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# Cayley, Sylvester, and Early Matrix Theory\*

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Tony Crilly. *Arthur Cayley: Mathematician Laureate of the Victorian Age*. Johns Hopkins University Press, Baltimore, MD, USA, 2006. xxi+610 pp. ISBN 0-8018-8011-4.

Karen Hunger Parshall. *James Joseph Sylvester. Jewish Mathematician in a Victorian World*. Johns Hopkins University Press, Baltimore, MD, USA, 2006. xiii+461 pp. ISBN 0-8018-8291-5.

The year 2008 marks the 150th anniversary of “A Memoir on the Theory of Matrices” by Arthur Cayley (1821–1895) [3]—the first paper on matrix algebra. Prior to this paper the theory of determinants was well developed, and Cauchy had shown that the eigenvalues of a real matrix are real (in the context of quadratic forms). Yet the idea that an array of numbers had algebraic properties that merited study in their own right was first put forward by Cayley. The term “matrix” had already been coined in 1850 by James Joseph Sylvester (1814–1897) [20]. The names of Cayley and Sylvester are of course well known to any student of linear algebra and matrix analysis, through eponymous objects such as the Cayley–Hamilton theorem, the Cayley transformation, Sylvester’s inertia theorem, and the Sylvester equation. The lives of these two mathematicians have been well documented during the century or so since their deaths, for example in [1, Chap, 21], [13], [14], [15], [18]. So what is the significance of these two new biographies, both published in 2006 by Johns Hopkins University Press? There are two answers.

First, both authors are the leading experts on the respective mathematicians, having spent much of their lives studying their work, their voluminous correspondence, and their place in the contemporary world of mathematics. Consequently, both biographies are authoritative and comprehensive, and they include new information that has come to light only in recent years. Second, as their titles indicate, both biographies put the lives of Cayley and Sylvester into their proper historical context, thereby giving insight into what it was like to be a mathematician in the 19th century and explaining the particular problems that Cayley, and more particularly Sylvester because of his Jewish religion, faced in carving out a career as a research mathematician.

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In several aspects, Cayley and Sylvester’s careers were remarkably similar:

	<b>Cayley</b>	<b>Sylvester</b>
Enter Cambridge University	Trinity College, 1838	St. John’s College, 1831
Wrangler in Tripos examinations	Senior Wrangler, 1842	Second wrangler, 1837
Work in London	Pupil barrister from 1846; called to the Bar in 1849	Actuary from 1844; pupil barrister from 1846; called to the Bar in 1850
Elected Fellow of the Royal Society	1852	1839
President of the London Mathematical Society (LMS)	1868–1869	1866–1867
Awarded Royal Society Copley Medal	1882	1880
Awarded LMS De Morgan Medal	1884	1887
British Association for the Advancement of Science	President, 1883	Vice President, 1863–1865; President of Section A, 1869

But this bare outline hides some important differences. While Cayley was elected to the Sadleirian Chair at Cambridge in 1863, aged 42, where he spent the rest of his career, Sylvester had a string of chairs, at University College London (1838, age 24), the University of Virginia, Charlottesville (1841–1842), The Royal Military Academy, Woolwich, London (1855–1870), Johns Hopkins University (1876–1883), and Oxford University (Savilian Chair of Geometry, 1883 until his death). The convoluted story of Sylvester’s appointment to, and tenure of, these chairs makes interesting reading. His happiest time was at the newly founded Johns Hopkins University, where he went at the age of 62. He was a trusted adviser to the president, established a research group, founded the American Journal of Mathematics, and helped to build up mathematical research in the USA [19].

While Cayley and Sylvester are both probably best known today for their work on matrices, this is just a small part of their work, represented in relatively few papers (especially on Cayley’s part), and during their lifetimes they were much better known for other things. Indeed, Crilly notes that (p. 520) “Statistically, Cayley’s attention to matrix algebra is even slighter than his attention to group theory and is insignificant when compared to the large corpus he produced on invariant theory.” In addition to the subjects just mentioned, Cayley contributed to geometry, elliptic functions, and graph theory, while Sylvester’s work was mainly algebraic.

Cayley and Sylvester led the British school of invariant theory, a subject about which they conversed and corresponded for many years. Indeed they were close friends, having met around 1847 when working near each other in the City of London. Despite, as Crilly puts it, “forming one of the most productive associations in the history of mathematics”, they never wrote a paper together—the convention at the time was for papers to be singly authored. Sylvester

was the more mercurial and temperamental of the two, and suffered periods of mathematical disillusionment, from which he credits Cayley with rescuing him. While Cayley read widely and was well aware of other research going on in Britain and on the continent, Sylvester paid little attention to others' work and became involved in a number of disputes, in which his often brazen defence is perfectly illustrated by his paper with the amazing title "Explanation of the Coincidence of a Theorem Given by Mr Sylvester in the December Number of This Journal, With One Stated by Professor Donkin in the June Number of the Same" [21].

We owe quite a lot of our linear algebra terminology to Sylvester, including the words "annihilator", "canonical form", "discriminant", "Hessian", "Jacobian", "minor", and "nullity". Sylvester coined "latent roots", and after reading his explanation of the term [22], one may wonder why "eigenvalue" has supplanted it:

"It will be convenient to introduce here a notion (which plays a conspicuous part in my new theory of multiple algebra), namely that of the *latent roots* of a matrix—latent in a somewhat similar sense as vapour may be said to be latent in water or smoke in a tobacco-leaf."

Sylvester also introduced the term "derogatory" for a matrix, which he also called "privileged"—suggesting that he regarded this as both a good and a bad property for a matrix to possess. The property arose in connection with determining all matrices that commute with a given matrix, a goal that was completely attained not by him but by his German contemporaries using a more sophisticated line of attack exploiting canonical forms.

Cayley introduced the vertical bar notation for determinants, in a paper written while he was an undergraduate. He also devised two different notations to distinguish between the products  $AB^{-1}$  and  $B^{-1}A$  in the context of groups:

$$\left. \begin{array}{c} A \\ B \end{array} \right\} , \quad \left[ \begin{array}{c} A \\ B \end{array} \right]$$

[2, p. 71], [5] and, in an 1860 letter to Sylvester [18, Letter 45],

$$\underset{\sim}{A} \underset{\sim}{B} , \quad \overset{\sim}{A} \overset{\sim}{B} .$$

Cayley was presumably not too concerned about the difficulty of typesetting his notation—even in L<sup>A</sup>T<sub>E</sub>X these expressions represent a challenge. Taber [26] later suggested

$$A:B, \quad \frac{A}{B}.$$

None of these notations caught on, but in 1928 Hensel [11] suggested the notation  $A/B$  and  $B \setminus A$ , which is nowadays used to great effect in MATLAB [16].

Both books are replete with many pages of notes, detailed reference lists, excellent indices, and, in Crilly’s case, appendices on “Arthur Cayley’s Social Circle” and “Glossary of Mathematical Terms”. Rummaging around these appurtenances reveals all sorts of interesting tidbits. My favourite concerns the Cayley–Hamilton theorem. Crilly points out on p. 470 that in a letter to Sylvester, Cayley actually stated a more general version of the theorem, which he did not mention in the subsequent paper. That version says that if the square matrices  $A$  and  $B$  commute and  $f(x, y) = \det(xA - yB)$  then  $f(B, A) = 0$ . I have not seen this result in any linear algebra or matrix theory textbook, but various generalizations of it have been derived over the years; for references, see [4].

My particular motivation for reading these books was my interest in Cayley and Sylvester’s roles in the development of matrix theory—and in particular in the theory of matrix functions, the history of which I summarize in [12]. Crilly notes (p. 230) that

“Cayley’s ‘Memoir,’ which could have been a useful starting point for further developments, went largely ignored . . . His habit of instant publication and not waiting for maturation had the effect of making the idea available even if it was effectively shelved.”

In fact, when Cayley visited Sylvester in Baltimore in the spring semester of 1882, Sylvester told Cayley of his own new discovery of a theory of matrices, only to be reminded by Cayley of the 1858 memoir (apropos of which Cayley had written to Sylvester concerning the Cayley–Hamilton theorem, as mentioned above). This did not stop Sylvester from explaining his rediscovery, in characteristically robust and prolix manner [23]:

“Much as I owe in the way of fruitful suggestion to Cayley’s immortal memoir, the idea of subjecting matrices to the additive process and of their consequent amenability to the laws of functional operation was not taken from it, but occurred to me independently before I had seen the memoir or was acquainted with its contents; and indeed forced itself upon my attention as a means of giving simplicity and generality to my formula for the powers or roots of matrices, published in the *Comptes Rendus* of the Institute for 1882 (Vol. 94, pp. 55, 396).”

It was Sylvester, not Cayley, who took up the development of matrix theory, towards the end of his time at Johns Hopkins. His enthusiasm is illustrated by his attempt to teach the theory of substitutions out of a new book by Netto. Sylvester

“lectured about three times, following the text closely and stopping sharp at the end of the hour. Then he began to think about matrices again. ‘I must give one lecture a week on those,’ he said. He could not confine himself to the hour, nor to the one lecture a week. Two weeks were passed, and Netto was forgotten entirely and never mentioned again.” (Parshall, p. 271, quoting Ellery W. Davis).

For the reader interested in the history of matrix theory these two biographies provide important information, but naturally there are other more concise and wide-ranging sources. For the complete picture I have found useful the papers by Grattan-Guinness and Ledermann [6], Hawkins [7] (essentially a shorter version of [9]), [8], [10], and Parshall [17] (a particular favourite, which is much broader than its title indicates). In these papers the important contributions of the Berlin school (Frobenius, Weierstrass, Kronecker) are explained.

Much of both books (particularly Crilly's) is concerned with Cayley and Sylvester's work on invariant theory, and I found these parts the least compelling. Crilly likens Cayley's search for invariants and covariants to Victorian naturalists' search for scientific specimens, and notes (p. 195) that his papers are "typically discursive, contain little formal proof, and many of them simply assemble the specimens."

Crilly and Parshall have both had a huge amount of material from which to weave their stories, and they have drawn on it to present an all-round picture of the extraordinary lives of these two mathematicians. Both books are superbly written and they are surely the definitive biographies of their subjects.

One hundred and fifty years after matrix theory began these books may spur the reader to consult some of the original papers. It is worth noting that the multi-volume collected works of Cayley and Sylvester are both freely available online at the University of Michigan Historical Mathematics Collection (<http://quod.lib.umich.edu/u/umhistmath/>), although this site is not easy to navigate.

Finally, the reader whose interest in these books has been piqued might ask "Which is the better?", or "Which should I read first?" My own answer has varied with time since I finished reading them; suffice it say I don't think either will disappoint.

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