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## A Porous-Medium Model for Vegetation in Bushfires

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### Abstract

The vegetation that burns in bushfires consists of water, pyrolysable material that can ultimately burn as a gas, solid combustible material (char) that is capable only of surface burning and unburnable material (ash). The vegetation is also a porous medium in which air, water vapour, pyrolysis products and combustion products flow past solid material. This flow is strongly influenced by and interacts with the flow of air above the layer of vegetation. As unburnt vegetation, the material is all contained in objects of different size and shape, such as leaves, stalks, branches and trunks, all exhibiting differing rates at which water and pyrolysis products can be generated through external heating, making some components much more susceptible to burning in a brief period while others would require longer time-scales; such material can often survive a fire. This article presents a sectional model for the vegetation in which different sizes of material are represented separately.

### 1 Introduction

Landscape fire models have been studied for many years and there is a range of formulations, as recently reviewed by Pastor et al [3]. Already in the 1980's there was interest in creating a comprehensive model that considered the combustible vegetation to be represented as a porous medium and that sought to write appropriate conservation equations for all of the key variables (e.g. Grishin et al [1]). A recent contribution to this type of landscape fire model is the multiphase approach of Larini et al [2] which numerically solves a set of equations to obtain propagating combustion fronts. In the present paper we revisit the formulation of models for fire spread in the landscape as a fundamental problem in porous medium combustion and describe a comprehensive sectional model for all of the key processes. We are able to account for all of the components of the vegetation as well as gaseous and surface chemistry leading to insights into the important relationships between the most significant dimensionless parameters in a way that Weber [4] foreshadowed, but that has not previously been possible.

### 2 Porous medium model for vegetation

A model for combustible vegetation needs to take into account solid and vapourisable material contained within the vegetation, inert vapour (water) and combustible vapour (oils and products of pyrolysis reactions in heated vegetation), the air and any products of the combustion. Part of the medium is solid and part of it is gaseous with transfer of matter from the solid to the gas phase. There may be some liquid present, such as water and sap, but because these would only remain liquid within the structure of the vegetation they can be considered to form part

of a lumped solid phase (the vegetation). Of course, while the solid material remains rigidly fixed, the gases can flow.

Heat and mass transfer in the gas operates through conduction, diffusion and advection. At high enough temperatures, hot premixed or non-premixed gaseous flames can arise through an exothermic reaction between oxygen and combustible vapour. Exchange of heat must take place between gaseous and solid material while radiation and absorption would account for transfer of heat from one part of the solid to another limited by the optical thickness of the vegetation.

To begin with, we can consider a more or less uniform distribution of vegetation of a similar thickness, such as grass or branches of similar diameter, within a moderately small test volume. In heating these branches by introducing hot air, or through radiation, a non-uniform temperature distribution must arise within each branch, as heat conducts inwards from the surfaces. As the temperature at any point in the branch rises above the boiling point of water  $T_{\bar{W}} \approx 373\text{ K}$  ( $100^\circ\text{C}$ ) or a temperature  $T_{\bar{F}} \approx 600\text{ K}$  ( $327^\circ\text{C}$ ) around which pyrolysis reactions occur, water or combustible vapour are driven out; the pyrolysis reaction can be modelled as a phase-change [2]. The mean temperature of the branch would also tend to increase with time.

The objective is to develop a workable model for describing the overall behaviour of vegetation in bushfires, representing reasonably well only the most essential phenomena. These include correctly describing the energetics of heating and vapourisation as well as a reasonable representation of the rates of these processes. Accounting for the temperature distribution within every branch would generate far too complex a model, even though this might be the most comprehensive approach.

#### 2.1 Energy and temperature of solid and gas

Let us suppose therefore that the material of the vegetation has density  $\bar{\rho}_{\bar{V}}$ . If the volume fraction of the branches is  $\phi$  then the overall density of the branches in the test volume is  $\rho_{\bar{V}} = \phi\bar{\rho}_{\bar{V}}$ ; a bar ( $\bar{\quad}$ ) over a symbol signifies measurement in the solid phase only; a tilde ( $\tilde{\quad}$ ) will signify the gas phase only; otherwise measurement is made per unit volume of the overall medium. For brevity, we shall write  $\zeta$  in place of  $\rho_{\bar{V}}$ .

The overall solid density  $\zeta$  is made up of partial densities  $\rho_{\bar{W}}$ ,  $\rho_{\bar{F}}$ ,  $\rho_{\bar{G}}$  and  $\rho_{\bar{A}}$  of water  $\bar{W}$ , pyrolysable combustible material  $\bar{F}$ , burnable char  $\bar{G}$  that remains solid after heating and unburnable solid  $\bar{A}$  (ash) having  $\rho_{\bar{W}} + \rho_{\bar{F}} + \rho_{\bar{G}} + \rho_{\bar{A}} = \zeta$ . Let us also define the mean temperature of this solid 'mixture' to be  $\theta$  such that the specific enthalpy (energy per unit mass)  $\bar{H}$  of the solid satisfies

$$\begin{aligned} \zeta \bar{H} &= \rho_{\bar{W}} \bar{h}_{\bar{W}} + \rho_{\bar{F}} \bar{h}_{\bar{F}} + \rho_{\bar{G}} \bar{h}_{\bar{G}} + \rho_{\bar{A}} \bar{h}_{\bar{A}} \\ &= \zeta \bar{C} \theta + \rho_{\bar{W}} \bar{h}_{0\bar{W}} + \rho_{\bar{F}} \bar{h}_{0\bar{F}} + \rho_{\bar{G}} \bar{h}_{0\bar{G}} + \rho_{\bar{A}} \bar{h}_{0\bar{A}} \end{aligned} \quad (1)$$

where any solid constituent  $\bar{\alpha} \in \mathbb{S} = \{\bar{W}, \bar{F}, \bar{G}, \bar{A}\}$  has specific enthalpy

$$\bar{h}_{\bar{\alpha}} = \bar{h}_{\bar{\alpha}}(\theta) = \bar{C}_{\bar{\alpha}}\theta + \bar{h}_{0\bar{\alpha}} \quad (2)$$

with specific heat  $\bar{C}_{\bar{\alpha}}$  and energy of formation  $\bar{h}_{0\bar{\alpha}}$ . The parameter  $\bar{C} = (\rho_{\bar{W}}\bar{C}_{\bar{W}} + \rho_{\bar{F}}\bar{C}_{\bar{F}} + \rho_{\bar{G}}\bar{C}_{\bar{G}} + \rho_{\bar{A}}\bar{C}_{\bar{A}})/\zeta$  represents the mean specific heat of the solid material.

We take the average temperature of the gas-phase within the test volume to be  $T$ . If, for simplicity, the gas-phase consists of water vapour W, pyrolysed combustible vapour F, oxygen X, combustion products P and an inert gas N, with overall density defined as  $\rho = \rho_W + \rho_F + \rho_X + \rho_P + \rho_N$ , then the specific enthalpy  $H$  in the gas phase can be taken to satisfy

$$\begin{aligned} \rho H &= \rho_W h_W + \rho_F h_F + \rho_X h_X + \rho_P h_P + \rho_N h_N \\ &= \rho C_p T + \rho_W h_{0W} + \rho_F h_{0F} + \rho_X h_{0X} + \rho_P h_{0P} + \rho_N h_{0N} \end{aligned} \quad (3)$$

where any gas constituent  $\alpha \in \mathbb{G} = \{W, F, X, P, N\}$  has specific enthalpy

$$h_{\alpha} = h_{\alpha}(T) = C_{p\alpha}T + h_{0\alpha} \quad (4)$$

with specific heat at constant pressure  $C_{p\alpha}$  and energy of formation  $h_{0\alpha}$ . The mean specific heat at constant pressure of the gas is  $C_p = (\rho_W C_{pW} + \rho_F C_{pF} + \rho_X C_{pX} + \rho_P C_{pP} + \rho_N C_{pN})/\rho$ .

Because we shall be considering the phase-changes and chemical reactions  $\bar{W} \rightarrow W$ ,  $\bar{F} \rightarrow F$ ,  $F+X \rightarrow 2P$  and  $\bar{G}+X \rightarrow P$ , ground-states may be set independently for W, X, N,  $\bar{G}$  and  $\bar{A}$ , other energies of formation being determined by the chemical and phase changes. It is convenient to select the values

$$h_{0W} = h_{0X} = h_{0N} = \bar{h}_{0\bar{G}} = \bar{h}_{0\bar{A}} = 0. \quad (5)$$

Converting unit mass of  $\bar{\alpha} \in \{\bar{W}, \bar{F}\}$  into the gas constituent  $\alpha$  at the vapourisation temperature  $T_{\bar{\alpha}}$  absorbs energy equal to the latent heat of vapourisation  $L_{\bar{\alpha}}$ . This requires that

$$\bar{h}_{0\bar{\alpha}} = h_{0\alpha} + (C_{p\alpha} - \bar{C}_{\bar{\alpha}})T_{\bar{\alpha}} - L_{\bar{\alpha}}. \quad (6)$$

As a result, we can write

$$\begin{aligned} \zeta \bar{H} &= \rho_{\bar{W}}(\bar{C}_{\bar{W}}(\theta - T_{\bar{W}}) + C_{pW}T_{\bar{W}} - L_{\bar{W}}) + \rho_{\bar{G}}\bar{C}_{\bar{G}}\theta \\ &+ \rho_{\bar{F}}(\bar{C}_{\bar{F}}(\theta - T_{\bar{F}}) + C_{pF}T_{\bar{F}} + h_{0F} - L_{\bar{F}}) + \rho_{\bar{A}}\bar{C}_{\bar{A}}\theta \end{aligned} \quad (7)$$

and

$$\begin{aligned} \rho H &= \rho_W C_{pW}T + \rho_N C_{pN}T + \rho_X C_{pX}T \\ &+ \rho_F(C_{pF}T + h_{0F}) + \rho_P(C_{pP}T + h_{0P}). \end{aligned} \quad (8)$$

Finally, if the solid volume fraction  $\phi$  is very small, then it can be noted that  $\rho_{\alpha}$  is very nearly equal to the concentration of  $\alpha$  in the gas phase alone, more strictly given by  $\tilde{\rho}_{\alpha} = \rho_{\alpha}/(1 - \phi)$ .

## 2.2 Vapourisation

A simple and reasonably representative model for the rate of evaporation of  $\bar{\alpha} \in \{\bar{W}, \bar{F}\}$ , as mass per unit volume and time, is

$$\bar{\omega}_{\bar{\alpha}} = \mu A_{\bar{\alpha}} \rho_{\bar{\alpha}}^{\nu} \frac{1}{2} (\theta - T_{\bar{\alpha}} + |\theta - T_{\bar{\alpha}}|) \quad (9)$$

where  $A_{\bar{\alpha}}$  is a constant and  $\mu$  is a rate constant that would vary with the size or thickness  $d$  of the branches, probably in proportion to  $d^{-2}$  since the time-scale for conductive heating within a branch varies in proportion to  $d^2$ . Likely values of the exponent  $\nu$  are 3/4 for cylindrical branches and 1/2 for predominantly planar or flat vegetation, such as leaves. Spherical vegetation would be likely to have  $\nu \approx 5/6$ . Each of these alternatives would halt evaporation abruptly (in finite time) when  $\rho_{\bar{\alpha}}$  reaches zero. Replacing  $\rho_{\bar{\alpha}}^{\nu}$  with a Heaviside function  $H(\rho_{\bar{\alpha}})$  would generate qualitatively similar overall results.

Because vapourisation generates the gas constituent  $\alpha$  at the temperature  $\theta$ , there is also a transfer of energy equal to  $h_{\alpha}(\theta)$  from solid to gas (per unit mass of vapourised material).

## 2.3 Chemical reactions

For simplicity, the single gas-phase reaction  $F + X \rightarrow 2P$  can be considered to oxidise the pyrolysed gas F and create reaction products P, with the rate

$$\omega = A \tilde{\rho}_F \tilde{\rho}_X \exp(-T_A/T) \quad (10)$$

where  $T_A$  is an activation temperature. The reaction can be considered to remove oxygen and pyrolysed gas at the respective rates  $\omega$  and  $\chi\omega$ , as mass per unit volume and time.

A solid-surface reaction can also be considered between the residual char and oxygen (producing gaseous products) in the form  $\bar{G} + X \rightarrow P$ . If this has a suitable rate-expression, such as

$$\bar{\omega} = \bar{\omega}(\rho_{\bar{G}}, \tilde{\rho}_X, \theta) \quad (11)$$

then oxygen and residual solids are removed at the rates  $\bar{\omega}$  and  $\bar{\chi}\bar{\omega}$ , respectively. For brevity, the details of this surface reaction will not be discussed further in this short article, apart from noting that it involves a transfer of energy between gas and solid, since gaseous oxidant and product carry their energies to and from the solid surface at the temperature  $\theta$ . This results in an overall energy transfer from gas to solid of

$$\begin{aligned} H_{\bar{G}}(\theta) &= h_X(\theta) - (1 + \bar{\chi})h_P(\theta) \\ &= C_{pX}\theta - (1 + \bar{\chi})(C_{pP}\theta + h_{0P}) \end{aligned} \quad (12)$$

per unit mass of reacted oxidant.

## 2.4 Exchange of heat between gas and solid

Heat transfer between the gas and the solid are very important. Of course the transfer should flow from the higher to the lower temperature and, if one assumes that the average solid temperature  $\theta$  is at least representative of the surface temperature of the solid, a suitable model assumes that the rate of heat transfer is proportional to  $T - \theta$ . The transfer-rate can also be enhanced by the flow-velocity  $\mathbf{u}$  of the gas-phase past the solid via the Peclet number  $|\mathbf{u}|d/\kappa$ , where  $\kappa$  is the gaseous thermal diffusivity.

Since the mean separation between branches and their mean surface area per unit volume should be proportional to  $d/\phi^{1/2}$  and  $\phi/d$ , respectively, for cylindrical vegetation, the heat-transfer rate from gas to solid can be estimated as

$$\tau = \rho C_p (\kappa g + h|\mathbf{u}|)(T - \theta) \quad (13)$$

per unit volume and unit time, where  $g \approx \phi/(d^2 \ln \phi^{-1/2})$  and  $h \approx \phi/d$  up to order one factors, at least for cylindrical vegetation. Thus convection-enhanced heat-transfer between gas and solid should dominate at large Peclet number. Different shapes and thicknesses  $d$  of the vegetation should lead to the same formula for heat transfer with different values of  $g$  and  $h$ .

As vapourisation and combustion take place, the mean diameter of the branches  $d$  is likely to decrease, with the mean separation remaining fixed. For cylindrical branches,  $d$  would probably vary in proportion to  $\rho_{\bar{G}}^{1/2}$  since the char  $\bar{G}$  is eroded by surface chemistry. The volume fraction  $\phi$  should then vary in proportion to  $d^2$  so that  $h$  would decrease approximately linearly with  $d$  while  $g$  varies (if at all) in proportion to  $1/|\ln d|$ .

## 2.5 Sectional characterisation of vegetation

If the structure of the vegetation includes a distribution of thicknesses  $d$  and shapes (such as cylindrical or flat) the solid material can be characterised by different 'sections' each representing a part of the vegetation that has more or less the same size

and shape, within a well-defined range if necessary. Each section can be denoted by a subscript (say)  $i \in \mathbb{V} = \{1, 2, \dots, N\}$  with corresponding definitions of

$$\phi_i, \rho_{\bar{W}i}, \rho_{\bar{F}i}, \rho_{\bar{G}i}, \rho_{\bar{A}i}, \zeta_i, \bar{H}_i, \theta_i, \mu_i, \nu_i, g_i \text{ and } h_i$$

giving corresponding formulae for the vapourisation rate  $\bar{\omega}_{\alpha i}$ , heat-transfer rate  $\tau_i$  and the surface-reaction rate  $\bar{\omega}_i$  for each component  $i \in \mathbb{V}$ . The overall volume fraction of all solid constituents is then  $\phi = \sum_{i \in \mathbb{V}} \phi_i$ .

## 2.6 Radiative transfer of energy

Radiation is emitted by each 'section' in proportion to its surface area and the fourth power of its temperature. If it is emitted isotropically, the emitted radiant intensity can be written as

$$I_e = \sum_{i \in \mathbb{V}} \sigma_i \theta_i^4 \quad (14)$$

where  $\sigma_i$  represents the emissivity multiplied by the surface area per unit volume of the section  $i$ . The actual intensity of radiation  $I$  might not be equal to  $I_e$  if the medium is not uniform, as in (18) below, and radiant energy can be taken to be reabsorbed by each section in proportion to both  $I$  and  $\sigma_i$ . Thus, in any one section, the nett rate of energy gain through radiation is

$$r_i = \sigma_i (I / \sigma - \theta_i^4) \quad \text{with} \quad \sigma = \sum_{i \in \mathbb{V}} \sigma_i. \quad (15)$$

If  $I = I_e$  then the values of  $r_i$  sum to zero.

The dependence  $\sigma_i(\phi_i, \rho_{\bar{W}i}, \rho_{\bar{F}i}, \rho_{\bar{G}i}, \rho_{\bar{A}i})$  of the radiant-surface on the properties of the solid section may be relatively complicated, but in most cases one would expect it to become zero as the remnants of the solid are burned away.

## 2.7 Spatial variation and conservation laws

If we now treat the vegetation as a porous continuum, with variations in space and time, conservation laws can be developed for the various constituents once fluxes are identified. Solid components are fixed and have no flux while gaseous constituents can be taken to have the fluxes

$$\rho_\alpha \mathbf{u}_\alpha = \rho_\alpha \mathbf{u} - D_\alpha \nabla \rho_\alpha \quad (16)$$

for the species  $\alpha$ . The flux of enthalpy in the gas-phase can be written as

$$\mathbf{q} = \sum_{\alpha \in \mathbb{G}} \rho_\alpha h_\alpha \mathbf{u}_\alpha - \rho C_p \kappa \nabla T. \quad (17)$$

Radiative fluxes arise where the emitted radiant intensity  $I_e$  varies with position. If  $\ell$  is the optical path-length, then absorption reduces the radiation arriving from any point, at a distance  $r$ , in proportion to  $\exp(-r/\ell)$ . If  $I_e$  varies quadratically with distance, having  $\nabla^2 I_e \neq 0$ , and if  $\ell$  is small enough then the radiant intensity at any point  $\mathbf{r}$  can be estimated as

$$I = \psi I_e + \psi \ell^2 \nabla^2 I_e \quad (18)$$

with  $\psi = \frac{1}{4\pi\ell} \iiint_{\Omega} \frac{1}{|\mathbf{x} - \mathbf{r}|^2} \exp(-|\mathbf{x} - \mathbf{r}|/\ell) dV(\mathbf{x})$

in which the volume integral ranges over the porous medium  $\Omega$ . In fact  $\psi$  only becomes non-unity if radiation is able to escape through any boundary. It is found that  $\psi \approx 1 - \frac{1}{2} \exp(-b/\ell)$  if just one planar boundary lies at a distance  $b$  from the position  $\mathbf{r} \in \Omega$ . Although it remains qualitatively correct, the diffusion-like term  $\psi \ell^2 \nabla^2 I_e$  loses accuracy when  $\psi$  differs significantly from unity and, in such cases, radiative heat-transfer within the medium becomes less effective while radiative losses increase.

Conservation laws can now be written, for any  $i \in \mathbb{V}$ , as

$$\begin{aligned} \partial_t \zeta_i \bar{H}_i &= H_{\bar{G}}(\theta) \bar{\omega}_i - h_{\bar{W}}(\theta) \bar{\omega}_{\bar{W}i} - h_{\bar{F}}(\theta) \bar{\omega}_{\bar{F}i} + \tau_i + r_i \\ \partial_t \rho_{\bar{W}i} &= -\bar{\omega}_{\bar{W}i} \\ \partial_t \rho_{\bar{F}i} &= -\bar{\omega}_{\bar{F}i} \\ \partial_t \rho_{\bar{G}i} &= -\bar{\chi} \bar{\omega}_i \end{aligned} \quad (19)$$

for components in the solid phase, and

$$\begin{aligned} \partial_t \rho H + \nabla \cdot \mathbf{q} &= \sum_{i \in \mathbb{V}} (r_i - \partial_t \zeta_i \bar{H}_i) \\ \partial_t \rho + \nabla \cdot \rho \mathbf{u} &= -\sum_{i \in \mathbb{V}} \partial_t (\rho_{\bar{W}i} + \rho_{\bar{F}i} + \rho_{\bar{G}i}) \\ \partial_t \rho_X + \nabla \cdot \rho_X \mathbf{u}_X &= -\omega - \sum_{i \in \mathbb{V}} \bar{\omega}_i \\ \partial_t \rho_F + \nabla \cdot \rho_F \mathbf{u}_F &= -\chi \omega - \sum_{i \in \mathbb{V}} \partial_t \rho_{\bar{F}i} \\ \partial_t \rho_W + \nabla \cdot \rho_W \mathbf{u}_W &= -\sum_{i \in \mathbb{V}} \partial_t \rho_{\bar{W}i} \\ \partial_t \rho_P + \nabla \cdot \rho_P \mathbf{u}_P &= (1 + \chi) \omega + \sum_{i \in \mathbb{V}} (1 + \bar{\chi}) \bar{\omega}_i \end{aligned} \quad (20)$$

for gas-phase components. These equations need to be supplemented by a gas equation of state

$$P = (C_p - C_v) \bar{\rho} T \quad (21)$$

where  $C_v$  is the mean gaseous specific heat at constant volume and by a suitable momentum conservation equation.

## 2.8 Further considerations

There are several ways in which particular elements of the model can be improved. These may include alternative formulae for vapourisation or pyrolysis rates and heat transfer, based on more detailed modelling of the vegetation within any section. For example, gas flow should enhance heat transfer and the supply of oxidant for surface reaction primarily on the upwind side, leading to nonuniform surface temperatures and a non-isotropic radiation field. The optical path length  $\ell$  is not necessarily isotropic and radiative absorption is more generally expressed as a volume integral that can include background radiation from outside the domain  $\Omega$  of the vegetation. Radiation associated with gaseous flames (especially if soot is produced) can also be very important in transferring heat to the vegetation. Nevertheless the model, as presented, is capable of representing many aspects of combustible vegetation.

## 3 Properties of the model

In general, the model could be applied within the broad context of a stratified layer of vegetation of finite height above ground, with atmospheric-scale modelling of the air-flow and hot combustion products above it. This article is restricted to the presentation of the model itself and to a discussion of its salient features. Solutions will be discussed in the verbal presentation.

### 3.1 Constant mean specific heats

To identify the essential nature of the model we can, firstly, assume that all solid specific heats are equal to the mean value  $\bar{C}$  and all gaseous specific heats equal  $C_p$ . The mean specific heats then do not change as the composition of the gas or the vegetation changes. After some algebraic manipulation, the solid energy equation can then be rewritten in the form

$$\begin{aligned} \zeta \bar{C} \partial_t \theta &= \tau + r + (\bar{Q} - (C_p - \bar{C}) \theta) \bar{\chi} \bar{\omega} \\ &\quad - (L_{\bar{W}} + (C_p - \bar{C})(\theta - T_{\bar{W}})) \bar{\omega}_{\bar{W}} \\ &\quad - (L_{\bar{F}} + (C_p - \bar{C})(\theta - T_{\bar{F}})) \bar{\omega}_{\bar{F}} \end{aligned} \quad (22)$$

for any section of the vegetation (omitting the subscripts  $i \in \mathbb{V}$ ), in which the heat of surface reaction is  $\bar{Q} = -(1 + \bar{\chi}) h_{0P} / \bar{\chi}$ .

Assuming that there is only one section for the vegetation (consisting of similarly sized and shaped components), conservation of gaseous energy takes the form

$$\rho C_p (\partial_t T + \mathbf{u} \cdot \nabla T - \kappa \nabla^2 T) = Q\omega - \tau - C_p (\theta - T) \partial_t (\rho_{\bar{W}} + \rho_{\bar{F}} + \rho_{\bar{G}}) \quad (23)$$

in which the thermal conductivity  $\rho C_p \kappa$  is taken to be constant and the heat of gaseous reaction is  $Q = \chi h_{0F} - (1 + \chi) h_{0P}$ ,

### 3.2 Dimensionless version with one section of vegetation

Still taking the specific heats to all equal  $\bar{C}$  in the solid and  $C_p$  in the gas, with only one section for the vegetation, it is useful to examine a dimensionless form for the model. This reveals some key parameters and dimensionless ratios which influence the nature of any solution. There are many ways to nondimensionalise and so only one representative version is given here.

A time-scale for heat transfer between solid and gas can be identified as

$$t_\tau = 1 / (\kappa g_c + h_c u_c) \quad (24)$$

where  $g_c$  and  $h_c$  are characteristic values of  $g$  and  $h$  and  $u_c$  is a suitable measure of the wind speed. A temperature  $T_c$  can also be identified at which stoichiometric gaseous chemistry occurs on the same time-scale  $t_\tau$  for heat transfer, such that

$$A\chi\rho_{Xc} \exp(-T_A/T_\tau) = \kappa g_c + h_c u_c = 1/t_\tau \quad (25)$$

where  $\rho_{Xc}$  is a characteristic value of the oxygen concentration. At higher temperatures the rate of gaseous chemical activity would outstrip heat transfer unless one of the reactants is in short supply. Two other characteristic temperature-scales can be identified from the solid and gas energy equations (22) and (23), namely

$$\theta_c = \frac{\rho_{\bar{G}c} \bar{Q}}{\rho_{\bar{G}c} C_p + \rho_{\bar{A}c} \bar{C}} \approx \frac{\bar{Q}}{C_p} \quad \text{and} \quad T_c = \frac{\rho_{Xc} Q}{\rho_c C_p} \quad (26)$$

where  $\rho_{\bar{A}c}$  is the density of ash (unburnable solid) and  $\rho_c$  and  $\rho_{\bar{G}c}$  characterise the density of gas and the overall density of char. If there is little ash, the final estimate for  $\theta_c$  holds. Thus  $T_c$  estimates the temperature rise through gaseous oxidation and  $\theta_c$ , the potential temperature rise through surface oxidation of char.

Dimensionless versions of the variables in the model and some dimensionless ratios can now be defined as follows

$$\begin{aligned} t &= t_\tau t' & \mathbf{x} &= t_\tau u_c \mathbf{x}' & \mathbf{u} &= u_c \mathbf{u}' & \kappa &= u_c^2 t_\tau \kappa' \\ \zeta &= \zeta_c \zeta' & \rho_{\bar{W}} &= \zeta_c \bar{W} & \rho_{\bar{F}} &= \zeta_c \bar{F} & \rho_{\bar{G}} &= \zeta_c \bar{G} \\ \rho &= \rho_c \rho' & \rho_X &= \rho_{Xc} X & \rho_F &= \chi \rho_{Xc} F & T &= T_\tau T' \\ \bar{Q} &= \bar{C} T_\tau \bar{Q}' & L_{\bar{\alpha}} &= \bar{C} T_\tau L'_{\bar{\alpha}} & T_{\bar{\alpha}} &= T_\tau T'_{\bar{\alpha}} & T_A &= T_\tau T'_A \\ \Phi &= \zeta_c / \rho_c & \Phi_{\bar{F}} &= \zeta_c / \chi \rho_{Xc} & \Phi_{\bar{G}} &= \zeta_c / \bar{\chi} \rho_{Xc} & \theta &= T_\tau \theta' \\ g' &= \kappa g t_\tau & h' &= h u_c t_\tau & Le_\alpha &= \kappa / D_\alpha & \gamma &= C_p / \bar{C} \\ \ell &= t_\tau u_c \ell' & \Gamma &= t_\tau \sigma T_\tau^3 / \zeta_c \bar{C} & \delta &= t_\tau \bar{\chi} \bar{\omega} (\zeta_c \bar{G}_0, \rho_{Xc}, T_\tau) \\ \varepsilon_{\bar{\alpha}} &= t_\tau \mu A_{\bar{\alpha}} \zeta_c^{V-1} & \rho_{Xc} Q &= \rho_c C_p Q' \end{aligned}$$

in which  $\zeta_c$  characterises the solid density and  $\bar{G}_0$  is the unburnt value of  $\bar{G}$ . Dropping the primes leads to the dimensionless equations

$$\begin{aligned} \rho (\partial_t T + \mathbf{u} \cdot \nabla T - \kappa \nabla^2 T) &= Q\omega - \tau \\ &\quad - (\theta - T) \Phi \partial_t (\bar{W} + \bar{F} + \bar{G}) \\ \partial_t \rho + \nabla \cdot \rho \mathbf{u} &= -\Phi \partial_t (\bar{W} + \bar{F} + \bar{G}) \\ \partial_t F + \nabla \cdot F \mathbf{u} - \kappa \nabla^2 F / Le_F &= -\omega - \Phi_{\bar{F}} \partial_t \bar{F} \\ \partial_t X + \nabla \cdot X \mathbf{u} - \kappa \nabla^2 X / Le_X &= -\omega + \Phi_{\bar{G}} \partial_t \bar{G} \end{aligned} \quad (27)$$

for temperature, mass, fuel vapour and oxidant in the gas phase, and

$$\begin{aligned} \zeta \partial_t \theta &= \gamma \tau / \Phi + \Gamma ((\psi - 1) \theta^4 + \psi \ell^2 \nabla^2 \theta^4) \\ &\quad + (L_{\bar{W}} - (1 - \gamma)(\theta - T_{\bar{W}})) \partial_t \bar{W} \\ &\quad + (L_{\bar{F}} - (1 - \gamma)(\theta - T_{\bar{F}})) \partial_t \bar{F} \\ &\quad - (\bar{Q} + (1 - \gamma)\theta) \partial_t \bar{G} \\ -\partial_t \bar{W} &= \varepsilon_{\bar{W}} \rho \bar{W}^V \frac{1}{2} (\theta - T_{\bar{W}} - |\theta - T_{\bar{W}}|) \\ -\partial_t \bar{F} &= \varepsilon_{\bar{F}} \rho \bar{F}^V \frac{1}{2} (\theta - T_{\bar{F}} - |\theta - T_{\bar{F}}|) \\ -\partial_t \bar{G} &= \delta \bar{\omega} \end{aligned} \quad (28)$$

describing the solid phase, along with the rate formulae

$$\begin{aligned} \omega &= FX \exp(T_A - T_A/T) \\ \tau &= \rho (g + h|\mathbf{u}|) (T - \theta) \end{aligned} \quad (29)$$

and a suitable formula for the dimensionless version of  $\bar{\omega}$ .

The constant  $\Phi$  would generally be large, relating the mass of solid to the mass of gas, while  $\Phi_{\bar{F}}$  and  $\Phi_{\bar{G}}$  would be larger still, relating only to the mass of oxidant in any volume. The ratio  $\Phi/\gamma$  measures the heat capacity of solid to that of gas in any volume, typically having  $\gamma \approx 3/4$  for dry wood (for pure water it would be about  $1/4$ ). The constants  $\Gamma$  and  $\ell^2$  offer a measure of the effectiveness of radiant heat-transfer while  $\kappa$  measures the effectiveness of thermal and mass diffusivity in the gas.

## 4 Conclusions

The model provides a basis for examining a variety of interactions in bushfires, including the effect of reduced moisture-content, the relative effects of radiant and convective heat transfer, and the effect of increased wind-speed. By employing a sectional approach, differing rates of heat-exchange, pyrolysis, etc., within the structure of the vegetation can be described.

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## 6 References

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