# *Periodic solutions of discrete Volterra equations*

Baker, Christopher and Song, Yihong

2004

MIMS EPrint: **2007.135**



ISSN 1749-9097

Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE DIRECT®

Mathematics and Computers in Simulation 64 (2004) 521–542

MATHEMATICS AND COMPUTERS IN SIMULATION

www.elsevier.com/locate/matcom

# Periodic solutions of discrete Volterra equations☆

Christopher T.H. Baker[a,*], Yihong Song[b]

[a] Department of Mathematics, The Victoria University of Manchester, Oxford Road, Manchester M13 9PL, UK
[b] c/o Department of Mathematics, Suzhou University, Suzhou, Jiangsu 215006, PR China

Received 19 December 2002; received in revised form 14 October 2003; accepted 14 October 2003

**Abstract**

In this paper, we investigate periodic solutions of linear and nonlinear discrete Volterra equations of convolution or non-convolution type with unbounded memory.

For linear discrete Volterra equations of convolution type, we establish Fredholm's alternative theorem and for equations of non-convolution type, and we prove that a unique periodic solution exists for a particular bounded initial function under appropriate conditions. Further, this unique periodic solution attracts all other solutions with bounded initial function. All solutions of linear discrete Volterra equations with bounded initial functions are asymptotically periodic under certain conditions. A condition for periodic solutions in the nonlinear case is established.


*Keywords:* Periodic; Asymptotically periodic solutions; Discrete Volterra equations; Resolvent matrices; Fredholm's alternative

## 1. Introduction

In the widest use of the term, "Volterra equations" are equations that are *causal* or *non-anticipative*. Discrete Volterra equations (DVEs) can be considered as the discrete analogue of classical Volterra integral equations such as

$$x(t) = f(t) + \int_{-\infty}^{t} B(t, s)x(s)\,\mathrm{d}s, \tag{1.1a}$$

and

$$x(t) = f(t) + \int_{0}^{t} B(t, s)x(s)\,\mathrm{d}s, \tag{1.1b}$$

☆ Work performed at The Victoria University of Manchester, supported by a URS award.
* Corresponding author. Tel.: +44-161-275-5800; fax: +44-161-275-5819.
*E-mail addresses:* cthbaker@ma.man.ac.uk, cthbaker@na-net.ornl.gov (C.T.H. Baker), ysong@ma.man.ac.uk, yihongsong@hotmail.com (Y. Song).

0378-4754/$30.00 © 2003 IMACS. Published by Elsevier B.V. All rights reserved.
doi:10.1016/j.matcom.2003.10.002

or corresponding integro-differential equations [12], and they arise, in particular, when one applies certain numerical methods [1,3] to Volterra integral (or integro-differential) equations. (They also arise directly, from modelling of discrete systems.)

The discrete analogues of (1.1a) could quite properly be described as Volterra *summation* equations.

In a preceding work [2], we established several fixed-point theorems for discrete Volterra operators and related existence results of solutions of discrete Volterra equations. This paper is devoted to the study of *periodic* or *asymptotically periodic* (see below) solutions of linear and nonlinear discrete Volterra "summation" equations with unbounded memory, which may be of convolution or non-convolution type. We first consider the linear system

$$x(n) = \sum_{j=-\infty}^{n} B(n, j)x(j), \quad n \geq 0, \tag{1.2a}$$

and [1] the corresponding inhomogeneous form

$$x(n) = f(n) + \sum_{j=-\infty}^{n} B(n, j)x(j), \quad n \geq 0. \tag{1.2b}$$

In addition, we consider (in Section 4) equations of the form

$$x(n) = f(n) + \sum_{0}^{n} B(n, j)x(j), \quad n \geq 0. \tag{1.2c}$$

We also discuss the corresponding convolution equations, e.g.,

$$x(n) = \sum_{j=-\infty}^{n} K(n-j)x(j), \quad n \geq 0, \tag{1.2d}$$

or

$$x(n) = f(n) + \sum_{j=-\infty}^{n} K(n-j)x(j), \quad n \geq 0, \tag{1.2e}$$

where we shall assume a summability condition $\sum_{n=0}^{\infty} |K(n)| < \infty$.

The above equations will be considered in $d$-dimensional Euclidean space $\mathbb{E}^d$, where we take $\mathbb{E}$ to be consistently either $\mathbb{R}$ or $\mathbb{C}$. We assume that $f(n) \in \mathbb{E}^d$ for $n = 0, 1, 2, \ldots$, that the matrices $B(n, j) \in \mathbb{E}^{d\times d}$, for $n, j \in \{0, 1, 2, \ldots\}$, satisfy $B(n, j) \equiv 0$ for $j > n$, or that $K(n) \in \mathbb{E}^{d\times d}$ for $n \in \{0, 1, 2, \ldots\}$ and we seek a solution $\{x(n)\}_{n\geq 0}$ in the appropriate space, with $x(n) \in \mathbb{E}^d$ for $n = 0, 1, 2, \ldots$, given appropriate $x(n) \in \mathbb{E}^d$ for $n = -1, -2, -3, \ldots$. A solution $\{x(n)\}_{n\geq 0}$ of Eq. (1.2a) or of (1.2b) is said to be a *periodic solution* if there exists $N \in \mathbb{Z}_+$ such that $x(n + N) = x(n)$ for $n \geq 0$. (When $\{x(n) \equiv x(n, \phi)\}_{n\geq 0}$ is a periodic solution, it does not follow that $x(n + N) = x(n)$ for $n < 0$.)

In [10], Elaydi et al. gave an overview of results, that have been obtained in the last two decades, on the existence of periodic solutions of difference equations. That survey covers both ordinary difference

[1] Eq. (1.2b) can be converted to an explicit problem, in simple cases; (1.2b) includes corresponding explicit problems as a special case. Generally, we cannot in practice convert implicit nonlinear discrete Volterra equations (see, e.g., Section 5) to an explicit form.

equations, such as $x(n+1) = A(n)x(n)$, and Volterra difference equations. Although several papers [6,8–10] relate to periodic solutions of systems of explicit difference equations, little work has been done on questions of the periodic solution of implicit Volterra summation equations, to the best of our knowledge.

The structure of this paper is as follows. In Section 2, we give a representation theorem for periodic sequences. In Section 3, we deal with periodic solutions of linear convolution discrete Volterra equations and give Fredholm's alternative for this system. In Section 4, we establish, for non-convolution discrete Volterra equations, the existence of a unique periodic solution of (1.2b). This result is obtained under more general conditions than those given in [6] for Volterra difference equations, and we prove the attractivity of the periodic solution. In fact, we prove that all solutions of (1.2b) with bounded initial functions are asymptotically periodic.

In Section 5, we discuss nonlinear discrete Volterra equations, which include nonlinear "ordinary" implicit difference equations as a special case, and prove the existence of a unique periodic solution under certain conditions. We investigate periodic solutions of linear and nonlinear discrete Volterra equations of convolution or non-convolution type with unbounded memory.

## 2. Preliminaries

We formalize our notation. Denote by $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{Z}^-$, respectively, the set of integers, the set of nonnegative integers, and the set of non-positive integers. In the following, we use $|\cdot|$ to denote a norm of a vector in $\mathbb{E}^d$ and also the corresponding subordinate norm of a matrix acting on the corresponding Banach space of vectors.

**Definition 2.1.** We denote by $\mathcal{S}(\mathbb{E}^d) = \{\xi : \xi = \{\xi_n\}_{n\geq 0}, \xi_n \in \mathbb{E}^d\}$ the linear space of sequences in $\mathbb{E}^d$. A sequence $\{\xi(n)\}_{n\geq 0} \in \mathcal{S}(\mathbb{E}^d)$ is *periodic of period* $N \in \mathbb{Z}_+$ if $\xi(n+N) = \xi(n)$ for $n \geq 0$, and for $N \geq 0$ we denote by $\mathcal{P}_N(\mathbb{E}^d)$ the space of $N$-periodic sequences in $\mathcal{S}(\mathbb{E}^d)$:

$$\mathcal{P}_N(\mathbb{E}^d) := \{\xi | \xi = \{\xi_n\}_{n\geq 0} \in \mathcal{S}(\mathbb{E}^d),\ \xi_{n+N} = \xi_n,\ n \geq 0\}, \tag{2.1a}$$

with norm $\|\xi\| = \sup_{0\leq n\leq N-1} |\xi_n|$. With this norm, $\mathcal{P}_N(\mathbb{E}^d)$ is a finite-dimensional Banach space whose elements are $N$-periodic sequences; further, $\mathcal{P}_N(\mathbb{E}^d)$ is a subspace of the Banach space $\boldsymbol{\ell}^\infty(\mathbb{E}^d)$ of bounded sequences,

$$\boldsymbol{\ell}^\infty(\mathbb{E}^d) := \{\xi \in \mathcal{S}(\mathbb{E}^d),\ \|\xi\|_\infty = \sup_{n\geq 0} |\xi_n| < \infty\}. \tag{2.1b}$$

To discuss the solution of equations such as (1.2a), we need to introduce the concept of an initial function.

**Definition 2.2.** We define an *initial function* $\phi(n)$ for (1.2a) or (1.2b) as a function from $\mathbb{Z}^-$ to $\mathbb{E}^d$, such that $x(n,\phi) = \phi(n)$ for $n \in \mathbb{Z}^-$. A *solution* $x = \{x(n) \equiv x(n,\phi)\}_{n\geq 0}$ of (1.2a) or (1.2b) is a sequence of elements $x(n) \in \mathbb{E}^d$ that satisfies (1.2a) or (1.2b) for $n \geq 0$ and

$$x(r,\phi) = \phi(r) \in \mathbb{E}^d \text{ for } r \in \mathbb{Z}^-. \tag{2.2}$$

**Definition 2.3.** We say that a homogeneous system (1.2a) or (1.2d) is *noncritical* with respect to $\mathcal{S}(\mathbb{E}^d)$ if the only solution is the trivial solution ($x(n) = 0$ for $n \geq 0$). It admits no nontrivial solution $\{x(n)\}_{n\geq 0}$ in $\mathcal{S}(\mathbb{E}^d)$. Otherwise, (1.2a) is said to be *critical* with respect to $\mathcal{S}(\mathbb{E}^d)$.

We take either $\mathbb{E} = \mathbb{C}$ or $\mathbb{E} = \mathbb{R}$. When $\phi(n) \in \mathbb{E}^d$ for $n \in \mathbb{Z}^-$, $K(n)$, $B(n, j) \in \mathbb{E}^d$, and $f(n) \in \mathbb{E}^d$ (for $n, j \in \mathbb{Z}_+$ and $j \leq n$), then any solution of (1.2a)–(1.2e) is also in $\mathbb{E}^d$. In consequence, results for $\mathbb{E} = \mathbb{R}$ can be deduced from results for $\mathbb{E} = \mathbb{C}$, and it will be sufficient to discuss the case $\mathbb{E} = \mathbb{C}$ and obtain the corresponding result for $\mathbb{E} = \mathbb{R}$ as an immediate corollary.

If we introduce the inner product for $\xi = \{\xi_n\}_{n\geq 0}$, $\eta = \{\eta_n\}_{n\geq 0} \in \mathcal{P}_N(\mathbb{C}^d)$, defined (where $\xi^* = \bar{\xi}^T$ denotes the conjugate transpose of $\xi$) by

$$\langle \xi, \eta \rangle := \sum_{n=0}^{N-1} \xi_n^* \eta_n, \quad \text{where } \xi_n, \eta_n \in \mathbb{C}^d, \tag{2.3}$$

then we can construct an orthogonal basis for $\mathcal{P}_N(\mathbb{C}^d)$, based upon the roots of unity and the columns of the identity matrix of order $d$. For simplicity, let us discuss the case $d = 1$. Let $\omega_j = \exp(2\mathrm{i}\pi j/N)$ be the $N$-th roots of unity, where $\mathrm{i}^2 = -1$, and let $\Omega_j = \{(\omega_j)^n\}_{n\geq 0}$ $j = 0, 1, \ldots, N-1$. It is clear that each $\Omega_j$ is an $N$-periodic sequence and

$$\langle \Omega_s, \Omega_j \rangle = \sum_{n=0}^{N-1} \exp\left[\frac{2\pi i(j-s)}{N}\right]^n = N\delta_{js},$$

where $\delta_{js} = 0$ if $j \neq s$ and $\delta_{js} = 1$ if $j = s$. Thus, the sequences $\Omega_j$, $j = 0, 1, \ldots, N-1$, form an orthogonal basis for $\mathcal{P}_N(\mathbb{C})$. Then any sequence $\{u(n)\}_{n\geq 0} \in \mathcal{P}_N(\mathbb{C})$ can be written in the form $u(n) = \sum_{s=0}^{N-1} c_s \omega_s^n$ for $n \geq 0$, where $c_s \in \mathbb{C}$, $s = 0, \ldots, N-1$. Since we can obviously extend the argument to the space $\mathcal{P}_N(\mathbb{C}^d)$ ($d > 1$), we have the following result.

**Lemma 2.4.** *Any* $\{u(n)\}_{n\geq 0} \in \mathcal{P}_N(\mathbb{E}^d)$ *can be represented*

$$u(n) = \sum_{s=0}^{N-1} c_s \omega_s^n, \quad n \geq 0, \tag{2.4}$$

*where*

$$\omega_s = \exp\left(\frac{2\mathrm{i}\pi s}{N}\right), \quad s = 0, \ldots, N-1.$$

*The vectors* $c_s \in \mathbb{C}^d$, $s = 0, \ldots, N-1$, *are uniquely determined by* $u(0), u(1), \ldots, u(N-1)$.

An explicit expression for $c_s \in \mathbb{C}^d$, $s = 0, \ldots, N-1$ can be given. For the space $\mathcal{P}_N(\mathbb{R}^d)$ of real sequences, it is clear that $\mathcal{P}_N(\mathbb{R}^d)$ can be embedded in $\mathcal{P}_N(\mathbb{C}^d)$, and we have the following result. Any $\{u(n)\}_{n\geq 0} \in \mathcal{P}_N(\mathbb{R}^d)$ can be represented for some $c_s \in \mathbb{C}^d$ by (2.4):

$$u(n) = \sum_{s=0}^{N-1} c_s \omega_s^n = \Re(u(n)) = \Re\left(\sum_{s=0}^{N-1} c_s \omega_s^n\right), \quad n \geq 0. \tag{2.5}$$

We shall employ (2.5) for convenience, but we can deduce the following result from (2.4).

**Lemma 2.5.** *Any* $\{x(n)\}_{n\geq 0} \in \mathcal{P}_N(\mathbb{R}^d)$ *can be represented, with corresponding* $a_j, b_j \in \mathbb{R}^d$*, as*

$$x(n) = a_0 + (-1)^n a_m + \sum_{j=1}^{m-1}\left(a_j \cos\left(\frac{nj\pi}{m}\right) + b_j \sin\left(\frac{nj\pi}{m}\right)\right), \quad \text{if } N = 2m, \tag{2.6a}$$

*or as*

$$x(n) = a_0 + \sum_{j=1}^{m}\left(a_j \cos\left(\frac{2\pi nj}{2m+1}\right) + b_j \sin\left(\frac{2\pi nj}{2m+1}\right)\right), \quad \text{if } N = 2m+1. \tag{2.6b}$$

Formulas (2.6a) and (2.6b) are from Carvalho's Lemma [4]. It is possible to use formulas (2.6) to treat the case where $\mathbb{E}$ is $\mathbb{R}$. However, it is more convenient to embed the real case $\mathbb{E} = \mathbb{R}$ in the case $\mathbb{E} = \mathbb{C}$, and employ (2.5); we mention (2.6) only for completeness.

## 3. Linear equations of convolution type

In this section, we consider the system

$$x(n) = \sum_{j=-\infty}^{n} K(n-j)x(j), \quad n \geq 0, \tag{3.1}$$

of discrete Volterra equations of convolution type, and its inhomogeneous analogue

$$x(n) = f(n) + \sum_{j=-\infty}^{n} K(n-j)x(j), \quad n \geq 0, \tag{3.2}$$

where $\{f(n)\}_{n\geq 0} \in \mathcal{S}(\mathbb{E}^d)$. Here, $\{K(n)\}_{n\geq 0} \in \mathcal{S}(\mathbb{E}^{d\times d})$ is a sequence of $d \times d$ matrices with entries in $\mathbb{E}$. We assume that $\{K(n)\}_{n\geq 0} \in \boldsymbol{\ell}^1(\mathbb{E}^{d\times d})$, that is:

**Assumption H1.**

$$\sum_{j=0}^{\infty} |K(j)| < \infty.$$

From (3.1) and (3.2), we can obtain the equivalent forms

$$x(n) = \sum_{j=0}^{\infty} K(j)x(n-j) \quad \text{and} \quad x(n) = f(n) + \sum_{j=0}^{\infty} K(j)x(n-j). \tag{3.3}$$

**Definition 3.1.** The $Z$-transform $\mathcal{Z}\{u\}(z)$ of a sequence $\{u(n)\}_{n\geq 0} \in \mathcal{S}(\mathbb{E}^d)$ is

$$\mathcal{Z}\{u\}(z) = \sum_{n=0}^{\infty} u(n)z^{-n}. \tag{3.4a}$$

Similarly, the $Z$-transform $\mathcal{Z}\{M\}(z)$ of a sequence $\{M(n)\}_{n\geq 0} \in \mathcal{S}(\mathbb{E}^{d\times d})$ is

$$\mathcal{Z}\{M\}(z) = \sum_{n=0}^{\infty} M(n)z^{-n}. \tag{3.4b}$$

**Remark 3.2.** Associated with (3.4a) and (3.4b) are the series

$$A(x)(\xi) := \sum_{n=0}^{\infty} x(n)\xi^n \quad \text{and} \quad A(M)(\xi) := \sum_{n=0}^{\infty} M(n)\xi^n, \qquad \xi \in \mathbb{C}.$$

If $R$ is the largest real number such that $A(x)(\xi)$ converges for $|\xi| \leq R$, then the series $\sum_{n=0}^{\infty} x(n)z^{-n}$ converges for $|z| \geq 1/R$. It is clear that if $\{x(n)\}_{n\geq 0} \in \boldsymbol{\ell}^1(\mathbb{E}^d)$, then (3.4a) converges at least for $|z| \geq 1$. If $\{x(n)\}_{n\geq 0} \in \boldsymbol{\ell}^{\infty}(\mathbb{E}^d)$, then (3.4a) converges for $|z| > 1$ (see [7]).

**Remark 3.3.** If $\{K(n)\}_{n\geq 0} \in \boldsymbol{\ell}^1$, $\sum_{j=0}^{\infty} K(j)\omega_s^{-j}$ converges for each $\omega_s$ $(0 \leq s \leq N-1)$, which implies that for sequences $\{K(n)\}_{n\geq 0} \in \boldsymbol{\ell}^1$ and $\{\omega_s^{-n}\}_{n\geq 0}$ we can define a matrix $\sum_{j=0}^{\infty} K(j)\omega_s^{-j}$ by the $Z$-transform. Naturally, for a vector sequence $\{c_s\omega_s^{-n}\}_{n\geq 0}$, we define a vector in $\mathbb{C}^d$ by

$$\sum_{j=0}^{\infty} K(j)c_s\omega_s^{-j} =: \left(\sum_{j=0}^{\infty} K(j)\omega_s^{-j}\right) c_s,$$

where $c_s \in \mathbb{C}^d$. Similarly, for a vector sequence $\{\sum_{s=0}^{N-1} c_s\omega_s^{-n}\}_{n\geq 0}$ we define

$$\sum_{j=0}^{\infty} K(j) \sum_{s=0}^{N-1} c_s\omega_s^{-j} =: \sum_{s=0}^{N-1} \left(\sum_{j=0}^{\infty} K(j)\omega_s^{-j}\right) c_s. \tag{3.5}$$

Thus, the left-hand side of (3.5) makes sense for any vectors $c_s \in \mathbb{C}^d$ for $0 \leq s \leq N-1$. We always assume the above definition in this section if needed.

Having presented our notation, we can now state the main purpose of this section. It is to establish the relationship between the periodic solutions of (3.1) or (3.2) and the roots of the corresponding equation

$$\det(I - \mathcal{Z}\{K\}(z)) = 0.$$

In fact, we show that any $N$-periodic solutions of (3.1) and (3.2) can be written down in the form of a $Z$-transform. The relation between periodic solutions of (3.2) and the associated equations

$$x(n) = f(n) + \sum_{j=0}^{n} K(n-j)x(j), \quad n \geq 0,$$

will be discussed in Section 4.

The main result of this section now follows. In the form stated, the results are apparently new but the method of proof is adapted from results already in the literature.

**Theorem 3.4.** *Suppose $K(n) \in \mathbb{E}^{d\times d}$ for $n \in \mathbb{Z}_+$. Under Assumption H1, (3.1) has a nontrivial periodic solution in $\mathcal{P}_N(\mathbb{E}^d)$ if and only if $\det(I - \mathcal{Z}\{K\}(z)) = 0$ has a root $z = \mathrm{e}^{\mathrm{i}\theta}$ on the unit complex circle for which $\theta/2\pi$ is rational. Specifically, if $\theta = 2\pi m/N$, then (3.1) has a N-periodic solution.*

**Proof.**

(i) Assuming that (3.1) has a nontrivial $N$-periodic solution $\{x(n)\}_{n\geq 0} \in \mathcal{P}_N(\mathbb{E}^d)$, by (2.4) we can write it in the form

$$x(n) = \sum_{s=0}^{N-1} c_s\omega_s^n, \quad n \geq 0, \tag{3.6}$$

where $c_s \in \mathbb{C}^d$ and $\omega_s = \exp(2\mathrm{i}\pi s/N)$, $s = 0, \ldots, N-1$. Thus, there exists at least one integer $q$ $(0 \leq q \leq N-1)$ such that $c_q \neq 0$. Substituting (3.6) into (3.3), yields

$$\sum_{s=0}^{N-1} c_s\omega_s^n = \sum_{j=0}^{\infty} K(j) \sum_{s=0}^{N-1} c_s\omega_s^{n-j} = \sum_{s=0}^{N-1}\left(\left(\sum_{j=0}^{\infty} K(j)\omega_s^{-j}\right)c_s\right)\omega_s^n.$$

Comparing both sides, we obtain (see Lemma 2.4) $c_s = (\sum_{j=0}^{\infty} K(j)\omega_s^{-j})c_s$; equivalently

$$[I - \mathcal{Z}\{K\}(\omega_s)]c_s = 0, \quad 0 \leq s \leq N-1, \tag{3.7}$$

where $\mathcal{Z}\{K\}(z)$ is the $Z$-transform of $\{K(n)\}_{n\geq 0}$, and $I$ is the $d \times d$ identity matrix. Notice that $c_q \neq 0$. It follows from $[I - \mathcal{Z}\{K\}(\omega_q)]c_q = 0$ that the equation $\det(I - \mathcal{Z}\{K\}(z)) = 0$ has a root $z = \omega_q = \mathrm{e}^{2\mathrm{i}\pi q/N}$.

(ii) Conversely, if $\det(I - \mathcal{Z}\{K\}(z)) = 0$ has a root $z = \omega_m = \mathrm{e}^{\mathrm{i}\theta}$ with $\theta = 2\pi m/N$, and $m$ is an integer $(0 \leq m \leq N-1)$, then there exists a vector $c_m \in \mathbb{C}^d$, $c_m \neq 0$, such that

$$[I - \mathcal{Z}\{K\}(\omega_m)]c_m = 0. \tag{3.8}$$

It is clear that the sequence $\{c_m\omega_m^n\}_{n\geq 0}$ is $N$-periodic. Let $x(n) = c_m\omega_m^n$ $(n \geq 0)$. From (3.8), we have $c_m = \mathcal{Z}\{K\}(\omega_m)c_m = \sum_{j=0}^{\infty} K(j)\omega_m^{-j}c_m$ and

$$x(n) = c_m\omega_m^n = \mathcal{Z}\{K\}(\omega_m)c_m\omega_m^n = \sum_{j=0}^{\infty} K(j)(c_m\omega_m^{n-j}) = \sum_{j=0}^{\infty} K(j)x(n-j),$$

which shows that $x(n) = c_m\omega_m^n$ $(n \geq 0)$ is a solution of (3.1). □

**Remark 3.5.** Let $S_N$ denote the set

$$S_N = \{s : \det(I - \mathcal{Z}\{K\}(\omega_s)) = 0\}. \tag{3.9}$$

If (3.1) is critical, then the set $S_N$ is not empty. By Theorem 3.4, $\{c_s\omega_s^n\}_{n\geq 0}$ is a non-trivial $N$-periodic solution of (3.1) for each $s \in S_N$, where $c_s \neq 0$ satisfies $(I - \mathcal{Z}\{K\}(\omega_s))c_s = 0$. Since any combination of such solutions $\{c_s\omega_s^n\}_{n\geq 0}$ $(s \in S_N)$ is also an $N$-periodic solution of (3.1), we conclude that any non-trivial $N$-periodic solution $\{x(n)\}_{n\geq 0}$ of (3.1) can be represented by

$$x(n) = \sum_{s\in S_N} \tau_s(c_s\omega_s^n), \quad \tau_s \in \mathbb{C}.$$

The periodic solution of (3.2) is closely related to that of (3.1):

**Theorem 3.6.** *Let Assumption H1 hold and let* $\{f(n)\}_{n\geq 0} \in \mathcal{P}_N(\mathbb{E}^d)$ *with* $f(n) = \sum_{s=0}^{N-1} f_s\omega_s^n$ *for* $n \geq 0$. *If (3.1) is noncritical with respect to* $\mathcal{P}_N(\mathbb{E}^d)$, *then (3.2) has a unique N-periodic solution* $\{x(n)\}_{n\geq 0}$ *given by formula*

$$x(n) = \sum_{s=0}^{N-1}([I - \mathcal{Z}\{K\}(\omega_s)]^{-1} f_s)\omega_s^n, \quad n \geq 0. \tag{3.10}$$

**Proof.** By Theorem 3.4, if (3.1) has no nontrivial $N$-periodic solution, then $I - \mathcal{Z}\{K\}(\omega_s)$ is nonsingular for $s = 0, 1, \ldots , N-1$. For given $f(n) = \sum_{s=0}^{N-1} f_s\omega_s^n$, let $x_s(n) = ([I - \mathcal{Z}\{K\}(\omega_s)]^{-1} f_s)\omega_s^n$ for $s = 0, 1, \ldots , N-1$ and $x(n) = \sum_{s=0}^{N-1} x_s(n)$. Obviously, $\{x_s(n)\}_{n\geq 0}$ and $\{x(n)\}_{n\geq 0}$ are $N$-periodic sequences. We show that $\{x(n)\}_{n\geq 0}$ is a solution of (3.2). Indeed, we have

$$\begin{aligned}
-\sum_{j=0}^{\infty} K(j)x_s(n-j) &= -\sum_{j=0}^{\infty} K(j)([I - \mathcal{Z}\{K\}(\omega_s)]^{-1} f_s)\omega_s^{n-j} \\
&= -\sum_{j=0}^{\infty} K(j)\omega_s^{-j}([I - \mathcal{Z}\{K\}(\omega_s)]^{-1} f_s)\omega_s^n \\
&= -\mathcal{Z}\{K\}(\omega_s)([I - \mathcal{Z}\{K\}(\omega_s)]^{-1} f_s)\omega_s^n \\
&= [I - \mathcal{Z}\{K\}(\omega_s)]([I - \mathcal{Z}\{K\}(\omega_s)]^{-1} f_s)\omega_s^n - ([I - \mathcal{Z}\{K\}(\omega_s)]^{-1} f_s)\omega_s^n \\
&= f_s\omega_s^n - x_s(n)
\end{aligned}$$

and

$$x(n) = \sum_{s=0}^{N-1} x_s(n) = \sum_{s=0}^{N-1} f_s\omega_s^n + \sum_{j=0}^{\infty} K(j)\sum_{s=0}^{N-1} x_s(n-j) = f(n) + \sum_{j=0}^{\infty} K(j)x(n-j).$$

Thus, $\{x(n)\}_{n\geq 0}$ is a solution of (3.2). It remains to prove that $x(n)$ in (3.10) is a unique solution of (3.2). Suppose that (3.2) has another $N$-periodic solution $\{y(n)\}_{n\geq 0}$. We can write it in the form $y(n) = \sum_{s=0}^{N-1} c_s\omega_s^n$ $(n \geq 0)$. Now substitution of $y(n) = \sum_{s=0}^{N-1} c_s\omega_s^n$ and $f(n) = \sum_{s=0}^{N-1} f_s\omega_s^n$ into (3.2) yields $[I - \mathcal{Z}\{K\}(\omega_s)]c_s = f_s$ $(0 \leq s \leq N-1)$. Consequently, $c_s = [I - \mathcal{Z}\{K\}(\omega_s)]^{-1} f_s$. Hence, $x(n) = \sum_{j=0}^{N-1}([I - \mathcal{Z}\{K\}(\omega_s)]^{-1} f_s)\omega_s^n$ $(n \geq 0)$ is a unique $N$-periodic solution of (3.2).

If (3.1) is critical, we have a result for (3.2) in the spirit of the Fredholm Alternative (see, e.g., [5], pp. 609–610): □

**Theorem 3.7.** *Suppose* $K(n) \in \mathbb{E}^{d\times d}$ $(n = 0, 1, \ldots)$ *and (3.1) is critical. Let Assumption H1 hold and suppose* $\{f(n)\}_{n\geq 0} \in \mathcal{P}_N(\mathbb{E}^d)$ *with* $f(n) = \sum_{j=0}^{N-1} f_j\omega_j^n$ *for* $n \geq 0$ $(f_j \in \mathbb{C}^d)$. *Then (3.2) has a nontrivial N-periodic solution in* $\mathcal{P}_N(\mathbb{E}^d)$ *if and only if*

$$\sum_{j=0}^{N-1} z^*(j)f(j) = 0, \tag{3.11}$$

*whenever* $\{z(n)\}_{n\geq 0} \in \mathcal{P}_N(\mathbb{E}^d)$ *is a nontrivial N-periodic solution of*

$$z(n) = \sum_{j=0}^{\infty} K^*(j)z(n-j), \quad n \geq 0. \tag{3.12}$$

*For* $f = \{f(n)\}_{n\geq 0}$ *that satisfy (3.11), (3.2) has N-periodic solutions in* $\mathcal{P}_N(\mathbb{E}^d)$ *given by*

$$x(n) = \sum_{s\in S_N} c_s\omega_s^n + \sum_{s\notin S_N}[(I - \mathcal{Z}\{K\}(\omega_s))^{-1} f_s]\omega_s^n, \quad n \geq 0, \tag{3.13}$$

*where* $c_s$ $(s \in S_N)$ *satisfies* $[I - \mathcal{Z}\{K\}(\omega_s)]c_s = f_s$, $s \in S_N$. *The second term,*

$$\sum_{s\notin S_N}[(I - \mathcal{Z}\{K\}(\omega_s))^{-1} f_s]\omega_s^n, \quad n \geq 0,$$

*of the solution (3.13) is orthogonal to the nontrivial solutions of (3.1) lying in* $\mathcal{P}_N(\mathbb{E}^d)$.

**Proof.**

(i) Suppose that (3.2) has an $N$-periodic solution $\{x(n)\}_{n\geq 0}$. It follows from (2.4) that $x(n)$ can be written as $x(n) = \sum_{s=0}^{N-1} c_s\omega_s^n$, $c_s \in \mathbb{C}^d$ $(n \geq 0)$. Substituting $x(n)$ and $f(n) = \sum_{j=0}^{n} f_j\omega_j^n$ into (3.2), yields both

$$[I - \mathcal{Z}\{K\}(\omega_s)]c_s = f_s, \quad s \in S_N \tag{3.14a}$$

and

$$[I - \mathcal{Z}\{K\}(\omega_s)]c_s = f_s, \quad s \notin S_N. \tag{3.14b}$$

Thus, from (3.14b), $c_s = (I - \mathcal{Z}\{K\}(\omega_s))^{-1} f_s$ $(s \notin S_N)$. Notice that (3.14a) has a solution $c_s$ for fixed $s \in S_N$ if and only if for all solutions $d_s$ of

$$[I - \mathcal{Z}\{K\}^*(\omega_s)]d_s = 0, \quad s \in S_N, \tag{3.15}$$

$f_s$ and $d_s$ are orthogonal ($d_s^* f_s = 0$). It is readily shown that $z(n) = \sum_{s\in S_N} d_s\omega_s^n$ is a solution of the adjoint Eq. (3.12) if $d_s$ satisfies (3.15). In addition, by Remark 3.5 any $N$-periodic solution $\{z(n)\}_{n\geq 0}$ of (3.12) is in the form $z(n) = \sum_{s\in S_N} d_s\omega_s^n$ and $d_s \in \mathbb{C}^d$ satisfies (3.15). Since $\sum_{n=0}^{N-1} \omega_i^n\bar{\omega}_j^n = 0$ if $i \neq j$ and $\sum_{n=0}^{N-1} \omega_i^n\bar{\omega}_i^n = N$, we have

$$\begin{aligned}\langle\{z(n)\}, \{f(n)\}\rangle &= \sum_{n=0}^{N-1} z^*(n) f(n) = \sum_{n=0}^{N-1}\sum_{s\in S_N}\sum_{j=0}^{N-1} d_s^* f_j\omega_j^n\bar{\omega}_s^n \\ &= \sum_{s\in S_N}\sum_{j=0}^{N-1} d_s^* f_j \sum_{n=0}^{N-1} \omega_j^n\bar{\omega}_s^n = N\sum_{s\in S_N} d_s^* f_s = 0,\end{aligned}$$

or $\{f(n)\}_{n\geq 0}$ is orthogonal to all solutions $\{z(n)\}_{n\geq 0}$ of (3.12) in the inner product defined by (2.3). Similarly, any nontrivial $N$-periodic solution $\{x_0(n)\}_{n\geq 0}$ of (3.1) is in the form $x_0(n) = \sum_{s\in S_N} e_s\omega_s^n$, where $e_s$ satisfies $[I - \mathcal{Z}\{K\}(\omega_s)]e_s = 0$, $s \in S_N$. If $x_\sharp(n) = \sum_{s\notin S_N}[(I - \mathcal{Z}\{K\}(\omega_s))^{-1} f_s]\omega_s^n$

$(n \geq 0)$, then $< \{x_0(n)\}, \{x_\sharp(n)\} >= 0$, which implies that $\{x_\sharp(n)\}_{n\geq 0}$ is orthogonal to the nontrivial $N$-periodic solutions of (3.1).

(ii) Conversely, suppose that $f(n) = \sum_{j=0}^{N_1} f_j \omega_j^n$ $(n \geq 0)$ satisfies (3.11). Note that any $N$-periodic solution $\{z(n)\}_{n\geq 0}$ of (3.12) is in the form $z(n) = \sum_{s\in S_N} d_s \omega_s^n$ and $d_s \in \mathbb{C}^d$ satisfies (3.15). It follows from $< \{z(n)\}, \{f(n)\} >= 0$ that $d_s^* f_s = 0$ for $s \in S_N$. Thus, (3.14a) has a solution $c_s$ for each fixed $s \in S_N$. Let

$$y(n) = \sum_{s\in S_N} c_s \omega_s^n + \sum_{s\notin S_N} ([I - \mathcal{Z}\{K\}(\omega_s)]^{-1} f_s)\omega_s^n, \quad n \geq 0,$$

where $c_s$ $(s \in S_N)$ satisfies (3.14a). Obviously, $\{y(n)\}_{n\geq 0}$ is $N$-periodic sequence. It remains to show that $\{y(n)\}_{n\geq 0}$ satisfies (3.2). Indeed, from (3.14a) we have

$$\begin{aligned}
-\sum_{j=0}^{\infty} K(j)y(n-j) &= -\sum_{s\in S_N}\sum_{j=0}^{\infty} K(j)c_s\omega_s^{n-j} - \sum_{s\notin S_N}\sum_{j=0}^{\infty} K(j)([I-\mathcal{Z}\{K\}(\omega_s)]^{-1} f_s)\omega_s^{n-j} \\
&= -\sum_{s\in S_N} \mathcal{Z}\{K\}(\omega_s)c_s\omega_s^n - \sum_{s\notin S_N} \mathcal{Z}\{K\}(\omega_s)([I-\mathcal{Z}\{K\}(\omega_s)]^{-1} f_s)\omega_s^n \\
&= \sum_{s\in S_N}(f_s - c_s)\omega_s^n + \sum_{s\notin S_N}[I-\mathcal{Z}\{K\}(\omega_s)]([I-\mathcal{Z}\{K\}(\omega_s)]^{-1} f_s)\omega_s^n \\
&\quad - \sum_{s\notin S_N}([I-\mathcal{Z}\{K\}(\omega_s)]^{-1} f_s)\omega_s^n \\
&= \sum_{s\in S_N}(f_s - c_s)\omega_s^n + \sum_{s\notin S_N} f_s\omega_s^n - \sum_{s\notin S_N}([I-\mathcal{Z}\{K\}(\omega_s)]^{-1} f_s)\omega_s^n \\
&= f(n) - y(n).
\end{aligned}$$

Thus, (3.13) is a solution of (3.2), and the proof is completed. □

## 4. Equations of non-convolution type

Consider the discrete Volterra equations of non-convolution type,

$$x(n) = f(n) + \sum_{j=0}^{n} B(n, j)x(j), \quad n \geq 0, \tag{4.1}$$

where $n$ and $j$ are integers, $x(n) \in \mathbb{E}^d$, $B(n, j) \in \mathbb{E}^{d\times d}$, $B(n, j) \equiv 0$ for $j > n$, and $\{f(n)\}_{n\geq 0} \in \mathbb{E}^d$ is a given sequence. We shall show that the boundedness of the solution of (4.1) is closely related to the existence of periodic solutions of (1.2b), namely,

$$z(n) = f(n) + \sum_{j=-\infty}^{n} B(n, j)z(j), \quad n \geq 0, \tag{4.2}$$

when

$$B(n + N, m + N) = B(n, m) \text{ for all } n, m \in \mathbb{Z}, \tag{4.3a}$$

$$f(n+N) = f(n) \text{ for all } n \in \mathbb{Z} \tag{4.3b}$$

where $N$ is some positive integer.

**Definition 4.1.** If $f(n+N) = f(n)$ for all $n \in \mathbb{Z}$ holds, we say that the term $f = \{f(n)\}_{n\in\mathbb{Z}}$ is $N$-periodic and if $\{B(n, j)\}_{n, j\in:\mathbb{Z}}$ (or $B : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{E}^{d\times d}$) satisfies (4.3a) we say it is $N$-periodic.

**Remark 4.2.** Although the $N$-periodic sequences $\{f(n)\}_{n\in\mathbb{Z}}$ and $\{B(n, j)\}_{n, j\in\mathbb{Z}}$ are defined on the set of integers by the above definition, we show in the following that any $N$-periodic sequence $\{f(n)\}_{n\in\mathbb{Z}}$ or $\{B(n, j)\}_{n, j\in\mathbb{Z}}$ is, in fact, uniquely determined by the restriction $\{f(n)\}_{n\in\mathbb{Z}_+}$ or $\{B(n, j)\}_{n, j\in\mathbb{Z}_+}$.

Suppose that $A : \mathbb{Z} \times \mathbb{Z} \to \mathbb{E}^{d\times d}$ is $N$-periodic. Denote by $A_+ : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{E}^{d\times d}$ the restriction of the mapping $A$ from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}_+ \times \mathbb{Z}_+$. Then, $A_+$ satisfies $A_+(n+N, m+N) = A_+(n, m)$ for all $n, m \in \mathbb{Z}_+$. In this case, we say that $A_+$ is $N$-periodic on $\mathbb{Z}_+$. We use the sequence $\{A_+(n, m)\}_{n,m\in\mathbb{Z}_+}$ to define a matrix mapping $D : \mathbb{Z} \times \mathbb{Z} \to \mathbb{E}^{d\times d}$, or $\{D(n, m)\}_{n,m\in\mathbb{Z}}$, as follows:

$$D(n, m) = \begin{cases} A_+(n, m) & \text{for all } n \geq \text{ and } m \geq 0, \\ A_+(n + lN, m + lN) & \text{if } n + lN \geq \text{ and } m + lN \geq 0, \end{cases} \tag{4.4}$$

where $l > 0$ is any positive integer. It is clear that $D(n, m)$ is well defined for any $n, m \in \mathbb{Z}$, and $\{D(n, m)\}_{n,m\in\mathbb{Z}}$ is also $N$-periodic. It is readily shown that

$$A(n, m) = D(n, m) \quad \text{for any } n, m \in \mathbb{Z}. \tag{4.5}$$

Since $D$ is constructed only by extending $\{A_+(n, m)\}_{n,m\in\mathbb{Z}_+}$ by the relation (4.4), we can conclude that any $N$-periodic matrix mapping $A_+ : \mathbb{Z}_+ \times \mathbb{Z}_+ \to \mathbb{E}^{d\times d}$ can be extended uniquely to be an $N$-periodic matrix mapping $D : \mathbb{Z} \times \mathbb{Z} \to \mathbb{E}^{d\times d}$ by the relation (4.4). Corresponding arguments apply to $f$.

**Definition 4.3.** The sequence $D = \{D(n, m)\}_{n,m\in\mathbb{Z}}$ defined by (4.4) is called the periodic extension of $\{A_+(n, m)\}_{n,m\in\mathbb{Z}_+}$ satisfying (4.3a).

There are some preliminary results that we need. We assume that $B(n, n)$ $(n \geq 0)$ satisfies

$$\det(I - B(n, n)) \neq 0 \quad \text{for all } n \geq 0. \tag{4.6}$$

Thus, the unique solution of (4.1) and (4.2) exists. For $B(n, m)$ in (4.2), we assume

**Assumption H2.**

$$\sum_{r=0}^{\infty} |B(n, n-r)| < \infty \quad \text{for each } n \in \mathbb{Z}_+.$$

Then, the sum $\sum_{j=-\infty}^{n} B(n, j)g(j)$ is bounded in the case $\sup_{n\in\mathbb{Z}}|g(n)| < \infty$.

For our main result we need the following crucial lemma.

**Lemma 4.4.** *If $\{x(n)\}_{n\geq 0}$ is a solution of (4.1) and $\{x(n)\}_{n\geq 0} \in \ell^{\infty}(\mathbb{E}^d)$, then there is a corresponding solution $\{z(n)\}$ of (4.2) such that (for every $n = 0, \pm 1, \pm 2, \ldots$) $z(n)$ is the limit of some subsequence of $x(n)$.*

**Proof.** Our proof is a more detailed version of that in [6, pp. 485–486]. Let $\{x(n)\}_{n\geq 0}$ be a bounded solution of (4.1). Then $\{x(jN)\}_{j=1}^{\infty}$, with $N$ as defined in (4.3a), is bounded and thus has a convergent subsequence $\{x(j_{i0})\}$ which convergences to a point in $\mathbb{E}^d$, say $z(0)$. Similarly, there is a subsequence $\{j_{i1}\}$ of $\{j_{i0}\}$ such that both subsequences $\{x(1+j_{i0}N)\}$ and $\{x(-1+j_{i0}N)\}$ converge to $z(1)$ and $z(-1)$, respectively. Inductively, one may show that for each nonnegative integer $n$, $\{x(\pm(n-1)+j_{i(n-1)}N)\}$ converge to $z(n-1)$ and $z(-(n-1))$, respectively, and $\{x(\pm n+j_{\mathrm{in}}N)\}$ converge to $z(n)$ and $z(-n)$, respectively, where $\{j_{\mathrm{in}}\}$ is a subsequence of $\{j_{i(n-1)}\}$.

We need to show that $\{z(n)\}_{-\infty}^{\infty}$ is actually a solution of (4.2). From (4.1) and (4.3a), we have

$$
\begin{aligned}
x(n+j_{\mathrm{in}}N) &= f(n+j_{\mathrm{in}}N) + \sum_{r=0}^{n+j_{\mathrm{in}}N} B(n+j_{\mathrm{in}}N, r)x(r) \\
&= f(n) + \sum_{r=-j_{\mathrm{in}}N}^{n} B(n+j_{\mathrm{in}}N, r+j_{\mathrm{in}}N)x(r+j_{\mathrm{in}}N) \\
&= f(n) + \sum_{r=-j_{\mathrm{in}}N}^{n} B(n, r)x(r+j_{\mathrm{in}}N).
\end{aligned}
\tag{4.7}
$$

Since $\{z(r)\}$ $(-\infty < r < \infty)$ is bounded, $\sum_{r=-\infty}^{n} B(n,r)z(r)$ is well defined for each $n \geq 0$. For fixed $n \geq 0$ and any $\varepsilon > 0$, it follows from Assumption H2 that there is a $j_{\mathrm{in}_0} > 0$ such that

$$
\sum_{r=-\infty}^{-j_{\mathrm{in}_0}N} |B(n,r)| < \frac{\varepsilon}{1+2M},
$$

where $M = \sup_{n\geq 0}|x(n)| < \infty$. Then for $j_{\mathrm{in}} > j_{\mathrm{in}_0}$, we have

$$
\begin{aligned}
&\left| \sum_{r=-\infty}^{n} B(n,r)z(r) - \sum_{r=-j_{\mathrm{in}}N}^{n} B(n,r)x(r+j_{\mathrm{in}}N) \right| \\
&\quad \leq \sum_{r=-\infty}^{-j_{\mathrm{in}_0}N} |B(n,r)||z(r) - x(r+j_{\mathrm{in}}N)| \leq 2M \sum_{r=-\infty}^{-j_{\mathrm{in}_0}N} |B(n,r)| < \varepsilon.
\end{aligned}
$$

Then, we can take the limit in (4.7) by letting $j_{\mathrm{in}} \to \infty$. In this case, the left-hand side of (4.7) converges to $z(n)$ and the right-hand side of (4.7) converges to $f(n) + \sum_{r=-\infty}^{n} B(n,r)z(r)$. Hence, $z(n) = f(n) + \sum_{r=-\infty}^{n} B(n,r)z(r)$ is a solution of (4.2). □

**Definition 4.5.** The resolvent $\{R(n,m)\}$ of the kernel $\{B(n,m)\}$ in (4.1) is defined by the solutions of the matrix equations

$$
R(n,m) = \sum_{j=m}^{n} R(n,j)B(j,m) - B(n,m), \quad 0 \leq m \leq n, \quad n \geq 0, \tag{4.8}
$$

with $R(n,m) \equiv 0$ for $n < m$.

The existence and uniqueness of $\{R(n, m)\}$ is guaranteed by (4.6). By using the resolvent $\{R(n, m)\}$, the solution of (4.1) can be given by the *variation of constants formula*

$$x(n) = f(n) - \sum_{j=0}^{n} R(n, j) f(j), \quad n \geq 0. \tag{4.9}$$

For details of resolvent matrices and variation of constants formula, see [11]. The periodic relation between the kernel $\{B(n, m)\}$ and the resolvent $\{R(n, m)\}$ is as follows.

**Lemma 4.6.** *If, for some positive integer $N$, $B(n + N, m + N) = B(n, m)$ for all $0 \leq m \leq n$, then the solution of (4.8) satisfies*

$$R(n + N, m + N) = R(n, m), \quad 0 \leq m \leq n. \tag{4.10}$$

**Proof.** For any $0 \leq m \leq n$, we have

$$\begin{aligned} R(n + N, m + N) &= \sum_{j=m+N}^{n+N} R(n + N, j)B(j, m + N) - B(n + N, m + N) \\ &= \sum_{i=m}^{n} R(n + N, i + N)B(i + N, m + N) - B(n, m) \\ &= \sum_{j=m}^{n} R(n + N, m + N)B(j, m) - B(n, m), \end{aligned}$$

implying that $R(n + N, m + N)$ is also a solution of (4.8). Hence, (4.10) follows. □

Lemma 4.6 shows that if the kernel $\{B(n, m)\}$ in (4.1) is $N$-periodic on $\mathbb{Z}_+$, then its resolvent $\{R(n, m)\}_{n,m\in\mathbb{Z}_+}$ is also $N$-periodic on $\mathbb{Z}_+$. From Remark 4.2, we can extend $\{R(n, m)\}_{n,m\in\mathbb{Z}_+}$ by the relation (4.4) to be $N$-periodic for $n, m \in \mathbb{Z}$. In this section, we always assume this extension.

To investigate periodic solutions, we assume the following conditions.

**Assumption H3.** The resolvent $\{R(n, m)\}$ satisfies $\lim_{n\to\infty} |R(n, r)| = 0$ for each $r \in \mathbb{Z}_+$.

**Assumption H4.** The resolvent $\{R(n, m)\}$ satisfies $\sum_{r=0}^{\infty} |R(n, n - r)| < \infty$ for each $n \in \mathbb{Z}_+$.

**Remark 4.7.** Suppose that $B(n, m) = K(n - m)$ (for all $0 \leq m \leq n$), then $R(n, m) = R^K(n - m)$ is a convolution type. In addition, if $\{K(n)\}_{n\geq0} \in \ell^1$, $\det(I - K(0)) \neq 0$ and $\det(I - \mathcal{Z}\{K\}(z)) \neq 0$ for $|z| \geq 1$, then $\{R^K(n)\}_{n\geq0} \in \ell^1$ by the discrete Paley-Wiener theorem (see, e.g., [13]). In this case, one readily sees that the Assumptions H2, H3 and H4 are satisfied. Thus, our discussion in this section includes corresponding convolution equations as a special case.

**Remark 4.8.** In [13], the discrete Paley-Wiener theorem for convolution equations has been extended to non-convolution equations. From the results in [13], we know that if $\sup_{n\geq0} \sum_{j=0}^{n} |B(n, j)| < 1$, then $\sup_{n\geq0} \sum_{j=0}^{n} |R(n, j)| < \infty$. If, in addition, $\lim_{n\to\infty} B(n, j) = 0$ for each $j \geq 0$, then $\lim_{n\to\infty} R(n, j) = 0$ for each $j \geq 0$. For details, see [13] or [15].

**Remark 4.9.** If $\{R(n, m)\}$ satisfies

$$|R(n, m)| \leq v^{n-m} \text{ for } 0 \leq m \leq n, \tag{4.11}$$

where $v \in (0, 1)$, then $\{R(n, m)\}$ satisfies conditions Assumption H3 and H4. Thus, conditions Assumption H3 and H4 include the condition (4.11) as a special case. In [6], Elaydi investigated periodic solutions of linear Volterra difference equations under the condition (4.11).

The remarks following Assumption H2 concerning the summability of $\{B(n, r)\}$ apply here equally for $\{R(n, r)\}$. The number $n$ in Assumption H2 and H4 can only be taken with $0 \leq n \leq N - 1$ if $B(n, m)$ satisfies (4.3a). We have the following.

**Lemma 4.10.** *Suppose that $B(n, m)$ satisfies (4.3a) and the Assumptions H2 and H4 hold. Then the following conditions are true:*

(i) $\sup_{n\in\mathbb{Z}_+} \sum_{r=0}^{\infty} |B(n, n - r)| = \sup_{n\in\mathbb{Z}_+} \sum_{r=-\infty}^{n} |B(n, r)| < \infty$ *and*

$$\sup_{n\geq 0} \sum_{r=0}^{n} |B(n, r)| < \infty;$$

(ii) *for any $\varepsilon > 0$ there exists a number $r_0 = r_0(\varepsilon) \in \mathbb{Z}_+$ such that, for all $n \in \mathbb{Z}_+$,*

$$\sum_{r=r_0}^{\infty} |B(n, n - r)| = \sum_{r=-\infty}^{n-r_0} |B(n, r)| < \varepsilon;$$

(iii) $\sup_{n\in\mathbb{Z}_+} \sum_{r=0}^{\infty} |R(n, n - r)| = \sup_{n\in\mathbb{Z}_+} \sum_{r=-\infty}^{n} |R(n, r)| < \infty$, *and*

$$\sup_{n\geq 0} \sum_{r=0}^{n} |R(n, r)| < \infty;$$

(iv) *for any $\varepsilon > 0$ there exists a number $r_0 = r_0(\varepsilon) \in \mathbb{Z}_+$ such that, for all $n \in \mathbb{Z}_+$,*

$$\sum_{r=r_0}^{\infty} |R(n, n - r)| = \sum_{r=-\infty}^{n-r_0} |R(n, r)| < \varepsilon.$$

**Proof.** We notice that each $n \geq 0$ can be written as $n = s + lN$, where $0 \leq s \leq N - 1$ and $l \in \mathbb{Z}_+$. Since $B(n + N, m + N) = B(n, m)$ for $n, m \in \mathbb{Z}$, then,

$$|B(n, n - r)| = |B(s + lN, s + lN - r)| = |B(s + lN, s - r + lN)| = |B(s, s - r)|.$$

Thus, $\sup_{n\in\mathbb{Z}_+} \sum_{r=0}^{\infty} |B(n, n - r)| = \max_{0\leq s\leq N-1} \sum_{r=0}^{\infty} |B(s, s - r)| < \infty$ by Assumption H2. Similarly, for any $\varepsilon > 0$ and $0 \leq s \leq N - 1$, it follows from Assumption H2 that there exists $r_0 \in \mathbb{Z}_+$ such that $\sum_{r=r_0}^{\infty} |B(s, s - r)| < \varepsilon$. Then for any $n \in \mathbb{Z}_+$, there exist $s$ $(0 \leq s \leq N - 1)$ and $l \in \mathbb{Z}_+$ such that $n = s + lN$. Thus,

$$\sum_{r=r_0}^{\infty} |B(n, n - r)| = \sum_{r=r_0}^{\infty} |B(s, s - r)| < \varepsilon.$$

As for the proof of statements (*iii*) and (*iv*), we notice that $R(n,m)$ satisfies (4.10). With the same technique, we can readily complete the proof. □

### 4.1. Existence and uniqueness of periodic solution

We are now in a position to state the main result in this section.

**Theorem 4.11.** *Let the Assumptions H2, H3 and H4 and (4.3a) and (4.3b) hold. Then (4.2) has the unique N-periodic solution*

$$z(n) = f(n) - \sum_{m=-\infty}^{n} R(n,m) f(m), \quad n \geq 0. \tag{4.12}$$

**Proof.** From Lemma 4.10 (*iii*) and formula (4.9), it can be readily shown that the solution of (4.1) is bounded. By Lemma 4.6, one can construct a sequence $x(n + r_{\text{in}}N)$ from a solution $x(n)$ of (4.1) such that $x(n + r_{\text{in}}N)$ converges to a solution $z(n)$ of (4.2). Formula (4.9) gives

$$\begin{aligned} x(n + r_{\text{in}}N) &= f(n + r_{\text{in}}N) - \sum_{j=0}^{n+r_{\text{in}}N} R(n + r_{\text{in}}N, j) f(j) \\ &= f(n) - \sum_{i=-r_{\text{in}}N}^{n} R(n + r_{\text{in}}N, i + r_{\text{in}}N) f(i + r_{\text{in}}N) \\ &= f(n) - \sum_{i=-r_{\text{in}}N}^{n} R(n,i) f(i). \end{aligned} \tag{4.13}$$

Note that $\sum_{i=-\infty}^{n} R(n,i) f(i)$ is well defined by Assumption H4. For fixed $n$ and any $\varepsilon > 0$, it follows from Assumption H4 that there exists $r_{\text{in}_0} > 0$ such that

$$\sum_{r=-\infty}^{-r_{\text{in}_0}N} |R(n,r)| < \frac{\varepsilon}{1 + \|f\|_\infty},$$

where $\|f\|_\infty = \sup_{n\in\mathbb{Z}} |f(n)|$. Then for $r_{\text{in}} > r_{\text{in}_0}$, we have

$$\left| \sum_{i=-\infty}^{n} R(n,i) f(i) - \sum_{i=-r_{\text{in}}N}^{n} R(n,r) f(i + r_{\text{in}}N) \right| \leq \sum_{r=-\infty}^{-r_{\text{in}_0}N} |R(n,r)||f(i)| \leq \|f\|_\infty \sum_{r=-\infty}^{-j_{\text{in}_0}N} |R(n,r)| < \varepsilon.$$

Thus, we can take the limit in (4.13) by letting $r_{\text{in}} \to \infty$ and obtain

$$z(n) = \lim_{r_{\text{in}}\to\infty} \left[ f(n) - \sum_{i=-r_{\text{in}}N}^{n} R(n,i) f(i) \right] = f(n) - \sum_{r=-\infty}^{n} R(n,r) f(r).$$

By Lemma 4.6, $z(n)$ is $N$-periodic.

It remains to show that $z(n)$ in (4.12) is the only $N$-periodic solution of (4.2). Let us assume that there is another $N$-periodic solution $\bar{z}(n)$ of (4.2). Then $\psi(n) = z(n) - \bar{z}(n)$ is an $N$-periodic solution of the equation

$$\psi(n) = \sum_{r=0}^{n} B(n,r)\psi(r) + \sum_{r=-\infty}^{-1} B(n,r)\psi(r).$$

By the variation of constants formula (4.9), we have

$$\begin{aligned}\psi(n) &= \sum_{r=-\infty}^{-1} B(n,r)\psi(r) - \sum_{j=0}^{n} R(n,j)\left[\sum_{r=-\infty}^{-1} B(j,r)\psi(r)\right] \\ &= \sum_{r=n+1}^{\infty} B(n,n-r)\psi(n-r) - \sum_{j=0}^{n} R(n,j)\left[\sum_{r=j+1}^{\infty} B(j,j-r)\psi(j-r)\right].\end{aligned}$$

Thus,

$$|\psi(n)| \le M \sum_{r=n+1}^{\infty} |B(n,n-r)| + M \sum_{j=0}^{n} |R(n,j)| \left[\sum_{r=j+1}^{\infty} |B(j,j-r)|\right],$$

where $M = \sup_{n\in\mathbb{Z}^-}\{|\psi(n)|\}$. For any $\varepsilon > 0$, it follows from Lemma 4.10 that there exists a number $r_0 = r_0(\varepsilon) \in \mathbb{Z}_+$ such that $\sum_{r=r_0}^{\infty} |B(n,n-r)| < \varepsilon$ for all $n \in \mathbb{Z}_+$. Thus, $\sum_{r=n+1}^{\infty} |B(n,n-r)| < \varepsilon$ for $n > r_0 - 1$. Let $C = \sup_{n\in\mathbb{Z}_+} \sum_{r=0}^{\infty} |B(n,n-r)|$ and $n > r_0$. (By Lemma 4.10, $C < \infty$.) From the above inequality, we obtain

$$\begin{aligned}|\psi(n)| &\le M \sum_{r=n+1}^{\infty} |B(n,n-r)| + M \sum_{j=0}^{n} |R(n,j)| \left[\sum_{r=j+1}^{\infty} |B(j,j-r)|\right] \\ &\le M\varepsilon + M \sum_{j=0}^{r_0-1} |R(n,j)| \sum_{r=j+1}^{\infty} |B(j,j-r)| + M \sum_{j=r_0}^{n} |R(n,j)| \sum_{r=j+1}^{\infty} |B(j,j-r)| \\ &\le M\varepsilon + MC \sum_{j=0}^{r_0-1} |R(n,j)| + MC \sum_{r=r_0+1}^{\infty} |B(j,j-r)|.\end{aligned}$$

It follows from Assumption H3 that for the above $\varepsilon > 0$, there exists a number $r_1 > 0$ such that $\sum_{j=0}^{r_0-1} |R(n,j)| < \varepsilon$ for $n > r_1$. Let $r_2 = \max(r_0, r_1)$. For $n > r_2$, we have

$$|\psi(n)| \le M\varepsilon + MC\varepsilon + MC\varepsilon = M(1 + C + C)\varepsilon.$$

Hence, $\lim_{n\to\infty} \psi(n) = 0$. Since $\psi(n)$ is periodic, it follows that $\psi(n) \equiv 0$. Consequently, $z(n) = \bar{z}(n)$. □

**Remark 4.12.** Suppose that system (4.1) has an $N$-periodic solution $\{x(n)\}_{n\ge0}$ and $\{z(n)\}_{n\ge0}$ is the unique $N$-periodic solution (4.12) of (4.2). Then, $\psi(n) = z(n) - x(n)$ is a solution of the equation

$$\psi(n) = \sum_{j=-\infty}^{-1} B(n, j)\phi(j) + \sum_{j=0}^{n} B(n, j)\psi(j), \tag{4.14}$$

where $\phi$ is the initial function of $z(n)$. By the variation of constants formula (4.9) we have

$$\psi(n) = \sum_{j=-\infty}^{-1} B(n, j)\phi(j) - \sum_{j=0}^{n} R(n, j)\left(\sum_{i=-\infty}^{-1} B(j, i)\phi(i)\right), \quad n \geq 0.$$

Using the same technique in the proof of the uniqueness of Theorem 4.11, we can show that $\lim_{n\to\infty} \psi(n) = \lim_{n\to\infty} (z(n) - x(n)) = 0$. Consequently, $z(n) \equiv x(n)$ by periodicity. It follows from (4.14) that $\sum_{j=-\infty}^{-1} B(n, j)\phi(j) = 0$.

### 4.2. The periodic attractor

From Theorem 4.11, we know that only one particular bounded initial function generates an $N$-periodic solution of (4.2). By the following Theorem 4.14, we show that all solutions of (4.2) with bounded initial functions are asymptotically $N$-periodic. More specifically, all these solutions are attracted by the $N$-periodic solution (4.12). Let us first give the definition of asymptotically $N$-periodic sequence.

**Definition 4.13.** Let $N$ be a positive integer. A sequence $\phi = \{\phi(n)\}_{n\geq 0} \in \mathcal{S}(\mathbb{E}^d)$ is called asymptotically $N$-periodic if and only if there exists a sequence $\psi = \{\psi(n)\}_{n\geq 0}$ which is $N$-periodic, namely, $\psi(n+N) = \psi(n)$ for each $n \in \mathbb{Z}_+$, such that $\lim(\phi(n) - \psi(n)) = 0$ as $n \to \infty$. In this case, we write $\phi \sim \psi$.

We are now in a position to give a theorem on periodic attraction.

**Theorem 4.14.** *Let the Assumptions H2, H3 and H4 and (4.3a) hold. Then all solutions of (4.2) with bounded initial function on $\mathbb{Z}^-$ are asymptotically $N$-periodic. In fact, all these solutions tend to the $N$-periodic solution (4.12) as $n \to \infty$.*

**Proof.** Suppose that $\{y(n)\}_{n\geq 0}$ is a solution of (4.2) with bounded initial data $\phi(r)$, $r \in \mathbb{Z}^-$. Then,

$$y(n) = f(n) + \sum_{r=-\infty}^{n} B(n, r)y(r) = f(n) + \sum_{j=-\infty}^{-1} B(n, r)\phi(r) + \sum_{r=0}^{n} B(n, r)y(r).$$

From the variation of constants formula, we obtain

$$\begin{aligned} y(n) &= f(n) + \sum_{r=-\infty}^{-1} B(n, r)\phi(r) - \sum_{r=0}^{n} R(n, r)\left(f(r) + \sum_{j=-\infty}^{-1} B(r, j)\phi(j)\right) \\ &= f(n) - \sum_{r=0}^{n} R(n, r)f(r) + \sum_{r=-\infty}^{-1} B(n, r)\phi(r) - \sum_{r=0}^{n} R(n, r)\sum_{j=-\infty}^{-1} B(r, j)\phi(j). \end{aligned}$$

The last term of the right-hand side of the above equation tends to zero as $n \to \infty$ by the proof of Theorem 4.11. It remains to show that $\sum_{r=-\infty}^{-1} B(n, r)\phi(r)$ tends to zero and $f(n) - \sum_{r=0}^{n} R(n, r)f(r)$

tends to the solution (4.12) as $n \to \infty$. To do this, we notice that every $n \geq 0$ can be written as $n = s + lN$, where $0 \leq s \leq N-1$ and $l \in \mathbb{Z}_+$. Therefore, $n \to \infty$ implies that $l \to \infty$ and vice versa. Let $n = s + lN$. Thus,

$$\sum_{r=-\infty}^{-1} B(n,r)\phi(r) = \sum_{r=-\infty}^{-1} B(s+lN, r)\phi(r) = \sum_{i=-\infty}^{-1-lN} B(s+lN, i+lN)\phi(i+lN)$$
$$= \sum_{i=-\infty}^{-1-lN} B(s,i)\phi(i+lN).$$

It follows from Assumption H2 that $\sum_{r=0}^{\infty} |B(n, n-r)| = \sum_{j=-\infty}^{n} |B(n, j)| < \infty$ for each $n$. Therefore, $|\sum_{i=-\infty}^{-1-lN} B(s,i)\phi(i+lN)| \leq M \sum_{i=-\infty}^{-1-lN} |B(s,i)| \to 0$ as $l \to \infty$, where $M = \sup_{n\leq 0} |\phi(n)|$. Similarly, let $n = s + lN$. Then,

$$f(n) - \sum_{r=0}^{n} R(n,r)f(r) = f(s+lN) - \sum_{r=0}^{s+lN} R(s+lN, r)f(r)$$
$$= f(s+lN) - \sum_{r=-lN}^{s} R(s+lN, r+lN)f(r+lN)$$
$$= f(s) - \sum_{r=-lN}^{s} R(s,r)f(r),$$

which tends to $f(s) - \sum_{r=-\infty}^{s} R(s,r)f(r)$. This proves Theorem 4.14. □

Theorem 4.14 shows that the $N$-periodic solution (4.12) is attractive.

**Remark 4.15.** If the kernel $B(n,m) = K(n-m)$ in (4.1) is of convolution type, then the resolvent $R(n,m) = R^K(n-m)$ is also of convolution type, $B(n+N, m+N) = K(n,m) = K(n-m)$ and $R(n+N, m+N) = R^K(n,m) = R^K(n-m)$ for all $N > 0$. If in addition $\{K(n)\}_{n\geq 0} \in \ell^1(\mathbb{R}^d)$, $\det(I - K(0)) \neq 0$ and $\det(I - \mathcal{Z}\{K\}(z)) \neq 0$ for $|z| \geq 1$, then $\{R^K(n)\}_{n\geq 0} \in \ell^1(\mathbb{R}^d)$, by the discrete Paley-Wiener theorem (see [13]). In this case, one readily sees that the Assumptions H2, H3 and H4 are satisfied. Thus, Theorem 4.11 and Theorem 4.14 include convolution equations as special cases.

## 5. The nonlinear case

Consider the nonlinear discrete Volterra equations on $\mathcal{P}_N(\mathbb{E}^d)$

$$z(n) = f(n) + \lambda \sum_{j=-\infty}^{n} B(n,j)z(j) + Q(n, z(n)) + G(z)(n), \tag{5.1}$$

where

$$f(n+N) = f(n), \quad B(n+N, m+N) = B(n,m), \tag{5.2}$$

for all $n, m \in \mathbb{Z}$ and for some positive integer $N$, $G : \mathcal{P}_N(\mathbb{E}^d) \to \mathcal{P}_N(\mathbb{E}^d)$ is a functional, $Q(n+N, \cdot) = Q(n, \cdot)$, $n \in \mathbb{Z}$, are functions from $\mathbb{E}^d$ to $\mathbb{E}^d$, and the parameter $\lambda$ is a small real number.

In this section, we assume that $G$ and $Q$ satisfy the following conditions:

**Assumption H5.** ($i$) $G(0) = 0$ and $Q(\cdot, 0) \equiv 0$, ($ii$) for each $\tau > 0$, there exists a number $\delta > 0$ such that

$$|G(\phi)(n) - G(\psi)(n)| \le \tau\|\phi - \psi\|_\infty \text{ for all } n \tag{5.3}$$

whenever $\phi, \psi \in \mathcal{P}_N(\mathbb{E}^d)$ with $\|\phi\|_\infty \le \delta$, $\|\psi\|_\infty \le \delta$, and ($iii$) for all $n$

$$|Q(n, x) - Q(n, y)| < \tau|x - y|, \quad \text{if} \quad |x| < \delta,\ |y| < \delta. \tag{5.4}$$

As examples of this kind of "small" functional $G$ we have those of the form

$$G(\phi)(n) = \sum_{j=-\infty}^{n} C(n, j)\phi^2(j) \quad \text{or} \quad G(\phi)(n) = \phi(n)\sum_{j=-\infty}^{n} C(n, j)\phi(j)$$

where $C(n + N, m + N) = C(n, m)$, which occur in many applications.

If $\{f(n)\}_{n\ge 0} \in \mathcal{P}_N(\mathbb{E}^d)$ is $N$-periodic, which means $f(n + N) = f(n)$ for every $n \ge 0$, we can readily extend it to be $N$-periodic in $-\infty < n < \infty$ by setting

$$f(n) = f(-n) \text{ if } n < 0. \tag{5.5}$$

In the sequel, we always assume this extension when needed and use the same notation $f$ for the doubly infinite sequence as for the original sequence. In this case, we note that $\mathcal{P}_N(\mathbb{E}^d)$ is the Banach space of doubly infinite sequences and the norm $\|\{f(n)\}\|_\infty = \sup_{n\in\mathbb{Z}}|f(n)|$ is the same as that of the one-side infinite sequences.

Let $S(\varepsilon) = \{\phi \in \mathcal{P}_N(\mathbb{E}^d) : \|\phi\|_\infty \le \varepsilon\}$ (a ball of radius $\varepsilon > 0$); it is a closed subset of $\mathcal{P}_N(\mathbb{E}^d)$.

**Theorem 5.1.** *Let Assumption H2 and Assumption H5 hold. Then there exists $\varepsilon_0 > 0$ such that for each $\varepsilon$, $0 < \varepsilon \le \varepsilon_0$, there exists a number $\eta > 0$ such that if $\|\{f(n)\}\|_\infty \le \eta$ and $|\lambda| < \eta$, then Eq. (5.1), namely*

$$z(n) = f(n) + \lambda\sum_{j=-\infty}^{n} B(n, j)z(j) + Q(n, z(n)) + G(z)(n),$$

*has a unique periodic solution $\{z(n)\}$ in $S(\varepsilon)$.*

**Proof.** By Lemma 4.10, there is a constant $C > 0$ such that $\sup_{n\in\mathbb{Z}}\sum_{r=0}^{\infty}|B(n, n-r)| \le C\ (< \infty)$. By AssumptionAH5, there exists a number $\delta > 0$ such that

$$|G\phi - G\psi| \le \frac{\{\|\phi - \psi\|_\infty\}}{\{4C\}} \quad \text{if} \quad \|\phi\|_\infty \le \delta, \quad \|\psi\|_\infty \le \delta$$

and

$$|Q(n, x) - Q(n, y)| < \frac{\{|x - y|\}}{\{4C\}} \quad \text{if} \quad |x| \le \delta \text{ and } |y| \le \delta$$

for all $n$. Let $\varepsilon_0 = \delta$. Given $\varepsilon$, $0 < \varepsilon \le \varepsilon_0$, define

$$\eta = \min\left\{\delta, \frac{\varepsilon}{\{4C\}}, \frac{\varepsilon}{\{4\}}, \frac{1}{\{4C\}}, 1\right\} \quad \text{and} \quad S(\varepsilon) = \{\phi \in \mathcal{P}_N(\mathbb{E}^d) : \|\phi\|_\infty \le \varepsilon\}.$$

For any $\phi \in S(\varepsilon)$, $\{f(n)\} \in \mathcal{P}_N(\mathbb{E}^d)$ with $\|f\|_\infty \le \eta$ and $\lambda$ satisfying $|\lambda| < \eta$, we define the operator $T$ on $S(\varepsilon)$ by the relation

$$(T\phi)(n) = f(n) + \lambda \sum_{j=-\infty}^{n} B(n, j)\phi(j) + Q(n, \phi(n)) + G(\phi)(n).$$

It is readily shown that $T\phi \in \mathcal{P}_N(\mathbb{E}^d)$. It follows from Assumption H2 and Assumption H5 that $T\phi$ is well defined and

$$|(T\phi)(n)| \le |f(n)| + |\lambda| \sum_{j=-\infty}^{n} |B(n, j)||\phi(j)| + |Q(n, \phi(n))| + |G(\phi)(n)|$$

for all $n \ge 0$. Thus, $\|T\phi\|_\infty \le \varepsilon$. Similarly, if $\phi, \psi \in S(\varepsilon)$, then

$$\begin{aligned}|T\phi(n) - T\psi(n)| \le |\lambda| \sum_{j=-\infty}^{n} |B(n, j)||\phi(j) - \psi(j)| + |Q(n, \phi(n)) - Q(n, \psi(n))| \\ + |G(\phi)(n) - G(\psi)(n)|\end{aligned}$$

and

$$\|T\phi - T\psi\|_\infty \le C\frac{\varepsilon}{4C}\|\phi - \psi\|_\infty + \frac{\varepsilon}{4C}\|\phi - \psi\|_\infty + \frac{\varepsilon}{4C}\|\phi - \psi\|_\infty \le \frac{3}{4}\varepsilon.$$

Hence, $T$ is a contraction mapping on $S(\varepsilon)$. By the contraction mapping theorem (see [16]), the system (5.1) has a unique solution in $\mathcal{P}_N(\mathbb{E}^d)$. The proof is completed. □

Next we represent this periodic solution using the resolvent of $B_\lambda = \lambda B(n, m)$. We assume $\det(I - \lambda B(n, n)) \ne 0$. As in Section 4, we define the resolvent $R_\lambda(n, m)$ as the unique solution of the following two equations

$$R_\lambda(n, m) = \sum_{j=m}^{n} R_\lambda(n, j)B_\lambda(j, m) - B_\lambda(n, m). \tag{5.6}$$

Thus, $R_\lambda(n, m)$ is $N$-periodic, and the solution $z(n)$ of (5.1) (if it exists) can be given by *variation of constants formula*

$$\begin{aligned}z(n) = f(n) - \sum_{j=0}^{n} R_\lambda(n, j)(f(j) + Q(j, z(j)) + G(z)(j)) + \sum_{j=-\infty}^{-1} B_\lambda(n, j)z(j) \\ - \sum_{j=0}^{n} R_\lambda(n, j)\left(\sum_{i=-\infty}^{-1} B_\lambda(j, i)z(i)\right).\end{aligned} \tag{5.7}$$

**Corollary 5.2.** *Suppose that $B_\lambda(n, m)$, $R_\lambda(n, m)$ satisfy Assumptions H2, H4 and H5. Then the unique solution $z(n)$ of (5.1) satisfies the equation*

$$z(n) = f(n) - \sum_{j=-\infty}^{n} R_\lambda(n, j)(f(j) + Q(j, z(j)) + G(z)(j)). \tag{5.8}$$

**Proof.** Since $z(n)$ is $N$-periodic, one gets for any integer $l > 0$

$$\begin{aligned} z(n) = z(n+lN) &= \left( f(n+lN) - \sum_{j=0}^{n+lN} R_\lambda(n+lN, j)(f(j) + Q(j, z(j)) + G(z)(j)) \right) \\ &\quad + \left( \sum_{j=-\infty}^{-1} B_\lambda(n+lN, j)z(j) - \sum_{j=0}^{n+lN} R_\lambda(n+lN, j) \left( \sum_{i=-\infty}^{-1} B_\lambda(j, i)z(i) \right) \right) \\ &= \left( f(n) - \sum_{j=-lN}^{n} R_\lambda(n, j)(f(j) + Q(j, z(j)) + G(z)(j)) \right) \\ &\quad + \sum_{j=-\infty}^{-1-lN} B_\lambda(n, j)z(j) - \sum_{k=-lN}^{n} R_\lambda(n, k) \left( \sum_{i=-\infty}^{-1-lN} B_\lambda(k+lN, i+lN)z(i+lN) \right) \\ &= \left( f(n) - \sum_{j=-lN}^{n} R_\lambda(n, j)(f(j) + Q(j, z(j)) + G(z)(j)) \right) \\ &\quad + \left( \sum_{j=-\infty}^{-1-lN} B_\lambda(n, j)z(j) - \sum_{k=-lN}^{n} R_\lambda(n, k) \left( \sum_{i=-\infty}^{-1-lN} B_\lambda(k, i)z(i) \right) \right). \end{aligned}$$

Let $l \to \infty$, the first term of the above equation tends to (5.8) and the second term to zero. Corollary 5.2 is proved. □

**Remark 5.3.** If the kernel $\{B(n, j) = K(n-j)\}$ is of convolution type and $\{K(n)\}_{n\geq 0}$ is in $\ell^1(\mathbb{R}^d)$, then the resolvent $R_\lambda(n, m) = R^K_\lambda(n-m)$ is also of convolution type and $\{R^K_\lambda(n)\} \in \ell^1(\mathbb{R}^d)$ if and only if $\det(I - \lambda K(0)) \neq 0$ and $\det(I - \lambda \mathcal{Z}\{K\}(z)) \neq 0$ for $|z| \geq 1$, by virtue of the discrete Paley-Wiener Theorem (see [13]), where $\mathcal{Z}\{K\}(z)$ is the $Z$-transform of $\{K(n)\}_{n\geq 0}$.

If $\lambda = 0$, the Eq. (5.1) reduces to $z(n) = f(n) + Q(n, z(n)) + G(z)(n)$, an implicit ordinary difference equation. Then Theorem 5.1 includes implicit ordinary difference equations as a special case.

**Remark 5.4.** We may relate our work to that in [14], concerning nonlinear equations, that are analogous to (4.2) but have the form

$$x(n) = f(n) + \sum_{j=0}^{n} B(n, j)\{x(j) + G_j(x(j))\}, \quad n \geq 0. \tag{5.9}$$

Notice that, in contrast to (4.2), the lower limit of summation in (5.9) is 0. We have discussed asymptotically periodic solutions for (5.9) in [14]. We show that if $\{f(n)\}_{n\geq 0}$ is an asymptotically periodic sequence, the Assumptions H3 and H4 and the condition (4.3a) are satisfied, then (5.9) has a unique asymptotically periodic solution under certain conditions for $G_j(\cdot)$ (see [14] for details).

## References


[1] C.T.H. Baker, A perspective on the numerical treatment of Volterra equations, J. Comput. Appl. Math. 125 (2000) 217–249.

[2] C.T.H. Baker, Y. Song, Discrete Volterra operators, fixed point theorems and their application, Nonlinear Stud. 10 (2003) 79–101.

[3] H. Brunner, P.J. van der Houwen, The Numerical Solution of Volterra Equations, North-Holland Publishing Co., Amsterdam, 1986. ISBN 0-444-70073-0.

[4] L.A.V. Carvalho, On a method to investigate bifurcation of periodic solutions in retarded differential equations, J. Differ. Equat. Appl. 4 (1998) 17–27.

[5] N. Dunford, J.T. Schwartz, Linear Operators. Part I. General Theory, Wiley, New York, 1988.

[6] S.N. Elaydi, Periodicity and stability of linear Volterra difference systems, J. Math. Anal. Appl. 181 (1994) 483–492.

[7] S.N. Elaydi, An Introduction to Difference Equations, Springer, New York, 1996.

[8] S.N. Elaydi, S. Zhang, Periodic solutions of Volterra difference equations with infinite delay. Part I. The linear case, in: Proceedings of the First International Conference on Difference Equations, San Antonio, TX, 1994 , pp. 163–174, Gordon and Breach, Luxembourg, 1995.

[9] S.N. Elaydi, S. Zhang, Periodic solutions of Volterra difference equations with infinite delay. Part II. The nonlinear case, in: Proceedings of the First International Conference on Difference Equations, San Antonio, TX, 1994, pp. 175–183, Gordon and Breach, Luxembourg, 1995.

[10] F. Dannan, S.N. Elaydi, P. Liu, Periodic solutions of difference equations, J. Differ. Equat. Appl. 6 (2000) 203–232.

[11] V.B. Kolmanovskii, A.D. Myshkis, J.-P. Richard, Estimate of solutions for some Volterra difference equations, Nonlinear Anal. TMA 40 (2000) 345–363.

[12] W. Pogorzelski, in: Jacques J. Schorr-Con, A. Kacner and Z. Olesiak (Eds.). Integral Equations and Their Applications, vol. I. Translated from the Polish, Pergamon Press, Oxford; PWN-Polish Scientific Publishers, Warsaw, 1966.

[13] Y. Song, Discrete Volterra Equations, Ph.D. Thesis, 2002, The University of Manchester, Manchester.

[14] Y. Song, C.T.H. Baker, Perturbation theory for discrete Volterra equations, J. Differ. Equat. Appl. 9 (2003) 969–987.

[15] Y. Song, Linear discrete Volterra equations—Discrete Paley–Wiener theorem and its generalization, submitted for publication.

[16] D.R. Smart, Fixed point theorems, Cambridge Tracts in Mathematics, No. 66. Cambridge University Press, London, 1974.