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Abstract. We establish an integral test involving only the distribution of the increments of a random walk S which determines whether $\limsup_{n\to\infty}(S_n/n^\kappa)$ is almost surely zero, finite or infinite when $1/2 < \kappa < 1$ and a typical step in the random walk has zero mean. This completes the results of Kesten and Maller [9] concerning finiteness of one-sided passage times over power law boundaries, so that we now have quite explicit criteria for all values of $\kappa \geq 0$. The results, and those of [9], are also extended to Lévy processes.

1. Random walks

In [9] an almost complete solution was given to the problem of finding analytic conditions, expressed directly in terms of the step distribution F of the random walk $S = (S_n, n \geq 0)$, for first-passage times over one-sided and two-sided power law boundaries of the random walk to be almost surely (a.s.) finite. The exception was for the one-sided passage time

$$T_\kappa^*(a) = \min\{n \geq 1 : S_n > an^\kappa\},\ a > 0, \tag{1.1}$$

in the case that

$$\frac{1}{2} < \kappa < 1,\ E|X| < \infty, EX = 0, E(X^+)^{\frac{1}{\kappa}} < \infty \text{ and } E(X^-)^{\frac{1}{\kappa}} = \infty. \tag{1.2}$$

(Here X denotes a generic step in the random walk.) This passage time is a.s. finite for all $a > 0$ if and only if

$$\limsup_{n\to\infty}\left(\frac{S_n}{n^\kappa}\right) = +\infty \text{ a.s.},$$

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and although Theorem 3 of [9] does give a necessary and sufficient condition for this when (1.2) holds, it is complicated and not expressed directly in terms of F.

In this section, Theorem 1.1 and Corollary 1.1 contain quite explicit necessary and sufficient conditions of the required form, and also give some additional information. In the following section, we extend the present results, and those of [9], to Lévy processes, again giving explicit criteria for finiteness of passage times above power law boundaries, in terms of the characteristics of the process.

To state our main results, define, for $y \geq 0$, the function

$$W(y) = \int_0^y \int_{-\infty}^{-z} |x| F(dx) dz = \int_0^y \int_x^\infty F(-z) dz dx + \int_0^y x F(-x) dx, \quad (1.3)$$

and note that $W(y) > 0$ for all $y > 0$ if F is not concentrated on $[0, \infty)$, thus, certainly if $E(X^-)^{\frac{1}{\kappa}} = \infty$ for some $\kappa > 0$, and $W(y) < \infty$ for all $y > 0$ if $EX^- < \infty$. When $W(y) > 0$ for all $y > 0$, define, for $\lambda > 0$,

$$I_\kappa(\lambda) := \int_1^\infty \exp\left\{ -\lambda \left(\frac{y^{\frac{2\kappa-1}{\kappa}}}{W(y)} \right)^{\frac{\kappa}{1-\kappa}} \right\} \frac{dy}{y} \leq \infty. \quad (1.4)$$

Theorem 1.1. *Assume (1.2) holds, and let*

$$\lambda^* = \inf\{\lambda > 0 : I_\kappa(\lambda) < \infty\}.$$

Then:

(i) if $\lambda^* = \infty$*, we have*

$$\limsup_{n\to\infty} \left(\frac{S_n}{n^\kappa} \right) = +\infty \text{ a.s.}; \quad (1.5)$$

(ii) if $\lambda^* = 0$*, we have*

$$\limsup_{n\to\infty} \left(\frac{S_n}{n^\kappa} \right) \stackrel{a.s.}{=} 0; \quad (1.6)$$

(iii) if $0 < \lambda^* < \infty$ *we have, for some* $0 < b < \infty$*,*

$$\limsup_{n\to\infty} \left(\frac{S_n}{n^\kappa} \right) \stackrel{a.s.}{=} b. \quad (1.7)$$

Applying Theorem 1.1 and the results of [9] gives the following criterion for (1.5):

Corollary 1.1. *Assume* $\frac{1}{2} < \kappa < 1$*,* $E|X| < \infty$*, and* $EX = 0$*. Then (1.5) holds if and only if (i)* $E(X^+)^{\frac{1}{\kappa}} = \infty$*, or (ii)* $E(X^+)^{\frac{1}{\kappa}} < \infty = E(X^-)^{\frac{1}{\kappa}}$ *and* $I_\kappa(\lambda) = \infty$ *for all* $\lambda > 0$*.*

Remark 1.1. If $F(-y) \sim 1/(y^{1/\kappa} L(y))$ as $y \to \infty$, with L slowly varying, it follows that $W(y) \sim c y^{2-1/\kappa}/L(y)$ for some $c > 0$, and the result of Theorem 1.1 holds with $I_\kappa(\lambda)$ replaced by

$$\int_1^\infty \exp\{-\lambda(L(y))^{\frac{\kappa}{1-\kappa}}\}\frac{dy}{y}.$$

So, writing ℓ_k for the k-th iterated logarithm, examples of the three possibilities in the theorem are got by taking (i) $L(y) = (\ell_3(y))^{\frac{1-\kappa}{\kappa}}$, (ii) $L(y) = (\ell_1(y))^{\frac{1-\kappa}{\kappa}}$, and (iii) $L(y) = (\ell_2(y))^{\frac{1-\kappa}{\kappa}}$.

It was observed in [9] that there is no loss of generality in assuming that the distribution of X^+ is concentrated at a single point, and it is not difficult to extend this argument to show that it is suffices to deal with the case that F is concentrated on $\{\cdots, -2, -1, 0, 1\}$. We refer to this as the u.s.f. (upwards skip free) case. Now the condition given in Theorem 3 of [9] is got by applying a result in Zhang [13] to the renewal process $(\sigma_n, n \geq 1)$ of increasing ladder times in S. Our contribution is to exploit the fact that in the u.s.f. case the Wiener-Hopf factorisation gives a simple analytic link between the tail behaviour of σ_1 and that of $-X$. (This observation goes back at least to Spitzer; see [12], p. 228.) We also notice that Zhang's result can be used to get (ii) and (iii) of Theorem 1.1. The final ingredient is some inequalities connecting the distribution tail of a non-negative random variable with its Laplace transform; these may have some independent interest, and are given in the Appendix. The proof of Theorem 1.1 is given in Section 3. In Section 2 we use the present ideas and those of Kesten and Maller [9] to give criteria for finiteness of passage times of Lévy processes above power law boundaries. Proofs of the Section 2 results are in Section 4.

2. Lévy processes

In this section $X = (X_t, t \geq 0)$ will be a Lévy process with characteristic triplet $(\gamma, \sigma, \Pi(\cdot))$. We use throughout similar notation to, and some of the results of, Doney [4] and Doney and Maller [5], [6]. In particular, we write

$$\overline{\Pi}^+(x) = \Pi((x,\infty)),\ \overline{\Pi}^-(x) = \Pi((-\infty,-x)),\ x > 0, \tag{2.1}$$

for the tails of $\Pi(\cdot)$, and assume X is not spectrally negative, so $\overline{\Pi}^+(x)$ is not identically zero. (For the spectrally negative case, see Remark 2.3 below.) By a rescaling which will not affect the results we can and will assume $\overline{\Pi}^+(1) > 0$. Let $\Pi^\#(\cdot)$ be the Lévy measure of $-X$, and define

$$J_-^\Pi = \int_{[1,\infty)} \left(\frac{x}{\overline{\Pi}^+(1) + \int_1^x \overline{\Pi}^+(y)dy}\right) \Pi^\#(dx). \tag{2.2}$$

If $\overline{\Pi}^-(1) > 0$ define

$$J_+^\Pi = \int_{[1,\infty)} \left(\frac{x}{\overline{\Pi}^-(1) + \int_1^x \overline{\Pi}^-(y)dy}\right) \Pi(dx).$$

If $\overline{\Pi}^-(1) = 0$ we set $J_+^\Pi = EX_1^+ \in (0,\infty]$ (thus if $\Pi(\cdot)$ is bounded on the left we can say that J_+^Π is finite or infinite according as EX_1^+ is finite or infinite).

From [7], [6] and [11], the following four equivalences can be deduced:

$$\lim_{t\to\infty}\left(\frac{X_t}{t}\right) = +\infty \text{ a.s.} \Longleftrightarrow J_-^{\Pi} < \infty = EX_1^+; \tag{2.3}$$

$$\lim_{t\to\infty} X_t = +\infty \text{ a.s.} \Longleftrightarrow J_-^{\Pi} < \infty = EX_1^+ \text{ or } 0 < EX_1 \le E|X_1| < \infty; \tag{2.4}$$

$$\limsup_{t\to\infty} X_t = +\infty \text{ a.s.} \Longleftrightarrow J_+^{\Pi} = \infty \text{ or } 0 \le EX_1 \le E|X_1| < \infty; \tag{2.5}$$

$$\limsup_{t\to\infty}\left(\frac{X_t}{t}\right) = +\infty \text{ a.s.} \Longleftrightarrow J_+^{\Pi} = \infty. \tag{2.6}$$

Next, let $c_1 = \overline{\Pi}^+(1) + \overline{\Pi}^-(1) > 0$, $\tilde{\mu} = \gamma/c_1$, and define, for $y > 0$,

$$W^{\Pi}(y) = \frac{1}{c_1}\int_0^y \int_x^{\infty} |z|\Pi(dz - \tilde{\mu})1_{\{|z-\tilde{\mu}|>1\}}dx. \tag{2.7}$$

Note that $W^{\Pi}(y)$ is positive for all $y > 0$ if $\overline{\Pi}^-(1) > 0$, which certainly holds if $E(X_1^-)^{\frac{1}{\kappa}} = \infty$ for any $\kappa > 0$, and is finite if $EX_1^- < \infty$.

Our main result gives necessary and sufficient conditions for

$$\limsup_{t\to\infty}\left(\frac{X_t}{t^{\kappa}}\right) = +\infty \text{ a.s.} \tag{2.8}$$

Theorem 2.1. *The following conditions are necessary and sufficient for (2.8).*

(a) For $\kappa > 1$*:*

$$\int_{[1,\infty)}\left(\frac{x^{\frac{1}{\kappa}}}{1 + x^{\frac{1}{\kappa}-1}\int_1^x \overline{\Pi}^-(y)dy}\right)\Pi(dx) = \infty. \tag{2.9}$$

(b) For $\kappa = 1$*:*

$$J_+^{\Pi} = \infty. \tag{2.10}$$

(c) For $\frac{1}{2} < \kappa < 1$ *and* $E|X_1| = \infty$*:*

$$J_+^{\Pi} = \infty. \tag{2.11}$$

(d) For $0 \le \kappa \le \frac{1}{2}$*:*

$$J_+^{\Pi} = \infty \text{ or } 0 \le EX_1 \le E|X_1| < \infty. \tag{2.12}$$

(e) For $\frac{1}{2} < \kappa < 1$*,* $E|X_1| < \infty$*, and* $EX_1 \ne 0$*:*

$$EX_1 > 0. \tag{2.13}$$

(f) For $\frac{1}{2} < \kappa < 1$, $E|X_1| < \infty$ *and* $EX_1 = 0$*:*

$$\text{(i) } E(X_1^+)^{\frac{1}{\kappa}} = \infty, \text{ or (ii) } E(X_1^+)^{\frac{1}{\kappa}} < \infty = E(X_1^-)^{\frac{1}{\kappa}}$$

and

$$I_\kappa^\Pi(\lambda) := \int_1^\infty \exp\left\{-\lambda\left(\frac{y^{\frac{2\kappa-1}{\kappa}}}{W^\Pi(y)}\right)^{\frac{\kappa}{1-\kappa}}\right\}\frac{dy}{y} = \infty \text{ for all } \lambda > 0. \quad (2.14)$$

Remark 2.1. The moment conditions on X in this theorem can easily be expressed in terms of the characteristics of X; see [10], p. 163.

Remark 2.2. Using

$$\frac{1}{1/a + 1/b} \le \min(a, b) = \frac{1}{\max(1/a, 1/b)} \le \frac{2}{1/a + 1/b}, \ a > 0, b > 0,$$

the integral condition in (2.9) can alternatively be written

$$\int_{[1,\infty)} \min\left(x^{\frac{1}{\kappa}}, \frac{x}{\int_1^x \overline{\Pi}^-(z)dz}\right)\Pi(dx) = \infty.$$

In this form it corresponds to condition (1.6) of [9] for random walks (the criterion due to Chow and Zhang 1986), and, similarly, that condition can be written

$$\int_{[1,\infty)} \left(\frac{x^{\frac{1}{\kappa}}}{1 + x^{\frac{1}{\kappa}-1}\int_0^x F(-z)dz}\right)F(dx) = \infty, \quad (2.15)$$

which is perhaps more convenient. Condition (2.15) is thus necessary and sufficient for $\limsup_{n\to\infty}(S_n/n^\kappa) = \infty$ a.s. (using the notation of Section 1) when $\kappa > 1$.

Remark 2.3. Suppose $\overline{\Pi}^+(x) = 0$ for all $x > 0$, so that $EX_1^+ < \infty$ and $\lim_{t\to\infty}(X_t/t) = EX_1 \in [-\infty, \infty)$ a.s. In this case we make the convention that J_+^Π and the integral in (2.9) converge regardless of whether $\overline{\Pi}^-(1) > 0$ or not. Then parts (a) and (b) of Theorem 2.1 remain true in the sense that neither (2.9) nor (2.10) holds. If in addition $E|X_1| = \infty$, so $EX_1^- = \infty$ and $\lim_{t\to\infty}(X_t/t) = -\infty$ a.s., part (c) of Theorem 2.1 remains true in the sense that (2.11) does not hold. If instead $E|X_1| < \infty$ then part (d) of Theorem 2.1 remains true since $\limsup_{t\to\infty}(X_t/t^\kappa) = +\infty$ a.s. if and only if $EX_1 = 0$, as is shown in the proof of (2.11). Part (e) obviously remains true. The proof of Part (f) of Theorem 2.1 does not depend on $\overline{\Pi}^+(1) > 0$, so it remains true in this spectrally negative case too.

The two-sided exit results of [9] for random walks can also be extended to Lévy processes as follows:

Theorem 2.2. *(a) If* $\kappa \ge 1$ *or if* $\frac{1}{2} < \kappa < 1$, $E|X_1| < \infty$ *and* $EX_1 = 0$*, then*

$$\limsup_{t\to\infty}\left(\frac{|X_t|}{t^\kappa}\right) = \infty \ \text{a.s. if and only if } E|X_1|^{1/\kappa} = \infty.$$

(b) In all other cases, we have

$$\limsup_{t\to\infty}\left(\frac{|X_t|}{t^\kappa}\right)=\infty \ \ a.s.$$

Theorem 2.2 is proved just as in Theorem 1 of [9], using the following version of the Marcinkiewicz—Zygmund strong law of large numbers for Lévy processes. We omit the details of these, as, for the two-sided case, they are almost the same as for the random walk proofs.

Proposition 2.1. *Fix $\kappa > 1/2$. Then*

$$\limsup_{t\to\infty}\left(\frac{|X_t-ct|}{t^\kappa}\right)<\infty \ \ a.s.$$

for some finite c implies $E|X_1|^{1/\kappa}<\infty$, and $E|X_1|^{1/\kappa}<\infty$ implies

$$\lim_{t\to\infty}\left(\frac{X_t-c't}{t^\kappa}\right)=0 \ \ a.s.$$

where $c'=EX_1$ if $\frac{1}{2}<\kappa\le 1$ and $c'=0$ if $\kappa>1$.

3. Proof of Theorem 1.1 and Corollary 1.2

3.1. The u.s.f. case

Throughout this sub-section we will assume that $S_n=\sum_1^n X_i$, where the X_i are independent and identically distributed copies of a random variable X whose distribution is given by

$$X=\begin{cases}1 & \text{with probability } p,\\ -Y & \text{with probability } q,\end{cases} \tag{3.1}$$

in which $p+q=1$, Y takes non-negative integer values only, and has finite mean μ, and $EX=p-q\mu=0$.

The crucial point is that if σ_n is the nth strict increasing ladder time of S then in this u.s.f. case we have $S_{\sigma_n}\equiv n$, so

$$\limsup_{n\to\infty}\left(\frac{S_n}{n^\kappa}\right)=\limsup_{n\to\infty}\frac{n}{(\sigma_n)^\kappa}=\left(\liminf_{n\to\infty}\left(\frac{\sigma_n}{n^{\frac{1}{\kappa}}}\right)\right)^{-\kappa}. \tag{3.2}$$

By specialising Theorem 1 of [13] to the case $\gamma(x)=x^{1/\kappa}$ we see that (1.5), (1.6) or (1.7) holds according as $\lambda_0^*=\infty$, $\lambda_0^*=0$, or $0<\lambda_0^*<\infty$, where $\lambda_0^*=\inf\{\lambda>0: I_\kappa^{(0)}(\lambda)<\infty\}$, and, writing $(1-\kappa)^{-1}=\gamma\in(2,\infty)$ and $A_\sigma(x)=\int_0^x P(\sigma>y)dy$,

$$I_\kappa^{(0)}(\lambda)=\int_1^\infty \exp\{-\lambda x^{-1}(A_\sigma(x))^\gamma\}\frac{dx}{x}. \tag{3.3}$$

Here we have written σ for a random variable with the distribution of σ_1. Now put $\psi_\sigma(\lambda) = E(e^{-\lambda\sigma})$; then Corollary 5.1 of the Appendix tells us that the ratio

$$\frac{A_\sigma(x)}{x(1-\psi_\sigma(1/x))}$$

is bounded above and below by positive constants for all $x > 0$. It follows that the previous statement holds good with λ_0^* replaced by λ_1^*, where the latter is defined by replacing $I_\kappa^{(0)}(\lambda)$ by

$$I_\kappa^{(1)}(\lambda) = \int_1^\infty \exp\{-\lambda x^{\gamma-1}(1-\psi_\sigma(1/x))^\gamma\}\frac{dx}{x}. \tag{3.4}$$

To relate this to the distribution of X, note that $-X$ is left-continuous, in the notation of p. 185 of Spitzer [12]. Let $Q(z) = E(z^{-X+1})$, which is the analogue in our notation of Spitzer's $P(z)$. Our σ, the first upwards ladder time of S, is Spitzer's T^*, the first downwards ladder time of $-S$, so our $\psi_\sigma(z) = E(e^{-z\sigma}) = E(t^{T^*})$, with $t = e^{-z}$, $0 < t < 1$, $z > 0$, in Spitzer's notation. Thus from p. 228 of [12] we have

$$Q(\psi_\sigma(z)) = e^z\psi_\sigma(z), \quad \text{for } 0 < z \le 1. \tag{3.5}$$

Now the fact that $EX = 0$ shows that $\Phi_X(\lambda) := E(e^{\lambda X}) - 1$ has a positive derivative on $(0, \infty)$ for $\lambda > 0$ and hence possesses a strictly increasing inverse function $\Phi_X^{\leftarrow}(\cdot)$ on $(0, \infty)$. From (3.5)

$$\Phi_X(-\log\psi_\sigma(z)) = \frac{Q(e^{-(-\log\psi_\sigma(z))})}{e^{-(-\log\psi_\sigma(z))}} - 1 = e^z - 1,$$

so, for all $z > 0$,

$$\psi_\sigma(z) = \exp\left(-\Phi_X^{\leftarrow}(e^z - 1)\right).$$

Thus, as $x \to \infty$,

$$1 - \psi_\sigma(1/x) \asymp -\log\psi_\sigma(1/x) = \Phi_X^{\leftarrow}(e^{1/x} - 1).$$

This allows us to replace $1 - \psi_\sigma(1/x)$ in $I_\kappa^{(1)}(\lambda)$ by $\Phi_X^{\leftarrow}(e^{1/x} - 1)$, and if we then write $e^{1/x} = 1 + 1/y$ in the resulting integral we see easily that we can replace λ_1^* by λ_2^*, defined by replacing $I_\kappa^{(1)}(\lambda)$ by

$$I_\kappa^{(2)}(\lambda) = \int_1^\infty \exp\left[-\lambda y^{\gamma-1}\{\Phi_X^{\leftarrow}(1/y)\}^\gamma\right]\frac{dy}{y}.$$

A further change of variable $z = 1/\Phi_X^{\leftarrow}(1/y)$ shows that we can replace λ_2^* by λ_3^*, defined by replacing $I_\kappa^{(2)}(\lambda)$ by

$$I_\kappa^{(3)}(\lambda) = \int_1^\infty \exp\{-\lambda(\Phi_X(1/z))^{1-\gamma}z^{-\gamma}\}\frac{\Phi_X'(1/z)dz}{z^2\Phi_X(1/z)}.$$

In the appendix we will prove the following result.

Lemma 3.1. *Assume (3.1) holds, and that* $EY^2 = \infty$. *Then, with* $\Phi_X(\lambda) = E(e^{\lambda X}) - 1$ *and*

$$V(x) = V_Y(x) = \int_0^x \int_y^\infty zP(Y \in dz)dy, \tag{3.6}$$

we have

$$\Phi_X(\lambda) \asymp \lambda^2 V(1/\lambda) \text{ and } \Phi'_X(\lambda) \asymp \lambda V(1/\lambda) \text{ as } \lambda \downarrow 0, \tag{3.7}$$

where "$\asymp$ *as* $\lambda \downarrow 0$" *means that the ratio of the two sides is bounded above and below by positive constants on some interval* $(0, \lambda_0]$.

Using this result, and noting that in this case $W(y) = qV(y)$, we see that we can replace λ_3^* by λ^*, as required.

3.2. The general case.

Now suppose that X has any distribution satisfying (1.2). Then, as pointed out in [9], the Marcinkiewicz-Zygmund law (see [2], p. 125) gives

$$\frac{\sum_{i=1}^n \{X_i^+ - E(X_i|X_i > 0)\mathbf{1}_{(X_i>0)}\}}{n^\kappa} \overset{a.s.}{\to} 0,$$

and this means can we replace X_i by the constant $E(X_i|X_i > 0)$ when $X_i > 0$, without affecting the value of $\limsup_{n\to\infty}(S_n/n^\kappa)$, or of course λ^*. In other words there is no loss of generality in assuming that X can take only one positive value; furthermore by scaling, we can take this value to be 1. Thus we almost have the situation discussed in the previous sub-section, the difference being that in the representation (3.1) the non-negative random variable Y has an arbitrary distribution, rather than one concentrated on the integers. However the random walk $S^{(0)}$ defined by

$$S_n^{(0)} = \sum_{i=1}^n X_i^{(0)}, \quad \text{where } X_i^{(0)} = \lfloor X_i \rfloor$$

and $\lfloor x \rfloor$ denotes the integer part of x, is u.s.f.. Moreover, the i.i.d. sequence $Z_i = X_i - X_i^{(0)}$ take values in $[0, 1)$, $X_i = 1$ implies $Z_i = 0$, and $p := E(Z_i|Z_i > 0) \in (0, 1)$ if S is not u.s.f., which we assume from now on. Let $I_1, I_2, \cdots$, denote a sequence of i.i.d. random variables, independent of S, with $P(I_r = 1) = p$ and $P(I_r = 0) = 1 - p$, and define another u.s.f. random walk by

$$S_n^{(1)} = \sum_{r=1}^n X_r^{(1)} \text{ where } X_r^{(1)} = X_r^{(0)} + I_r\mathbf{1}_{\{Z_r \neq 0\}}.$$

The point is that

$$EX_r^{(1)} = EX_r^{(0)} + pP\{Z_r \neq 0\} = EX_r^{(0)} + EZ_r = EX_r = 0,$$

and

$$S_n = S_n^{(1)} + S_n^{(2)}, \text{ where } S_n^{(2)} = \sum_{r=1}^{n} V_r \text{ and } V_r = Z_r - I_r \mathbf{1}_{\{Z_r \neq 0\}}.$$

Now $S^{(2)}$ is a random walk whose steps take values in $[-1, 1]$ and have zero mean. Thus by the Marcinkiewicz-Zygmund law

$$\frac{S_n^{(2)}}{n^\kappa} \to 0 \text{ a.s.},$$

and consequently (1.5) holds for S if and only if it holds for $S^{(1)}$. By construction (1.2) holds for $X^{(1)}$, and we finish the proof by showing that the λ^* of Theorem 1.1 evaluated for S and for $S^{(1)}$ are the same. Since $E(X^-)^{\frac{1}{\kappa}} = \infty$ implies $W(y) \to \infty$ as $y \to \infty$ this will certainly follow if we can show that, in the obvious notation,

$$\sup_{y>0} |W(y) - W^{(1)}(y)| < \infty.$$

However, using the representations

$$W(y) = \int_0^y E\{|X|; X \leq -z\} dz,$$
$$W^{(1)}(y) = \int_0^y E\{|X + V|; X + V \leq -z\} dz,$$

where $|V| \leq 1$ a.s., this is easy to check.

Proof of Corollary 1.1. Fix $\frac{1}{2} < \kappa < 1$, $E|X| < \infty$, and $EX = 0$. Suppose (1.5) holds. If $E(X^+)^{\frac{1}{\kappa}} < \infty$ and $E(X^-)^{\frac{1}{\kappa}} < \infty$, that is, $E|X|^{\frac{1}{\kappa}} < \infty$, then $\lim_{n\to\infty}(S_n - nEX)/n^\kappa = \lim_{n\to\infty}(S_n/n^\kappa) = 0$ a.s. by the Marcinkiewicz-Zygmund law, a contradiction. If $E(X^+)^{\frac{1}{\kappa}} < \infty = E(X^-)^{\frac{1}{\kappa}}$, then also $I_\kappa(\lambda) = \infty$ for all $\lambda > 0$ or else we get a contradiction from Theorem 1.1.

Conversely, if $E(X^+)^{\frac{1}{\kappa}} = \infty$ then (1.5) holds by Part (f) of Theorem 2 of [9], while if $E(X^+)^{\frac{1}{\kappa}} < \infty = E(X^-)^{\frac{1}{\kappa}}$ and $I_\kappa(\lambda) = \infty$ for all $\lambda > 0$ then (1.5) follows from Part (i) of Theorem 1.1. □

4. Proof of Theorem 2.1

We will make repeated use of results of [4], [5] and [6] to transfer between the Lévy process and approximating random walks. For the "large time" ($t \to \infty$) results we are concerned with here, "small jumps" in X can be neglected, with some care. Most useful is a representation in [4] which leads to inequality (4.6) below. Let J_n be the n –th jump in X with absolute value exceeding 1, occurring at time τ_n, say. Since we assume $c_1 = \overline{\Pi}^+(1) + \overline{\Pi}^-(1) > 0$, there are such jumps and the $(J_i)_{i=1,2,\cdots}$ are i.i.d. with distribution $\Pi(dx)1_{\{|x|>1\}}/c_1$.

(a) Keep $\kappa > 1$. Suppose the integral in (2.9) is infinite. Then

$$\int_{[1,\infty)} \left(\frac{x^{\frac{1}{\kappa}}}{1 + x^{\frac{1}{\kappa}-1} \int_0^x P(J_1 \le -z)dz} \right) dP(J_1 \le x) = \infty, \tag{4.1}$$

so from (1.6) of [9] (and see also Remark 2.2) we get that $\limsup_{n\to\infty}(\sum_1^n J_i/n^\kappa) = +\infty$ a.s. Since $\kappa > 1$ this means $\limsup_{n\to\infty}(\sum_1^n (J_i + \tilde{\mu})/n^\kappa) = +\infty$ a.s. for any $\tilde{\mu} \in \mathbb{R}$. Now by (2.19) of [6],

$$\hat{S}_n = \sum_{i=1}^n (J_i + \tilde{\mu}) + \sum_{i=1}^n W_i = S_n^* + \sum_{i=1}^n W_i, \text{ say}, \tag{4.2}$$

where $\tilde{\mu} = \gamma/c_1$, and the W_i are i.i.d. random variables with expectation 0 and a finite moment generating function (m.g.f.). So $\limsup_{n\to\infty}(S_n^*/n^\kappa) = +\infty$ a.s. We can write (see the display after (2.19) of [6])

$$X_t = S_{N_t}^* - \gamma(N_t - c_1 t)/c_1 + \overline{X}_t, \tag{4.3}$$

where $N_t = \max\{n : \tau_n \le t\}$ is a Poisson process of rate c_1 , and $\overline{X}$ is a Lévy process with $E\overline{X}_t = 0$, having a finite m.g.f. for each $t > 0$, and being independent of $S.$ and $N.$, which are also independent processes. Since $\lim_{t\to\infty}(N_t/t) = c_1$ a.s. and both $N_t - c_1 t$ and $\overline{X}_t$ are $o(t)$ a.s. as $t \to \infty$ by the strong law of large numbers, we get

$$\limsup_{t\to\infty} \left(\frac{X_t}{t^\kappa} \right) = \infty \text{ a.s.}, \tag{4.4}$$

as required. Conversely, still with $\kappa > 1$, let (4.4) hold. Then (4.3) and the strong law give $\limsup_{t\to\infty}(S_{N_t}^*/N_t^\kappa) = +\infty$ a.s., so

$$\limsup_{t\to\infty} \left(\sum_1^{N_t} (J_i + \tilde{\mu})/N_t^\kappa \right) = +\infty \text{ a.s.},$$

thus $\limsup_{n\to\infty}(\sum_1^n J_i/n^\kappa) = +\infty$ a.s. Thus (4.1) holds by (1.6) of [9]. But (4.1) is equivalent to (2.9).

(b) (2.10) is immediate from (2.6).

(c) Keep $\frac{1}{2} < \kappa < 1$ and $E|X_1| = \infty$. Then (4.4) implies $\limsup_{t\to\infty} X_t = +\infty$ a.s. and hence $J_+^\Pi = \infty$ by (2.5) and the fact that $E|X_1| = \infty$. Conversely, suppose $J_+^\Pi = \infty$. Then for all constants $\tilde{\mu} \in \mathbb{R}$,

$$J_+(\tilde{\mu}) := \int_{[\tilde{\mu},\infty)} \left(\frac{x dP((J_1 + \tilde{\mu})^+ \le x)}{\int_0^x P(J_1 + \tilde{\mu} \le -y)dy} \right) = \infty. \tag{4.5}$$

To show this, suppose $J_+(\tilde{\mu}) < \infty$ for some $\tilde{\mu} \in \mathbb{R}$. If $E(J_1 + \tilde{\mu})^- < \infty$, or, equivalently, $EJ_1^- < \infty$, then the denominator of (4.5) is bounded as $x \to \infty$ and so $J_+^\Pi < \infty$, a contradiction. Consequently $E(J_1+\tilde{\mu})^- = \infty$, so $EJ_1^- = \infty$ and thus, as $x \to \infty$,

$$\int_1^x P(J_1 + \tilde{\mu} \le -y)dy = \int_{1+\tilde{\mu}}^{x+\tilde{\mu}} P(J_1 \le -y)dy \sim \int_0^x P(J_1 \le -y)dy.$$

This means that

$$\begin{aligned} J_+(\tilde{\mu}) &= \int_{(0,\infty)} \left(\frac{x+\tilde{\mu}}{\int_{\tilde{\mu}}^{x+2\tilde{\mu}} P(J_1 \le -y)dy} \right) dP(J_1^+ \le x) \\ &\sim \int_{(0,\infty)} \left(\frac{x}{\int_1^x P(J_1 \le -y)dy} \right) dP(J_1^+ \le x) \end{aligned}$$

is also infinite, thus $J_+^\Pi < \infty$, a contradiction. So (4.5) holds.

Now (4.5) together with $E|J_1 + \tilde{\mu}| = \infty$ gives $\limsup_{n\to\infty}(S_n^*/n^\kappa) = +\infty$ a.s., where $S_n^* = \sum_1^n (J_i + \tilde{\mu})$, by [9], equation (1.8). Thus by (4.2) and the law of the iterated logarithm, $\limsup_{n\to\infty}(\hat{S}_n/n^\kappa) = +\infty$ a.s.. From equations (1.3) and (1.8) of [4] we see that

$$S_n^{(-)} + \tilde{i}_0 \le X_t \le S_n^{(+)} + \tilde{m}_0, \ \text{for } \tau_n \le t < \tau_{n+1}, \tag{4.6}$$

where $S^{(-)}$ and $S^{(+)}$ are random walks with the same distribution as $\hat{S}$, and $\tilde{i}_0$ and $\tilde{m}_0$ are finite random variables. Thus

$$\limsup_{n\to\infty} \left(\frac{S_n^{(-)}}{n^\kappa} \right) = \limsup_{n\to\infty} \left(\frac{\hat{S}_n}{n^\kappa} \right) = +\infty \text{ a.s.}$$

Then (4.6) together with $\tau_n \sim c_1 n$ a.s. gives

$$\limsup_{n\to\infty} \left(\frac{X_{\tau_n}}{\tau_n^\kappa} \right) = \frac{1}{c_1^\kappa} \limsup_{n\to\infty} \left(\frac{S_n^{(-)}}{n^\kappa} \right) = +\infty \text{ a.s.,}$$

so (4.4) holds for this case.

(d) Keep $0 \le \kappa \le \frac{1}{2}$. Then (4.4) implies (4.4) for $\kappa = 0$, so $J_+^\Pi = \infty$ or $0 \le EX_1 \le E|X_1| < \infty$ by (2.5). If $J_+^\Pi = \infty$ then (4.4) holds for $\kappa = 1$ hence for $0 \le \kappa \le 1/2$ by (2.6). So suppose $0 \le EX_1 \le E|X_1| < \infty$. If $EX_1 > 0$ we have $\lim_{t\to\infty}(X_t/t) = EX_1 > 0$ by the strong law, so (4.4) holds for $0 \le \kappa \le 1/2$. Finally, suppose $EX_1 = 0$. Now $X_n = \sum_1^n \Delta X_i$, where $\Delta X_i = X_i - X_{i-1}$ are i.i.d. as X_1, thus $0 \le E\Delta X_1 \le E|\Delta X_1| < \infty$, and so $\limsup_{n\to\infty}(X_n/\sqrt{n}) = +\infty$ a.s. by [9], Eq. (1.9). But this implies (4.4) for $0 \le \kappa \le 1/2$.

(e) Keep $\frac{1}{2} < \kappa < 1$, $E|X_1| < \infty$, $EX_1 \ne 0$. Clearly $EX_1 > 0$ implies (4.4) by the strong law, while (4.4) implies $J_+^\Pi = \infty$ or $0 \le EX_1 \le E|X_1| < \infty$ by (2.5). In the present case, since $E|X_1| < \infty$, $EX_1 \ne 0$, we must have $EX_1 > 0$.

(f) Keep $\frac{1}{2} < \kappa < 1$, $E|X_1| < \infty$, and $EX_1 = 0$. Suppose $E(X_1^+)^{\frac{1}{\kappa}} = \infty$. Then from Corollary 1.1, we get $\limsup_{n\to\infty}(X_n/n^\kappa) = +\infty$ a.s., hence $\limsup_{t\to\infty}(X_t/t^\kappa) = +\infty$ a.s.

Alternatively, suppose $E(X_1^+)^{\frac{1}{\kappa}} < \infty = E(X_1^-)^{\frac{1}{\kappa}}$ and (2.14) holds. From the definition of $W^\Pi(y)$ in (2.7) we see that

$$W^\Pi(y) = \int_0^y \int_{-\infty}^{-x} |z| P(J_1 + \tilde{\mu} \in dz) dx \tag{4.7}$$

coincides with the W of Section 1 evaluated for $J_1 + \tilde{\mu}$. Recall from (4.2) that $S_n^* = \sum_1^n (J_i + \tilde{\mu})$, where $\tilde{\mu} = \gamma/c_1$. Now $E|X_1| < \infty$ gives $E|J_1 + \tilde{\mu}| < \infty$, while, from (4.3), $EX_1 = 0$ implies $ES_n^* = 0$ hence $E(J_1 + \tilde{\mu}) = 0$. Also, $E(X_1^+)^{\frac{1}{\kappa}} < \infty$ implies $E((J_1 + \tilde{\mu})^+)^{\frac{1}{\kappa}} < \infty$, and, similarly, $E(X_1^-)^{\frac{1}{\kappa}} = \infty$ implies $E((J_1 + \tilde{\mu})^-)^{\frac{1}{\kappa}} = \infty$. Thus, with (4.7), we can apply Corollary 1.2 to get

$$\limsup_{n\to\infty} \left(\frac{\sum_{i=1}^n (J_i + \tilde{\mu})}{n^\kappa} \right) = \limsup_{n\to\infty} \left(\frac{S_n^*}{n^\kappa} \right) = +\infty \text{ a.s.}$$

So $\limsup_{n\to\infty}(S_n^{(-)}/n^\kappa) = +\infty$ a.s. and (4.4) holds, by (4.6).

Conversely, assume (4.4). Then (4.3) and the law of the iterated logarithm give $\limsup_{n\to\infty}(S_n^*/n^\kappa) = +\infty$ a.s. As above, we have $E(J_1 + \tilde{\mu}) = 0$, so we can apply Corollary 2 to see that $E((J_1 + \tilde{\mu})^+)^{\frac{1}{\kappa}} = \infty$, hence $E(J_1^+)^{\frac{1}{\kappa}} = \infty$ and $E(X_1^+)^{\frac{1}{\kappa}} = \infty$, or else $E(J_1^+)^{\frac{1}{\kappa}} < \infty = E(J_1^-)^{\frac{1}{\kappa}}$, thus $E(X_1^+)^{\frac{1}{\kappa}} < \infty = E(X_1^-)^{\frac{1}{\kappa}}$ and (by virtue of (4.7)), (2.14) holds. □

5. Appendix

We want to establish some inequalities involving $\phi_Y(\lambda) = E(e^{-\lambda Y})$, where Y is an arbitrary non-negative random variable, but we start with a more general set-up.

Lemma 5.1. *Suppose that for* $\lambda > 0$

$$f(\lambda) = \lambda \int_0^\infty e^{-\lambda y} U(y) dy = \int_0^\infty e^{-y} U(y/\lambda) dy, \tag{5.1}$$

where U is non-negative, non-decreasing, and such that there is a positive constant c with

$$U(2x) \le cU(x) \textit{ for all } x > 0. \tag{5.2}$$

Then

$$U(x) \asymp f(1/x), \tag{5.3}$$

where $\asymp$ means that the ratio of the two sides is bounded above and below by positive constants for all $x > 0$.

Proof. (This proof is similar to that of Proposition 1, p. 74 of Bertoin [1]. Since (5.2) implies that U is O-regularly varying, it is also related to a Tauberian theorem for functions of dominated variation, due to de Haan and Stadtmūller, [8].) It is immediate from (5.1) that for any $k > 0$, $\lambda > 0$,

$$U(k/\lambda) = e^k U(k/\lambda) \int_k^\infty e^{-y} dy \le e^k \int_k^\infty e^{-y} U(y/\lambda) dy \le e^k f(\lambda), \quad (5.4)$$

and with $k = 1$ this is one of the required bounds. Next, condition (5.2) gives

$$f(\lambda/2) = \int_0^\infty e^{-y} U(2y/\lambda) dy \le c \int_0^\infty e^{-y} U(y/\lambda) dy = c f(\lambda).$$

Using this and rewriting (5.4) as

$$U(y/\lambda) = U((y/2)/(\lambda/2)) \le e^{y/2} f(\lambda/2)$$

gives, for any $x > 0$,

$$\begin{aligned} f(\lambda) &\le U(x/\lambda) \int_0^x e^{-y} dy + f(\lambda/2) \int_x^\infty e^{y/2} e^{-y} dy \\ &= (1 - e^{-x}) U(x/\lambda) + 2 f(\lambda/2) e^{-x/2} \\ &\le (1 - e^{-x}) U(x/\lambda) + 2 c f(\lambda) e^{-x/2}. \end{aligned}$$

Assuming, with no loss of generality, that $c > 1/4$, and choosing $x = x_0 := 2 \log 4c$ and an integer n_0 with $2^{n_0} \ge x_0$ we deduce, using (5.2) again, that

$$f(\lambda) \le 2(1 - \frac{1}{16c^2}) U(x_0/\lambda) \le 2c^{n_0}(1 - \frac{1}{16c^2}) U(1/\lambda),$$

and this is the other bound. □

Corollary 5.1. *Let $\phi_Y(\lambda) = E(e^{-\lambda Y})$, where Y is an arbitrary non-negative random variable, and for $x > 0$ put $A_Y(x) = \int_0^x P(Y > y) dy$. Then*

$$1 - \phi_Y(x) \asymp x A_Y(1/x). \quad (5.5)$$

Proof. Two integrations by parts give

$$f(\lambda) := \frac{1 - \phi_Y(\lambda)}{\lambda} = \int_0^\infty e^{-\lambda y} P(Y > y) dy = \lambda \int_0^\infty e^{-\lambda y} A_Y(y) dy,$$

which is (5.1) with $U(y) = A_Y(y)$, and it is obvious that (5.2) holds for A_Y with $c = 2$. □

Corollary 5.2. *Let $\phi_Y(\lambda) = E(e^{-\lambda Y})$, where Y is an arbitrary non-negative random variable with finite mean $\mu > 0$, and for $x > 0$ put*

$$V_Y(x) = \int_0^x \int_y^\infty z P(Y \in dz) dy.$$

Then

$$\mu + \phi'_Y(x) \asymp x V_Y(1/x), \quad (5.6)$$

and consequently

$$\phi_Y(x) - 1 + \mu x \asymp x^2 V_Y(1/x). \quad (5.7)$$

Proof. Note that $V_Y(x) = \mu A_Z(x)$, where the distribution of Z is given by

$$P(Z \in dy) = \frac{yP(Y \in dy)}{\mu}.$$

Since $\phi_Z(\lambda) := E(e^{-\lambda Z}) = -\phi'_Y(\lambda)/\mu$, (5.6) follows from (5.5), and then (5.7) follows by integration. □

We can now prove Lemma 3.1:

Proof. Just note that with $\phi_Y(\lambda) = E(e^{-\lambda Y})$ we have

$$\begin{aligned}\Phi_X(\lambda) - 1 &= p(e^{\lambda} - 1) + q(\phi_Y(\lambda) - 1)\\ &= p(e^{\lambda} - 1 - \lambda) + q(\phi_Y(\lambda) - 1 + \mu\lambda)\\ &= q(\phi_Y(\lambda) - 1 + \mu\lambda) + O(\lambda^2),\end{aligned}$$

and

$$\Phi'_X(\lambda) = p(e^{\lambda} - 1) + q(\phi'_Y(\lambda) + \mu) = q(\phi'_Y(\lambda) + \mu) + O(\lambda),$$

and use (5.6) and (5.7). □

References

1. Bertoin, J.: Lévy Processes. Cambridge University Press, Cambridge, 1996
2. Chow, Y.S., Teicher, H.: Probability Theory. Independence, Interchangeability, Martingales, 2nd edition, Springer-Verlag, New York-Berlin, 1988
3. Chow, Y.S., Zhang, C.-H.: A note on Feller's strong law of large numbers. Ann. Probab., **14**, 1088–1094 (1986)
4. Doney, R.A.: Stochastic bounds for Lévy processes. Ann. Probab. **32**, 1545–1552 (2004)
5. Doney, R.A., Maller, R.A.: Stability and attraction to Normality for Lévy processes at zero and infinity. J. Theoretical Probab. **15**, 751–792 (2002)
6. Doney, R.A., Maller, R.A.: Moments of passage times for Lévy processes. Ann. Inst. Henri Poincaré. **40**, 279–297 (2004)
7. Erickson, K.B.: The strong law of large numbers when the mean is undefined. Trans. Am. Math Soc. **185**, 371–381 (1973)
8. de Haan, L., Stadtmūller, U.: Dominated variation and related concepts and Tauberian theorems for Laplace transforms. J. Math. Anal. Appl. **108**, 344–365 (1985)
9. Kesten, H., Maller, R.A.: Random walks crossing power law boundaries. Studia Sci. Math. Hung. **34**, 219–252 (1998)
10. Sato, K-I.: Lévy Processes and Infinitely Divisible Distributions. Cambridge University Press, Cambridge, 1999
11. Sato, K-I.: Basic results on Lévy processes. In: Lévy Processes, Theory and Applications. O.E. Barndorff-Nielsen, T. Mikosch, S. Resnick (eds.), Birkhäuser, Boston, 2001
12. Spitzer, F.: Principles of Random Walk. 2nd edition, Springer-Verlag, New York, 1976
13. Zhang, C-H.: The lower limit of a normalized random walk. Ann. Probab. **14**, 560–581 (1986)