic reflection algebras $H_c(G) = H_{1,c}(G)$ associated with $((\mathbb{C}^2)^n, G)$ to give non-commutative deformations of those Hilbert schemes. Even when there is no crepant resolution of such a singularity (by Ginzburg and Kaledin [GK] this happens for Weyl groups G of types other than A and B) the $\mathbb{Z}$-algebra associated to $H_c(G)$

will still contain interesting information, as Gordon [Go2] demonstrates. For a Weyl group, the analogue of Theorem 1.6 is at least known for sufficiently large values of the defining parameter c [BEG2, Proposition 4.3], but little is known for small values of c.

The translation principle obviously holds for factors of enveloping algebras of semi-simple Lie algebras and we can prove an analogue of Theorem 1.4 in this case. However, the proof uses non-trivial Lie theoretic results, notably the Beilinson–Bernstein equivalence of categories, and it is unclear whether this approach carries information that cannot be obtained from that equivalence. It would be interesting to see if the recent work [BK,Ta] on the Beilinson–Bernstein equivalence for quantised enveloping algebras can be understood in a $\mathbb{Z}$-algebra framework.

1.12. The paper is organised as follows. In Section 2 we recall the needed facts about rational Cherednik algebras, while in Section 3 we prove Theorem 1.6. In Section 4 we describe some of Haiman's work on Hilbert schemes, adapted to the variety $\mathrm{Hilb}(n)$, and use it to describe various Poincaré series that will be needed in the proof of Theorem 1.4. Section 5 proves the results about $\mathbb{Z}$-algebras that were mentioned earlier in this introduction. Section 6 is the heart of the paper: in it we prove Theorem 1.4(2). This is derived from an analogous result about the associated graded module of $B_{k0} \otimes_{U_c} eH_c$ that also implies Corollary 1.9. Section 7 then gives a reinterpretation of Theorem 1.4 in terms of a tensor product filtration of B_{ij}. In Appendix A we prove the following result that may be of independent interest: *suppose that $R = \bigoplus_{i \geqslant 0} R_i$ is an $\mathbb{N}$-graded algebra over a field k, with $R_0 = k$. If P is a right R-module that is both graded and projective, then P is graded-free in the sense that P has a free basis of homogeneous elements*. This is a graded analogue of a classic result from [Ka] for which we do not know a reference.

2. Rational Cherednik algebras

2.1. In this section, we define the rational Cherednik algebras (which will always be of type A in this paper) and give some of the basic properties that will be needed in the body of the paper.

Let $W = \mathfrak{S}_n$ be the *symmetric group* on n letters, regarded as the Weyl group of type A_{n-1} acting on its $(n-1)$-dimensional representation $\mathfrak{h} \subset \mathbb{C}^n$ by permutations. Let $\mathcal{S} = \{s = (i, j) \text{ with } i < j\} \subset W$ denote the reflections, with reflecting hyperplanes $\alpha_s = 0$. We make similar definitions for $\mathfrak{h}^*$ and normalise $\alpha_s^\vee \in \mathfrak{h}$ so that $\alpha_s(\alpha_s^\vee) = 2$.

Given $c \in \mathbb{C}$, *the rational Cherednik algebra of type A_{n-1}* is the $\mathbb{C}$-algebra H_c generated by the vector spaces $\mathfrak{h}$ and $\mathfrak{h}^*$ and the group W with defining relations

$$wxw^{-1} = w(x), \quad wyw^{-1} = w(y) \qquad \text{for all } y \in \mathfrak{h}, x \in \mathfrak{h}^*, w \in W,$$

$$x_1x_2 = x_2x_1, \quad y_1y_2 = y_2y_1 \qquad \text{for all } y_i \in \mathfrak{h}, x_j \in \mathfrak{h}^*,$$

$$yx - xy = x(y) - \sum_{s \in \mathcal{S}} c\alpha_s(y)x(\alpha_s^\vee)s \qquad \text{for all } y \in \mathfrak{h}, x \in \mathfrak{h}^*.$$

We should note that the definition of the Cherednik algebra is not uniform throughout the literature. The definition we are using agrees with that in [BEG1,BEG2,EG,Gu] but *not* that from [GGOR] where our constant c equals $-k_1$ for their constant k_1 (see [GGOR, Remark 3.1]).

2.2. We write the coordinate ring of an affine variety V as $\mathbb{C}[V]$. By Etingof and Ginzburg [EG, Theorem 1.3], the subalgebra of H_c generated by $\mathfrak{h}^*$ can and will be identified with $\mathbb{C}[\mathfrak{h}]$, while $\mathfrak{h}$ generates a copy of $\mathbb{C}[\mathfrak{h}^*]$ inside H_c and the elements $w \in W$ span a copy of the group algebra $\mathbb{C}W$ in H_c. Fix once and for all dual bases $\{x_i\}$ and $\{y_i\}$ of $\mathfrak{h}^*$ and $\mathfrak{h}$, respectively; thus $\mathbb{C}[\mathfrak{h}] = \mathbb{C}[x_1, \dots, x_{n-1}]$ and $\mathbb{C}[\mathfrak{h}^*] = \mathbb{C}[y_1, \dots, y_{n-1}]$.

By Etingof and Ginzburg [EG, Theorem 1.3] there is a Poincaré–Birkhoff–Witt isomorphism of $\mathbb{C}$-vector spaces

$$\mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}} \mathbb{C}W \otimes_{\mathbb{C}} \mathbb{C}[\mathfrak{h}^*] \xrightarrow{\sim} H_c. \tag{2.2.1}$$

Filter H_c by $\mathsf{ord}^0 H_c = \mathbb{C}[\mathfrak{h}] * W$, $\mathsf{ord}^1 H_c = \mathfrak{h} + \mathsf{ord}^0 H_c$ and $\mathsf{ord}^i H_c = (\mathsf{ord}^1 H_c)^i$ for $i > 1$, and define the *associated graded ring* to be $\mathsf{ogr}\, H_c = \bigoplus \mathsf{ogr}^n H_c$, where $\mathsf{ogr}^n H_c = \mathsf{ord}^n H_c/\mathsf{ord}^{n-1} H_c$. Then (2.2.1) is equivalent to the assertion that $\mathsf{ogr}\, H_c$ is isomorphic to the skew group ring $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] * W$ defined by $\sigma f = \sigma(f)\sigma$, for $\sigma \in W$ and $f \in \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]$.

2.3. The Dunkl–Cherednik representation. Let $\delta \in \mathbb{C}[\mathfrak{h}]$ denote the discriminant polynomial $\delta = \prod_{s \in \mathcal{S}} \alpha_s$. Thus δ transforms under W by the sign representation and $\mathfrak{h}^{\mathrm{reg}} = \mathfrak{h} \setminus \{\delta = 0\}$ is the subset of $\mathfrak{h}$ on which the action of W is free. By Etingof and Ginzburg [EG, Proposition 4.5] there is an injective algebra morphism $\theta_c : H_c \to D(\mathfrak{h}^{\mathrm{reg}}) * W$, where $D(Z)$ denotes the *ring of differential operators* on an affine variety Z. Under θ_c the elements of $\mathbb{C}[\mathfrak{h}]$ are identified with the multiplication operators while, by Etingof and Ginzburg [EG, p. 280] and in the notation of (2.2), $y_i \in \mathfrak{h}$ is sent to the *Dunkl operator*

$$\theta_c(y_i) = \partial_i - \sum_{s \in S} c\alpha_s(y_i)\alpha_s^{-1}(1 - s), \quad \text{where } \partial_i = \partial/\partial x_i. \tag{2.3.1}$$

Since δ acts ad-nilpotently on $D(\mathfrak{h}^{\mathrm{reg}}) * W$, the set $\{\delta^n\}$ forms an Ore set in that ring. As observed in [BEG1, p. 288], θ_c becomes an isomorphism on inverting δ; that is,

$$H_c^{\mathrm{reg}} = H_c[\delta^{-1}] \cong D(\mathfrak{h}^{\mathrm{reg}}) * W. \tag{2.3.2}$$

For any variety Z, there is a natural filtration on $D(Z)$ by order of operators and this induces a filtration on $D(\mathfrak{h}^{\mathrm{reg}}) * W$ and its subalgebras by defining elements of W to have order zero. If R is a subalgebra (or submodule) of $D(\mathfrak{h}^{\mathrm{reg}}) * W$, we write the operators of order $\leqslant n$ as $\mathsf{ord}^n(R)$. When $R = H_c$, ord is clearly the same filtration as that defined in (2.2). The associated graded ring of R will be written $\mathsf{ogr}(R) = \bigoplus \mathsf{ogr}^n(R)$, where $\mathsf{ogr}^n(R) = \mathsf{ord}^n(R)/\mathsf{ord}^{n-1}(R)$, and the resulting graded structure of $\mathsf{ogr}(R)$ will be called the *order* or ogr *gradation*. (This will be only one of several filtrations used in this paper.)

2.4. The rings of differential operators $D(\mathfrak{h})$ and $D(\mathfrak{h}^{\mathrm{reg}})$ also have a graded structure given by the adjoint action $[\mathbf{E}, -]$ of the *Euler operator* $\mathbf{E} = \sum x_i \partial_i \in D(\mathfrak{h})$. We will call this the *Euler grading* and write $\mathbf{E}$-deg for the corresponding degree function; thus $\mathbf{E}\text{-deg}\, x_i = 1$ and $\mathbf{E}\text{-deg}\, \partial_i = -1$. Since $\mathbf{E} \in D(\mathfrak{h})^W$, $\mathbf{E}$ commutes with W in $D(\mathfrak{h}^{\mathrm{reg}}) * W$ and so this grading extends to that ring with $\mathbf{E}\text{-deg}\, W = 0$. By inspection (2.3.1) implies that the y_i also have degree -1 and so each H_c is also graded under $[\mathbf{E}, -]$ and we continue to call this the Euler grading.

It is well-known and easy to check that the $\mathbf{E}$-grading is compatible with the order filtration on $D(\mathfrak{h}^{\mathrm{reg}}) * W$, in the sense that $[\mathbf{E}, \mathsf{ord}^n\, D(\mathfrak{h}^{\mathrm{reg}}) * W] \subseteq \mathsf{ord}^n\, D(\mathfrak{h}^{\mathrm{reg}}) * W$ for all $n \geqslant 0$. We therefore obtain an induced grading, again called the $\mathbf{E}$-grading, on the associated graded ring $\mathsf{ogr}\, D(\mathfrak{h}^{\mathrm{reg}}) * W \cong \mathbb{C}[\mathfrak{h}^{\mathrm{reg}} \oplus \mathfrak{h}^*] * W$. Clearly this is again given by $\mathbf{E}\text{-deg}\, \mathfrak{h}^* = 1$ (which we define to mean that $\mathbf{E}\text{-deg}(x) = 1$ for every $0 \neq x \in \mathfrak{h}^*$) while $\mathbf{E}\text{-deg}\, \mathfrak{h} = -1$ and $\mathbf{E}\text{-deg}\, W = 0$.

One should note that, in general, $\mathbf{E} \notin H_c$. However, there is a natural element in H_c that has the same adjoint action. Indeed, let

$$\mathbf{h} = \mathbf{h}_c = \frac{1}{2} \sum_{i=1}^{n-1} (x_i y_i + y_i x_i) \in H_c. \tag{2.4.1}$$

This is independent of the choice of basis. By Berest et al. [BEG1, (2.6)] we have

$$[\mathbf{h}, x] = x, \;\; [\mathbf{h}, y] = -y, \;\; \text{and } [\mathbf{h}, w] = 0 \quad \text{for all } x \in \mathfrak{h}^*, y \in \mathfrak{h} \text{ and } w \in W. \tag{2.4.2}$$

Thus commutation with $\mathbf{h}$ also induces the Euler grading on H_c.

2.5. The spherical subalgebra. Let $e \in \mathbb{C}W$ denote the trivial idempotent and let $e_- \in \mathbb{C}W$ be the sign idempotent; thus $e = |W|^{-1} \sum_{w \in W} w$ and $e_- = |W|^{-1} \sum_{w \in W} \mathrm{sign}(w) w$. The main algebra of study in this paper is not the Cherednik algebra itself, but its *spherical subalgebra* $U_c = eH_ce$ and the related algebra $U_c^- = e_- H_c e_-$. We will use frequently and without comment that δ is a W-anti-invariant and so $e_-\delta = \delta e$. Also, as $\mathbf{E}\text{-deg}\, W = 0$, both U_c and U_c^- have an induced $\mathbf{E}$-graded structure.

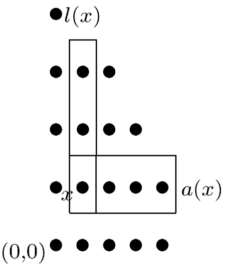
2.6. Partitions. The rest of this section is devoted to the definition and basic properties of category $\mathcal{O}_c$. Since its structure depends upon the combinatorics of W-representations, we begin with the relevant notions from that theory.

We write a partition of n as $\mu = (\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_l > 0)$, with the understanding that $\mu_i = 0$ for $i > l$. The *Ferrers' diagram* of μ is the set of lattice points

$$d(\mu) = \{(i, j) \in \mathbb{N} \times \mathbb{N} : j < \mu_{i+1}\}.$$

Following the French style, the diagram is drawn with the i-axis vertical and the j-axis horizontal, so the parts of μ are the lengths of the rows, and $(0, 0)$ is the lower left corner. The *arm* $a(x)$ and the *leg* $l(x)$ of a point $x \in d(\mu)$ denote the number of points strictly to the right of x and above x, respectively. The *hook length* $h(x)$ is $1 + a(x) + l(x)$. For example:

$\mu = (5, 5, 4, 3, 1)$



$$a(x) = 3,\quad l(x) = 2,\quad h(x) = 6. \tag{2.6.1}$$

The *transpose partition* μ^t is obtained from μ by exchanging the rows and columns of μ.

We will always use the *dominance ordering* of partitions as in [Mac, p. 7]; thus if λ and μ are partitions of n then $\lambda \geqslant \mu$ if and only if $\sum_{i=1}^k \lambda_i \geqslant \sum_{i=1}^k \mu_i$ for all $k \geqslant 1$.

Let $\mathsf{Irrep}(W)$ denote the set of simple W-modules, up to isomorphism. As usual, irreducible representations of W will be parametrised by partitions of n. We use the ordering on $\mathsf{Irrep}(W)$ arising from the dominance ordering; thus, as in [Mac, Example 1, p. 116], the trivial representation triv is labelled by (n) while the sign representation sign is parametrised by (1^n) and so $\mathsf{triv} > \mathsf{sign}$. Note that the operation on $\mathsf{Irrep}(W)$ given by tensoring by sign corresponds to the transposition of partitions.

2.7. Category $\mathcal{O}_c$. (*See* [GGOR,BEG1, *Definition* 2.4].) Let $\mathcal{O}_c$ be the abelian category of finitely generated H_c-modules M which are locally nilpotent for the subalgebra $\mathbb{C}[\mathfrak{h}^*] \subset H_c$. By Guay [Gu, Theorem 3] $\mathcal{O}_c$ is a highest weight category.

Given $\mu \in \mathsf{Irrep}(W)$, we define $\Delta_c(\mu)$, an object of $\mathcal{O}_c$ called the *standard module*, to be the induced module $\Delta_c(\mu) = H_c \otimes_{\mathbb{C}[\mathfrak{h}^*]*W} \mu$, where $\mathbb{C}[\mathfrak{h}^*]*W$ acts on μ by $pw \cdot m = p(0)(w \cdot m)$ for $p \in \mathbb{C}[\mathfrak{h}^*]$, $w \in W$ and $m \in \mu$. It is shown in [BEG1, Section 2] that each $\Delta_c(\mu)$ has a unique simple quotient $L_c(\mu)$, that the set $\{L_c(\mu) : \mu \in \mathsf{Irrep}(W)\}$ provides a complete list of non-isomorphic simple objects in $\mathcal{O}_c$ and that every object in $\mathcal{O}_c$ has finite length. Note that it follows from the PBW Theorem 2.2.1 that the standard module $\Delta_c(\mu)$ is a free left $\mathbb{C}[\mathfrak{h}]$-module of rank $\dim(\mu)$.

2.8. The KZ functor. Let $M \in \mathcal{O}_c$. Then its localisation $M^{\mathrm{reg}} = M[\delta^{-1}]$ at the powers of δ is a W-equivariant D-module on $\mathfrak{h}^{\mathrm{reg}}$ in the sense that M^{reg} is a W-equivariant vector bundle on $\mathfrak{h}^{\mathrm{reg}}$ with a flat W-equivariant connection. On taking the germs of horizontal sections on $\mathfrak{h}^{\mathrm{reg}}/W$ we get a representation of the braid group $B_n = \pi_1(\mathfrak{h}^{\mathrm{reg}}/W)$. This representation factors through the *Hecke algebra* $\mathcal{H}_q$ of W with parameter $q = \exp(2\pi i c)$ [GGOR, Theorem 5.13]. In this way we have the *Knizhnik–Zamolodchikov functor* $\mathsf{KZ} : \mathcal{O}_c \to \mathcal{H}_q\text{-}\mathsf{mod}$. There is an anti-involution ι on $\mathcal{H}_q$ induced by $\iota(T_w) = T_{w^{-1}}$. Given a module V for $\mathcal{H}_q$, the space $V^* = \mathrm{Hom}_{\mathcal{H}_q}(V, \mathbb{C})$ becomes an $\mathcal{H}_q$-module via the rule $h \cdot f(v) = f(\iota(h)v)$.

The images of the standard modules under KZ are known [GGOR, Remark 6.9 and Corollary 6.10]. For $c \in \mathbb{R}_{\geqslant 0}$ and $\mu \in \mathsf{Irrep}(W)$

$$\mathsf{KZ}(\Delta_c(\mu)) \cong \mathrm{Sp}_q(\mu)^* \tag{2.8.1}$$

where $\mathrm{Sp}_q(\mu)$ is the so-called *Specht module* associated to μ. (The dual module appears since the defining relations for the rational Cherednik algebra given in [GGOR] are normalised differently to (2.1); as remarked in (2.1), our parameter c corresponds to $-k_1$ in [GGOR].) Now suppose that $M \in \mathcal{O}_c$ has a filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{t-1} \subset M_t = M$$

such that M_i/M_{i-1} is a standard module for all $1 \leqslant i \leqslant t$. If $N \in \mathcal{O}_c$ and $c \notin \frac{1}{2} + \mathbb{Z}$ then [GGOR, Proposition 5.9] implies that

$$\mathrm{Hom}_{H_c}(N, M) = \mathrm{Hom}_{\mathcal{H}_q}(\mathsf{KZ}(N), \mathsf{KZ}(M)). \tag{2.8.2}$$

3. Morita equivalence of Cherednik algebras

3.1. A powerful technique in the theory of semisimple Lie algebras is the translation principle, given by tensoring with a finite-dimensional module, in part because it gives an equivalence of categories between the $\mathcal{O}$ categories (and the Harish-Chandra categories) corresponding to distinct central characters [BG]. One interpretation of this is that tensoring with a module of $\mathfrak{k}$-finite vectors gives a Morita equivalence between the corresponding factors of the enveloping algebra [JS, Corollary 4.13].

Although it does not involve finite dimensional modules, there is a natural analogue of this procedure for Cherednik algebras, given by the Heckman–Opdam shift functors defined in (3.2). These functors have proved useful in a number of papers (see, for example, [BEG1,BEG2,Go2]) and for particular values of c these functors are known to give equivalences of categories between H_c, U_c and U_{c+1} (see, for example, [BEG1, Theorem 8.1] and [BEG2, Proposition 4.3]). It is an open problem to determine precisely when these equivalences exist [GGOR, Remark 5.17] and this question is crucial to our $\mathbb{Z}$-algebra approach to Cherednik algebras. We give an essentially complete answer to this question in Corollary 3.13 and Remark 3.14. We also prove that the equivalence $H_c \to H_{c+1}$ maps category $\mathcal{O}_c$ to $\mathcal{O}_{c+1}$ and sends the standard module $\Delta_c(\mu)$ to $\Delta_{c+1}(\mu)$, see Proposition 3.16.

3.2. Fix $c \in \mathbb{C}$ and keep the notation of (2.5). If we identify H_c with its image in $D(\mathfrak{h}^{\mathrm{reg}}) * W$ via the Dunkl operator (2.3.1) then, by Berest et al. [BEG2, Proposition 4.1], there is an identity

$$U_c \;=\; \delta^{-1} U^{-}_{c+1} \delta \;=\; e\delta^{-1} H_{c+1} \delta e. \tag{3.2.1}$$

In particular, this implies that $Q_c^{c+1} = eH_{c+1}e_{-}\delta = eH_{c+1}\delta e$ is a (U_{c+1}, U_c)-bi-submodule of $D(\mathfrak{h}^{\mathrm{reg}}) * W$. The shift functors mentioned above are given by

$$S_c : U_c\text{-}\mathsf{mod} \to U_{c+1}\text{-}\mathsf{mod} : \qquad N \mapsto Q_c^{c+1} \otimes_{U_c} N$$

and

$$\widetilde{S}_c : H_c\text{-}\mathsf{mod} \to H_{c+1}\text{-}\mathsf{mod} : \qquad M \mapsto H_{c+1}e_{-}\delta \otimes_{U_c} eM.$$

3.3. When c is a positive real number, the Morita equivalence between U_c and U_{c+1} is given by S_c and we begin with that case. The general case, proved in Corollary 3.13, will be an easy consequence.

Theorem. *Assume that* $c \in \mathbb{R}_{\geqslant 0}$ *with* $c \notin \frac{1}{2} + \mathbb{Z}$. *Then both shift functors* $\widetilde{S}_c : H_c\text{-}\mathsf{mod} \to H_{c+1}\text{-}\mathsf{mod}$ *and* $S_c : U_c\text{-}\mathsf{mod} \to U_{c+1}\text{-}\mathsf{mod}$ *are Morita equivalences.*

Moreover, the idempotent functor $E_c : H_c\text{-}\mathsf{mod} \to U_c\text{-}\mathsf{mod}$ *given by* $M \mapsto eM$ *is a Morita equivalence.*

Proof. To prove that S_c is an equivalence we need to show that $Q = Q_c^{c+1}$ is a projective generator for U_{c+1}-mod, with endomorphism ring $\mathrm{End}_{U_{c+1}}(Q) = U_c$. Arguing as in [EG, Theorem 1.5(iv)] the dual $Q^* = \mathrm{Hom}_{U_{c+1}}(Q, U_{c+1})$ is $P = \delta^{-1} e_{-} H_{c+1} e$. By the dual basis lemma, Q is a projective U_{c+1}-module with $\mathrm{End}_{U_{c+1}}(Q) = U_c$ if and only if $PQ = U_c$ while Q is a generator if and only if $QP = U_{c+1}$. Substituting in the given formulæ for Q and P shows that we need to prove that

$$H_{c+1}e_{-}H_{c+1} = H_{c+1} \quad \text{and} \quad H_{c+1}eH_{c+1} = H_{c+1} \quad \text{for } c \geqslant 0. \tag{3.3.1}$$

Similarly, as $H_c e$ is a projective left H_c-module, E_c will be a Morita equivalence if we prove that

$$H_c e H_c = H_c \quad \text{for } c \geqslant 0. \tag{3.3.2}$$

Since $\widetilde{S}_c = E_{c+1}^{-1} \circ S_c \circ E_c$, Eqs. (3.3.1) and (3.3.2) will suffice to prove the theorem.

The proof of Theorem 3.3 will be through a series of lemmas and we begin with the first equality in (3.3.1). Set $d = c + 1$; thus $d \in \mathbb{R}_{\geqslant 1}$, with $d \notin \frac{1}{2} + \mathbb{Z}$.

3.4. Reduction to category $\mathcal{O}$. If $H_d e_- H_d$ is a *proper* two-sided ideal of H_d it must be contained in a primitive ideal, and hence, by Ginzburg [Gi, Generalised Duflo Theorem], annihilate an object from category $\mathcal{O}_d$. Thus, it is enough to show that e_- does not annihilate any simple module belonging to $\mathcal{O}_d$.

To do this we first show in Corollary 3.6 that the composition factors of $\Delta_d(\mu)$ are of the form $L_d(\lambda)$ for $\lambda \leqslant \mu$. Under the $\mathbb{Z}$-strings ordering such a result is proved in [Gu] but as we work with the dominance ordering of partitions and representations, as defined in (2.6), this definitely requires work, see also (3.7). We then show that the lowest weight copy of the sign module for W in $\Delta_d(\mu)$ does not occur in any standard module $\Delta_d(\lambda)$ for $\lambda < \mu$. Since $L_d(\mu)$ is the head (that is, the unique simple factor module) of $\Delta_d(\mu)$ it will follow that $e_- L_d(\mu) \neq 0$.

3.5. Lemma. *Let $c \in \mathbb{R}_{\geqslant 0}$ with $c \notin \frac{1}{2} + \mathbb{Z}$. If $\mathrm{Hom}_{H_c}(\Delta_c(\lambda), \Delta_c(\mu)) \neq 0$ for $\lambda, \mu \in \mathsf{Irrep}(W)$, then $\lambda \leqslant \mu$ in the dominance ordering.*

Proof. Let $S_q = S_q(n, n)$ be the q-Schur algebra defined in [DJ2, Section 1], where $q = \exp(2\pi i c)$. It is conjectured in [GGOR, Remark 5.17] that S_q-mod is equivalent to $\mathcal{O}_c$. We cannot prove this, but we will show that there is a relationship which implies the lemma.

For each $\mu \in \mathsf{Irrep}(W)$ there is an S_q-module $W_q(\mu)$, called the *q-Weyl module*. By Dipper and James [DJ2, Corollary 8.6], there is an isomorphism

$$\mathrm{Hom}_{\mathcal{H}_q}(\mathrm{Sp}_q(\mu), \mathrm{Sp}_q(\lambda)) \cong \mathrm{Hom}_{S_q}(W_q(\lambda), W_q(\mu)). \tag{3.5.1}$$

On the other hand, by (2.8.1) and (2.8.2) we have

$$\begin{aligned} \mathrm{Hom}_{H_c}(\Delta_c(\lambda), \Delta_c(\mu)) &\cong \mathrm{Hom}_{\mathcal{H}_q}(\mathrm{Sp}_q(\lambda)^*, \mathrm{Sp}_q(\mu)^*) \\ &\cong \mathrm{Hom}_{\mathcal{H}_q}(\mathrm{Sp}_q(\mu), \mathrm{Sp}_q(\lambda)). \end{aligned} \tag{3.5.2}$$

Each $W_q(v)$ has a simple head $F_q(v)$, [DJ2, Theorem 4.6] and $\{F_q(v) : v \in \mathsf{Irrep}(W)\}$ is a complete, repetition-free list of the simple S_q-modules up to isomorphism, [DJ2, Theorem 8.8]. Furthermore, $F_q(\lambda)$ is a composition factor of $W_q(\mu)$ only if $\lambda \leqslant \mu$, [DJ2, Corollary 8.9]. By (3.5.1) and (3.5.2) a non-zero homomorphism $\phi : \Delta_c(\lambda) \to \Delta_c(\mu)$ implies the existence of a non-zero homomorphism $\phi' : W_q(\lambda) \to W_q(\mu)$. Thus $F_q(\lambda)$ must be a composition factor of $W_q(\mu)$ and so $\lambda \leqslant \mu$. $\square$

3.6. Corollary. *Assume that $c \in \mathbb{R}_{\geqslant 0}$, with $c \notin \frac{1}{2} + \mathbb{Z}$. If $[\Delta_c(\mu) : L_c(\lambda)] \neq 0$ for $\lambda, \mu \in \mathsf{Irrep}(W)$, then $\lambda \leqslant \mu$ in the dominance ordering.*

Remarks. (1) For arbitrary c and μ, the unique occurrence of $L_c(\mu)$ as a composition factor of $\Delta_c(\mu)$ is as its head—see, for example, the discussion after Lemma 7 in [Gu, Section 2].

(2) Since sign is minimal in the dominance ordering, the lemma and the above remark imply that $\Delta_c(\mathsf{sign})$ is irreducible for all $c \in \mathbb{R}_{\geqslant 0}$. This can also be deduced from [Gu].

Proof. We argue by induction on μ. More precisely, suppose that $[\Delta_c(\mu) : L_c(\lambda)] \neq 0$ for some $\mu \neq \lambda$ and that the lemma holds for any $v < \mu$. (The induction starts since there are only finitely many σ with $\sigma < \mu$.) Let $P_c(\lambda)$ denote the projective cover of $\Delta_c(\lambda)$, as in [GGOR, Section 3.5], and write K for the kernel of the associated homomorphism $\phi : P_c(\lambda) \to \Delta_c(\mu)$. By Guay [Gu, Proposition 13] there is a Δ-filtration of $P_c(\lambda)$

$$P_c(v) = M_0 \supset M_1 \supset \cdots \supset M_t = 0$$

with each factor M_j/M_{j+1} of the form $\Delta_c(\lambda_j)$ for some $\lambda_j \in \mathsf{Irrep}(W)$. Thus there exists i such that $M_i + K/K \neq 0$ but $M_{i+1} + K/K = 0$. This gives a non-zero composition

$$\psi : \Delta_c(\lambda_i) \cong M_i/M_{i+1} \longrightarrow (M_i + K)/K \longrightarrow P_c(\lambda)/K \longrightarrow \Delta_c(\mu).$$

By Lemma 3.5, $\lambda_i \leqslant \mu$. If $\lambda_i = \mu$ then the first remark after the statement of the lemma would imply that ψ and hence ϕ are surjective, contradicting the fact that $\lambda \neq \mu$. Thus $\lambda_i < \mu$. By BGG reciprocity [Gu, Theorem 19], $[P_c(\lambda) : \Delta_c(\lambda_i)] = [\Delta_c(\lambda_i) : L_c(\lambda)] \neq 0$ and so, by induction, $\lambda \leqslant \lambda_i$. Thus $\lambda < \mu$. $\square$

3.7. A result analogous to Corollary 3.6 is proved as part of the proof of Guay [Gu, Proposition 13]. However, the $\mathbb{Z}$-strings ordering used in [Gu] is different from the dominance ordering. An explicit example where the orderings differ can be found when $n = 8$, by taking $\lambda = (6, 1, 1)$ and $\mu = (4, 4)$. In this case λ and μ are incomparable in the dominance ordering, but comparable in the $\mathbb{Z}$-strings ordering.

3.8. The canonical grading on $\mathcal{O}_c$. The final ingredient we need for the proof of Theorem 3.3 is a canonical grading on $\mathcal{O}_c$. Let $\mathbf{h}_c \in H_c$ be defined as in (2.4.1). Then, for $M \in \mathcal{O}_c$ and $\alpha \in \mathbb{C}$, set

$$W_\alpha(M) = \{m \in M : (\mathbf{h}_c - \alpha)^k m = 0 \text{ for } k \gg 0\}.$$

By Ginzburg et al. [GGOR, (2.4.1)] this gives the *canonical grading* $M = \sum_{\alpha \in \mathbb{C}} W_\alpha(M)$.

This observation has two useful consequences. First, if $\theta : M_1 \to M_2$ is an H_c-module homomorphism with $M_i \in \mathcal{O}_c$, then $\theta(W_\alpha(M_1)) \subseteq W_\alpha(M_2)$ for each $\alpha \in \mathbb{C}$. Secondly, if $p \in H_c$ has $\mathbf{E}$-deg $p = t$, then (2.4.2), implies that $p \cdot W_\alpha(M) \subseteq W_{\alpha+t}(M)$. Note that the standard module $\Delta_c(\mu)$ is therefore a lowest weight module since it is generated as a $\mathbb{C}[\mathfrak{h}]$-module by the space $1 \otimes \mu$.

3.9. To describe the graded structure of the standard modules we need a little notation. Recall that the space of coinvariants $\mathbb{C}[\mathfrak{h}]^{\mathrm{co}\,W} = \mathbb{C}[\mathfrak{h}]\big/\mathbb{C}[\mathfrak{h}]_+^W\mathbb{C}[\mathfrak{h}]$ is a finite-dimensional graded algebra isomorphic as a W-module to the regular representation. As in [Op], the polynomials

$$f_\mu(v) = \sum_{i \geqslant 0}[\mathbb{C}[\mathfrak{h}]_i^{\mathrm{co}\,W} : \mu]v^i \tag{3.9.1}$$

are called the *fake degrees* of $\mu \in \mathsf{Irrep}(W)$. We define $n(\mu)$ to be the lowest power of v appearing in $f_\mu(v)$; thus, $f_\mu(v) = av^{n(\mu)} +$ higher-order terms. In the notation of Haiman [Ha3], $n(\mu)$ is equal to the *partition statistic* $\sum_i \mu_i(i-1)$ (see the proof of Gordon [Go1, Theorem 6.4]). Finally, (3.9.1) implies that

$$f_{\mu^t}(1) = \dim\, \mu^t = \dim\, \mu = f_\mu(1) \quad \text{for } \mu \in \mathsf{Irrep}(W). \tag{3.9.2}$$

3.10. Given a graded W-module $M = \sum_{\alpha\in\mathbb{C}} W_\alpha(M)$ we define its *graded Poincaré series* to be

$$p(M, v, W) = \sum_{\alpha\in\mathbb{C}} v^\alpha \sum_{\lambda\in\mathsf{Irrep}(W)} [W_\alpha(M) : \lambda][\lambda].$$

This is easily determined for standard modules.

Proposition. (1) *Under the canonical grading, the subspace* $1\otimes\mu$ *of* $\Delta_c(\mu)$ *has weight* $m + c(n(\mu) - n(\mu^t))$, *where* $m = (n-1)/2$.

(2) *The Poincaré series of* $\Delta_c(\mu)$ *as a graded W-module is*

$$p(\Delta_c(\mu), v, W) = v^{m+c(n(\mu)-n(\mu^t))}\,\frac{\sum_\lambda f_\lambda(v)[\lambda\otimes\mu]}{\prod_{i=2}^n(1-v^i)}. \tag{3.10.1}$$

Proof. (1) We need to compute the action of $\mathbf{h} = \frac{1}{2}\sum_{i=1}^{n-1}(x_iy_i + y_ix_i)$ on the space $1\otimes\mu$. By the defining relations of H_c from (2.1), and the fact that the $\{x_i\}$ and $\{y_i\}$ are dual bases, we obtain

$$\begin{aligned}\mathbf{h} &= \textstyle\sum_i x_iy_i + (n-1)/2 - \frac{1}{2}\sum_{s\in\mathcal{S}}\sum_i c\alpha_s(y_i)x_i(\alpha_s^\vee)s\\ &= \textstyle\sum x_iy_i + (n-1)/2 - \frac{c}{2}\sum_{s\in\mathcal{S}}\alpha_s(\alpha_s^\vee)s = \sum x_iy_i + m - c\sum_{s\in\mathcal{S}} s.\end{aligned}$$

The action of $\sum(1-s)$ on $\lambda \in \mathsf{Irrep}(W)$ can be derived from [BM,Lu]. More precisely, λ is special by Lusztig [Lu, (4.2.2)] and so $n(\lambda) = b_\lambda = a_\lambda$ in the notation of Lusztig [Lu]. Therefore, by Broué and Michel [BM, Section 4.21] and Lusztig [Lu, Section 4.1 and (5.11.5)], $\sum_s(1-s)$ acts on $\lambda \in \mathsf{Irrep}(W)$ with weight $N+n(\lambda)-n(\lambda^t)$, where $N = n(n-1)/2$ is the cardinality of $\mathcal{S}$. Thus $\sum_s s$ acts on $1\otimes\mu$ with weight $-(n(\mu)-n(\mu^t))$ and hence $\mathbf{h}$ acts with weight $m + c(n(\mu) - n(\mu^t))$.

(2) As graded W-modules, $\Delta(\mu) \cong (\mathbb{C}[\mathfrak{h}] \otimes \mu)[k]$ for $k = m + c(n(\mu) - n(\mu^t))$. The shift arises from the fact that, by (1), the generator $1 \otimes \mu$ of $\Delta_c(\mu)$ lives in degree k. The Chevalley–Shephard–Todd Theorem implies that, as graded W-modules, $\mathbb{C}[\mathfrak{h}] \cong \mathbb{C}[\mathfrak{h}]^W \otimes \mathbb{C}[\mathfrak{h}]^{\mathrm{co}\,W}$. Now $\mathbb{C}[\mathfrak{h}]^W$ is a polynomial ring with generators in degrees $2 \leqslant i \leqslant n$ and so its Poincaré polynomial is $\prod_{i=2}^n (1 - v^i)^{-1}$. On the other hand, the coinvariant ring $\mathbb{C}[\mathfrak{h}]^{\mathrm{co}\,W}$ has graded Poincaré polynomial $\sum_\lambda \sum_i [\mathbb{C}[\mathfrak{h}]_i^{\mathrm{co}\,W} : \lambda][\lambda]v^i$. By definition, this is just $\sum_\lambda f_\lambda(v)[\lambda]$. Combining these observations gives (3.10.1).

□

3.11. Completion of the proof of Theorem 3.3. We will first prove that $H_d e_- H_d = H_d$ (where $d = c + 1$, as before). Since $\mu \cong \mu^*$ for symmetric groups, the sign representation is a direct summand of $\mu \otimes \nu$ if and only if $\nu = \mu^t$. Thus (3.10.1) implies that sign first appears in $\Delta_d(\mu)$ in the weight space

$$m + d(n(\mu) - n(\mu^t)) + n(\mu^t) = m + dn(\mu) - (d-1)n(\mu^t),$$

where $m = (n-1)/2$. If $\lambda \leqslant \mu$ then $n(\lambda) \geqslant n(\mu)$ by Shi [Sh, Theorem B and Proposition 1.6]. Moreover, as $\lambda^t \geqslant \mu^t$, we have $n(\lambda^t) \leqslant n(\mu^t)$. Since $d \in \mathbb{R}_{\geqslant 1}$,

$$m + dn(\lambda) - (d-1)n(\lambda^t) \geqslant m + dn(\mu) - (d-1)n(\mu^t)$$

with equality if and only if $\lambda = \mu$.

It follows that the copy of sign appearing in the lowest possible weight space of $\Delta_d(\mu)$ is never a weight of $\Delta_d(\lambda)$ for $\lambda < \mu$. By Corollary 3.6, this means that this copy of sign is a weight for $L_d(\mu)$ and hence that $e_- L_d(\mu) \neq 0$. By (3.4) this implies that $H_d e_- H_d = H_d$, and so the first equality of (3.3.1) is proven.

It remains to show that $H_c e H_c = H_c$ for $c \in \mathbb{R}_{\geqslant 0}$. The argument of (3.4) shows that we need to prove that e does not annihilate any simple module from $\mathcal{O}_c$. In this case triv appears in $\mu \otimes \nu$ if and only if $\nu = \mu$. Therefore, (3.10.1) now implies that triv first appears in $\Delta_c(\mu)$ in degree $m + c(n(\mu) - n(\mu^t)) + n(\mu)$. Let $\lambda \leqslant \mu$. Then

$$\begin{aligned} m + c(n(\lambda) - n(\lambda^t)) + n(\lambda) &= m + (c+1)n(\lambda) - cn(\lambda^t) \\ &\geqslant m + c(n(\mu) - n(\mu^t)) + n(\mu) \end{aligned}$$

with equality if and only if $\lambda = \mu$. This means that triv appears in $\Delta_c(\lambda)$ in a higher degree than its first appearance in $\Delta_c(\mu)$. In particular, the simple quotient $L_c(\mu)$ of $\Delta_c(\mu)$ contains a copy of triv and so it cannot be annihilated by e. This therefore completes the proof of (3.3.1) and (3.3.2) and hence proves the theorem. □

3.12. General equivalences. We now give the promised extension of Theorem 3.3 to more general values of c. Since it requires no extra work, and it is put to crucial use in [BFG], we will also prove the result over more general base fields. Thus if k is a subfield of $\mathbb{C}$, with $c \in k$, let $H(k)_c$ denote the k-algebra defined by the generators and relations from (2). We write $U(k)_c$, $Q(k)_c^{c+1}$, etc., for the corresponding objects defined over k.

Hypothesis. Set $\mathcal{C} = \{z : z = \frac{m}{d}$ *where* $m, d \in \mathbb{Z}$ with $2 \leqslant d \leqslant n$ and $z \notin \mathbb{Z}\}$. Assume that $c \in \mathbb{C}$ is such that $c \notin \frac{1}{2} + \mathbb{Z}$. If c is a rational number with $-1 < c < 0$ assume further that $c \notin \mathcal{C}$.

3.13. Corollary. *Let $k \subseteq \mathbb{C}$ be a field and assume that $c \in k$ satisfies Hypothesis* 3.12.

(1) *$U(k)_c$ and $H(k)_c$ are Morita equivalent. If*

$$c \notin (-2, -1)_{\mathcal{C}} = \{z \in \mathcal{C} : -2 < z < -1\},$$

then $U(k)_c$ is Morita equivalent to $U(k)_{c+1}$.

(2) *Let $a = -c$. Then $H(k)_a$ is Morita equivalent to $U(k)_a^- = e_-H(k)_a e_-$. If $a \notin (1, 2)_{\mathcal{C}}$, then $U(k)_a^-$ is Morita equivalent to $U(k)_{a-1}^-$.*

Proof. (1) We start with the case $U_c = U(\mathbb{C})_c$. If $c \notin \mathcal{C}$ then it follows from [BEG1, Theorem 8.1] and [DJ1, Theorem 4.3] that H_c, U_c and U_c^- are simple, Morita equivalent rings (see the introduction to [BEG3] for the details). Since this also applies to H_{c+1} the conditions (3.3.1) are trivially satisfied and the result follows.

Thus we may assume that $c \in \mathcal{C}$. If $c \geqslant -1$, then necessarily $c \geqslant 0$ and so the result follows from Proposition 3.3. Otherwise $c \leqslant -1$. In this case the discussion before [De, Remark 2.2] shows that there is an isomorphism $\chi : H_c \to H_{-c}$ satisfying $\chi(e_-) = e$. Thus, for any c, (3.2.1) implies that $U_c \cong U_{-c}^- \cong eH_{-c-1}e = U_{-c-1}$. The result for $c \leqslant -1$ therefore follows from the cases already discussed.

Finally, let k be an arbitrary subfield of $\mathbb{C}$ and consider $U(k)_c$. In order to prove, for example, that $U(k)_c$ is Morita equivalent to $U(k)_{c+1}$ we need to prove that $Q(k)P(k) = U(k)_{c+1}$ and $P(k)Q(k) = U(k)_c$. By construction, $Q(\mathbb{C}) = Q(k) \otimes_k \mathbb{C}$, and similarly for $P(\mathbb{C})$. By the earlier part of the proof, $U(\mathbb{C})_c/P(\mathbb{C})Q(\mathbb{C}) = 0$. The faithful flatness of $U(\mathbb{C})_c = U(k)_c \otimes_k \mathbb{C}$ as a $U(k)_c$-module then ensures that $U(k)_c/P(k)Q(k) = 0$. All the other steps in the proof follow in exactly the same way.

(2) Using the identity $U_c \cong U_{-c}^-$, this follows from part (1). $\square$

3.14. Remarks. (1) The condition that $c \notin \frac{1}{2} + \mathbb{Z}$ is needed in Theorem 3.3 and Corollary 3.13 in order to apply (2.8.2) and may be unnecessary. This is the case when $n = 2$ as U_c is Morita equivalent to U_{c+1} if and only if $c \neq -\frac{3}{2}, -\frac{1}{2}$ (see, for example, [EG, Proposition 8.2]). The point about the excluded cases is that $U_{-\frac{1}{2}}$ is simple but the two neighbouring algebras, $U_{\frac{1}{2}}$, $U_{-\frac{3}{2}}$ are not. Combining [EG, Proposition 8.2] with [St, Theorem B] shows that $U_{-\frac{1}{2}}$ has infinite global dimension, and so the next Corollary 3.15 also fails for this value of c.

(2) This also shows that the hypothesis $c \notin (-2,0)_{\mathcal{C}}$ is serious. Indeed, for any $n \geqslant 2$, let $c = -m/n \in (-1,0)_{\mathcal{C}}$. Then one can prove that the factor module $V_c = \Delta_c(\mathsf{sign})/I_c$ considered in [CE, Theorem 3.2] does not contain a copy of the W-module triv (we thank Pavel Etingof for this fact). In particular $eV_c = 0$ and so *the functor E_c is not an equivalence*. If we further assume that $(m,n)=1$, then V_c is the unique irreducible finite-dimensional H_c-module by Chmutova and Etingof [CE, Corollary 3.3] and Berest et al. [BEG2, Theorem 1.2(ii)]. Since $U_c = \mathrm{End}_{H_c}(eH_c)$, this implies that U_c has no finite-dimensional modules. However, by Corollary 3.13(1) and Berest et al. [BEG2, Theorem 1.2] $U_{c\pm 1}$-mod does have such modules and so *there is no equivalence between U_c and $U_{c\pm 1}$*.

3.15. Corollary. *Assume that $c \in \mathbb{C}$ satisfies Hypothesis* 3.12. *Then H_c and U_c have finite homological global dimension and satisfy the Auslander–Gorenstein conditions and Cohen–Macaulay conditions of Levasseur* [Le].

Proof. Since this result takes us a little far afield, the details of the proof are left to the interested reader. Standard techniques show that $\mathsf{ogr}\, H_c \cong \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] * W$ and hence H_c have the given properties (see, for example, [Br, Theorem 4.4]). By Corollary 3.13, U_c is Morita equivalent to H_c and it follows that U_c also has these properties. $\square$

3.16. The shift functor on $\mathcal{O}_c$. Many computations for U_c reduce to computations in category $\mathcal{O}$ and so it is important to know that, under the hypotheses of Theorem 3.3, S_c does provide an equivalence between the corresponding categories. This is the point of the next result.

Proposition. *Assume that $c \in \mathbb{C}$ satisfies Hypothesis* 3.12 *and that $c \notin \mathbb{Q}_{\leqslant -1}$. Then the shift functor $\widetilde{S}_c$ restricts to an equivalence between $\mathcal{O}_c$ and $\mathcal{O}_{c+1}$ such that $\widetilde{S}_c(\Delta_c(\lambda)) \cong \Delta_{c+1}(\lambda)$ for all partitions λ of n. Thus $S_c(e\Delta_c(\lambda)) = e\Delta_{c+1}(\lambda)$.*

Remark. By Corollary 3.13(2), an analogue of the proposition also holds when $c \in \mathbb{Q}_{\leqslant -1}$, provided that one shifts in a negative direction.

Proof. The final assertion of the proposition is immediate from the previous one combined with Corollary 3.13(1).

We begin by showing that $\widetilde{S}_c$ restricts to an equivalence between $\mathcal{O}_c$ and $\mathcal{O}_{c+1}$. Fix $M \in \mathcal{O}_c$. Let $\mathcal{I}_t = \mathbb{C}[\mathfrak{h}^*]^W_{\geqslant t}$ denote the W-invariant elements of $\mathbb{C}[\mathfrak{h}^*]$ of degree at least t and set $I_t = \mathcal{I}_t\mathbb{C}[\mathfrak{h}^*]$, Then $\mathbb{C}[\mathfrak{h}^*]/I_t$ is a finite-dimensional algebra and so all homogeneous elements of $\mathbb{C}[\mathfrak{h}^*]$ of large degree belong to I_t. Thus it is enough to show that, if $\widetilde{m} = he_{-}\delta \otimes em \in \widetilde{S}_c(M) = H_{c+1}e_{-}\delta \otimes_{U_c} eM$, for some $h \in H_{c+1}$ and $m \in M$, then $\widetilde{m}$ is annihilated by $\mathcal{I}_t$ for $t \gg 0$.

Recall the $\mathbf{E}$-grading on H_c from (2.4). Since $\mathbb{C}[\mathfrak{h}^*]$ acts locally nilpotently on M, the PBW isomorphism (2.2.1) shows that any homogeneous element of H_c of sufficiently large negative $\mathbf{E}$-degree annihilates $m \in M$. Thus, assume that $qm = 0$ for all $q \in H_c$

with $\mathbf{E}\text{-deg}(q) \leqslant -t$ and let $p \in \mathbb{C}[\mathfrak{h}^*]^W_{\geqslant t}$. Then

$$\begin{aligned} phe_-\delta \otimes em &= [p,h]e_-\delta \otimes em + h\delta\delta^{-1}pe_-\delta \otimes em \\ &= [p,h]e_-\delta \otimes em + he_-\delta \otimes \delta^{-1}p\delta em. \end{aligned}$$

Since $\mathbf{E}\text{-deg}\, \delta^{-1}p\delta = \mathbf{E}\text{-deg}\, p \leqslant -t$, we have $\delta^{-1}p\delta em = 0$ by the hypothesis on t.

Therefore $p(he_-\delta \otimes em) = [p,h]e_-\delta \otimes em$ for any such p. Since the choice of t was independent of h, this implies that $p^r(he_-\delta \otimes em) = \mathrm{ad}(p)^r(h)(e_-\delta \otimes em)$, for any $r > 0$. Now, p commutes with both $\mathbb{C}[W]$ and $\mathbb{C}[\mathfrak{h}^*]$, and so the defining relations of H_{c+1} from (2.1) ensure that the adjoint action of $p \in \mathbb{C}[\mathfrak{h}^*]^W$ on H_{c+1} is locally nilpotent (see also [BEG1, Lemma 3.3(v)]). Therefore a sufficiently large power of p annihilates $he_-\delta \otimes em$. Thus $\widetilde{S}_c(M) \in \mathcal{O}_{c+1}$ and $\widetilde{S}_c$ does restrict to the desired equivalence.

It remains to compute $\widetilde{S}_c(\Delta_c(\lambda))$ and we begin with the analogous problem on H^{reg}_{c+1}. In the notation of (2.3.2), $\widetilde{S}_c(\Delta_c(\lambda))^{\mathrm{reg}} = H^{\mathrm{reg}}_{c+1}e_-\delta \otimes_{\delta^{-1}U^-_{c+1}\delta} e\Delta_c(\lambda)$. By (2.3.2), $H^{\mathrm{reg}}_{c+1} \cong A = D(\mathfrak{h}^{\mathrm{reg}}) * W$ and so $\widetilde{S}_c(\Delta_c(\lambda))^{\mathrm{reg}} \cong Ae_-\delta \otimes_B e\Delta_c(\lambda)^{\mathrm{reg}}$, where $B = \delta^{-1}e_-Ae_-\delta$. On the other hand, (3.2.1) induces an isomorphism

$$\theta : Ae_-\delta \otimes_B e\Delta_c(\lambda)^{\mathrm{reg}} \longrightarrow Ae \otimes_{eAe} e\Delta_c(\lambda)^{\mathrm{reg}}; \quad ae_-\delta \otimes em \mapsto a\delta e \otimes em.$$

Combined with the identity $H_c eH_c = H_c$ from Corollary 3.13(1), this implies that

$$\widetilde{S}_c(\Delta_c(\lambda))^{\mathrm{reg}} \cong Ae \otimes_{eAe} e\Delta_c(\lambda)^{\mathrm{reg}} \cong \left(H_ce \otimes_{U_c} e\Delta_c(\lambda)\right)^{\mathrm{reg}} \cong \Delta_c(\lambda)^{\mathrm{reg}} \neq 0. \quad (3.16.1)$$

If $c \notin \mathcal{C}$, we are done. Indeed, in this case [BEG1, Corollary 2.11] implies that $\Delta_{c+1}(\lambda)$, $\Delta_c(\lambda)$ and hence $\widetilde{S}_c(\Delta_c(\lambda))$ are all simple modules. The isomorphism (3.16.1) implies that $\widetilde{S}_c(\Delta_c(\lambda)) \hookrightarrow \Delta_{c+1}(\lambda)^{\mathrm{reg}}$. Under this embedding, $\widetilde{S}_c(\Delta_c(\lambda)) \cap \Delta_{c+1}(\lambda) \neq 0$ and hence $\widetilde{S}_c(\Delta_c(\lambda)) = \Delta_{c+1}(\lambda)$.

We may therefore assume that $c \in \mathcal{C}$, in which case Hypothesis 3.12 implies that $c \geqslant 0$ and we can use the KZ-functor from (2.8). By (3.16.1) and (2.8.1), $\mathsf{KZ}(\widetilde{S}_c(\Delta_c(\lambda))) \cong \mathsf{KZ}(\Delta_c(\lambda)) \cong \mathrm{Sp}_q(\lambda)^*$. By (2.8.2) and (3.5.1) we therefore have

$$\mathrm{Hom}_{H_{c+1}}(\widetilde{S}_c(\Delta_c(\lambda)), \Delta_{c+1}(\lambda)) \cong \mathrm{Hom}_{S_q}(W_q(\lambda), W_q(\lambda)) = \mathbb{C}. \quad (3.16.2)$$

It follows from Corollary 3.6 that the composition factors of $\Delta_{c+1}(\lambda)$ are of the form $L_{c+1}(\nu)$ with $\nu \leqslant \lambda$ in the dominance ordering. We will show by an ascending induction on this ordering that $\widetilde{S}_c(\Delta_c(\lambda)) \cong \Delta_{c+1}(\lambda)$.

If λ is minimal in the dominance ordering then $\lambda = \mathsf{sign}$ and so both $\Delta_{c+1}(\lambda)$ and $\widetilde{S}_c(\Delta_c(\lambda))$ are simple by Remark 3.6. By (3.16.2) there is a non-zero map from $\widetilde{S}_c(\Delta_c(\lambda))$ to $\Delta_{c+1}(\lambda)$ which therefore must be an isomorphism. This begins the induction.

Let λ be arbitrary and suppose that, for all $v < \lambda$ in the dominance ordering, we have $\widetilde{S}_c(\Delta_c(v)) \cong \Delta_{c+1}(v)$, and hence that $\widetilde{S}_c(L_c(v)) \cong L_{c+1}(v)$. Since $\widetilde{S}_c$ is an equivalence, $\widetilde{S}_c(\Delta_c(\mu))$ has simple head $\widetilde{S}_c(L_c(\mu))$ for each μ. By (3.16.2) $\widetilde{S}_c(L_c(\lambda))$ is therefore isomorphic to a composition factor of $\Delta_{c+1}(\lambda)$. But, by Corollary 3.6 and the remark thereafter, the composition factors of $\Delta_{c+1}(\lambda)$, except for the head, are of the form $L_{c+1}(v)$ for $v < \lambda$. Thus the non-zero map (3.16.2) must send the head of $\widetilde{S}_c(\Delta_c(\lambda))$ to the head of $\Delta_{c+1}(\lambda)$ and so induce an isomorphism $\widetilde{S}_c(\Delta_c(\lambda)) \xrightarrow{\sim} \Delta_{c+1}(\lambda)$. This completes the inductive step, and hence the proof of the proposition. $\square$

4. The Hilbert scheme

4.1. Haiman's work on Hilbert schemes gives detailed information about their structure, in particular as "Proj" of appropriate Rees rings. The resulting formulæ for the Poincaré series of these rings will be crucial to the proof of the main theorem in Section 6. In this section, we briefly describe the relevant results from the literature and use this to derive the appropriate Poincaré series.

4.2. Let $\mathrm{Hilb}^n \mathbb{C}^2$ be *the Hilbert scheme of n points on the plane*, which we realise as the set of ideals of colength n in the polynomial ring $\mathbb{C}[\mathbb{C}^2]$. Similarly, identify the variety $S^n\mathbb{C}^2$ of *unordered n*-tuples of points in $\mathbb{C}^2$ with the categorical quotient $\mathbb{C}^{2n}/W$ under the diagonal action of W on $\mathbb{C}[\mathbb{C}^{2n}]$. Then the map $\tau : \mathrm{Hilb}^n \mathbb{C}^2 \to S^n\mathbb{C}^2 = \mathbb{C}^{2n}/W$ which sends an ideal to its support (counted with multiplicity) is a resolution of singularities (see, for example, [Na, Theorem 1.15]).

We will actually be interested in a resolution of singularities for $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W$ rather than $\mathbb{C}[\mathbb{C}^{2n}]^W$, simply because the associated graded ring of U_c is $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W$. The results we need follow easily from the corresponding results on $\mathrm{Hilb}^n \mathbb{C}^2$ and so we begin with the latter.

4.3. The isospectral Hilbert scheme. Following Haiman [Ha3, Definition 3.2.4] *the isospectral Hilbert scheme* $\mathbb{X}_n$ is the reduced fibre product

$$\begin{array}{ccc} \mathbb{X}_n & \xrightarrow{f_1} & \mathbb{C}^{2n} \\ {\scriptstyle\rho_1}\big\downarrow & & \big\downarrow \\ \mathrm{Hilb}^n \mathbb{C}^2 & \xrightarrow{\tau} & \mathbb{C}^{2n}/W. \end{array}$$

It is a highly non-trivial fact (see [Ha3, Theorem 3.1 and the proof of Proposition 3.7.4]) that ρ_1 is a flat map of degree $n!$.

Haiman has given a description of both $\mathrm{Hilb}^n \mathbb{C}^2$ and $\mathbb{X}_n$ as Proj of appropriate graded rings and we recall this description since it will be extremely important to us. Let $\mathbb{A}^1 = \mathbb{C}[\mathbb{C}^{2n}]^{\varepsilon}$ be the space of W-alternating polynomials in $\mathbb{C}[\mathbb{C}^{2n}]$ and write $\mathbb{J}^1 = \mathbb{C}[\mathbb{C}^{2n}]\mathbb{A}^1$ for the ideal generated by $\mathbb{A}^1$. For $d \geqslant 1$ define $\mathbb{A}^d$ and $\mathbb{J}^d$ to be the respective dth powers of $\mathbb{A}^1$ and $\mathbb{J}^1$ using multiplication in $\mathbb{C}[\mathbb{C}^{2n}]$; thus $\mathbb{J}^d = \mathbb{C}[\mathbb{C}^{2n}]\mathbb{A}^d$. Finally, set $\mathbb{J}^0 = \mathbb{C}[\mathbb{C}^{2n}]$, $\mathbb{A}^0 = \mathbb{C}[\mathbb{C}^{2n}]^W$ and $\mathbb{A} = \bigoplus_{d\geqslant 0} \mathbb{A}^d \cong \mathbb{A}^0[t\mathbb{A}^1]$. Then [Ha1, Proposition 2.6] proves that

$$\mathrm{Hilb}^n\,\mathbb{C}^2 \cong \mathrm{Proj}\,\mathbb{A} \text{ as a scheme over } \mathrm{Spec}\,\mathbb{A}^0 = \mathbb{C}^{2n}/W. \tag{4.3.1}$$

Similarly, $\mathbb{X}_n \cong \mathrm{Proj}\,\mathbb{S}$, where $\mathbb{S} = \mathbb{C}[\mathbb{C}^{2n}][t\mathbb{J}^1]$, is the blowup of $\mathbb{C}^{2n}$ at $\mathbb{J}^1$ [Ha3, Proposition 3.4.2].

4.4. Observe that $\mathbb{J}^d$ is generated by its W-alternating or W-invariant elements, respectively, depending on whether d is odd or even. Following Haiman we refer to these elements as having *correct parity*.

Lemma. (1) *For any $d\geqslant 0$, $\mathbb{A}^d$ consists of the elements of $\mathbb{J}^d$ with the correct parity.*
(2) *If $\mathbb{C}^n$ denotes the first copy of that space in $\mathbb{C}^{2n}$, then $\mathbb{J}^d$ is a free module over both $\mathbb{C}[\mathbb{C}^n]$ and $\mathbb{C}[\mathbb{C}^n]^W$.*

Proof. (1) The statement is clearly true for $d = 0, 1$. Assume, by induction, that it is true for $d - 1$. We will suppose that d is even, the argument in the odd case being similar. Since $\mathbb{A}^1$ generates the ideal $\mathbb{J}^1$, any element $x \in \mathbb{J}^d$ can be decomposed as $x = \sum_i p_i q_i$ where $p_i \in \mathbb{J}^{d-1}$ and $q_i \in \mathbb{A}^1$. Since $q_i e = e_- q_i$ we have $(p_i q_i)e = (p_i e_-)q_i$ for all i. If x has the correct parity then $x = xe = \sum_i (p_i q_i)e = \sum_i (p_i e_- q_i)$. But $\mathbb{J}^{d-1}e_-$ is the subset of W-alternating elements of $\mathbb{J}^{d-1}$ and so $\mathbb{J}^{d-1}e_- = \mathbb{A}^{d-1}$ by induction. Thus $x \in \mathbb{A}^{d-1}\mathbb{A}^1 = \mathbb{A}^d$.

(2) By Haiman [Ha3, Proposition 4.11.1] $\mathbb{J}^d$ is a projective module over $\mathbb{C}[\mathbb{C}^n]$ and hence over $\mathbb{C}[\mathbb{C}^n]^W$. Since $\mathbb{C}[\mathbb{C}^n]$ and $\mathbb{C}[\mathbb{C}^n]^W$ are polynomial rings, any such projective module is free by the Quillen–Suslin theorem. $\square$

4.5. Geometric interpretation. There is a geometric description of both $\mathbb{A}^d$ and $\mathbb{J}^d$. Let $\mathcal{B}_1$ be the *tautological rank n vector bundle* on $\mathrm{Hilb}^n\,\mathbb{C}^2$ and let $\mathcal{P}_1 = (\rho_1)_*\mathcal{O}_{\mathbb{X}_n}$ denote the *Procesi bundle* of rank $n!$ arising from the map $\rho_1 : \mathbb{X}_n \to \mathrm{Hilb}^n\,\mathbb{C}^2$. Write $\mathcal{L}_1 = \bigwedge^n \mathcal{B}_1$ for the determinant bundle of $\mathcal{B}_1$. By Haiman [Ha1, Proposition 2.12] $\mathcal{L}_1$ is also the canonical ample line bundle $\mathcal{O}_{\mathrm{Hilb}^n\,\mathbb{C}^2}(1)$ associated to the presentation $\mathrm{Hilb}^n\,\mathbb{C}^2 \cong \mathrm{Proj}\,\mathbb{A}$.

4.6. Set $l = dn$ for some $d\geqslant 1$ and write $\mathbb{R}(n,l) = H^0(\mathrm{Hilb}^n\,\mathbb{C}^2, \mathcal{P}_1 \otimes \mathcal{B}_1^l)$. One should note that $\mathbb{R}(n,l)$ is defined in [Ha4] to be the coordinate ring of the polygraph $Z(n,l)$ but, by Haiman [Ha4, Theorem 2.1], it is also isomorphic to the given ring of global sections. There is an action of $W \times W^d$ on $\mathcal{P}_1 \otimes \mathcal{B}_1^l$, with W acting fibrewise on $\mathcal{P}_1$ and $W^d \subset \mathfrak{S}_l$ acting on $\mathcal{B}_1^l$ by permutations. By construction, $(\mathcal{P}_1)^W = \mathcal{O}_{\mathrm{Hilb}^n\,\mathbb{C}^2}$ and $(\mathcal{B}_1^l)^{\varepsilon_d} = \mathcal{L}_1^d$, where ε_d denotes the sign representation of W^d.

The proof of Haiman [Ha3, Proposition 4.11.1] shows that $\mathbb{J}^d \cong \mathbb{R}(n,l)^{\varepsilon_d}$. On the other hand, the action of W^d is trivial on $\mathrm{Hilb}^n\,\mathbb{C}^2$, so $\mathcal{P}_1 \otimes \mathcal{B}_1^l$ is a direct sum of its isotypic components. Hence

$$\mathbb{J}^d \cong \mathbb{R}(n,l)^{\varepsilon_d} = H^0(\mathrm{Hilb}^n\,\mathbb{C}^2, (\mathcal{P}_1 \otimes \mathcal{L}_1)^{\varepsilon_d}) = H^0(\mathrm{Hilb}^n\,\mathbb{C}^2, \mathcal{P}_1 \otimes \mathcal{L}_1^d). \tag{4.6.1}$$

It is not true, however, that the natural W-action on the two sides agrees. Indeed, thanks to the proof of Haiman [Ha2, Proposition 4.2] the isomorphism written W-equivariantly is

$$\mathbb{J}^d \otimes \varepsilon^{\otimes d} \cong H^0(\mathrm{Hilb}^n\,\mathbb{C}^2, \mathcal{P}_1 \otimes \mathcal{L}_1^d). \tag{4.6.2}$$

The reason for this is that the isomorphism in (4.6.1) is given by the $\mathbb{C}[\mathbb{C}^{2n}]$-module homomorphism sending (in the notation of Haiman [Ha2]) the generators $\Delta_{t_1}(\mathbf{a},\mathbf{b})\cdots\Delta_{t_d}(\mathbf{a},\mathbf{b})$ on the right-hand side to their evaluations on the left hand side: $\mathbf{a} \mapsto \mathbf{x}$, $\mathbf{b} \mapsto \mathbf{y}$. The element $\Delta_{t_j}(\mathbf{a},\mathbf{b})$ has a trivial W-action as no $\mathbf{x}$'s or $\mathbf{y}$'s are involved, whereas its specialisation has a W-action of $\varepsilon^{\otimes d}$ since that specialisation is the product of d determinants.

As a result, (4.6.2) and Lemma 4.4 combine to prove:

Lemma. *There is an isomorphism* $\mathbb{A}^d \cong \mathbb{R}(n,l)^{W\times\varepsilon_d} = H^0(\mathrm{Hilb}^n\,\mathbb{C}^2, \mathcal{L}_1^d)$ *of* $\mathbb{A}^0$-*modules.*

4.7. (Bi)graded characters. There is a bigrading on $\mathbb{C}[\mathbb{C}^{2n}] = \mathbb{C}[\mathbf{x},\mathbf{y}]$ with $\deg x_i = (1,0)$ and $\deg y_j = (0,1)$ which, as in [Ha4, (12)], arises from the action of $\mathbb{T}^2 = (\mathbb{C}^*)^2$ on the plane $\mathbb{C}^2$ given by $(\alpha,\beta)\cdot(u,v) = (\alpha^{-1}u, \beta^{-1}v)$ for $(u,v)\in\mathbb{C}^2$. This action extends to $\mathrm{Hilb}^n\,\mathbb{C}^2$, and the bundles $\mathcal{P}_1$, $\mathcal{B}_1$, $\mathcal{L}_1$ are naturally $\mathbb{T}^2$-equivariant. The isomorphisms from (4.6.1) and Lemma 4.6 respect the induced bigradings. Of course, the sections M of any one of these modules obtains an induced action of $\mathbb{T}^2$ and this is equivalent to a $\mathbb{Z}^2$-grading $M = \bigoplus M_{i,j}$; explicitly, an element $f\in M$ is homogeneous of weight (i,j) if $(\alpha,\beta)f = \alpha^i\beta^j f$.

The $\mathbb{T}^2$–fixed points of $\mathrm{Hilb}^n\,\mathbb{C}^2$ are precisely the ideals I_μ that are associated to partitions μ of n by

$$I_\mu = \mathbb{C}\cdot\{x^r y^s : (r,s)\notin d(\mu)\} \subseteq \mathbb{C}[x,y],$$

see [Ha4, Proposition 3.1]. The set of monomials $\mathcal{B}_\mu = \{x^r y^s : (r,s)\in d(\mu)\}$ that are not in I_μ form a natural $\mathbb{C}$-basis of $\mathbb{C}[x,y]/I_\mu$.

4.8. For a bigraded space $V = \sum_{i,j} V_{i,j}$ with finite-dimensional weight spaces we define the *bigraded Poincaré series* of V to be

$$p(V,s,t) = \sum_{i,j} \dim(V_{i,j}) s^i t^j.$$

Haiman has calculated the bigraded Poincaré series of $\mathbb{R}(n,l)$ and a similar calculation will allow us to find the bigraded Poincaré series of $\mathbb{J}^d$. For a pair of partitions λ, μ let $K_{\lambda\mu}(t,s)$ be the *Kostka–Macdonald coefficients* defined in [Mac, VI, (8.11)]. Set

$$\Omega(\mu) = \prod_{x \in d(\mu)} (1 - s^{1+l(x)}t^{-a(x)})(1 - s^{-l(x)}t^{1+a(x)})$$

and

$$P_\mu(s,t) = \sum_\lambda s^{n(\mu)} K_{\lambda\mu}(t, s^{-1}) f_\lambda(1).$$

We remark that many of the formulæ we cite from Haiman's papers are given in terms of Frobenius series $\mathcal{F}_M(z; s, t)$ but, as in [Ha2, (6.5)], we can always specialise these to Hilbert series $p(M, s, t)$ by specialising $s_\lambda(z)$ to $f_\lambda(1) = \dim \lambda$.

Proposition. *Under the* $\mathbb{T}^2$*-bigraded structure, the bigraded Poincaré series of* $\mathbb{J}^d$ *is*

$$p(\mathbb{J}^d, s, t) = \sum_\mu P_\mu(s,t)\, \Omega(\mu)^{-1} s^{dn(\mu)} t^{dn(\mu^t)}.$$

Proof. By Haiman [Ha4, Theorem 2.1] $H^i(\mathrm{Hilb}^n\, \mathbb{C}^2, \mathcal{P}_1 \otimes \mathcal{L}_1^d) = 0$ for $i > 0$, while (4.6.1) implies that $H^0(\mathrm{Hilb}^n\, \mathbb{C}^2, \mathcal{P}_1 \otimes \mathcal{L}_1^d) = \mathbb{J}^d$. Thus, in the notation of Haiman [Ha4, Section 3], $p(\mathbb{J}^d, s, t) = \chi_{\mathcal{P}_1 \otimes \mathcal{L}_1^d}(s,t)$ and so, by Haiman [Ha4, Proposition 3.2],

$$\begin{aligned} p(\mathbb{J}^d, s, t) &= \textstyle\sum_\mu p(\mathcal{P}_1 \otimes \mathcal{L}_1^d(I_\mu), s, t)\ \Omega(\mu)^{-1} \\ &= \textstyle\sum_\mu p(\mathcal{P}_1(I_\mu), s, t) p(\mathcal{L}_1(I_\mu), s, t)^d\ \Omega(\mu)^{-1}. \end{aligned} \tag{4.8.1}$$

Here we have used the fact that, as I_μ defines a finite dimensional scheme, we can identify the sheaf $\mathcal{P}_1 \otimes \mathcal{L}_1^d(I_\mu)$ with its global sections, and so $p(\mathcal{P}_1 \otimes \mathcal{L}_1^d(I_\mu), s, t)$ is naturally defined.

We now evaluate the final term of (4.8.1). It is proved in [Ha1, (3.9)], using the notation of Haiman [Ha1, (1.9)], that $p(\mathcal{L}_1(I_\mu), s, t) = \prod_{x \in d(\mu)} s^{l'(x)} t^{a'(x)} = s^{n(\mu)} t^{n(\mu^t)}$. On the other hand, by Haiman [Ha4, Proposition 3.4] (which is proved in [Ha3, Section 3.9] and uses the notation of Haiman [Ha4, (46)]), $p(\mathcal{P}_1(I_\mu), s, t) = P_\mu(s,t)$. Substituting these observations into (4.8.1) shows that

$$p(\mathbb{J}^d, s, t) = \sum_\mu P_\mu(s,t)\, \Omega(\mu)^{-1} s^{dn(\mu)} t^{dn(\mu^t)},$$

as required. $\square$

4.9. Blowing up $(\mathfrak{h}\oplus\mathfrak{h}^*)/W$**.** All the results described so far have natural analogues for the subvariety $\mathfrak{h}\oplus\mathfrak{h}^*$ of $\mathbb{C}^{2n}$. Geometrically, this follows from the observation that the natural additive action of $\mathbb{C}^2$ by translation on $\mathrm{Hilb}^n\,\mathbb{C}^2$ gives a decomposition $\mathrm{Hilb}^n\,\mathbb{C}^2 = \mathbb{C}^2 \times \left(\mathrm{Hilb}^n\,\mathbb{C}^2\right)/\mathbb{C}^2$ into a product of varieties [Na, p. 10]. Unravelling the actions shows that $\mathrm{Hilb}^n\,\mathbb{C}^2/\mathbb{C}^2$ provides a resolution of singularities for $\mathfrak{h}\oplus\mathfrak{h}^*$. However, since we need the algebraic consequences of Haiman's results, we will take a more algebraic approach.

We emphasise that the embedding $\mathfrak{h}\oplus\mathfrak{h}^*\hookrightarrow\mathbb{C}^{2n}$ is always given by embedding $\mathfrak{h}$ into the first copy of $\mathbb{C}^n$ and $\mathfrak{h}^*$ into the second copy. To fix notation, let $\mathfrak{h}$ be the hypersurface $\mathbf{z}=0$ in $\mathbb{C}^n$ and similarly let $\mathfrak{h}^*$ be the hypersurface $\mathbf{z}^*=0$ in the second copy of $\mathbb{C}^n$; thus $\mathbb{C}[\mathbb{C}^{2n}]=\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*][\mathbf{z},\mathbf{z}^*]$. Since $\mathbf{z},\mathbf{z}^*\in\mathbb{C}[\mathbb{C}^{2n}]^W$, this induces the decomposition $\mathbb{C}[\mathbb{C}^{2n}]^W=\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]^W[\mathbf{z},\mathbf{z}^*]$. Following the lead of (4.3), we set

$$A^1=\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]^{\varepsilon}\ \subset\ \mathbb{A}^1=\mathbb{C}[\mathbb{C}^{2n}]^{\varepsilon}\ \text{and}\ J^1=\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]A^1\ \subset\ \mathbb{J}^1=\mathbb{C}[\mathbb{C}^{2n}]\mathbb{A}^1.$$

We then define $A^0=\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]^W$, $J^0=\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]$ and, for $d>1$, take $A^d=(A^1)^d$ and $J^d=(J^1)^d$ for the respective dth powers using the multiplication in $\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]$. Finally, we write

$$A=\bigoplus_{i\geqslant 0}A^i\cong A^0[A^1t]\quad\text{and}\quad S=\bigoplus_{i\geqslant 0}J^i\cong\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*][J^1t]$$

for the corresponding Rees rings. The next result is basic observation about these objects.

Lemma. (1) *For* $d\geqslant 0$, $\mathbb{A}^d=A^d[\mathbf{z},\mathbf{z}^*]$ *is the set of polynomials with coefficients from* A^d. *Similarly*, $\mathbb{J}^d=J^d[\mathbf{z},\mathbf{z}^*]$.

(2) *Each* J^d *is a free module over* $\mathbb{C}[\mathfrak{h}]$ *and* $\mathbb{C}[\mathfrak{h}]^W$.

Proof. (1) By construction, $\mathbb{A}^1=\left(\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*][\mathbf{z},\mathbf{z}^*]\right)^{\varepsilon}=\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]^{\varepsilon}[\mathbf{z},\mathbf{z}^*]=A^1[\mathbf{z},\mathbf{z}^*]$ as polynomial extensions.Thus $\mathbb{A}^d=(A^1[\mathbf{z},\mathbf{z}^*])^d=(A^1)^d[\mathbf{z},\mathbf{z}^*]=A^d[\mathbf{z},\mathbf{z}^*]$ and $\mathbb{J}^d=A^d[\mathbf{z},\mathbf{z}^*]\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]=J^d[\mathbf{z},\mathbf{z}^*]$.

(2) By (1) and Lemma 4.4, $\mathbb{J}^d=J^d[\mathbf{z},\mathbf{z}^*]$ is a free module over $\mathbb{C}[\mathfrak{h}][\mathbf{z}]$ and hence over $\mathbb{C}[\mathfrak{h}]$. Therefore, so is its $\mathbb{C}[\mathfrak{h}]$-module summand J^d. $\square$

4.10. Recall the resolution of singularities $\tau:\mathrm{Hilb}^n\,\mathbb{C}^2\to\mathbb{C}^{2n}/W$ defined in (4.2) and define $\mathrm{Hilb}(n)=\tau^{-1}(\mathfrak{h}\oplus\mathfrak{h}^*/W)$, with the resulting morphism $\tau:\mathrm{Hilb}(n)\to\mathfrak{h}\oplus\mathfrak{h}^*/W$. Using the identifications of (4.9), the basic properties of $\mathrm{Hilb}(n)$ are easy to determine.

Corollary. (1) $\mathrm{Hilb}(n)=\mathrm{Proj}(A)$ *and* $\tau:\mathrm{Hilb}(n)\to\mathfrak{h}\oplus\mathfrak{h}^*/W$ *is a resolution of singularities*.

(2) *Moreover* τ *is a crepant resolution*: *that is* $\omega_{\mathrm{Hilb}(n)}\cong\mathcal{O}_{\mathrm{Hilb}(n)}$.

(3) *Set* $X_n = \mathrm{Proj}(S)$. *Then* X_n *is the reduced fibre product*

$$\begin{array}{ccc} X_n & \longrightarrow & \mathfrak{h} \oplus \mathfrak{h}^* \\ {\scriptstyle\rho}\big\downarrow & & \big\downarrow \\ \mathrm{Hilb}(n) & \xrightarrow{\tau} & \mathfrak{h} \oplus \mathfrak{h}^*/W. \end{array}$$

and the map ρ *is flat of degree* $n!$.

Proof. (1) Recall from (4.3.1) that $\mathrm{Hilb}^n\,\mathbb{C}^2 = \mathrm{Proj}(\mathbb{A})$. By Lemma 4.9, $\mathbb{A} = A[\mathbf{z}, \mathbf{z}^*]$. The maps $A \hookrightarrow \mathbb{A}$ and $\mathbb{C}[\mathbf{z}, \mathbf{z}^*] \hookrightarrow \mathbb{A}$ give maps $\mathrm{Hilb}^n\,\mathbb{C}^2 \to \mathrm{Proj}(A)$ and $\mathrm{Hilb}^n\,\mathbb{C}^2 \to \mathrm{Spec}(\mathbb{C}[\mathbf{z}, \mathbf{z}^*]) \cong \mathbb{C}^2$ and hence, by universality, a map $\mathrm{Hilb}^n\,\mathbb{C}^2 \to \mathrm{Proj}(A) \times \mathbb{C}^2$. It is easy to check that this is an isomorphism locally and hence globally. The identification of $\mathfrak{h} \oplus \mathfrak{h}^*$ with the subvariety $\mathbf{z} = 0 = \mathbf{z}^*$ of $\mathbb{C}^{2n}$ easily yields $\mathrm{Hilb}(n) = \mathrm{Proj}(A)$ and so $\mathrm{Hilb}^n\,\mathbb{C}^2 = \mathrm{Hilb}(n) \times \mathbb{C}^2$. Since $\mathrm{Hilb}^n\,\mathbb{C}^2$ is a resolution of singularities of $\mathbb{C}^2/W$, the result follows.

(2) By Hartshorne [Hr, Exercise II.8.3(b)] $\omega_{\mathrm{Hilb}^n\,\mathbb{C}^2} \cong \omega_{\mathrm{Hilb}(n)} \boxtimes \omega_{\mathbb{C}^2}$, the external tensor product on $\mathrm{Hilb}^n\,\mathbb{C}^2 = \mathrm{Hilb}(n) \times \mathbb{C}^2$. Now (2) follows since $\omega_{\mathrm{Hilb}^n\,\mathbb{C}^2} \cong \mathcal{O}_{\mathrm{Hilb}^n\,\mathbb{C}^2}$ by Haiman [Ha3, Proposition 3.6.3].

(3) As in (1), $\mathbb{S} = \bigoplus \mathbb{J}^d = S[\mathbf{z}, \mathbf{z}^*]$ and $\mathrm{Proj}(\mathbb{S}) \cong \mathrm{Proj}(S) \times \mathbb{C}^2$. The assertions of the corollary now follow from the corresponding results for $\mathbb{X} = \mathrm{Proj}(\mathbb{S})$ that were stated in (4.3). $\square$

We also have analogues of $\mathcal{P}_1$ and $\mathcal{L}_1$ for $\mathrm{Hilb}(n)$. These are defined in the same way: $\mathcal{P} = \rho_* \mathcal{O}_{X_n}$ is the *Procesi bundle* on $\mathrm{Hilb}(n)$ of rank $n!$ arising from the map $\rho : X_n \to \mathrm{Hilb}(n)$ while $\mathcal{L}$ is the canonical ample line bundle $\mathcal{O}_{\mathrm{Hilb}(n)}(1)$ associated to the presentation $\mathrm{Hilb}(n) \cong \mathrm{Proj}\, A$.

4.11. Since $\mathbf{z}$ and $\mathbf{z}^*$ are bihomogeneous, the bigradings of (4.7) to pass $\mathrm{Hilb}(n)$. Thus, Lemma 4.9(1) implies that $p(J^m, s, t) = (1-s)(1-t)p(\mathbb{J}^m, s, t)$. Substituting this formula into Proposition 4.8 gives

Corollary. *The bigraded Poincaré series of* J^d *is*

$$p(J^d, s, t) = \sum_{\mu} P_\mu(s,t)(1-s)(1-t)\,\Omega(\mu)^{-1} s^{dn(\mu)} t^{dn(\mu^t)}. \qquad \square$$

4.12. In Corollary 4.13 we will give a singly graded analogue of Corollary 4.11 that will be needed in the proof of the Theorem 1.4. In the proof we will need the following combinatorial formulæ for the fake degrees $f_\mu(v)$, as defined in (3.9.1).

Lemma. *Let* $\mu \in \mathsf{Irrep}(W)$. *Then*

(1) $f_\mu(v) = v^N f_{\mu^t}(v^{-1})$, *where* $N = n(n-1)/2$,

(2) $f_\mu(v)\prod_{x\in d(\mu)}(1-v^{h(x)}) = v^{n(\mu)}\prod_{i=1}^n(1-v^i)$, *where* $h(x)=1+a(x)+l(x)$ *as in* (2.6),
(3) $\sum_\lambda v^{n(\mu)}K_{\lambda\mu}(v^{-1},v^{-1})f_\mu(v^{-1})f_\lambda(1) = \sum_\lambda f_\lambda(v^{-1})f_\mu(1)f_\lambda(1)$.

Proof. (1) This is a well-known formula (see, for example, [Op, p. 453]).

(2,3) Up to a change of notation, these are both proved within the proof of Gordon [Go1, Theorem 6.4]—see the displayed equations immediately after, respectively immediately before [Go1, (18)]. $\square$

4.13. The **E**-grading from (2.4) descends naturally to $\operatorname{ogr} D(\mathfrak{h}) \cong \mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]$ and we will use the same notation there; thus **E**-deg $\mathfrak{h}^* = 1$ and **E**-deg $\mathfrak{h} = -1$. For an **E**-graded module (or, indeed, any $\mathbb{Z}$-graded module) $M = \bigoplus_{i\in\mathbb{Z}} M_i$, we write the corresponding Poincaré series as $p(M,v) = \sum v^i \dim_{\mathbb{C}} M_i$. Set

$$[n]_v! = \frac{\prod_{i=1}^n(1-v^i)}{(1-v)^n}. \tag{4.13.1}$$

Corollary. *Under the* **E**-grading, *the module* $\overline{J^d} = J^d/\mathbb{C}[\mathfrak{h}]_+^W J^d$ *has Poincaré series*

$$p(\overline{J^d},v) = \frac{\sum_\mu f_\mu(1)f_\mu(v^{-1})v^{-d(n(\mu)-n(\mu^t))}[n]_v!}{\prod_{i=2}^n(1-v^{-i})}. \tag{4.13.2}$$

Proof. Since $\mathbb{C}[\mathfrak{h}]_+^W$ is **E**-graded, so is $\overline{J^d}$, and so the result does make sense. By Lemma 4.4(2), the fundamental invariants of $\mathbb{C}[\mathfrak{h}]^W$ form an r-sequence in J^d for any $d\geqslant 0$. Since these elements have degrees $2\leqslant r\leqslant n$, Corollary 4.11 implies that $\overline{J^d}$ has Poincaré series

$$p(\overline{J^d},v) = \left((1-t)\prod_{i=1}^n(1-s^i)\sum_\mu P_\mu(s,t)\,\Omega(\mu)^{-1}s^{dn(\mu)}t^{dn(\mu^t)}\right)_{s=v,t=v^{-1}}, \tag{4.13.3}$$

where P_μ and $\Omega(\mu)$ are defined in (4.8). Lemma 4.12(2) implies that

$$\Big(\Omega(\mu)\Big)_{s=v,t=v^{-1}} = f_\mu(v)^{-1}f_\mu(v^{-1})^{-1}\prod_{i=1}^n(1-v^i)(1-v^{-i}).$$

This gives

$$p(\overline{J^d},v) = \frac{\sum_\mu P_\mu(v,v^{-1})f_\mu(v)f_\mu(v^{-1})v^{dn(\mu)}v^{-dn(\mu^t)}}{\prod_{i=2}^n(1-v^{-i})}. \tag{4.13.4}$$

By Lemma 4.12(3) the numerator of this expression can be described as

$$\sum_{\mu}\left(\sum_{\lambda} f_\lambda(v^{-1})f_\lambda(1)\right) f_\mu(1)f_\mu(v)v^{d(n(\mu)-n(\mu^t))}. \tag{4.13.5}$$

Applying Lemma 4.12(1) and using the equality $f_\mu(1) = f_{\mu^t}(1)$ from (3.9.2) we find that (4.13.5) equals

$$\sum_{\mu}\left(\sum_{\lambda} f_\lambda(v^{-1})f_\lambda(1)\right) f_{\mu^t}(1)f_{\mu^t}(v^{-1})v^N v^{-d(n(\mu^t)-n(\mu))}. \tag{4.13.6}$$

The standard formula $\sum \dim \mathbb{C}[\mathfrak{h}]_i^{\mathrm{co}\,W} v^{-i} = [n]_{v^{-1}}!$ shows that the fake degrees satisfy the identity

$$\sum_{\lambda} f_\lambda(v^{-1})f_\lambda(1) = \frac{\prod_{i=1}^n(1-v^{-i})}{(1-v^{-1})^n} = [n]_{v^{-1}}!.$$

Applying this and (4.13.6) to (4.13.4) we find that

$$p(\overline{J^d}, v) = \frac{\sum_\mu f_{\mu^t}(1)f_{\mu^t}(v^{-1})v^{-d(n(\mu^t)-n(\mu))}v^N[n]_{v^{-1}}!}{\prod_{i=2}^n(1-v^{-i})}. \tag{4.13.7}$$

After changing the order of summation from μ to μ^t and using the equality

$$v^N[n]_{v^{-1}}! = v^N\frac{\prod_{i=1}^n(1-v^{-i})}{(1-v^{-1})^n} = \frac{\prod_{i=1}^n(1-v^{i})}{(1-v)^n} = [n]_v!,$$

(4.13.7) becomes the required equality (4.13.2), and the corollary is proved. $\square$

5. $\mathbb{Z}$–algebras

5.1. Typically in non-commutative algebra—and certainly in our case—one cannot apply the Rees ring construction since one is working with just right modules or homomorphism groups rather than bimodules. One way round this is to use $\mathbb{Z}$-algebras and in this section we describe the basic properties that we need from this theory. The reader is referred to [BP] or [SV, Section 11] for the more general theory and to [BGS, Section 3] for applications of $\mathbb{Z}$-algebras to Koszul duality.

Throughout this paper a *$\mathbb{Z}$-algebra* will mean a *lower triangular $\mathbb{Z}$-algebra*. By definition, this is a (non-unital) algebra $B = \bigoplus_{i\geqslant j\geqslant 0} B_{ij}$, where multiplication is defined in matrix fashion: $B_{ij}B_{jk} \subseteq B_{ik}$ for $i \geqslant j \geqslant k \geqslant 0$ but $B_{ij}B_{\ell k} = 0$ if $j \neq \ell$. Although B cannot have a unit element, we do require that each subalgebra B_{ii} has a unit element 1_i such that $1_i b_{ij} = b_{ij} = b_{ij}1_j$, for all $b_{ij} \in B_{ij}$.

5.2. Let B be a $\mathbb{Z}$-algebra. We define the category B-Grmod to be the category of $\mathbb{N}$-graded left B-modules $M = \bigoplus_{i\in\mathbb{N}} M_i$ such that $B_{ij}M_j \subseteq M_i$ for all $i \geqslant j$ and $B_{ij}M_k = 0$ if $k \neq j$. Homomorphisms are defined to be graded homomorphisms of degree zero. The subcategory of noetherian graded left B-modules will be denoted B-grmod. In all examples considered in this paper B-grmod will consist precisely of the finitely generated graded left B-modules.

A module $M \in B$-Grmod is *bounded* if $M_n = 0$ for all but finitely many $n \in \mathbb{Z}$ and *torsion* if it is a direct limit of bounded modules. We let B-Tors denote the full subcategory of torsion modules in B-Grmod and write B-tors for the analogous subcategory of B-qgr. The corresponding quotient categories are written B-Qgr = B-Grmod/B-Tors and B-qgr = B-grmod/B-tors. Write $\pi(M)$ for the image in B-Qgr of $M \in B$-Grmod.

5.3. There are two basic examples of $\mathbb{Z}$-algebras that will interest us. For the first, suppose that $S = \bigoplus_{n\geqslant 0} S_n$ is an $\mathbb{N}$-graded algebra. As in [BGS, Example 3.1.3] we can canonically associate a $\mathbb{Z}$-algebra $\widehat{S} = \bigoplus_{i\geqslant j\geqslant 0} \widehat{S}_{ij}$ to S by setting $\widehat{S}_{ij} = S_{i-j}$ with multiplication induced from that in S. Define categories S-$\mathsf{Grmod}, \dots, S$-qgr in the usual manner. In particular, S-Grmod denotes the category of $\mathbb{Z}$-graded S-modules, from which the other definitions follow as in the last paragraph. We then let S-$\mathsf{Grmod}_{\geqslant 0}$ denote the full subcategory of S-Grmod consisting of $\mathbb{N}$-graded S-modules $M = \bigoplus_{i\in\mathbb{N}} M_i$. It is immediate from the definitions that the identity map $\imath : M = \bigoplus_{i\in\mathbb{N}} M_i \mapsto M = \bigoplus_{i\in\mathbb{N}} M_i$ gives equivalences of categories S-$\mathsf{Grmod}_{\geqslant 0} \simeq \widehat{S}$-$\mathsf{Grmod}$ and S-$\mathsf{grmod}_{\geqslant 0} \simeq \widehat{S}$-$\mathsf{grmod}$. For any module $M \in S$-Grmod, one has $\pi(M) = \pi(M_{\geqslant 0})$ in S-Qgr and so $\imath$ induces category equivalences

$$S\text{-}\mathsf{Qgr} \simeq \widehat{S}\text{-}\mathsf{Qgr} \quad \text{and} \quad S\text{-}\mathsf{qgr} \simeq \widehat{S}\text{-}\mathsf{qgr}. \tag{5.3.1}$$

5.4. For the second class of examples, suppose that we are given noetherian algebras R_n for $n \in \mathbb{N}$ with (R_i, R_j)-bimodules R_{ij}, for $i > j \geqslant 0$. Assume, moreover, that there are morphisms $\theta_{ij}^{jk} : R_{ij} \otimes_{R_j} R_{jk} \to R_{ik}$ satisfying the obvious associativity conditions. Then we can define a $\mathbb{Z}$-algebra $R_{\mathbb{Z}}$ by $R_{\mathbb{Z}} = \bigoplus_{i\geqslant j\geqslant 0} R_{ij}$, where $R_{ii} = R_i$ for all i.

A particular example of this construction is the one that interests us. Suppose that $\{R_n : n \in \mathbb{N}\}$ are Morita equivalent algebras, with the equivalence induced from the progenerative (R_{n+1}, R_n)-bimodules P_n. Define $R_{ij} = P_{i-1} \otimes_{R_{i-1}} \cdots \otimes_{R_{j+2}} P_{j+1} \otimes_{R_{j+1}} P_j$ and $R_{jj} = R_j$, for $i > j \geqslant 0$. Tensor products provide the isomorphisms $\theta_{\bullet}^{\bullet}$ and associativity is automatic. The corresponding $\mathbb{Z}$-algebra $R_{\mathbb{Z}} = \bigoplus_{i\geqslant j\geqslant 0} R_{ij}$ will be called the *Morita $\mathbb{Z}$-algebra associated to the data* $\{R_n, P_n : n \in \mathbb{N}\}$.

5.5. Write R-mod for the category of finitely generated left modules over a noetherian ring R. Although easy, the next result provides the foundation for our approach to U_c: in order to study U_c-mod it suffices to study $R_{\mathbb{Z}}$-qgr, for any Morita $\mathbb{Z}$-algebra $R_{\mathbb{Z}}$ with $R_0 \cong U_c$.

Lemma. *Suppose that $R_{\mathbb{Z}}$ is the Morita $\mathbb{Z}$-algebra associated to the data $\{R_n, P_n : n \in \mathbb{N}\}$, where R_0 is noetherian.*

(1) *Each finitely generated graded left $R_{\mathbb{Z}}$-module is noetherian.*
(2) *The association $\phi : M \mapsto \bigoplus_{n\in\mathbb{N}} R_{n0} \otimes_{R_0} M$ induces an equivalence of categories between R_0-mod and $R_{\mathbb{Z}}$-qgr.*

Proof. (1) Any finitely generated graded left $R_{\mathbb{Z}}$-module M is a graded image of $\bigoplus_{a_i}\left(\bigoplus_{j\geqslant a_i} R_{ja_i}\right)\otimes_{R_{a_i}} R_{a_i}$, for some $a_i \in \mathbb{N}$ and so we may assume that $M = \bigoplus_{j\geqslant a} R_{ja}$, for some $a \geqslant 0$. Let $L \subseteq M$ be a graded submodule and write R^*_{ij} for the dual of the progenerator R_{ij}. Then

$$X(j) = R^*_{ja} \otimes_{R_j} L_j \subseteq R^*_{ja} \otimes_{R_j} M_j = R^*_{ja} \otimes R_{ja} \xrightarrow{\sim} R_a \quad \text{for } j \geqslant a.$$

Since R_a is Morita equivalent to R_0, it is noetherian and therefore $\sum_{j\geqslant a} X(j) = \sum_{i=a}^{b} X(i)$, for some $b \geqslant a$. Now,

$$L_k = R_{ka}X(k) \subseteq \sum_{i=a}^{b} R_{ka}X(i) = \sum_{i=a}^{b} R_{ki}R_{ia}X(i) = \sum_{i=a}^{b} R_{ki}L_i \quad \text{for } k \geqslant a.$$

Thus L is generated by L_j for $b \geqslant j \geqslant a$. Finally, as each L_i is a submodule of the noetherian left R_i-module R_{ia}, it is finitely generated and hence so is L.

(2) Certainly $\phi(M) \in R_{\mathbb{Z}}$-Grmod and, as $\phi(M)$ is finitely generated by the generators of ${}_{R_0}M$, one has $\phi(M) \in R_{\mathbb{Z}}$-grmod. Thus $\Phi(M) = \pi\phi(M) \in R_{\mathbb{Z}}$-qgr. Since Φ sends R_0-module homomorphisms to graded $R_{\mathbb{Z}}$-module homomorphisms, Φ is a functor.

Conversely, suppose that $\widetilde{N} \in R_{\mathbb{Z}}$-qgr and pick a preimage $N \in R_{\mathbb{Z}}$-grmod. Then N is generated by $\bigoplus_{i=0}^{a} N_i$, for some a, and so $N_j = R_{ja}N_a$, for all $j \geqslant a$. For $j \geqslant i \geqslant a$ we have natural maps of R_a-modules

$$\theta_{ji} : R^*_{ia} \otimes N_i \cong R^*_{ia} \otimes R^*_{ji} \otimes R_{ji} \otimes N_i \cong R^*_{ja} \otimes (R_{ji} \otimes N_i) \twoheadrightarrow R^*_{ja} \otimes N_j,$$

where the tensor products are over the appropriate R_k. By the associativity of tensor products, $\theta_{ki} = \theta_{kj}\theta_{ji}$, for all $k \geqslant j \geqslant i \geqslant a$. Since each N_i is a noetherian R_i-module, each $R^*_{ia} \otimes N_i$ is a noetherian R_a-module and so θ_{ji} is an isomorphism for all $j \geqslant i \gg 0$. Equivalently, $N_j \cong R_{ji} \otimes N_i$ for all such $j \geqslant i$.

Set $\Theta(\widetilde{N}) = R^*_{j0} \otimes N_j \in R_0$-mod for some $j \gg 0$. Since any two preimages of $\widetilde{N}$ in $R_{\mathbb{Z}}$-grmod agree in high degree, $\Theta(\widetilde{N})$ is independent of the choice of N. Moreover, as $R^*_{j0} = R^*_{k0} R_{kj}$,

$$\phi(\Theta(\widetilde{N}))_{\geqslant j} \cong \bigoplus_{k \geqslant j} R_{k0} \otimes R^*_{j0} \otimes N_j \cong \bigoplus_{k \geqslant j} R_{kj} \otimes N_j = \bigoplus_{k \geqslant j} N_k$$

and so $\Phi\Theta(\widetilde{N}) = \widetilde{N}$. Checking that Θ and Φ are inverse equivalences is now routine. $\square$

5.6. We remark that many of the standard techniques and results concerned with associated graded modules for unital algebras extend routinely to $\mathbb{Z}$-algebras. These only appear in peripheral ways in this paper and so we refer the reader to [GS] for a discussion of these results.

6. The main theorem

6.1. In this section, we prove the main theorem of the paper by proving Theorem 1.4 from the introduction. Indeed we will prove more generally that a version of that theorem holds for all values of $c \in \mathbb{C}$ that satisfy Hypothesis 3.12. As was true with Corollary 3.13 and Proposition 3.16, the theorem will take slightly different forms depending on whether $c \in \mathbb{Q}_{\leqslant -1}$ or not, so it is convenient to separate the cases with

6.2. Hypothesis The element $c \in \mathbb{C}$ satisfies Hypothesis 3.12 but $c \notin \mathbb{Q}_{\leqslant -1}$.

6.3. Assume that Hypothesis 6.2 holds. By Corollary 3.13 there is a Morita equivalence $S_c : U_c\text{-mod} \to U_{c+1}\text{-mod}$ given by $S_c(M) = Q_c^{c+1} \otimes_{U_c} M$, where $Q_c^{c+1} = eH_{c+1}e_{-}\delta \subset D(\mathfrak{h}^{\text{reg}}) * W$ is considered as a right U_c-module via (3.2.1). Following (5.4) we can therefore define a Morita $\mathbb{Z}$-algebra $B(c) = B = U_{\mathbb{Z}}$ associated to the data $\{U_{c+i},\ Q_{c+i}^{c+i+1};\ i \in \mathbb{N}\}$; thus $B = \bigoplus_{i \geqslant j \geqslant 0} B_{ij}$ where, for integers $i > j \geqslant 0$,

$$B_{jj} = U_{c+j} \quad \text{and} \quad B_{ij} = Q_{c+i-1}^{c+i} Q_{c+i-2}^{c+i-1} \cdots Q_{c+j}^{c+j+1}, \tag{6.3.1}$$

where the multiplication in taken in $D(\mathfrak{h}^{\text{reg}}) * W$. Note that, by Corollary 3.13, we have a natural isomorphism

$$B_{ij} \cong Q_{c+i-1}^{c+i} \otimes_{U_{c+i-1}} Q_{c+i-2}^{c+i-1} \otimes_{U_{c+i-2}} \cdots \otimes_{U_{c+j+1}} Q_{c+j}^{c+j+1} \tag{6.3.2}$$

and so this does accord with the definition in (5.4).

6.4. The Main Theorem. Assume that $c \in \mathbb{C}$ satisfies Hypothesis 6.2. The differential operator filtration ord on $D(\mathfrak{h}^{\mathrm{reg}}) * W$, as defined in (2.3), induces filtrations on the subspaces B_{ij} and hence on B, which we will again write as ord. The fact that these filtrations are induced from that of $D(\mathfrak{h}^{\mathrm{reg}}) * W$ ensures that the associated graded object

$$\mathsf{ogr}\, B = \bigoplus_{i \geqslant j \geqslant 0} \mathsf{ogr}\, B_{ij} \; * * * *$$

is also a $\mathbb{Z}$-algebra. Similarly, recall from (4.9) the $\mathbb{N}$-graded algebra $A = \bigoplus_{i \geqslant 0} A^i$ associated to $\mathrm{Hilb}(n)$. In this section, it is more convenient to use the isomorphic algebra $A = \bigoplus_{i \geqslant 0} A^i \delta^i$ to which we canonically associate the $\mathbb{Z}$-algebra $\widehat{A} = \bigoplus_{i \geqslant j \geqslant 0} A^{i-j} \delta^{i-j}$, in the notation of (5.3).

Theorem. *Assume that $c \in \mathbb{C}$ satisfies Hypothesis* 6.2 *and define B and $\widehat{A}$ as above. Then*:

(1) *There is an equivalence of categories $U_c\text{-}\mathsf{mod} \xrightarrow{\sim} B\text{-}\mathsf{qgr}$.*
(2) *There is an equality $\mathsf{ogr}\, B = e\widehat{A}e$ and hence a graded $\mathbb{Z}$-algebra isomorphism $\mathsf{ogr}\, B \cong \widehat{A}$.*
(3) $\mathsf{ogr}\, B\text{-}\mathsf{qgr} \simeq \mathsf{Coh}\,\mathrm{Hilb}(n)$.

Combining Theorem 6.4 with Corollary 3.13 and the isomorphism $U_c \cong U_{-c-1}$ from the proof of that result gives

Corollary. (1) *Assume that $c \in \mathbb{C}$ satisfies Hypothesis* 3.12. *Then there exists a $\mathbb{Z}$-algebra B' such that $U_c\text{-}\mathsf{mod} \simeq B'\text{-}\mathsf{qgr}$ and $\mathsf{ogr}\, B \cong \widehat{A}$. Thus $\mathsf{ogr}\, B'\text{-}\mathsf{qgr} \simeq \mathsf{Coh}(\mathrm{Hilb}(n))$.*

(2) *Let $c \in \mathbb{C}$ with $c \notin \frac{1}{2} + \mathbb{Z}$. Then $H_c\text{-}\mathsf{mod} \simeq B''\text{-}\mathsf{qgr}$ and $\mathsf{ogr}\, B''\text{-}\mathsf{qgr} \simeq \mathsf{Coh}(\mathrm{Hilb}(n))$ for some $\mathbb{Z}$-algebra B''.* $\square$

6.5. Analogues of Theorem 6.4 also hold for certain important U_{c+k}-modules and we will derive the theorem from one of these. The module in question is the (U_{c+k}, H_c)-bimodule $N(k) = B_{k0}eH_c$ with the induced ord filtration coming from the inclusion $N(k) \subset D(\mathfrak{h}^{\mathrm{reg}}) * W$. Recall the definition of J^d from (4.9).

Proposition. *Assume that $c \in \mathbb{C}$ satisfies Hypothesis* 6.2 *and let $k \in \mathbb{N}$. Then $\mathsf{ogr}\, N(k) = eJ^k\delta^k$ as submodules of $\mathsf{ogr}\, D(\mathfrak{h}^{\mathrm{reg}}) * W = \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] * W$.*

6.6. Outline of the proof of the theorem and proposition. *For the rest of the section, we will assume that $c \in \mathbb{C}$ satisfies Hypothesis* 6.2. Thus the notation from (6.3) and (6.4) is available and, by Corollary 3.13, $N(k) \cong B_{k0} \otimes_{U_c} eH_c$ is a progenerative (U_{c+k}, H_c)-bimodule. As will be shown in (6.20), Theorem 6.4 follows easily from Proposition 6.5, so we need only discuss the proof of the latter result. This is non-trivial and will take most of the section but, in outline, is as follows.

It is easy to see that $eJ^k\delta^k \subseteq \mathsf{ogr}\, N(k)$ (see Lemma 6.9). The other inclusion is considerably harder. The philosophy behind the proof is to note that we can grade both $J^k\delta^k$ and $N(k)$ by the **E**-gradation. This is not immediately useful since the graded pieces of the two sides are infinite dimensional but both sides have factor modules for which the graded pieces are finite dimensional. For $eJ^k\delta^k \cong J^k\delta^k$ the factor is the module $\overline{J^k\delta^k}$ described by Corollary 4.13, while the analogous factor $\overline{N(k)}$ of $\mathsf{ogr}\, N(k)$ is described in (6.11) and Corollary 6.14 and is related to the standard modules $\Delta_{c+k}(\mu)$. The key observation is that these factors have the same Poincaré series and so they are naturally isomorphic as graded vector spaces. The proof of the theorem then amounts to lifting this isomorphism to give the desired equality $eJ^k\delta^k = \mathsf{ogr}\, N(k)$.

This also shows that the result has to be non-trivial. Indeed, an alternative proof of the proposition (or the theorem) would also provide an alternative proof to a number of the results from [Ha3].

6.7. We start with two elementary observations that will be used frequently. If $R = \bigcup F^i R$ is a filtered ring and $r \in F^m R \setminus F^{m-1} R$, we write $\sigma(r) = [r + F^{m-1}R] \in \mathrm{gr}_F^m R$ for the *principal symbol* of r.

Lemma. *Let $R = \bigcup F^i R$ be a filtered k-algebra, for a field k.*

(1) *Let A, B be subspaces of R and give A, B and AB the induced filtration F. Then $(\mathrm{gr}_F A)(\mathrm{gr}_F B) \subseteq \mathrm{gr}_F AB$, as subspaces of $\mathrm{gr}_F R$. Indeed, if $a \in A$ and $b \in B$ satisfy $\sigma(a)\sigma(b) \neq 0$, then $\sigma(a)\sigma(b) = \sigma(ab)$.*

(2) *Suppose that $A = \bigcup F^i A$ is a filtered right R-module and that $B = \bigcup F^i B$ is a filtered left R-module and give the vector space $A \otimes_R B$ the* tensor product filtration: *$F^n(A \otimes B) = \sum_j F^j A \otimes F^{n-j} B$. Then there is a natural surjection $\mathrm{gr}_F A \otimes_{\mathrm{gr}\, R} \mathrm{gr}_F B \twoheadrightarrow \mathrm{gr}_F(A \otimes_R B)$.*

Proof. (1) Identify $\mathrm{gr}_F A = \bigoplus (F^n A + F^{n-1}R)/F^{n-1}R \subseteq \mathrm{gr}_F R$ so that the result makes sense. Suppose that $\bar{a} \in \mathrm{gr}_F^n A$ and $\bar{b} \in \mathrm{gr}_F^m B$ are such that $\bar{a}\bar{b} \neq 0$ in $\mathrm{gr}_F R$. Lift $\bar{a}$ to $a \in F^n A$ and $\bar{b}$ to $b \in F^m B$. Then, as elements of $\mathrm{gr}_F R$, one has $\bar{a}\bar{b} = [a + F^{n-1}R][b + F^{m-1}R] \subseteq [ab + F^{n+m-1}R]$. Since $\bar{a}\bar{b} \neq 0$, $ab \in F^{n+m}R \setminus F^{n+m-1}R$, whence $\bar{a}\bar{b} = \sigma(ab)$ is the image of ab in $\mathrm{gr}_F(AB)$.

(2) Define a map $\rho : \mathrm{gr}_F A \times \mathrm{gr}_F B \to \mathrm{gr}_F(A \otimes_R B)$ by $\rho(\bar{a}, \bar{b}) = [a \otimes b + F^{n+m-1}(A \otimes B)]$, for $\bar{a} \in \mathrm{gr}_F^n A$, $\bar{b} \in \mathrm{gr}_F^m B$ and where the rest of the notation is the same as for part (1). This clearly defines a $\mathbb{C}$-bilinear map that is $\mathrm{gr}_F R$-balanced in the sense that $\rho(\bar{a}\bar{r}, \bar{b}) = \rho(\bar{a}, \bar{r}\bar{b})$ for $\bar{r} \in \mathrm{gr}_F^s R$. By universality, ρ therefore induces a map $\mathrm{gr}_F A \otimes_{\mathrm{gr}\, R} \mathrm{gr}_F B \to \mathrm{gr}_F(A \otimes_R B)$. It is surjective since $F^{n+m}(A \otimes B)/F^{n+m-1}(A \otimes B)$ is spanned by elements of the given form $[a \otimes b + F^{n+m-1}(A \otimes B)]$. $\square$

6.8. Lemma. *Let $R = \bigcup_{i \geqslant 0} F^i R$ be a filtered ring, pick $r \in R$ and let I be a subset of R. Under the induced filtrations, $\mathrm{gr}_F(rI) = \sigma(r)\mathrm{gr}_F(I)$ in the following cases:*

(1) *$\sigma(r)$ is regular in $\mathrm{gr}_F R$;*

(2) *$r = r^2 \in F^0(R)$ and $rI \subseteq I$.*

Proof. Assume that $r \in F^s R \setminus F^{s-1} R$. We claim that, in both cases, it suffices to prove that $F^n(rI) = rF^{n-s}I$ for all $n \geqslant s$. Indeed, if this is true then the identity $F^m(rI) = rI \cap F^m R$ implies that the nth summand of $\mathrm{gr}(rI)$ equals

$$\frac{F^n(rI)}{F^{n-1}(rI)} = \frac{F^n(rI)}{F^n(rI) \cap F^{n-1}R} \cong \frac{F^n(rI) + F^{n-1}R}{F^{n-1}R} = \frac{rF^{n-s}I + F^{n-1}R}{F^{n-1}R},$$

which is the nth summand of $\sigma(r)gr(I)$.

(1) In this case, $rt \in F^n(rI) = rI \cap F^n(R) \Leftrightarrow t \in I$ and $t \in F^{n-s}R$, as required.
(2) Here, $rF^nI \subseteq F^n(rI)$ whence $rF^nI = r^2F^nI \subseteq rF^n(rI) \subseteq rF^nI$. Since $rF^n(rI) = F^n(rI)$ this implies that $rF^n(I) = F^n(rI)$. $\square$

Example. It is easy to check that some hypotheses are required for the lemma to hold. For example, filter the polynomial ring $R = \mathbb{C}[x, y]$ by $x, xy \in F^0R$ but $y \in F^1R$. Then $x, xy \in F^0(xR)$, yet $xy \notin \sigma(x)\mathrm{gr}_F R$.

6.9. We now turn to the proof of Proposition 6.5. As was mentioned in (6.6) the inclusion $J^k\delta^k e \subseteq \mathsf{ogr}\, N(k)$ is easy.

Lemma. (1) *For $i \geqslant j \geqslant 0$ we have $e(A^{i-j}\delta^{i-j})e \subseteq \mathsf{ogr}\, B_{ij}$.*

(2) *The inclusion of* (1) *is an equality for $i = j$ and for $i = j + 1$.*

(3) *For $k \geqslant 0$ there is an inclusion $eJ^k\delta^k \subseteq \mathsf{ogr}\, N(k)$ of left eA^0e-modules. This is an equality for $k = 0$.*

Proof. (2) By the PBW Theorem 2.2.1, $\mathsf{ogr}\, B_{ii} = e(\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] * W)e$ and so the claim holds for $i = j$. Similarly, since $e, \delta \in \mathsf{ord}^0(D(\mathfrak{h}^{\mathrm{reg}}) * W)$ and δ is regular in $\mathsf{ogr}(D(\mathfrak{h}^{\mathrm{reg}}) * W)$, Lemma 6.8 implies that

$$\begin{aligned} \mathsf{ogr}\, B_{j+1,j} &= \mathsf{ogr}(eH_{c+j+1}e_-\delta) = \mathsf{ogr}(eH_{c+j+1}e_-)\delta = e(\mathsf{ogr}\, H_{c+j+1})e_-\delta \\ &= e(\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] * W)e_-\delta = eA^1\delta e. \end{aligned}$$

(1) Combining (2) with Lemma 6.7(1) and induction shows that

$$\begin{aligned} (eA^1\delta^1 e)^{i-j} &= \mathsf{ogr}\, B_{i,i-1}\mathsf{ogr}\, B_{i-1,i-2} \cdots \mathsf{ogr}\, B_{j+1,j} \\ &\subseteq \mathsf{ogr}\left(B_{i,i-1} \cdots B_{j+1,j}\right) = \mathsf{ogr}\, B_{ij}. \end{aligned}$$

(3) When $k = 0$, the assertion $eJ^k\delta^k = \mathsf{ogr}\, N(k)$ is just the statement that $e\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] = e(\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] * W)$. When $k > 0$, (1) and Lemma 6.7 give

$$eJ^k\delta^k = eA^k\delta^k e\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] * W \subseteq \mathsf{ogr}\, B_{k0}\, \mathsf{ogr}(eH_c) \subseteq \mathsf{ogr}\, N(k). \quad \square$$

6.10. The next several results will be aimed at getting a more detailed understanding of the bimodule structure of $N(k)$ and its factors. For the most part we are interested in their graded structure for which the actions of the elements $\mathbf{h}_{c+t} \in H_{c+t}$ from (2.4.1) are particularly useful. Given an (U_{c+s}, U_{c+t})-bimodule M, define

$$\mathbf{h} \bullet m = \mathbf{h}_{c+s} m - m\mathbf{h}_{c+t} \text{ for any } m \in M.$$

When $s = t$ this is just the adjoint action of $\mathbf{h}_{c+s}$ on M.

Lemma. (1) $e\mathbf{h}_{c+t-1}e = \delta^{-1}e_{-}\mathbf{h}_{c+t}e_{-}\delta$.

(2) *The action of* $\mathbf{h}$ *is diagonalisable on the modules* $N(i)$, B_{ij} *and* $M(i) = H_{c+i}eB_{i0}$, *for any* $i \geqslant j \geqslant 0$.

Proof. (1) Use the first paragraph of the proof of Gordon [Go2, Theorem 4.10].

(2) We start with the B_{ij}. If $b_1 \in B_{i\ell}$ and $b_2 \in B_{\ell j}$, then $\mathbf{h} \bullet (b_1 b_2) = (\mathbf{h} \bullet b_1)b_2 + b_1(\mathbf{h} \bullet b_2)$. Thus, by induction, it suffices to prove the result for each $B_{t,t-1} = eH_{c+t}\delta e$. Clearly $e\mathbf{h} = \mathbf{h}e$. Thus, by (1), for any $m \in H_{c+t}$ we have

$$\begin{aligned} \mathbf{h} \bullet em\delta e &= \mathbf{h}_{c+t}em\delta e - em\delta e\mathbf{h}_{c+t-1} \\ &= e\mathbf{h}_{c+t}m\delta e - em\delta(\delta^{-1}e_{-}\mathbf{h}_{c+t}e_{-}\delta) = e([\mathbf{h}_{c+t}, m])\delta e. \end{aligned} \tag{6.10.1}$$

By (2.4.2) H_{c+t} is diagonalisable under the adjoint $\mathbf{h}_{c+t}$-action and so the result for B_{ij} follows. The same argument works for the modules $N(i)$ and $M(i)$ if one uses the decompositions $N(i) = (B_{i0})(eH_c)$ and $M(i) = (H_{c+i}e)(B_{i0})$. $\square$

6.11. The factors of $N(k)$ that most interest us are defined as follows. Since $N(k)$ is a (U_{c+k}, H_c)-bimodule, the embeddings $\mathbb{C}[\mathfrak{h}]^W \hookrightarrow U_{c+k}$ and $\mathbb{C}[\mathfrak{h}^*] \hookrightarrow H_c$ make $N(k)$ into a $(\mathbb{C}[\mathfrak{h}]^W, \mathbb{C}[\mathfrak{h}^*])$-bimodule. Let $\mathbb{C}$ be the trivial module over either $\mathbb{C}[\mathfrak{h}]^W$ or $\mathbb{C}[\mathfrak{h}^*]$ and set $\overline{N(k)} = \mathbb{C} \otimes_{\mathbb{C}[\mathfrak{h}]^W} N(k)$ and $\underline{N(k)} = N(k) \otimes_{\mathbb{C}[\mathfrak{h}^*]} \mathbb{C}$. As $\mathbb{C}$ is a graded $\mathbf{h}$-module, the adjoint action of $\mathbf{h}$ on $N(k)$ from Lemma 6.10 induces a $\mathbb{Z}$-grading, again called the $\mathbf{h}$-grading, on both $\overline{N(k)}$ and $\underline{N(k)}$. If an element b from any of these three modules has degree n in this grading we write $\mathbf{h}\text{-deg}(b) = n$. The reader should note that, as will be explained in (6.14), this is *not* the same as the $\mathbf{E}$-gradation on these modules.

The next result gives the elementary properties of these modules.

Lemma. (1) *For any* $i \geqslant j \geqslant 0$, $B_{ij} \subseteq U_{c+i} \cap U_{c+j}$.

(2) *For* $k \geqslant 0$, *both* $N(k)$ *and* U_{c+k} *are free left* $\mathbb{C}[\mathfrak{h}]^W$-*modules*, *while* $N(k)$ *is a free right* $\mathbb{C}[\mathfrak{h}^*]$-*module*.

(3) $\underline{N(k)}$ *is a finitely generated*, *free left* $\mathbb{C}[\mathfrak{h}]^W$-*module*.

(4) *Similarly*, $\overline{N(k)}$ *is a finitely generated*, *free right* $\mathbb{C}[\mathfrak{h}^*]$-*module*.

Proof. We will use frequently and without comment the fact that $\mathbb{C}[\mathfrak{h}^*]$ is a free $\mathbb{C}[\mathfrak{h}^*]^W$-module. Moreover, as $\mathbb{C}[\mathfrak{h}^*]^W$ is a polynomial ring, any projective $\mathbb{C}[\mathfrak{h}^*]^W$ is free by the Quillen–Suslin theorem.

(1) By induction, we may assume that $i = j+1$. The inclusion $B_{ij} = eH_{c+i}\delta e \subseteq U_{c+i}$ is immediate. If $p \in H_{c+i}$ then, by (3.2.1),

$$epe_-\delta = e\delta^{-1}\delta pe_-\delta = \delta^{-1}e_-\delta pe_-\delta \in \delta^{-1}e_-H_{c+i}e_-\delta = U_{c+j}.$$

(2) By the PBW Theorem 2.2.1, each H_d is free as a left $\mathbb{C}[\mathfrak{h}]$-module and as a right $\mathbb{C}[\mathfrak{h}^*]$-module. Therefore, H_d is a free left $\mathbb{C}[\mathfrak{h}]^W$-module as is its summand H_de. Under the left action of W, $(H_de)^W = eH_de$ since, if $fe \in (H_de)^W$, then $fe = |W|^{-1}\sum_{w\in W} wfe = efe$. But $(H_de)^W$ is a W-module summand of H_de, while the actions of W and $\mathbb{C}[\mathfrak{h}]^W$ commute. Thus $U_d = (H_de)^W$ is a $\mathbb{C}[\mathfrak{h}]^W$-module summand of H_de and hence is free. By Corollary 3.13, $N(k) \cong B_{k0} \otimes_{U_c} eH_c$ is a projective left U_{c+k}-module and hence a free left $\mathbb{C}[\mathfrak{h}]^W$-module.

On the other hand, $N(k)$ is a projective right H_c-module and hence a projective right $\mathbb{C}[\mathfrak{h}^*]$-module.

(3) Set $X = H_c \otimes_{\mathbb{C}[\mathfrak{h}^*]} \mathbb{C}$. Clearly $X \in \mathcal{O}_c$ in the sense of (2.7) and, by (2.2.1), $X \cong \mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}} \mathbb{C}W$ as left $\mathbb{C}[\mathfrak{h}] * W$-modules. Thus X is a finitely generated free left $\mathbb{C}[\mathfrak{h}]$-module and so, by Ginzburg et al. [GGOR, Proposition 2.21], X has a filtration whose factors are standard modules.

By definition, $\underline{N(k)} = eM$ where $M = \widetilde{S}_{c+k-1} \circ \cdots \circ \widetilde{S}_c(X)$, in the notation of (3.2). By Proposition 3.16 M also has a finite filtration by standard modules and so [GGOR, Proposition 2.21] shows that M is a finitely generated free module over $\mathbb{C}[\mathfrak{h}]$ and hence over $\mathbb{C}[\mathfrak{h}]^W$. Thus, so is its summand eM.

(4) We first show that $N(k)$ is finitely generated as a right module over $R = (\mathbb{C}[\mathfrak{h}]^W)^{\mathrm{op}} \otimes_{\mathbb{C}} \mathbb{C}[\mathfrak{h}^*]$. By (1), $B_{k0} \subseteq U_c$ and so $N(k) \subseteq eH_c$. Thus $\mathsf{ogr}\, N(k) \subseteq \mathsf{ogr}\, H_c = \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] * W$, which is certainly a noetherian $\mathbb{C}[\mathfrak{h}]^W \otimes \mathbb{C}[\mathfrak{h}^*]$-module. Since the ord filtration on $N(k)$ is the one induced from $D(\mathfrak{h}^{\mathrm{reg}}) * W$, the actions of $\mathbb{C}[\mathfrak{h}]^W$ and $\mathbb{C}[\mathfrak{h}^*]$ on $\mathsf{ogr}\, N(k)$ are the natural ones induced from the actions of those rings on $N(k) \subset D(\mathfrak{h}^{\mathrm{reg}}) * W$. In other words, the given R-module structure of $\mathsf{ogr}\, N(k)$ is the one induced from the R-module structure of $N(k)$. Since the former module is finitely generated, so is the latter.

Let $y_1, \ldots, y_{n-1}$ be the generators of $\mathbb{C}[\mathfrak{h}^*]$ and let $q_1, \ldots, q_{n-1}$ be the fundamental invariants of $\mathbb{C}[\mathfrak{h}]^W$. By (2), the $\{y_j\}$ form an r-sequence in $N(k)$, while (3) implies that the $\{q_j\}$ form an r-sequence in the factor $\underline{N(k)} = N(k)/\sum N(k)y_j$ as a module over $\mathbb{C}[\mathfrak{h}]^W = R/\sum y_jR$. It therefore follows that $\Sigma = \{y_\ell, q_m : 1 \leqslant \ell, m \leqslant n-1\}$ is a regular sequence for the right R-module $N(k)$. In particular, if $\mathfrak{n} = \sum y_iR + q_jR$, then Σ is an r-sequence for the $R_\mathfrak{n}$-module $N(k)_\mathfrak{n}$. By the Auslander–Buchsbaum formula [Mt, Example 4, p. 114], $N(k)_\mathfrak{n}$ is therefore free as a $R_\mathfrak{n}$-module.

Finally, consider $\overline{N(k)} = N(k)/\sum q_jN(k)$. Under the induced $\mathbf{h}$-grading, $\overline{N(k)}$ is a finitely generated, graded $\mathbb{C}[\mathfrak{h}^*]$-module and so corresponds to a $\mathbb{C}^*$-equivariant coherent sheaf on $\mathfrak{h}^*$. As a result the locus where $\overline{N(k)}$ is not free is a $\mathbb{C}^*$-stable closed subvariety of $\mathfrak{h}^*$. If this locus is non-empty it must contain the unique $\mathbb{C}^*$-fixed point

$\mathfrak{p} = (y_1, \ldots, y_{n-1})$ for this expanding $\mathbb{C}^*$-action. But then $(\overline{N(k)})_{\mathfrak{p}}$ would not be free, contradicting the conclusion of the last paragraph. $\square$

6.12. We next need to understand the graded structure of the modules $\overline{N(k)}$ and $\underline{N(k)}$ under the $\mathbf{h}$-grading. To do this, we express $\underline{N(0)}$ as a weighted sum of standard modules in the Grothendieck group $G_0(U_c)$ and then to use Proposition 3.16 to write $\underline{N(k)} = B_{k0} \otimes \underline{N(0)}$ in a similar manner. This is quite delicate since there are some subtle shifts involved and we first want to understand these shifts for $B_{ij} \otimes \Delta_c(\mu)$.

We will need to work with the following graded version $\widetilde{\mathcal{O}}_d$ of $\mathcal{O}_d$ constructed in [GGOR, Section 2.4]. The objects M in $\widetilde{\mathcal{O}}_d$ are finitely generated H_d-modules on which $\mathbb{C}[\mathfrak{h}^*]$ acts locally nilpotently and which come equipped with a $\mathbb{Z}$-grading $M = \bigoplus_{r\in\mathbb{Z}} M_r$ such that $pM_r \subseteq M_{r+\ell}$ for each $p \in H_d$ with $\mathbf{E}$-$\deg(p) = \ell$. The morphisms are homogeneous H_d-module homomorphisms of degree zero. A *graded standard module* $\widetilde{\Delta}_d(\mu)$, isomorphic to $\Delta_d(\mu)$ as an ungraded module, is given by setting $\widetilde{\Delta}_d(\mu)_r = \mathbb{C}[\mathfrak{h}]_r \otimes \mu$. By local nilpotence and finite generation, each weight space of a module $M \in \widetilde{\mathcal{O}}_d$ is finite dimensional and so M has a well-defined Poincaré series. There is a degree shift functor [1] in $\widetilde{\mathcal{O}}_d$ defined by $M[1]_r = M_{r-1}$. By abuse of notation, $\widetilde{\mathcal{O}}_d$ will also denote the corresponding category of graded U_d-modules.

Lemma. *Fix* $i \geqslant j \geqslant 0$ *and* $\mu \in \mathsf{Irrep}(W)$. *Give* B_{ij} *the adjoint* $\mathbf{h}$-*grading and let* $B_{ij} \otimes_{U_{c+j}} e\widetilde{\Delta}_{c+j}(\mu)$ *have the grading this induces. Then* $B_{ij} \otimes_{U_{c+j}} e\widetilde{\Delta}_{c+j}(\mu) \in \widetilde{\mathcal{O}}_{c+i}$ *and, as elements of that category,*

$$B_{ij} \otimes_{U_{c+j}} e\widetilde{\Delta}_{c+j}(\mu) \cong e\widetilde{\Delta}_{c+i}[(i-j)(n(\mu) - n(\mu^t))].$$

Proof. Write $\nabla = B_{ij} \otimes_{U_{c+j}} e\widetilde{\Delta}_{c+j}(\mu)$ and let $\deg_{c+u}$ denote the degree function in $\widetilde{\mathcal{O}}_{c+u}$. By hypothesis, the graded structure of an element $b \otimes v \in \nabla$ is given by $\deg(b \otimes v) = \mathbf{h}\text{-}\deg(b) + \deg_{c+j}(v)$. Proposition 3.16 implies that (as ungraded modules)

$$\nabla = S_{c+i} \circ \cdots \circ S_{c+j+1}(e\Delta_{c+j}(\mu)) \cong e\Delta_{c+i}(\mu). \tag{6.12.1}$$

Thus, under its given grading, $\nabla \in \widetilde{\mathcal{O}}_{c+i}$.

Unfortunately, it is not easy to write the generator $e \otimes \mu$ of $e\Delta_{c+i}(\mu)$ as an element of ∇ and for this reason the shift in the grading in (6.12.1) is subtle. In order to understand this we will use the canonical grading from (3.8) and we write the corresponding degree function as $\deg_{\text{can}}$. The advantage of this grading is that it is simply given by the left multiplication of $\mathbf{h}_{c+i}$. Thus, as (6.12.1) is an isomorphism of left U_{c+i}-modules and hence of left $\mathbb{C}[\mathbf{h}_{c+i}]$-modules, it is automatically a graded isomorphism under the canonical grading.

Since $\mathfrak{h}^*$ has $\mathbf{E}$-degree 1, the canonical grading on $\Delta_d(\mu)$, for any $d \in \mathbb{C}$, is a shift of the grading on $\widetilde{\Delta}_d(\mu)$. The shift is easy to compute. By definition, the generator $1 \otimes \mu$ of $\widetilde{\Delta}_d(\mu)$ has $\deg_d(1 \otimes \mu) = 0$ whereas, by Proposition 3.10, the generator $1 \otimes \mu$

of $\Delta_d(\mu)$ has

$$\deg_{\mathrm{can}}(1 \otimes \mu) = D(d, \mu) = (n-1)/2 + d(n(\mu) - n(\mu^t)).$$

We may therefore regard $\Delta_d(\mu)$ as being in $\widetilde{\mathcal{O}}_d$, in which case

$$\Delta_d(\mu) = \widetilde{\Delta}_d(\mu)[D(d, \mu)]. \tag{6.12.2}$$

Let $b \in B_{ij}$ with $\mathbf{h}$-$\deg(b) = r$ and suppose that $v \in e\Delta_{c+j}(\mu)$ has $\deg_{\mathrm{can}}(v) = s$. Then

$$\mathbf{h}_{c+i} \cdot b \otimes v = (\mathbf{h} \bullet b) \otimes v + b\,\mathbf{h}_{c+j} \otimes v \;\;=\;\; (\mathbf{h} \bullet b) \otimes v + b \otimes \mathbf{h}_{c+j} v = (r+s) b \otimes v.$$

Thus $\deg_{\mathrm{can}}(b \otimes v) = \mathbf{h}\text{-}\deg(b) + \deg_{\mathrm{can}}(v)$. Finally, (6.12.2) implies that

$$\begin{aligned} \deg_{c+i}(b \otimes v) &= \deg_{\mathrm{can}}(b \otimes v) - D(c+i, \mu) \\ &= \mathbf{h}\text{-}\deg(b) + \deg_{\mathrm{can}}(v) - D(c+i, \mu) \\ &= \mathbf{h}\text{-}\deg(b) + \deg_{c+j}(v) + D(c+j, \mu) - D(c+i, \mu) \\ &= \deg(b \otimes v) + (j-i)(n(\mu) - n(\mu^t)), \end{aligned}$$

as required. □

6.13. Given a $\mathbb{Z}$-graded complex vector space $M = \bigoplus_{r \in \mathbb{Z}} M_r$ such that $\dim_{\mathbb{C}} M_r$ is finite for all r then, as in (4.13), we define the Poincaré series of M to be $p(M, v) = \sum v^r \dim_{\mathbb{C}} M_r$. Each $N(k)$ is graded via the adjoint $\mathbf{h}$ action from (6.11), although of course the summands are infinite dimensional. Thus, in order to understand the more detailed structure of $N(k)$ and $\mathsf{ogr}\, N(k)$ we will consider the Poincaré series of the factor modules $\overline{N(k)}$ and $\underline{N(k)}$.

Proposition. *If* $\overline{N(k)}$ *as graded via the adjoint* $\mathbf{h}$ *action on* $N(k)$, *then its Poincaré series is*

$$p(\overline{N(k)}, v) = \frac{\sum_\mu f_\mu(1) f_\mu(v^{-1}) v^{-k(n(\mu) - n(\mu^t))} [n]_v!}{\prod_{i=2}^n (1 - v^{-i})}. \tag{6.13.1}$$

Proof. We first calculate the Poincaré series for $\underline{N(k)}$, and we begin with $\underline{N(0)}$. As in the proof of Lemma 6.11(3), $X = H_c \otimes_{\mathbb{C}[\mathfrak{h}^*]} \mathbb{C}$ is an object of $\widetilde{\mathcal{O}}_c$, where the grading is the natural one defined by $\deg(1 \otimes 1) = 0$. By construction, $eX \cong \underline{N(0)}$ and this is a *graded* isomorphism since the adjoint $\mathbf{h}$-graded structure of $\underline{N(0)} = U_c/I$ is simply defined by $\mathbf{h}$-$\deg(e) = 0$. Thus, as elements of the Grothendieck group $G_0(\widetilde{\mathcal{O}}_c)$, we can write $[X] = \sum_\mu p_\mu[\widetilde{\Delta}_c(\mu)]$ for some $p_\mu \in \mathbb{Z}[v, v^{-1}]$. By (2.2.1) we have a graded isomorphism $X \cong \mathbb{C}[\mathfrak{h}] \otimes \mathbb{C}W$. Applying $(\mathbb{C} \otimes_{\mathbb{C}[\mathfrak{h}]} -)$ to the formula $[X] = \sum_\mu p_\mu[\widetilde{\Delta}_c(\mu)]$ therefore yields $\mathbb{C}W = \sum_\mu p_\mu[\mu]$. It follows from (3.9.2) that $p_\mu = f_\mu(1)$ and so $[\,\underline{N(0)}\,] = \sum_\mu f_\mu(1)[e\widetilde{\Delta}_c(\mu)]$. Combining this formula with Lemma 6.12 shows that

$$[\,\underline{N(k)}\,] = \sum_\mu f_\mu(1) v^{k(n(\mu)-n(\mu^t))}[e\widetilde{\Delta}_{c+k}(\mu)]. \tag{6.13.2}$$

The Poincaré series of $N(k)$ is now easy to compute. First, in the *canonical grading*, (3.10.1) shows that

$$p(\Delta_d(\mu), v, W) = v^{D(d,\mu)} \frac{\sum_\lambda f_\lambda(v)[\lambda \otimes \mu]}{\prod_{i=2}^n (1 - v^i)}$$

and so

$$p(e\Delta_d(\mu), v) = v^{D(d,\mu)} \frac{f_\mu(v)}{\prod_{i=2}^n (1 - v^i)}$$

for any $d \in \mathbb{C}$. Thus (6.12.2) implies that $p(e\widetilde{\Delta}_d(\mu), v) = f_\mu(v) \prod_{i=2}^n (1 - v^i)^{-1}$ in the graded category $\widetilde{\mathcal{O}}_d$. Combined with (6.13.2) this shows that

$$p(\underline{N(k)}, v) \;=\; \frac{\sum_\mu f_\mu(1) f_\mu(v) v^{k(n(\mu)-n(\mu^t))}}{\prod_{i=2}^n (1 - v^i)}. \tag{6.13.3}$$

Finally, we calculate the Poincaré series of $\overline{N(k)}$. By Lemma 6.11(2,3), an $\mathbf{h}$-homogeneous basis for this module is given by lifting a homogeneous $\mathbb{C}$-basis from $\overline{N(k)} \otimes_{\mathbb{C}[\mathfrak{h}^*]} \mathbb{C} = \mathbb{C} \otimes_{\mathbb{C}[\mathfrak{h}]^W} \underline{N(k)}$. Thus, combining (6.13.3) with the formulæ $p(\mathbb{C}[\mathfrak{h}]^W, v) = \prod_{i=2}^n (1 - v^i)^{-1}$ and $p(\mathbb{C}[\mathfrak{h}^*], v) = (1 - v^{-1})^{n-1}$ gives

$$p(\overline{N(k)}, v) = \frac{\sum_\mu f_\mu(1) f_\mu(v) v^{k(n(\mu)-n(\mu^t))}}{(1 - v^{-1})^{n-1}}. \tag{6.13.4}$$

It remains to adjust (6.13.4) to yield (6.13.1). Set $N = n(n-1)/2$. Then Lemma 4.12(1) and (3.9.2) combine to show that

$$\begin{aligned} \sum_\mu f_\mu(1) f_\mu(v) v^{k(n(\mu)-n(\mu^t))} &= \sum_\mu f_{\mu^t}(1) f_{\mu^t}(v^{-1}) v^{k(n(\mu)-n(\mu^t))} \\ &= v^N \sum_\lambda f_\lambda(1) f_\lambda(v^{-1}) v^{k(n(\lambda^t)-n(\lambda))}. \end{aligned}$$

Moreover, rearranging (4.13.1) gives

$$[n]_v! \;=\; \frac{\prod_{i=1}^n(1-v^i)}{(1-v)^n} \;=\; v^N\,\frac{\prod_{i=1}^n(1-v^{-i})}{(1-v^{-1})^n}.$$

Combining these formulæ with (6.13.4) gives (6.13.1). $\square$

6.14. Recall the Euler gradation $\mathbf{E}$-deg on $D(\mathfrak{h}^{\mathrm{reg}}) * W$ and its subrings from (2.4). Since e, e_- and δ are homogeneous under this action, each $Q_{c+\ell}^{c+\ell+1}$ and hence each B_{ij} and $N(k)$ is also graded under this action. As in (2.4), this induces a graded structure, again called $\mathbf{E}$-deg, on $\mathsf{ogr}\, B_{ij}$ and $\mathsf{ogr}\, N(k)$. Since the fundamental invariants of $\mathbb{C}[\mathfrak{h}]^W$ are $\mathbf{E}$-homogeneous, the $\mathbf{E}$-grading on $N(k)$ descends to gradings on $\overline{N(k)}$ and $\underline{N(k)}$. Similarly, each $A^u\delta^u$ and $J^u\delta^u$ has an $\mathbf{E}$-grading induced from that on $\mathbb{C}[\mathfrak{h}\oplus\mathfrak{h}^*]$ and hence so does $A = \bigoplus_{u\geqslant 0} A^u\delta^u$.

However, the $\mathbf{E}$-grading on B_{k0} and hence on $N(k)$ is *not* equal to the adjoint $\mathbf{h}$-grading. The problem is that, in (6.10.1), the adjoint $\mathbf{h}$ action does not "see" the element δ. Thus if we wish to relate the Poincaré series of $N(k)$ to that of $J^k\delta^k$ we need the following slight modification of Proposition 6.13.

Corollary. *Let $k\geqslant 0$, set $N = n(n-1)/2$ and write $K = kN$.*

(1) *If $b \in B_{ij}$ for $i\geqslant j\geqslant 0$ is homogeneous under the* $\mathbf{h}$*-grading then it is homogeneous in the* $\mathbf{E}$*-grading and* $\mathbf{E}\text{-deg}\, b = (i-j)N + \mathbf{h}\text{-deg}\, b$.
(2) *Under the* $\mathbf{E}$*-grading,* $\overline{N(k)}$ *has Poincaré series*

$$p(\overline{N(k)}, v) = v^K\,\frac{\sum_\mu f_\mu(1)f_\mu(v^{-1})v^{-k(n(\mu)-n(\mu^t))}[n]_v!}{\prod_{i=2}^n(1-v^{-i})}$$

while $\underline{N(k)}$ *has Poincare series*

$$p(\underline{N(k)}, v) = v^K\,\frac{\sum_\mu f_\mu(1)f_\mu(v)v^{k(n(\mu)-n(\mu^t))}}{\prod_{i=2}^n(1-v^i)}.$$

Proof. (1) If $b_1 \in B_{ik}$ and $b_2 \in B_{kj}$ then $\mathbf{h}\bullet(b_1b_2) = (\mathbf{h}\bullet b_1)b_2 + b_1(\mathbf{h}\bullet b_2)$ and $[\mathbf{E}, b_1b_2] = [\mathbf{E}, b_1]b_2 + b_1[\mathbf{E}, b_2]$. By induction, it therefore suffices to prove the result when $b = em\delta e \in B_{k,k-1} = eH_{c+k}\delta e$, for some $k>0$. By (6.10.1) we see that $\mathbf{h}\bullet b = e[\mathbf{h}_{c+k}, m]\delta e$ whereas $[\mathbf{E}, b] = e[\mathbf{E}, m]\delta e + em[\mathbf{E}, \delta]e$. By (2.4), $[\mathbf{h}_{c+k}, m] = [\mathbf{E}, m]$ and so the two gradings differ by $\mathbf{E}\text{-deg}\,\delta = N$.

(2) This follows from (1) combined with Proposition 6.13, respectively (6.13.3). $\square$

6.15. Fix $k\geqslant 0$ and for notational simplicity write $\mathcal{J} = eJ^k\delta^k$ and $\mathcal{N} = N(k)$. The final step in the proof of Proposition 6.5 is to show that the inclusion $\Theta : \mathcal{J} \hookrightarrow \mathsf{ogr}\,\mathcal{N}$

from Lemma 6.9(3) is surjective. In order to effectively use Corollary 6.14, we do this by lifting Θ to a $\mathbb{C}[\mathfrak{h}]^W$-module map $\theta : \mathcal{J} \to \mathcal{N}$.

The order filtration on $D(\mathfrak{h}^{\text{reg}}) * W$ induces a graded structure on the ring $\mathsf{ogr}\, D(\mathfrak{h}^{\text{reg}}) * W \cong \mathbb{C}[\mathfrak{h}^{\text{reg}} \oplus \mathfrak{h}^*] * W$ and hence on the module $\mathsf{ogr}\,\mathcal{N}$, which we call the *order gradation*; thus $\deg_{\mathsf{ord}}(\mathbb{C}[\mathfrak{h}] * W) = 0$, while $\deg_{\mathsf{ord}} \mathfrak{h} = 1$. We will use the same terminology for the induced grading on the rings $A^0 = \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W$ and A and the module $\mathcal{J}$.

Let $\mathcal{N}^m = \mathsf{ord}^m \mathcal{N}$ denote the elements in $\mathcal{N}$ of order $\leqslant m$. Similarly, write $\mathcal{J} = \bigoplus_{m \geqslant 0} \mathsf{ogr}^m \mathcal{J}$ for the graded structure of $\mathcal{J}$ under the ord gradation and write the induced order filtration as $\mathcal{J} = \bigcup \mathcal{J}^m$, for $\mathcal{J}^m = \mathsf{ord}^m \mathcal{J} = \bigoplus_{0 \leqslant i \leqslant m} \mathsf{ogr}^i \mathcal{J}$.

Lemma. *There exists an injective map $\theta : \mathcal{J} \hookrightarrow \mathcal{N}$ of left $\mathbb{C}[\mathfrak{h}]^W$-modules such that*

(1) *θ is a graded homomorphism under the* **E**-*gradation and is a filtered homomorphism under the order filtration.*
(2) *The associated graded map $\mathsf{ogr}\,\theta : \mathcal{J} \to \mathsf{ogr}\,\mathcal{N}$ induced by θ is precisely $\mathsf{ogr}\,\theta = \Theta$.*

Proof. Trivially, Θ is an **E**-graded map (by which we always mean a graded map of degree zero), as well as being graded under the ord gradation. For any m, $\mathsf{ogr}^m \mathcal{J}$ is an **E**-graded $\mathbb{C}[\mathfrak{h}]^W$-module. By Corollary 4.9(2) $\mathcal{J}$ is a free left $\mathbb{C}[\mathfrak{h}]^W$-module, and hence so is each summand $\mathsf{ogr}^m \mathcal{J}$. Thus we may pick an **E**-homogeneous free basis $\{a_{jm}\}$ for $\mathsf{ogr}^m \mathcal{J}$. Now $a_{jm} = \Theta(a_{jm}) \in \mathsf{ogr}^m \mathcal{N} = \mathcal{N}^m / \mathcal{N}^{m-1}$ and the surjection $\pi_m : \mathcal{N}^m \to \mathcal{N}^m / \mathcal{N}^{m-1}$ is an **E**-graded surjection. Thus, for each j, m we can pick an **E**-homogeneous preimage $\theta(a_{jm}) \in \mathcal{N}^m$ of $\Theta(a_{jm})$.

Define θ to be the $\mathbb{C}[\mathfrak{h}]^W$-module map induced by the map $a_{jm} \mapsto \theta(a_{jm})$ on basis elements. Since π_m is a left $\mathbb{C}[\mathfrak{h}]^W$-module map, a straightforward induction on orders of elements ensures that the $\theta(a_{jm}) \in \mathcal{N}^m$ are a free basis for the module they generate. The other conclusions of the lemma follow automatically from the construction of θ. $\square$

6.16. As happens with many questions about W-invariants, it is easy to prove that Θ is surjective on $\mathfrak{h}^{\text{reg}}$. Given a left $\mathbb{C}[\mathfrak{h}]^W$-module M, we will write $M[\delta^{-2}]$ for the localisation $\mathbb{C}[\mathfrak{h}]^W[\delta^{-2}] \otimes_{\mathbb{C}[\mathfrak{h}]^W} M$. Clearly, when M is a left $\mathbb{C}[\mathfrak{h}]$-module, $M[\delta^{-2}]$ is naturally isomorphic to $\mathbb{C}[\mathfrak{h}][\delta^{-1}] \otimes_{\mathbb{C}[\mathfrak{h}]} M$.

Lemma. (1) *The inclusion $\Theta[\delta^{-2}] : \mathcal{J}[\theta^{-2}] \hookrightarrow (\mathsf{ogr}\,\mathcal{N})[\delta^{-2}]$ is an equality.*

(2) *The induced map $\theta[\delta^{-2}] : \mathcal{J}[\theta^{-2}] \to \mathcal{N}[\delta^{-2}]$ is an isomorphism. This map is graded under the* **E**-*grading and is a filtered isomorphism under the order filtration, in the sense that $\theta[\delta^{-2}]$ maps $\mathsf{ord}^n \mathcal{J}[\delta^{-2}]$ isomorphically to $\mathsf{ord}^n \mathcal{N}[\delta^{-2}]$ for each n.*

Proof. (1) By (2.3.2) $B_{k,k-1}[\delta^{-2}] = eH_{c+k}\delta[\delta^{-2}]e = e(D(\mathfrak{h}^{\text{reg}}) * W)e$, for any $k \in \mathbb{C}$. Repeated application of this shows that $B_{ij}[\delta^{-2}] = e(D(\mathfrak{h}^{\text{reg}}) * W)e$ and so, by Corollary 3.13, that $\mathcal{N}[\delta^{-2}] = e(D(\mathfrak{h}^{\text{reg}}) * W)eH_c = e(D(\mathfrak{h}^{\text{reg}}) * W)$. Since $\mathsf{ord}(\delta^2) =$

0, we deduce that $(\mathsf{ogr}\,\mathcal{N})[\delta^{-2}] = e(\mathbb{C}[\mathfrak{h}^{\mathrm{reg}} \oplus \mathfrak{h}^*] * W)$. On the other hand, since $\delta^{2k} \in J^k\delta^k \subseteq \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]$, certainly

$$\mathcal{J}[\delta^{-2}] = e\mathbb{C}[\mathfrak{h}^{\mathrm{reg}} \oplus \mathfrak{h}^*] = e(\mathbb{C}[\mathfrak{h}^{\mathrm{reg}} \oplus \mathfrak{h}^*] * W).$$

Since Θ is given by inclusion, $\Theta[\delta^{-2}]$ is therefore an isomorphism.

(2) By Lemma 6.15, θ and hence $\theta[\delta^{-2}]$ are graded maps under the **E**-gradation and filtered under the order filtration. Since $\mathrm{gr}(\theta[\delta^{-2}]) = \Theta[\delta^{-2}]$ is an isomorphism, necessarily $\theta[\delta^{-2}]$ is a filtered isomorphism. $\square$

6.17. Notation. As in (6.15), we set $\mathcal{J} = eJ^k\delta^k$, $\mathcal{N} = N(k)$ and write $\theta(\mathcal{J})^m = \mathsf{ord}^m\,\theta(\mathcal{J}) = \theta(\mathcal{J})\cap\mathcal{N}^m$ for all $m\geqslant 0$. We rewrite the $\mathbb{C}[\mathfrak{h}]^W$-basis of $\theta(\mathcal{J})$ constructed in the proof of Lemma 6.15 as $\{a_{g\ell m}\}$, where each $a_{g\ell m}$ is g-homogeneous under the **E**-gradation and has order exactly ℓ. Since these were induced from the bases $\{a_{c\ell}\}$ of $\mathsf{ogr}^\ell\,\mathcal{J}$, the set $\{a_{g\ell m} : \ell \leqslant t\}$ does give a basis of $\theta(\mathcal{J})^t$.

By Lemma 6.11(2), $\mathcal{N}$ is a free left $\mathbb{C}[\mathfrak{h}]^W$-module and it is certainly graded. Thus, by Theorem A.1, it is graded-free. We may therefore pick a $\mathbb{C}[\mathfrak{h}]^W$-basis $\{b_{gu}\}$ of $\mathcal{N}$ where, again, each b_{gu} is **E**-homogeneous of degree g but of unspecified order. This basis is far from unique and one cannot expect that $\{b_{gu} : b_{gu} \in \mathcal{N}^m\}$ forms a basis of $\mathcal{N}^m$; indeed at this stage we do not even know that $\mathcal{N}^m$ is a free $\mathbb{C}[\mathfrak{h}]^W$-module.

6.18. We are now ready to put these observations together to prove the hard part of Proposition 6.5.

Proposition. *Fix $k\geqslant 0$ and set $\mathcal{J} = eJ^k\delta^k$ and $\mathcal{N} = N(k)$. Then the map $\theta : \mathcal{J} \to \mathcal{N}$ is an isomorphism.*

Proof. Set $\mathfrak{m} = \mathbb{C}[\mathfrak{h}]^W_+$ and note that $\mathcal{N}/\mathfrak{m}\mathcal{N} = \overline{N(k)}$. On the other hand, in the notation of Corollary 4.13, $\mathcal{J}/\mathfrak{m}\mathcal{J} \cong \overline{J^k}[K]$ is the shift of $\overline{J^k}$ by $\deg \delta^k = K = kn(n-1)/2$. By Corollaries 4.13 and 6.14, we therefore have an equality of Poincaré series under the **E**-gradation

$$p(\mathcal{J}/\mathfrak{m}\mathcal{J}, v) = v^K \frac{\sum_\mu f_\mu(1) f_\mu(v^{-1}) v^{-k(n(\mu)-n(\mu^t))}[n]_v!}{\prod_{i=2}^n (1 - v^{-i})} = p(\mathcal{N}/\mathfrak{m}\mathcal{N}, v). \qquad (6.18.1)$$

Keep the $\mathbb{C}[\mathfrak{h}]^W$-bases of $\theta(\mathcal{J}) \cong \mathcal{J}$ and $\mathcal{N}$ described in Notation 6.17. Write $a(g\ell m) = g$ whenever $a_{g\ell m}$ exists for that choice of g, ℓ, m; in particular $\sum_{g\ell m} v^{a(g\ell m)}$ denotes the sum $\sum v^g$, where one has one copy of v^g for each ℓ, m for which $a_{g\ell m}$ exists. Define $b(gu)$ analogously. Since the bases $\{a_{g\ell m}\}$ and $\{b_{gu}\}$ induce $\mathbb{C}$-bases of $\mathcal{J}/\mathfrak{m}\mathcal{J}$, respectively, $\overline{N(k)}$, (6.18.1) can be reinterpreted as

$$\sum_{g,\ell,m} v^{a(g\ell m)} = v^K \frac{\sum_\mu f_\mu(v^{-1}) f_\mu(v) v^{-k(n(\mu)-n(\mu^t))}[n]_v!}{\prod_{i=2}^n (1 - v^{-i})} = \sum_{g,u} v^{b(gu)}. \qquad (6.18.2)$$

We note that (6.18.2) has several consequences for the $a(g\ell m)$ and $b(gu)$.

(†1) For fixed g, there exist only finitely many elements $a_{g\ell m}$ and b_{gu}. This is because the middle expression in (6.18.2) is a well-defined series.

(†2) There exists a universal upper bound $a(g\ell m)\leqslant T$. This is because the numerator in the middle expression in (6.18.2) is a finite sum of polynomials. However, there is no universal lower bound.

(†3) For any g_0, the number of $a_{g\ell m}$ with $g = g_0$ equals the number of b_{gu} with $g = g_0$. This is simply because $\sum v^{a(g\ell m)} = \sum v^{b(gu)}$ and the numbers are finite by (†1).

We aim to adjust the basis $\{b_{gu}\}$ to be equal to the basis $\{a_{g\ell m}\}$, and we achieve this by a downwards induction on g. The induction starts since, by (†3), there are no basis elements b_{gu} with $g > T$.

Let $-\infty < G\leqslant T$ and, by induction, suppose that $\{b_{gu} : u \in \mathbb{Z}\} = \{a_{g\ell m} : \ell, m \in \mathbb{Z}\}$ for all $g > G$. Suppose that there exists a basis element $b_{Gw} \notin \{a_{G\ell m}\}$. By Lemma 6.16(2), $\theta(\mathcal{J})[\delta^{-2}] = \mathcal{N}[\delta^{-2}]$ and so there exists a homogeneous element $\mathbf{x}^m \in \mathbb{C}[\mathfrak{h}]^W$ of $\mathbf{E}$-degree m such that $\mathbf{x}^m b_{Gw} \in \theta(\mathcal{J})$. Thus we have the $\mathbf{E}$-homogeneous equation

$$\mathbf{x}^m b_{Gw} = \sum_{g<G} c_{gfh}a_{gfh} + \sum c_{Gfh}a_{Gfh} + \sum_{g>G} c'_{gz}b_{gz}, \tag{6.18.3}$$

where $c_{gfh}, c'_{gz} \in \mathbb{C}[\mathfrak{h}]^W$ and summation over f, h, z is suppressed. As $\theta(\mathcal{J}) \subseteq \mathcal{N}$, we may write each a_{gfh} as an $\mathbf{E}$-homogeneous sum $a_{gfh} = \sum d_\bullet b_{uz}$ for some $d_\bullet = d_{fghuz} \in \mathbb{C}[\mathfrak{h}]^W$ and obtain

$$\mathbf{x}^m b_{Gw} = \sum_{g<G} c_{gfh}d_\bullet b_{uz} + \sum c_{Gfh}d_\bullet b_{uz} + \sum_{g>G} c'_{gz}b_{gz}. \tag{6.18.4}$$

Both the last two displayed equations are $\mathbf{E}$-homogeneous of $\mathbf{E}$-degree $G+m$ and so, by (6.18.3), each element c_{gfh} must have $\mathbf{E}$-degree $\geqslant m$. Thus, the b_{uz} appearing in the first two terms on the right-hand side of (6.18.4) must have $\mathbf{E}$-degree $\leqslant G$. Thus the only appearance of b_{gz} with $g > G$ is in the third sum. Since the b_{uz} are a $\mathbb{C}[\mathfrak{h}]^W$-basis of $\mathcal{N}$, that third term $\sum_{g>G} c'_{gz}b_{gz}$ is actually zero.

Now consider where the specific term b_{Gw} appears on the right-hand side of (6.18.4). For $g < G$, (6.18.3) implies that $\mathbf{E}\text{-deg}\, c_{gfh} > m$ for each f, h and so b_{Gw} cannot appear in the first sum. Thus it must appear non-trivially in some term $c_{Gf'h'}d'b_{Gw}$ in the second sum. In this case, (6.18.3) implies that $\mathbf{E}\text{-deg}\, c_{Gf'h'} = m$. Hence $d' \in \mathbb{C}\setminus\{0\}$ and

$$a_{Gf'h'} = d'b_{Gw} + \sum_{(uz)\neq(Gw)} d_{uz}{}''b_{uz}.$$

Thus we can replace b_{Gw} by $a_{Gf'h'}$ in our basis for $\mathcal{N}$. By (†3), the sets $\{a_{G\ell m} : \ell, m \in \mathbb{Z}\}$ and $\{b_{Gu} : u \in \mathbb{Z}\}$ have equal finite cardinality. After a finite number of steps we therefore have $\{b_{Gu}\} \subseteq \{a_{G\ell m}\}$ and hence $\{b_{Gu}\} = \{a_{G\ell m}\}$. This completes the inductive step and hence the proof of the lemma. $\square$

We now pull everything together to prove Theorem 6.4 and Proposition 6.5.

6.19. Proof of Proposition 6.5. Recall that $\Theta : eJ^k\delta^k \to \mathsf{ogr}\, N(k)$ is the natural inclusion (see Lemma 6.9). For any $k \geqslant 0$, Proposition 6.18 implies that the map $\theta : eJ^k\delta^k \to N(k)$ is an isomorphism. Therefore, Lemma 6.15(2) implies that $\mathrm{gr}_\Lambda N(k) = \mathsf{ogr}\, \theta(eJ^k\delta^k) = \Theta(eJ^k\delta^k) = eJ^k\delta^k$. $\square$

6.20. Proof of Theorem 6.4. (1) This follows from Corollary 3.13(1) and Lemma 5.5.

(2) Fix $i \geqslant j \geqslant 0$. Since $c+j$ still satisfies Hypothesis 6.2, Proposition 6.5 implies that $\mathsf{ogr}\, B_{ij}eH_{c+j} = eJ^{i-j}\delta^{i-j}$. Multiplying this identity on the right by e and applying Lemma 6.8 and Corollary 3.13(1) gives

$$eJ^{i-j}\delta^{i-j}e = \mathsf{ogr}(B_{ij}eH_{c+j})e = \mathsf{ogr}(B_{ij}eH_{c+j}e) = \mathsf{ogr}\, B_{ij}.$$

Since δ transforms under W by the sign representation, Lemma 4.4(1) shows that $eJ^{i-j}\delta^{i-j}e = eA^{i-j}\delta^{i-j}e$. Combining these observations gives $\mathsf{ogr}\, B_{ij} = eA^{i-j}\delta^{i-j}e$. Therefore, $\mathsf{ogr}\, B = \bigoplus \mathsf{ogr}\, B_{ij} = e\widehat{A}e \cong \widehat{A}$, as graded vector spaces. In order to ensure that this is an isomorphism of graded $\mathbb{Z}$-algebras we need to check that the multiplication in $\mathsf{ogr}\, B$ coming from the tensor product multiplication in B is the same as the natural multiplication in $\widehat{A}$. This follows from Lemma 6.7(1).

(3) The equivalences $\mathsf{ogr}(B)\text{-}\mathsf{qgr} \simeq A\text{-}\mathsf{qgr} \simeq \mathsf{Coh}(\mathrm{Hilb}(n))$ follow from (2) combined with (5.3), respectively, Corollary 4.10(1). $\square$

6.21. Corollary. *Assume that $c \in \mathbb{C}$ satisfies Hypothesis 6.2 and pick $i \geqslant j \geqslant 0$. Then, for $m \geqslant 0$, each of the modules $\mathsf{ord}^m N(i)$, $\mathsf{ogr}^m N(i)$, $\mathsf{ord}^m B_{ij}$ and $\mathsf{ogr}^m B_{ij}$ is free as a left $\mathbb{C}[\mathfrak{h}]^W$-module.*

Proof. By construction and Proposition 6.5, the map $\Theta : \mathsf{ogr}\, N(i) \to eJ^i\delta^i$ is an isomorphism of ord-graded modules. Thus $\mathsf{ogr}^m N(i) \cong \mathsf{ogr}^m eJ^i\delta^i$ is a free $\mathbb{C}[\mathfrak{h}]^W$-module by Lemma 4.9. By induction on m, it follows that $\mathsf{ord}^m N(i)$ is also free. The analogous results for B_{ij} follow by multiplying everything on the right by e. $\square$

6.22. We end the section by noting that Proposition 6.5 provides an interesting connection between H_c-modules and the isospectral scheme X_n defined in (4.10). Adjusting to the conventions of this section, we identify $\mathrm{Hilb}(n) = \mathrm{Proj}\, \widetilde{A}$, for $\widetilde{A} = \bigoplus A^k\delta^k$. By construction, the Procesi bundle $\mathcal{P} = \rho_*\mathcal{O}_{X_n}$ from (4.10) is then just the image in $\mathsf{Coh}\,\mathrm{Hilb}(n)$ of the $\widetilde{A}$-module $\bigoplus J^k\delta^k$. Thus the next result is an immediate consequence of Proposition 6.5.

Corollary. *Assume that $c \in \mathbb{C}$ satisfies* (6.2). *Let $e\widetilde{H}_c = \bigoplus_{k \geqslant 0} B_{k0} \otimes_{U_c} e\widetilde{H}_c$ be the B-module associated to the U_c-module eH_c and filter each $B_{k0} \otimes_{U_c} eH_c \cong B_{k0}eH_c$ by the* ord *filtration. Set $\mathsf{ogr}\, e\widetilde{H}_c = \bigoplus \mathsf{ogr}\, B_{k0}eH_c$. Then the sheaf associated to $\mathsf{ogr}\, e\widetilde{H}_c$ in $\mathsf{Coh}\, \mathrm{Hilb}(n)$ is the Procesi bundle $\mathcal{P}$.* $\square$

6.23. Just as Theorem 6.4 can be interpreted as saying that U_c provides a non-commutative model for $\mathrm{Hilb}(n)$, so Corollary 6.22 can be interpreted as saying that the algebra H_c provides a non-commutative model for X_n. Here is one aspect of this analogy. It follows from [BKR,Ha4] that there is an equivalence ξ of derived categories between $\mathfrak{h} \oplus \mathfrak{h}^*/W$ and $\mathrm{Hilb}(n)$ that is induced by a Fourier–Mukai transform over $\mathcal{P}$. Now pass to the non-commutative situation, replacing $\mathfrak{h} \oplus \mathfrak{h}^*/W$, $\mathrm{Hilb}(n)$ and $\mathcal{P}$ by H_c-mod, B-qgr and eH_c, respectively. Then Corollary 3.13 shows that eH_c still induces a derived equivalence between the two categories. Indeed, it is even a equivalence of categories. The fact that derived equivalences in the commutative case can become full equivalences in the non-commutative case happens elsewhere and is in accord with the philosophy behind [GK, Conjecture 1.6]; see [GK, Remark 1.7] for more details.

As will be justified in [GS], Corollary 6.22 therefore "sees" the equivalence ξ and this provides some intriguing connections between sheaves on $\mathrm{Hilb}(n)$ and modules over H_c.

6.24. If one considers Cherednik algebras in characteristic $p > 0$, where H_c is a finite module over its centre, then the relationship between H_c and $\mathrm{Hilb}(n)$ becomes closer still. For example, [BFG] shows that there is even a derived equivalence between H_c and an Azumaya algebra over a Frobenius twist of $\mathrm{Hilb}(n)$. Similarly in characteristic zero, symplectic reflection algebras with parameter $t = 0$ are finite modules over their centre, and [GSm, Theorem 1.2] shows that there are often derived equivalences between these algebras and varieties that deform Hilbert schemes.

7. Tensor product filtrations

7.1. The tensor product decomposition (6.3.2) of the B_{ij} can be used to give a second filtration on that module by inducing a filtration on B_{ij} from the ord filtration on the tensorands. It turns out that the main theorem is essentially equivalence to the assertion that the two filtrations are equal. In this short section we give the details behind this assertion. Analogues of this result also hold for the module $N(k)$ defined in (6.5) and the module $M(k) = H_{c+k}eB_{k0} = H_{c+k}\delta e B_{k-1,0}$ defined in (B.1) and so we begin by giving a general context for all three results.

7.2. For fixed $i \geqslant j \geqslant 0$ we are interested in the following tensor product decompositions

$$B_{ij} \cong Q^{c+i}_{c+i-1} \otimes Q^{c+i-1}_{c+i-2} \otimes \cdots \otimes Q^{c+j+1}_{c+j}, \tag{7.2.1}$$

$$N(i) \cong Q^{c+i}_{c+i-1} \otimes \cdots \otimes Q^{c+1}_{c} \otimes eH_c \qquad \text{or} \qquad N(i) \cong B_{i0} \otimes eH_c \tag{7.2.2}$$

and

$$M(i) \cong H_{c+i}\delta e \otimes B_{i-1,i-2} \otimes \cdots \otimes B_{10} \quad \text{or} \quad M(i) \cong H_{c+i}\delta e \otimes_{U_{c+i-1}} B_{i-1,0}, \tag{7.2.3}$$

where the tensor products are over the appropriate rings U_k. Corresponding to these decompositions we have the *tensor product filtration* ten defined by the following convention: given a module $C = C_1 \otimes \cdots \otimes C_r$, where each C_j is filtered by the ord filtration, define

$$\mathsf{ten}^n(C) = \left\{ \sum c_1 \otimes \cdots \otimes c_r, \ : \ c_m \in \mathsf{ord}^{\ell(m)}(C_m) \text{ with } \sum_{m=1}^{r} \ell(m) \leqslant n \right\}. \tag{7.2.4}$$

As usual, the associated graded module is written $\mathsf{tgr}\, C = \bigoplus \mathsf{ten}^n C/\mathsf{ten}^{n-1} C$.

Lemma. *Assume that $c \in \mathbb{C}$ satisfies Hypothesis* 6.2. *Let C denote one of the objects B_{ij}, $N(i)$ or $M(i)$ and consider the tensor product filtrations induced from one of the tensor product decompositions* (7.2.1–7.2.3). *Then $\mathsf{ord}^m C = \mathsf{ten}^m C$, for all $m \geqslant 0$.*

Proof. We will prove the result for the decomposition (7.2.1) and the first decomposition in each of (7.2.2) and (7.2.3). The proof in the remaining cases is left to the reader as it uses essentially the same argument, although one needs to use the conclusion of the lemma for (7.2.1).

In each of the three cases we are given a decomposition $C = C_1 \otimes \cdots \otimes C_r$, say with $\mathsf{ogr}\, C_j = D_j$ and $\mathsf{ogr}\, C = D$. Moreover, by Theorem 6.4, respectively, Proposition 6.5 combined with Lemma 6.9, respectively, Proposition B.1 combined with Lemma B.2, there is an equality $D_1 \cdots D_r = D$ given by multiplication in $D(\mathfrak{h}^{\text{reg}}) * W$. Equivalently, the natural multiplication map $\chi : D_1 \otimes \cdots \otimes D_r \to D$ is surjective. Consider the graded map χ in more detail. Given elements $\bar{\alpha}_j \in \mathsf{ogr}^{m(j)} D_j$, with $m = \sum m(j)$, lift the $\bar{\alpha}_j$ to elements $\alpha_j \in \mathsf{ord}^{m(j)} C_j$. Then χ is defined by

$$\chi(\bar{\alpha}_1 \otimes \cdots \otimes \bar{\alpha}_r) = \left(\alpha_1 \cdots \alpha_r + \mathsf{ord}^{m-1} C \right) / \mathsf{ord}^{m-1} C.$$

By the definition of the ten filtration, this says that image of χ is contained in (and indeed equal to) $\bigoplus_m \left(\mathsf{ten}^m C + \mathsf{ord}^{m-1} C \right) / \mathsf{ord}^{m-1} C$. But χ is surjective. By induction on m we therefore have $\mathsf{ord}^m C = \mathsf{ten}^m C + \mathsf{ord}^{m-1} C = \mathsf{ten}^m C$. $\square$

7.3. The equality of filtrations given by Lemma 7.2 is not merely a formality; indeed the result for B_{ij} is essentially the same result as Theorem 6.4. To see this, suppose that $\mathsf{ogr}\, B_{ij} = \mathsf{tgr}\, B_{ij}$ for all $i \geqslant j \geqslant 0$. As Lemma 6.9(2) shows, $\mathsf{ogr}\, B_{\ell+1,\ell} = A^1\delta$ for each ℓ and so, by Lemma 6.7(2), we get a surjection χ from $E = (A^1\delta)^{\otimes(i-j)}$ onto $\mathsf{tgr}\, B_{ij} = \mathsf{ogr}\, B_{ij}$.

The multiplication map $\phi : E \to (A^1\delta)^{i-j}$ is surjective and its kernel is the largest torsion A^0-submodule of $(A^1\delta)^{i-j}$. On the other hand $\mathsf{ogr}\, B_{ij} \subseteq e\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W$ is a torsion-free A^0-module and so $\ker(\phi) \subseteq \ker(\chi)$. Thus $\mathsf{ogr}\, B_{ij} = E/\ker(\chi)$ is a homomorphic image of $(A^1\delta)^{i-j}$. Since $(A^1\delta)^{i-j}$ is a right ideal of the domain A^0, any proper factor of $(A^1\delta)^{i-j}$ will be torsion. Thus $\ker(\phi) = \ker(\chi)$ and $\mathsf{ogr}\, B_{ij} \cong (A^1\delta)^{i-j}$.

7.4. The observation in (7.3) suggests that Lemma 7.2 will only hold for very special decompositions and this is indeed the case. In essence, Theorem 6.4 says that the identity $B_{ij} \cong B_{i,i-1} \otimes \cdots \otimes B_{j+1,j}$ is a filtered isomorphism. On the other hand, an identity like $H_c \cong H_c e \otimes_{U_c} eH_c$ from Theorem 3.3 is clearly not filtered; in writing the element 1 as an element of $H_c e \otimes eH_c$ an easy computation shows that one needs to use commutators of elements from $\mathbb{C}[\mathfrak{h}]$ and $\mathbb{C}[\mathfrak{h}^*]$ and so $1 \notin \mathsf{ten}^0(H_c)$. However, $ge = ge \cdot 1 \in \mathsf{ten}^0(H_c)$ for any $0 \neq g \in \mathbb{C}[\mathfrak{h}]^W$ and so $\sigma(ge)\sigma(1) = 0$ in $\mathsf{tgr}\, H_c$. On the other hand, as 1 is a regular element of $\mathsf{ogr}\, H_c$, no such equation is possible $\mathsf{ogr}\, H_c$. Thus $\mathsf{ten}\, H_c \ncong \mathsf{ogr}\, H_c$.

As a second example, it is easy to check that Lemma 7.2 will fail for $M(i)$ if one introduces one more tensor product, $M(i) \cong H_{c+i} e \otimes_{U_{c+i}} B_{i0}$. Indeed, Lemma B.2 implies that $\mathsf{ogr}\, M(1) = \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]\delta e$. On the other hand, for the given decomposition Lemmas 6.9 and 6.7 imply that $\mathsf{tgr}\, H_c$ is a homomorphic image of $T = \mathsf{ogr}\, H_{c+1} e \otimes_{U_{c+1}} \mathsf{ogr}\, Q_c^{c+1} \cong \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]e \otimes_{A^0} A^1\delta e$. Clearly the image of T in $\mathsf{ogr}\, M(1)$ is just $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]eA^1\delta e = J^1\delta e$. By the argument of the second paragraph of (7.3), this is also the image of $\mathsf{tgr}\, M(1)$ in $\mathsf{ogr}\, M(1)$.

Acknowledgments

We thank Victor Ginzburg for bringing his conjecture to our attention, since it really formed the starting point for this work. We also thank Tom Nevins and Catharina Stroppel for suggesting many improvements to us. Gordon thanks the University of Washington and the University of California at Santa Barbara for their hospitality while parts of this paper were written. Part of this research was conducted while Stafford was visiting the Mittag-Leffler Institute and he thanks them for their hospitality and financial support.

Appendix A. Graded projective modules

A.1. The aim of this appendix is to prove the following graded analogue of a well-known result of Kaplansky [Ka, Theorem 2], for which we do not know a reference.

Theorem. *Let $A = \bigoplus_{i \geqslant 0} A_i$ be a connected $\mathbb{N}$-graded k-algebra (thus $A_0 = k$). Let P be a right A-module that is both graded and projective. Then P is a* graded-free *A-module in the sense that P has a free basis of homogeneous elements.*

Proof. Throughout this proof all graded maps are graded maps of degree zero. We will write the degree of a homogeneous element $x \in P$ as $|x|$.

An observation of Eilenberg [Ei, Section 1] shows that P is graded projective in the sense that there is a graded isomorphism $F \cong P \oplus Q$, for some A-module Q and graded-free A-module F. We need a minor variant on this result, so we give the proof. Take a graded surjection $\phi : F = \bigoplus f_i A \twoheadrightarrow P$ and an ungraded splitting $\theta : P \to F$. If $p_i = \phi(f_i)$, then write $\theta(p_i) = g_i + h_i$, where g_i is the homogeneous component of $\theta(p_i)$ with $|g_i| = |p_i|$. Then check that the map $p_i \mapsto g_i$ also splits ϕ. This proof also shows that, if P is countably generated, then we can take F to be a countably generated graded-free module.

The heart of the proof of the theorem is contained in the next two sublemmas.

A.2. Sublemma. *Under the hypotheses of the theorem, P is a graded direct sum of countably generated A-modules.*

Proof. The proof of Kaplansky [Ka, Theorem 1] also works in the category of graded modules. $\square$

A.3. Sublemma. *Keep the hypotheses of the theorem and assume that P is countably generated. If $x \in P$ then there exists a graded-free direct summand G of P such that $x \in G$.*

Proof. By the result of Eilenberg described above, we may pick a graded isomorphism $F \cong P \oplus Q$, for some A-module Q and countably generated graded-free A-module F. Select a homogeneous basis $\{u_i : i \in \mathbb{N}\}$ for F such that there is a graded expression $x = \sum_{i=1}^n u_i a_i$, with $a_i \in A$ and n as small as possible.

We first claim that no a_j can be written as a *left* linear combination of the other a_ℓ. Indeed, suppose that $a_n = \sum_{i=1}^{n-1} r_i a_i$, for some $r_i \in A$. By taking the appropriate component we may assume that each r_i is homogeneous with $|r_i| = |a_n| - |a_i|$. It follows that $|u_n r_i| = |u_i|$ and hence that $u_i' = u_i + u_n r_i$ is homogeneous. However

$$\sum_{i=1}^{n-1} u_i' a_i = \sum_{i=1}^{n-1} u_i a_i + u_n \left(\sum_{i=1}^{n-1} r_i a_i \right) = x.$$

This contradicts the minimality of n and proves the claim.

Reorder the basis $\{u_\ell\}$ so that $|u_i| \leqslant |u_{i+1}|$ for $1 \leqslant i \leqslant n$ and write $u_i = p_i + q_i$, for $p_i \in P$, $q_i \in Q$, all of the same degree. Notice that $P \ni x = \sum u_i a_i = \sum p_i a_i + \sum q_i a_i$ and so $\sum q_i a_i \in P \cap Q = 0$. Hence

$$x = \sum_{i=1}^{n} u_i a_i = \sum_{i=1}^{n} p_i a_i. \tag{A.3.1}$$

Next write each p_i as a homogeneous sum $p_i = \sum_{j=1}^{n} u_j c_{ji} + t_i$, where $t_i \in \sum_{i>n} u_i A$. Then

$$x = \sum_{i=1}^{n} u_i a_i = \sum p_i a_i = \sum_{i,j=1}^{n} u_j c_{ji} a_i + \sum_{i=1}^{n} t_i a_i.$$

Since $\{u_i\}$ is a basis,

$$a_j = \sum_{i=1}^{n} c_{ji} a_i \quad \text{for} \quad 1 \leqslant j \leqslant n. \tag{A.3.2}$$

We claim that $c_{ji} = 0$ for $i < j$ and that $|c_{ji}| > 0$ whenever $i > j$ (and $c_{ji} \neq 0$). Since $|u_i| \leqslant |u_{i+1}|$, we have $|a_i| \geqslant |a_{i+1}|$ for each i. Also $|c_{ji}| = |u_i| - |u_j|$ for all i, j and so $c_{ji} = 0$ if $|u_i| < |u_j|$. Thus both parts of the claim are clear when $|u_i| \neq |u_j|$; equivalently, when $|a_i| \neq |a_j|$. So, suppose that $|a_i| = |a_j|$, for some $i \neq j$ and that $c_{ji} \neq 0$. Then $c_{ji} \in k^*$ and so (A.3.2) expresses a_i as a left linear combination of the other a_ℓ. This contradicts the initial minimality assumption on n and proves the claim. Note that $c_{jj} = 1$ for all j, since otherwise (A.3.2) would express a_j as a left linear combination of the other a_ℓ.

The last paragraph implies that $C = (c_{ji})$ is an upper triangular matrix, with units on the diagonal and so it is invertible. In particular, $\{p_1, \ldots, p_n\} \cup \{u_{n+\ell} : \ell > 0\}$ is a basis for F. Thus $G = \sum_{i=1}^{n} p_i A$ is a graded-free direct summand of F contained in P. Thus G is also a graded-free direct summand of P which, by (A.3.1), contains x. $\square$

A.4. The proof of the theorem follows from the sublemmas by an easy induction. By Sublemma A.2 we may assume that P is countably generated, say by homogeneous elements z_i for $i \in \mathbb{N}$. By induction, suppose that there is a graded decomposition $P = Q_1 \oplus \cdots \oplus Q_n \oplus R_n$, where each Q_i is graded-free and $z_i \in Q_1 \oplus \cdots \oplus Q_i$, for $1 \leqslant i \leqslant n$. By Sublemma A.3 this does hold when $n = 1$. Write $z_{n+1} = q + r$ as a homogeneous sum, where $q \in \sum Q_j$ and $r \in R_n$. Since R_n also satisfies the hypotheses of Sublemma A.3, R_n has a graded-free summand Q_{n+1} containing r, completing the inductive step. Finally,

$$\widetilde{P} \;=\; \lim_{n \to \infty} \big(Q_1 \oplus \cdots \oplus Q_n\big) \;\cong\; \bigoplus_{i=1}^{\infty} Q_i$$

is a graded-free submodule of P that contains each z_i. Therefore $P = \widetilde{P}$. $\square$

Appendix B. Another module

B.1. Fix $c \in \mathbb{C}$ that satisfies Hypothesis 6.2 and an integer $k \geqslant 0$. For applications in [GS] we will need an analogue of Proposition 6.5 for the left H_{c+k}-module

$M(k) = H_{c+k}eB_{k0} \subseteq D(\mathfrak{h}^{\mathrm{reg}}) * W$. As before, we filter $M(k)$ by the induced order filtration ord, so that $\mathsf{ogr}\, M(k) \subseteq \mathsf{ogr}\, D(\mathfrak{h}^{\mathrm{reg}}) * W = \mathbb{C}[\mathfrak{h}^{\mathrm{reg}} \oplus \mathfrak{h}^*] * W$. The aim of this appendix is then to prove:

Proposition. *The left H_{c+k}-module $M(k) = H_{c+k}eB_{k0}$ satisfies $\mathsf{ogr}\, M(k) = J^{k-1}\delta^k e$.*

Recall that Proposition 6.5 showed that the module $N(k) = B_{k0} \otimes eH_c$ had associated graded ring $eJ^k\delta^k$. In a sense, Proposition B.1 is just a left-right analogue of that result and so much of the present proof is formally very similar to that of Proposition 6.5.

We should first explain why the two results involve different powers of J^1. The reason is that one can write $M(k) = H_{c+k}eH_{c+k}\delta eB_{k-1,0}$. By Corollary 3.13 and (3.3.2) the left hand end of this expression collapses to give $M(k) = H_{c+k}\delta eB_{k-1,0}$. In particular, $M(1) = H_{c+1}\delta e$. A routine computation using Lemmas 6.7 and 6.8 then gives

B.2. Lemma. *The associated graded module $\mathsf{ogr}\, M(1)$ equals $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]\delta e$ while $J^{k-1}\delta^k e \subseteq \mathsf{ogr}\, M(k)$ for all $k \geqslant 1$.* $\square$

It takes considerably more work to show that $J^{k-1}\delta^k e$ actually equals $\mathsf{ogr}\, M(k)$ for $k > 1$. The proofs of the first few steps in this argument are very similar to those of Lemmas 6.11, 6.15 and 6.16 in the proof of Proposition 6.5 and so we will just indicate how to modify the earlier proofs to work here.

B.3. Since $M(k)$ is a (H_{c+k}, U_c)-bimodule, the embeddings $\mathbb{C}[\mathfrak{h}] \hookrightarrow H_{c+k}$ and $\mathbb{C}[\mathfrak{h}^*]^W \hookrightarrow U_c$ make $M(k)$ into a $(\mathbb{C}[\mathfrak{h}], \mathbb{C}[\mathfrak{h}^*]^W)$-bimodule. Let $\mathbb{C}$ be the trivial module over either $\mathbb{C}[\mathfrak{h}]$ or $\mathbb{C}[\mathfrak{h}^*]^W$ and set $\overline{M(k)} = \mathbb{C} \otimes_{\mathbb{C}[\mathfrak{h}]} M(k)$ and $\underline{M(k)} = M(k) \otimes_{\mathbb{C}[\mathfrak{h}^*]^W} \mathbb{C}$.

Lemma. (1) *$M(k)$ is free as a left $\mathbb{C}[\mathfrak{h}]$-module and a right $\mathbb{C}[\mathfrak{h}^*]^W$-module.*

(2) *$\underline{M(k)}$ is a finitely generated, free left $\mathbb{C}[\mathfrak{h}]$-module.*

(3) *Analogously, $\overline{M(k)}$ is a finitely generated, free right $\mathbb{C}[\mathfrak{h}^*]^W$-module.*

Proof. (1) By Corollary 3.13, $M(k)$ is projective as a left H_{c+k}-module and as a right U_c-module. By (2.2.1), H_{c+k} and hence $M(k)$ is free as a left $\mathbb{C}[\mathfrak{h}]$-module. Similarly, the argument of Lemma 6.11(2) shows that U_c and hence $M(k)$ are free right $\mathbb{C}[\mathfrak{h}^*]^W$-modules.

(2) This is contained in the proof of Lemma 6.11(3).

(3) Mimic the proof of Lemma 6.11(4). $\square$

B.4. Using the conventions from (6.14), each $M(k)$ and $J^{k-1}\delta^k e$ is $\mathbf{E}$-graded. Since $\mathbb{C}[\mathfrak{h}]_+$ is $\mathbf{E}$-graded, the $\mathbf{E}$-grading on $M(k)$ descends to one on $\overline{M(k)}$. Similarly, $J^{k-1}\delta^k e$ has the order grading ogr from (6.15). Write $\Theta : J^{k-1}\delta^k e \hookrightarrow \mathsf{ogr}\, M(k)$ for the inclusion from Lemma B.2.

Lemma. *There exists an injective map $\theta : J^{k-1}\delta^k e \hookrightarrow M(k)$ of left $\mathbb{C}[\mathfrak{h}]$-modules such that*:

(1) *θ is an $\mathbf{E}$-graded homomorphism and is a filtered homomorphism under the order filtration.*

(2) *The associated graded map* $\mathsf{ogr}\,\theta : J^{k-1}\delta^k e \to \mathsf{ogr}\,M(k)$ *induced by* θ *is precisely* $\mathsf{ogr}\,\theta = \Theta$.
(3) *In the notation of* (6.16), *the map* $\theta[\delta^{-2}] : (J^{k-1}\delta^k e)[\delta^{-2}] \to M(k)[\delta^{-2}]$ *is an isomorphism. This map is* **E**-*graded and is a filtered isomorphism under the order filtration.*

Proof. (1,2) As in the proof of Lemma 6.15, one constructs θ by lifting a **E**-homogeneous basis of the free $\mathbb{C}[\mathfrak{h}]$-module $\mathsf{ogr}^n(J^{k-1}\delta^k)e$ to a set of **E**-homogeneous elements in $\mathsf{ord}^n\,M(k)$.

(3) This is essentially the same as the proof of Lemma 6.16. □

B.5. By Lemma 6.10, $M(k)$ is graded under the adjoint **h**-action and, as both copies of $\mathbb{C}$ are **h**-graded modules, this grading restricts to one on $\overline{M(k)}$ and $\underline{M(k)}$. In each case, we call this *the* **h**-*grading*. For the reasons given in (6.14), this does not equal the **E**-grading.

Proposition. *If* $\overline{M(k)}$ *is graded via the adjoint* **h** *action, it has Poincaré series*

$$p(\overline{M(k)}, v) = \frac{\sum_\mu f_\mu(1) f_\mu(v^{-1}) v^{-(k-1)(n(\mu)-n(\mu^t))}}{\prod_{i=2}^n (1-v^{-i})}.$$

Proof. This is similar to the proof of Proposition 6.13 except that we use the module $Y = H_c e \otimes_R \mathbb{C}$, where $R = e\mathbb{C}[\mathfrak{h}^*]^W e$, in place of $X = H_c \otimes_{\mathbb{C}[\mathfrak{h}^*]} \mathbb{C}$. As in that proposition, Y is an object in $\widetilde{\mathcal{O}}_c$ and so we can write $[Y] = \sum_\mu p_\mu[\widetilde{\Delta}_c(\mu)]$ for some $p_\mu \in \mathbb{Z}[v, v^{-1}]$. To calculate the p_μ note that, by (2.2.1), $Y \cong \mathbb{C}[\mathfrak{h}] \otimes \mathbb{C}[\mathfrak{h}^*]^{\mathrm{co}\,W}$. Applying $(\mathbb{C} \otimes_{\mathbb{C}[\mathfrak{h}]} -)$ to the equation $[Y] = \sum p_\mu[\widetilde{\Delta}_c(\mu)]$ therefore yields $[\mathbb{C}[\mathfrak{h}^*]^{\mathrm{co}\,W}] = \sum_\mu p_\mu[\mu]$. Thus (3.9.1) implies that $p_\mu = f_\mu(v^{-1})$ (this is a polynomial in v^{-1} rather than v since $\mathbb{C}[\mathfrak{h}^*]$ is negatively **E**-graded) and so, as an element of $G_0(\widetilde{\mathcal{O}}_c)$,

$$[Y] = \sum_\mu f_\mu(v^{-1})[\widetilde{\Delta}_c(\mu)]. \tag{B.5.1}$$

Now consider $\underline{M(k)}$, which we can write as $H_{c+k}e \otimes_{U_{c+k}} B_{k0} \otimes_{U_c} eY$. By (3.3.2) and Corollary 3.13, $H_{c+k}e \otimes_{U_{c+k}} e\widetilde{\Delta}_{c+k}(\lambda) \cong \widetilde{\Delta}_{c+k}(\lambda)$. Thus (B.5.1) and Lemma 6.12 combine to show that

$$[\underline{M(k)}] = \sum_\mu f_\mu(v^{-1}) v^{k(n(\mu)-n(\mu^t))}[\widetilde{\Delta}_{c+k}(\mu)].$$

As graded vector spaces, $\widetilde{\Delta}_{c+k}(\mu) \cong \mathbb{C}[\mathfrak{h}] \otimes \mu$ and so (3.9.2) implies that $p(\widetilde{\Delta}_{c+k}(\mu), v) = f_\mu(1)(1-v)^{-(n-1)}$. Therefore,

$$p(\underline{M(k)}, v) = \frac{\sum_\mu f_\mu(1) f_\mu(v^{-1}) v^{k(n(\mu)-n(\mu^t)}}{(1-v)^{(n-1)}}. \tag{B.5.2}$$

By (2) and (3) of Lemma B.3, a homogeneous basis for $\overline{M(k)}$ is given by lifting a homogeneous $\mathbb{C}$-basis for $\overline{M(k)} \otimes_{\mathbb{C}[\mathfrak{h}^*]^W} \mathbb{C} = \mathbb{C} \otimes_{\mathbb{C}[\mathfrak{h}]} \underline{M(k)}$. Thus, combining (B.5.2) with the formulæ $p(\mathbb{C}[\mathfrak{h}^*]^W, v) = \prod_{i=2}^n (1-v^{-i})^{-1}$ and $p(\mathbb{C}[\mathfrak{h}], v) = (1-v)^{n-1}$ gives

$$p(\overline{M(k)}, v) = \frac{\sum_\mu f_\mu(v^{-1}) f_\mu(1) v^{k(n(\mu)-n(\mu^t))}}{\prod_{i=2}^n (1-v^{-i})}. \tag{B.5.3}$$

By Opdam [Op, Theorem 8] the fake degrees satisfy $f_\mu(v^{-1}) = f_{\mu^t}(v^{-1}) v^{n(\mu^t)-n(\mu)}$. Combined with (3.9.2) this implies that

$$f_\mu(v^{-1}) f_\mu(1) v^{k(n(\mu)-n(\mu^t))} = f_{\mu^t}(v^{-1}) f_{\mu^t}(1) v^{-(k-1)(n(\mu^t)-n(\mu))}.$$

Substituting this into (B.5.3) gives the stated formula for $p(\overline{M(k)}, v)$. $\square$

B.6. As was true for Corollary 6.14, we need to slightly modify Proposition B.5 in order to compute the Poincaré series for $\overline{M(k)}$ under the **E**-grading.

Corollary. *Set $K = kn(n-1)/2$ and $\mathfrak{n} = \mathbb{C}[\mathfrak{h}]_+$. Under the* **E**-*grading there is an equality of Poincaré series*

$$p(\overline{M(k)}, v) = v^K \frac{\sum_\mu f_\mu(1) f_\mu(v^{-1}) v^{-(k-1)(n(\mu)-n(\mu^t))}}{\prod_{i=2}^n (1-v^{-i})} = p(J^{k-1}\delta^k/\mathfrak{n}J^{k-1}\delta^k,\ v). \tag{B.6.1}$$

Proof. Eq. (6.10.1) continues to hold if we replace $em\delta e$ by $m\delta e$. Thus the argument of Corollary 6.14(1) combined with Proposition B.5 and the formula $M(k) = H_{c+k}\delta e B_{k-1,0}$ gives the first equality of (B.6.1).

In order to obtain the second equality in (B.6.1), note that

$$p(J^{k-1}\delta^k/\mathfrak{n}J^{k-1}\delta^k,\ v) = v^K p(J^{k-1}/\mathfrak{n}J^{k-1},\ v).$$

Set $p(v) = p(J^{k-1}/\mathfrak{n}J^{k-1},\ v)$ and $q(v) = p(J^{k-1}/\mathfrak{m}J^{k-1},\ v)$, where $\mathfrak{m} = \mathbb{C}[\mathfrak{h}]_+^W$ The Poincaré series $q(v)$ has been computed in Corollary 4.13. Since that series was obtained by specialising the bigraded Poincaré series $p(J^d, s, t)$ from Corollary 4.11, it follows immediately that

$$p(v) = \frac{p(\mathbb{C}[\mathfrak{h}], v)}{p(\mathbb{C}[\mathfrak{h}]^W, v)}\, q(v) = \frac{(1-v)^{n-1}}{\prod_{i=2}^n (1-v^i)}\, q(v) = \frac{q(v)}{[n]_v!}$$

where the final equality uses (4.13.1). Substituting these observations into Corollary 4.13 gives the second equality in (B.6.1). $\square$

B.7. Proof of Proposition B.1. We need to show that the map $\theta : J^{k-1}\delta^k e \to M(k)$ is an isomorphism for all $k \geqslant 1$. This is analogue of Proposition 6.18. In that case, a purely formal argument showed that Proposition 6.18 followed from (6.18.1). The same argument can be used, essentially without change, to show that the bijectivity of θ follows from (B.6.1).

Combined with Lemma B.4(ii) this says that $\mathsf{ogr}\, M(k) = \mathsf{ogr}\, \theta(J^{k-1}\delta^k e) = J^{k-1}\delta^k e$, as required. $\square$

Appendix C. Index of Notation

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