Infinite dimensional second order differential equations via T^2M

Aghasi, M and Dodson, CTJ and Galanis, GN and Suri, A

2007

MIMS EPrint: **2005.46**

ISSN 1749-9097

NONLINEAR ANALYSIS 67 (2007) 2829-2838

INFINITE DIMENSIONAL SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS VIA T^2M

M. AGHASI, C.T.J. DODSON, G.N. GALANIS, AND A. SURI

Abstract. The vector bundle structure obtained on the second order (acceleration) tangent bundle T^2M of a smooth manifold M by means of a linear connection on the base provides an alternative way for the study of second order ordinary differential equations on manifolds of finite and infinite dimension. Second order vector fields and their integral curves could provide a new way of representing and solving a wide class of evolutionary equations for states on Fréchet manifolds of sections that arise naturally as inequivalent configurations of a physical field. The technique is illustrated by examples in the framework of Banach and Fréchet spaces, and on Lie groups, in particular discussing the case of autoparallel curves, which include geodesics if the connection is induced by a Riemannian structure.

Introduction

Second order ordinary differential equations on manifolds have received renewed geometric attention in recent years from interactions with jet fields, linear and non-linear connections, Lagrangians and Finsler structures (cf., for instance, [2], [3], [5], [25]). Applications in physical field theory have a central role in the theory of time-dependent Lagrangian particle systems (see [23], [24]). Sufficient methods for the study of equations of such type have so far been developed only for those known as sprays, which correspond to linear connections.

We propose an alternative way of studying second order ordinary differential equations on a smooth manifold M. This approach uses the second order tangent bundle T^2M of M, consisting of all equivalence classes of curves in M that agree up to their acceleration. Then T^2M can be endowed with a vector bundle structure in the presence of a linear connection on M (see [6], [7]). This vector bundle structure is strongly dependent on the choice of the linear connection on the base manifold and thereby differs from the classical fibre bundle of 2-jets; the local sections of T^2M can be used to describe in detail second order ordinary differential equations on M.

Our methodology is suitable for Banach modelled manifolds, and serves also as a basis for the study of second order ordinary differential equations on a wide class of Fréchet manifolds. Fréchet spaces of sections arise naturally as configurations of a physical field and evolution equations naturally involve

2000 Mathematical Subject Classification. Primary 58B25; Secondary 58A05.

second order operators. The moduli space of inequivalent configurations of a physical field is the quotient of the infinite-dimensional configuration space $\mathcal{X}$ by the appropriate symmetry gauge group. Typically, $\mathcal{X}$ is modelled on a Frechét space of smooth sections of a vector bundle over a closed manifold and is a Hilbert Lie group. Inverse limit Hilbert manifolds and inverse limit Hilbert groups, introduced by Omori [21, 22], provide an appropriate setting for the study of the Yang-Mills and Seiberg-Witten field equations.

Let M be a finite-dimensional path-connected Riemannian manifold. The free loop space of all smooth maps from the circle group S^1 to M is a Fréchet manifold ΛM, cf. Manoharan [18, 19]. A string structure is defined as a lifting of the structure group to an S^1-central extension of the loop group. Suppose that $\widetilde{G} \to \widetilde{P} \to X$ is a lifting of a principal Fréchet bundle $G \to P \to X$ over a Fréchet manifold X and further that $S^1 \to \widetilde{G} \to G$ is an S^1-central extension of G. Manoharan showed that every connection on the principal bundle $G \to P \to X$ together with a $\widetilde{G}$-invariant connection on $S^1 \to \widetilde{P} \to P$ defines a connection on $\widetilde{G} \to \widetilde{P} \to X$.

The Fréchet problem is complicated by lack of a general solvability theory for ordinary differential equations on the models; that inhibits the establishment of existence and uniqueness of solutions from initial conditions, analogous to the cases of finite dimensional and Banach spaces. Neeb [20] discusses this in his Monastir Lecture Notes on infinite-dimensional Lie groups. However, if one restricts to the category of Fréchet manifolds that can be viewed as projective limits of Banach manifolds, then the difficulty is eased. This approach proves to be compatible with the taking of projective limits, so leading to a new way of studying and solving second order differential equations on Fréchet manifolds.

Our approach is illustrated by some examples in the last section of the paper. We show that second order vector fields may be used for the description of a class of autoparallel curves on infinite dimensional manifolds, including geodesics in the Riemannian case. Additionally, we illustrate the case of a Banach or Fréchet space endowed with the canonical flat connection and that of a smooth Lie group with the flat or direct connection.

1. Second Order Vector Fields

In this section we define and study the basic notion for the description of second order differential equations on a smooth manifold M of finite or infinite dimension: That of *second order vector fields*.

The second order tangent bundle of M, T^2M, is the set of all classes $[(c,x)]_2$ of smooth curves $c : (-\varepsilon, \varepsilon) \to M$, $\varepsilon > 0$, with respect to the equivalence:

$$c_1 \approx_x c_2 \Leftrightarrow c_1(0) = c_2(0),\ \ c_1'(0) = c_2'(0) \text{ and } c_1''(0) = c_2''(0).$$

In contrast to the case of TM, T^2M fails to be vector bundle over M as a result of the incompatibilities between the nonlinearity of acceleration and the structure of a vector bundle. However, the presence of a linear connection

$$\nabla : T(TM) \longrightarrow TM$$

on the base manifold M, circumvents this difficulty, endowing T^2M with a natural vector bundle structure.

To be more precise, let $\mathbb{E}$ be the (finite dimensional or Banach) space model of M, $\{(U_\alpha,\phi_\alpha)\}_{\alpha\in I}$ an atlas, $\{(\pi_M^{-1}(U_\alpha),\tau_\alpha)\}_{\alpha\in I}$ the associated atlas of the tangent bundle TM of M and $\{(\pi_{TM}^{-1}(\pi_M^{-1}(U_\alpha)),\widetilde{\tau}_\alpha)\}_{\alpha\in I}$ the analogous trivialization of $T(TM)$. Then, adopting the formalism of Vilms [28], the local expressions

$$\nabla_\alpha := \tau_\alpha\circ\nabla\circ(\widetilde{\tau}_\alpha)^{-1} : \phi_\alpha(U_\alpha)\times\mathbb{E}\times\mathbb{E}\times\mathbb{E}\longrightarrow\phi_\alpha(U_\alpha)\times\mathbb{E},$$

of ∇ have the form

$$\nabla_\alpha(y,u,v,w) = (y, w+\Gamma_\alpha(y)(u,v)),\quad a\in I.$$

Here $\{\Gamma_\alpha\}_{\alpha\in I}$ is the family of Christoffel symbols of ∇:

$$\Gamma_\alpha : \phi_\alpha(U_\alpha)\longrightarrow\mathcal{L}_s^2(\mathbb{E}\times\mathbb{E},\mathbb{E});\quad \alpha\in I,$$

$\mathcal{L}_s^2(\mathbb{E}\times\mathbb{E},\mathbb{E})$ denoting the space of bilinear symmetric mappings from $\mathbb{E}\times\mathbb{E}$ to $\mathbb{E}$. Based on the above we have defined in [6] a vector bundle structure on T^2M over M with fiber type $\mathbb{E}\times\mathbb{E}$. The corresponding local trivializations have the form:

$$\begin{aligned}\Phi_\alpha \ &:\ (\pi_M^2)^{-1}(U_\alpha)\longrightarrow U_\alpha\times\mathbb{E}\times\mathbb{E}\\ &:\ [(c,x)]_2\mapsto\big(x,(\phi_\alpha\circ c)'(0),(\phi_\alpha\circ c)''(0)\\ &\qquad\qquad +\Gamma_\alpha(\phi_\alpha(x))((\phi_\alpha\circ c)'(0),(\phi_\alpha\circ c)'(0))\big)\end{aligned}$$

if π_M^2 stands for the projection $\pi_M^2([(c,x)]_2)=x$.

It is obvious that there is a strong dependence of the vector bundle structure defined on T^2M on the choice of the linear connection ∇ of M. However, these structures are classified via the notion of conjugate connections. More precisely, the vector bundle structures induced on T^2M by two linear connections ∇, ∇' of M are isomorphic if the connections are conjugate by means of a diffeomorphism of M (i.e. the connections commute with the first and second differential of the diffeomorphism, see [7] for details).

Taking into account this characterization, we may proceed with the definition of the notion of second order vector fields.

Definition 1.1. A section $\xi : M\to T^2M$ of the second order vector bundle T^2M, i.e. a smooth map satisfying

$$\pi_M^2\circ\xi = id_M,$$

where id_M denotes the identity map of M, is called a *second order vector field* on the base manifold M.

Of course, this property is sensitive also to the choice of the initial connection ∇, and a change of choice causes corresponding changes in the set of second order vector fields.

The second order vector fields may be viewed also as derivations in the following way: We consider the set of real numbers $\mathbb{R}$ as a 1-dimensional smooth manifold endowed with the identity global chart and the canonical flat connection with Christoffel symbols:

$$\Gamma:\mathbb{R}\longrightarrow\mathcal{L}(\mathbb{R},\mathcal{L}(\mathbb{R},\mathbb{R}))$$

which are vanishing everywhere $\Gamma(y) = 0$, $y \in \mathbb{R}$. Then, the corresponding second order tangent bundle $T^2\mathbb{R}$ becomes a vector bundle with global vector chart

$$\Psi : T^2\mathbb{R} \longrightarrow \mathbb{R}^3 : [(c, x)]_2 \mapsto \big(x, c'(0), c''(0)\big).$$

Based on this construction we may let each second order vector field on M act as a derivation on the set of smooth functions $C^\infty(M, \mathbb{R})$ as follows:

$$\xi : C^\infty(M, \mathbb{R}) \longrightarrow C^\infty(M, \mathbb{R}^2) : f \mapsto \xi(f),$$

where

$$\xi(f)(x) = T_x^2 f(\xi(x)).$$

Here $T_x^2 f$ denotes the second order differential of f on the fiber over x:

$$\begin{aligned} T_x^2 f \quad &: \quad T_x^2 M \longrightarrow T^2_{f(x)}\mathbb{R} \equiv \mathbb{R}^2 : \\ &: \quad [(c, x)]_2 \mapsto [(f \circ c, f(x))]_2 \equiv \big((f \circ c)'(0), (f \circ c)''(0)\big). \end{aligned}$$

The above functor is well-defined and independent of the choice of the curve c as one may easily check. However, although the previous definition is a natural extension of the classical (first-order) case, the existence of a corresponding Lie bracket operator seems to be unreachable due to the fact that the result of this derivation does not remain in the same space.

2. Ordinary Differential Equations of Second Order on a Banach manifold

Having established in the previous section all the necessary background, we proceed here to the study of second order ordinary differential equations on a smooth manifold M modelled on a Banach space $\mathbb{E}$.

Let ξ be a second order vector field on M.

Definition 2.1. An *integral curve* of ξ is a smooth map $\theta : J \to M$, defined on an open interval J of $\mathbb{R}$, if it satisfies the condition

$$T_t^2\theta(\partial_t) = \xi(\theta(t)), \tag{1}$$

where ∂_t is the second order tangent vector of $T_t^2\mathbb{R}$ induced by a curve $c : \mathbb{R} \to \mathbb{R}$ with $c'(0) = 1, c''(0) = 1$.

Note that if we restrict ourselves to the case where the base manifold M is a Banach space $\mathbb{E}$ with differential structure induced by the global chart $(\mathbb{E}, id_{\mathbb{E}})$, then the first part of the above condition reduces to the second derivative of θ:

$$T_t^2\theta(\partial_t) = \theta''(t) = D^2\theta(t)(1, 1).$$

In other words, our definition gives a natural generalization of the notion of second derivative on a manifold M. On the other hand, it offers the opportunity to approach ordinary differential equations of order two on M. Namely, the next result holds.

Theorem 2.2. *Let ξ be a second order vector field on M. Then, the existence of an integral curve θ of ξ is equivalent to the solution of a system of second order differential equations on $\mathbb{E}$.*

Proof. Keeping the formalism of Section 1, we consider $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in I}$ a smooth atlas of M and $\{(U_\alpha, \Phi_\alpha)\}_{\alpha\in I}$ the corresponding local trivialization of T^2M. Then, the local expression of the second derivative $T_t^2\theta(\partial_t)$ takes the form

$$\begin{aligned}\Phi_\alpha(T_t^2\theta(\partial_t)) &= \Phi_\alpha([(\theta\circ c, \theta(t))]_2) \\ &= \big(\theta(t), (\phi_\alpha\circ\theta\circ c)'(0), (\phi_\alpha\circ\theta\circ c)'(0)''(0) + \\ &\quad \Gamma_a((\phi_\alpha\circ\theta)(t))((\phi_\alpha\circ\theta\circ c)'(0), (\phi_\alpha\circ\theta\circ c)'(0))\big)\end{aligned}$$

However,

$$\begin{aligned}(\phi_\alpha\circ\theta\circ c)'(0) &= D(\phi_\alpha\circ\theta)(c(0))(c'(0)) = T_t(\phi_\alpha\circ\theta)(1) = (\phi_\alpha\circ\theta)'(t), \\ (\phi_\alpha\circ\theta\circ c)''(0) &= D^2(\phi_\alpha\circ\theta)(c(0))(c'(0), c'(0)) + D(\phi_\alpha\circ\theta)(c(0))(c''(0)) \\ &= D^2(\phi_\alpha\circ\theta)(t)(1,1) + D(\phi_\alpha\circ\theta)(t)(1) \\ &= (\phi_\alpha\circ\theta)''(t) + (\phi_\alpha\circ\theta)'(t)\end{aligned}$$

As a result,

$$\begin{aligned}\Phi_\alpha(T_t^2\theta(\partial_t)) &= \big(\theta(t), (\phi_\alpha\circ\theta)'(t), (\phi_\alpha\circ\theta)''(t) + (\phi_\alpha\circ\theta)'(t) + \\ &\quad \Gamma_a((\phi_\alpha\circ\theta)(t))((\phi_\alpha\circ\theta)'(t), (\phi_\alpha\circ\theta)'(t))\big)\end{aligned}$$

and the local expression of equation (1) takes the form

$$(\phi_\alpha\circ\theta)'(t) = \Phi_\alpha^{(2)}(\xi(\theta(t))), \tag{2}$$

$$\begin{aligned}(\phi_\alpha\circ\theta)''(t) + (\phi_\alpha\circ\theta)'(t) &+ \Gamma_a((\phi_\alpha\circ\theta)(t))((\phi_\alpha\circ\theta)'(t), (\phi_\alpha\circ\theta)'(t)) \\ &= \Phi_\alpha^{(3)}(\xi(\theta(t))), \quad a\in I,\end{aligned} \tag{3}$$

where $\Phi_\alpha^{(2)}, \Phi_\alpha^{(3)}$ stand for the projection of Φ_α to the second and third factor respectively. □

We have proved in this way that integral curves of second order vector fields generalize the notion of second order ordinary differential equations on manifolds.

However, it is important to notice here that equations (2), (3) do not possess always common solutions, as we show below in the Lie group example of Section 5.5. Therefore, we do not have a general Cauchy type theorem although each equation independently admits a local solution for any initial condition, since the differentiability of all the involved functions guarantees the satisfaction of the necessary Lipschitz conditions.

Nevertheless, it is clear from the proof of Theorem 2.2 that the second order differential equations described by our approach depend not only on the choice of the second order vector field but also on the host geometric background of the manifold, as expressed by the chosen linear connection.

3. The Case of a Lie Group

In this Section we consider the case where M is not an arbitrary Banach modelled manifold but a Lie group $M = (G, \gamma)$. We may generalize in this framework the classical notion of left invariant vector fields:

Theorem 3.1. *Let v be any vector of the second order tangent space of G over the unitary element. Then, a corresponding left invariant second order vector field ξ of G may be constructed.*

Proof. Keeping the formalism of the previous sections, let $c : J \subset \mathbb{R} \to G$ be the curve that realizes v as a second order vector over the unitary element e: $v = [(c, e)]_2$. Then, the desired second order vector field may be defined following the classical pattern:

$$\xi^2 : G \to T^2 G : x \mapsto T_e^2 L_x(v),$$

where L_x denotes the left translation of G through $x \in G$. Obviously ξ^2 is a section of the natural second order projection of G.

The necessary differentiability may be checked through the local charts $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in I}$ of G and $\{(U_\alpha, \Phi_\alpha)\}_{\alpha \in I}$ of T^2G, as defined in Section 1. Indeed, the local expression F of ξ^2 takes the form:

$$\begin{aligned} F(h) &= (\Phi_\alpha \circ \xi^2 \circ \phi_\alpha^{-1})(h) = \Phi_\alpha([(L_x \circ c, x)]_2) = \\ &= (h, (\phi_\alpha \circ L_x \circ c)'(0), (\phi_\alpha \circ L_x \circ c)''(0) + \\ &\quad \Gamma_\alpha(h)((\phi_\alpha \circ L_x \circ c)'(0), (\phi_\alpha \circ L_x \circ c)'(0)), \end{aligned}$$

if $\phi_\alpha(x) = h$. The second variable of the above mapping is the local expression of the classical (first order) left invariant vector field of G that corresponds to the tangent vector realized by the curve c. Using the Leibnitz rule, one may check that

$$(\phi_\alpha \circ L_x \circ c)''(0) = f''(h, 0) \cdot ((0, 1), (0, 1)),$$

where $f = \phi_\alpha \circ \gamma \circ (\phi_\alpha^{-1} \circ c)$ and $\gamma : G \times G \to G$ the multiplication of the Lie group. Therefore, F is a differentiable function of h, a fact that proves the local differentiability of ξ^2.

Finally, it is easy to check that ξ^2 commutes with the left translations of G and their second differentials, i.e. $T^2 L_x \circ \xi = \xi \circ L_x$, $x \in G$, thus the term left invariant vector field of second order for ξ^2 is fully justified. □

A question that naturally rises now is whether these left invariant vector fields of second order admit integral curves, like in the case of TM. *This is not always true*, since, as explained above, the corresponding differential system cannot always be solved. However, the monoparameteric subgroups of G are always such integral curves:

Proposition 3.2. *Every monoparametric subgroup $\beta : \mathbb{R} \to G$ is an integral curve of the second order left invariant vector field ξ^2 of G that corresponds to $\ddot{\beta}(0)$.*

Proof. If we denote by c the curve $c(t) = t + \frac{t^2}{2}$, $t \in \mathbb{R}$, that realizes the basic second order vector field ∂_0 and by μ_t the additive translation of $\mathbb{R}$ by t ($\mu_t(s) = t + s$), then

$$\begin{aligned} \xi^2(\beta(t)) &= T_e^2 L_{\beta(t)}(T_0^2 \beta(\partial_0)) = T_0^2(L_{\beta(t)} \circ \beta)(\partial_0) = T_0^2(\beta \circ \mu_t)(\partial_0) = \\ &= T_t^2 \beta([\mu_t \circ c, t]_2) = T_t^2 \beta(\partial_t) \end{aligned}$$

which proves that β is indeed an integral curve of ξ^2. We have proved in this way, that this type of classical (first order) integral curve of G is an integral curve of order two as well. □

4. Generalization to Fréchet manifolds

In this section we extend our methodology to the framework of Fréchet modelled manifolds. The general case of a second (or even first) order differential equation on such manifolds cannot be confronted successfully using the classical pattern of Banach modelled manifolds. For, on the model spaces an ordinary differential equation may admit no, one or multiple solutions for the same initial condition.

These analytical problems with several applications in theoretical physics (see, e.g. [1], [21, 22], [26]) led a number of authors to propose different methods for the study of certain types of differential equations in Fréchet spaces (see [4], [9], [14], [15]).

In a series of previous papers of the third author ([10], [11], [12]) a new way leading to the solution of a wide class of such types of equations is proposed. This stems from the fact that every Fréchet space is isomorphic to a projective limit of Banach spaces, and the taking of projective limits is compatible with differentiation. These techniques can also be combined with our approach to second order differential equations described above, to provide a way out of the analytic difficulties.

More precisely, let M be a smooth manifold modelled on the Fréchet space $\mathbb{F}$. Since always $\mathbb{F}$ can be realized as a projective limit of Banach spaces $\mathbb{F} \simeq \varprojlim\{\mathbb{E}^i; \rho^{ji}\}_{i,j\in\mathbb{N}}$, we assume that the manifold itself is obtained as the limit of a projective system of Banach modelled manifolds $\{M^i; \varphi^{ji}\}_{i,j\in\mathbb{N}}$ and that is covered by a system of 'projective limit' charts:

For each $x = (x^i) \in M$ there exists a projective system of local charts $\{(U^i_\alpha, \phi^i_\alpha)\}_{i\in\mathbb{N}}$ such that $x^i \in U^i_\alpha$ and the corresponding limit $\varprojlim U^i_\alpha$ is open in M.

Let ∇ be a linear connection on M realized also as a projective limit of connections on the factors M^i. This is equivalent to the fact that the corresponding Christoffel symbols commute with the connecting morphisms of the tangent bundles of M which have the form:

$$g^{ji}_k : T^kM^j \to T^kM^i\colon\ [f,x]^j_k \longmapsto [\phi^{ji}\circ f, \phi^{ji}(x)]^i_k,$$

where $k = 1, 2$ denotes the order of the tangent bundle.

Under these conditions, M can be endowed with a Fréchet manifold structure modelled on $\mathbb{F}$ via the charts $\{(\varprojlim U^i_\alpha, \varprojlim \phi^i_\alpha)\}_{a\in I}$. For the differentiability of mappings in this framework we adopt the definition of Leslie ([16, 17]).

In consequence, the tangent bundles TM and T^2M of M are endowed also with Fréchet manifold structures of the same type modelled on $\mathbb{F}^2$, $\mathbb{F}^4$ respectively. The corresponding local structures are defined by the differentials $\{\varprojlim(T\phi^i_\alpha)\}_{a\in I}$ for the first order tangent bundle and by the projective limits of the trivializations

$$\begin{aligned}
\Phi^i_\alpha \ &: \ (\pi^2_{M^i})^{-1}(U^i_\alpha) \longrightarrow U^i_\alpha \times \mathbb{E}^i \times \mathbb{E}^i \\
&: \ [(c,x)]_2 \longmapsto \big(x, (\phi^i_\alpha\circ c)'(0), (\phi^i_\alpha\circ c)''(0) \\
&\qquad\qquad + \Gamma^i_\alpha(\phi^i_\alpha(x))((\phi^i_\alpha\circ c)'(0), (\phi^i_\alpha\circ c)'(0))\big),
\end{aligned}$$

for T^2M.

Based on the above constructions, we may prove the following main result.

Theorem 4.1. *Every second order vector field ξ on M obtained as projective limit of second order vector fields $\{\xi^i$ on $M^i\}_{i\in\mathbb{N}}$ admits locally a unique integral curve θ satisfying an initial condition of the form $\theta(0) = x$ and $T_t\theta(\partial_t) = y$, $x \in M$, $y \in T_{\theta(t)}M$, provided that the components ξ^i admit also integral curves of second order.*

Proof. Since each ξ^i is a second order vector field on the Banach modelled manifold M^i, the assumption of existence of an integral curve θ^i is equivalent to the system of differential equations:

$$\begin{aligned}
(\varphi^i_\alpha o\theta^i)'(t) \quad &= \quad \Phi^{2,i}_\alpha(\xi^i(\theta^i(t))),\\
(\varphi^i_\alpha o\theta^i)''(t) + (\varphi^i_\alpha o\theta^i)'(t) \quad &+ \quad \Gamma^i_\alpha(\varphi^i_\alpha o\theta^i)(t)[(\varphi^i_\alpha o\theta^i)'(t), (\varphi^i_\alpha o\theta^i)'(t)]\\
&= \quad \Phi^{3,i}_\alpha(\xi^i(\theta^i(t))),
\end{aligned}$$

under the initial conditions $\theta^i(0) = x^i := \varphi^i(x)$, and $T_t\theta^i(\partial_t) = y^i := T_{\theta(t)}\varphi^i(y)$, if $\varphi^i : M = \varprojlim M^i \to M^i$, $i \in \mathbb{N}$, are the canonical projections of the projective limit.

We claim that $\theta := \varprojlim \theta^i$ exists and fulfils the conditions of the theorem. Indeed, we initially observe that for each pair of indices $j \geq i$, $\varphi^{ji} \circ \theta^j$ is also an integral curve of ξ^i since:

$$\begin{aligned}
(\varphi^i_\alpha \circ (\varphi^{ji} \circ \theta^j))'(t) &= (\rho^{ji} \circ \varphi^j_\alpha \circ \theta^j)'(t) = \rho^{ji}((\varphi^j_\alpha \circ \theta^j)'(t)) =\\
&= \rho^{ji}(\Phi^{2,j}_\alpha(\xi^j(\theta^j(t)))) = \Phi^{2,i}_\alpha(g^{ji}_2(\xi^j(\theta^j(t))))\\
&= \Phi^{2,i}_\alpha(\xi^i(\varphi^{ji} o\theta^j(t)))\ ;
\end{aligned}$$

and

$$\begin{aligned}
&(\varphi^i_\alpha \circ (\varphi^{ji} \circ \theta^j))''(t) + (\varphi^i_\alpha \circ (\varphi^{ji} \circ \theta^j))'(t)\\
&+\Gamma^i_\alpha(\varphi^i_\alpha \circ (\varphi^{ji} o\theta^j)(t))[(\varphi^i_\alpha \circ (\varphi^{ji} o\theta^j))'(t), (\varphi^i_\alpha \circ (\varphi^{ji} o\theta^j))'(t))]\\
&= (\rho^{ji} \circ \varphi^j_\alpha \circ \theta^j)''(t) + (\rho^{ji} \circ \varphi^j_\alpha \circ \theta^j)'(t)+\\
&+\Gamma^i_\alpha((\rho^{ji} \circ \varphi^j_\alpha \circ \theta^j)(t))[(\rho^{ji} \circ \varphi^j_\alpha \circ \theta^j)'(t), (\rho^{ji} \circ \varphi^j_\alpha \circ \theta^j)'(t))]\\
&= \rho^{ji}((\varphi^j_\alpha \circ \theta^j)''(t) + (\varphi^j_\alpha \circ \theta^j)'(t)+\\
&+\Gamma^j_\alpha((\varphi^j_\alpha \circ \theta^j)(t))[(\varphi^j_\alpha \circ \theta^j)'(t), (\varphi^j_\alpha \circ \theta^j)'(t)])\\
&= \rho^{ji}(\Phi^{3,j}_\alpha(\xi^j(\theta^j(t)))) = \Phi^{3,i}_\alpha(g^{ji}_2(\xi^j(\theta^j(t))))\\
&= \Phi^{3,i}_\alpha(\xi^i(\varphi^{ji} o\theta^j(t))) \qquad .
\end{aligned}$$

Next we have $\theta^i(0) = x^i$ and $T_t\theta^i(\partial_t) = y^i$, give $(\varphi^{ji} \circ \theta^j)(0) = x^i$ and $T_{\theta^j(t)}\varphi^{ji} o\theta^j(\partial_t) = y^i$. As a result, $\varphi^{ji} \circ \theta^j$ and θ^i will coincide as integral curves of the same second order vector fields over the same initial conditions. Therefore, $\theta = \varprojlim \theta^i$ exists and is smooth as a projective limit of smooth functions (see [13]).

Moreover,

$$(\varphi_\alpha \circ \theta)'(t) = ((\varphi^i_\alpha \circ \theta^i)'(t))_{i\in N} = (\Phi^{2,i}_\alpha(\xi^i(\theta^i(t)))_{i\in N} = \Phi^2_\alpha(\xi(\theta(t)))$$

and

$$
\begin{aligned}
&(\varphi_\alpha \circ \theta)''(t) + (\varphi_\alpha \circ \theta)'(t) + \Gamma_\alpha((\varphi_\alpha \circ \theta)(t))[(\varphi_\alpha \circ \theta)'(t), (\varphi_\alpha \circ \theta)'(t)] \\
&= ((\varphi_\alpha^i \circ \theta^i)''(t) + (\varphi_\alpha^i \circ \theta^i)'(t) + \Gamma_\alpha^i((\varphi_\alpha^i \circ \theta^i)(t))[(\varphi_\alpha^i \circ \theta^i)'(t), (\varphi_\alpha^i \circ \theta^i)'(t)])_{i\in N} \\
&= (\Phi_\alpha^{3,i}(\xi^i(\theta^i(t))))_{i\in N} = \Phi_\alpha^3(\xi(\theta(t)))
\end{aligned}
$$

We have proved in this way that θ is the desired integral curve of the second order vector field ξ. The uniqueness of it under the given initial conditions is obtained following similar reasoning and by checking that each projection of θ via the canonical mappings $\varphi^i : M \to M^i$ is the unique integral curve of ξ^i satisfying $\theta^i(0) = x^i$ and $T_t\theta^i(\partial_t^1) = y^i$. □

5. Applications - Examples.

5.1. **Autoparallel curves and Riemannian geodesics.** Our approach to second order differential equations on manifolds with connection gives also a very simple way to describe the important class of *autoparallel curves*, which include in particular the geodesics in a Riemannian manifold. To be more precise let (M, ∇) be an infinite dimensional manifold endowed with a smooth atlas $\{(U_\alpha, \phi_\alpha)\}_{\alpha\in I}$ and a linear connection ∇ with Christoffel symbols

$$\{\Gamma_\alpha : \phi_\alpha(U_\alpha) \longrightarrow \mathcal{L}_s^2(\mathbb{E}\times\mathbb{E};\mathbb{E})\}_{\alpha\in I}.$$

Keeping the formalism of Sections 1, 2, let ξ be a second order vector field on M induced by a constant curve:

$$\xi(x) = [(c_x, x)]_2,$$

where $c_x(t) = x$, $t \in [0,1]$. Then, the local expression of ξ takes the form:

$$
\begin{aligned}
\Phi_\alpha([(c_x, x)]_2) &= \big(x, (\phi_\alpha \circ c_x)'(0), (\phi_\alpha \circ c_x)''(0) \\
&\qquad + \Gamma_\alpha(\phi_\alpha(x))[(\phi_\alpha \circ c_x)'(0), (\phi_\alpha \circ c_x)'(0)]\big) \\
&= (x, 0, 0).
\end{aligned}
$$

As a result, equations (2),(3) that provide the corresponding integral curves θ of ξ through x (see Theorem 2.2) will reduce to

$$(\phi_\alpha \circ \theta)'(t) = 0, \quad a \in I,$$

which ensures that θ is an autoparallel curve of M. We have proved in this way that *the integral curves of second order vector fields induced by constant functions are autoparallel curves.*

This result holds also for every second order vector field ξ that locally fulfils

$$\Phi_\alpha^2 \circ \xi = \Phi_\alpha^3 \circ \xi, \quad a \in I.$$

In this case, equation (3) reduces to

$$(\phi_\alpha \circ \theta)''(t) + \Gamma_a((\phi_\alpha \circ \theta)(t))((\phi_\alpha \circ \theta)'(t), (\phi_\alpha \circ \theta)'(t)) = 0, \quad a \in I.$$

This is exactly the classical local condition that an arbitrary geodesic of M has to satisfy in the case that M is a Riemannian manifold with Levi-Civita connection ∇.

5.2. **Second order differential equations and autoparallel curves.** In the non-Riemannian case we do not have geodesics, but with a connection we still have autoparallel curves. Via the linear connection ∇, T^2M becomes a vector bundle isomorphic to $TM \times TM$ under the isomorphism,

$$h : T^2M \longrightarrow TM \times TM : [f,x]_2 \longmapsto ([f,x],[\nabla_{Tf}Tf,x]).$$

As shown in [8], we can define the covariant derivative

$$\begin{array}{rcl}\widetilde{\nabla} : C^\infty(TM) \times C^\infty(T^2M) & \longrightarrow & C^\infty(T^2M) \\ (s,\eta) & \longmapsto & h^{-1}(\nabla_s h_1(\eta) \oplus \nabla_s h_2(\eta)),\end{array}$$

where $h = h_1 \oplus h_2$.

Now we are ready to characterize the differential equations associated with an autoparallel curve $\gamma : J \longrightarrow M$.

$$\widetilde{\nabla}_{T\gamma}T^2\gamma = 0 \iff \nabla_{T\gamma}h_1(T^2\gamma) = 0\ ,\ \nabla_{T\gamma}h_2(T^2\gamma) = 0.$$

(i) $\nabla_{T\gamma}h_1(T^2\gamma) = \nabla_{T\gamma}T\gamma$ and according to [28], $\nabla_{T\gamma}T\gamma = \nabla T(T\gamma)$. Hence $\nabla_{T\gamma}h_1(T^2\gamma) = 0$ *iff* for each $\alpha \in I$:

$$\begin{array}{rcl}\tau_\alpha^{-1} \circ \nabla_\alpha \circ \tilde{\tau}_\alpha(T(T\gamma(\partial_t^1)) & = & \tau_\alpha^{-1}((\phi_\alpha \circ \gamma)(t), (\phi_\alpha \circ \gamma)''(t) \\ & + & \Gamma_\alpha((\phi_\alpha \circ \gamma)(t))[(\phi_\alpha \circ \gamma)'(t), (\phi_\alpha \circ \gamma)'(t)]) \\ & = & 0.\end{array}$$

That is $\nabla_{T\gamma}h_1(T^2\gamma) = 0$ *iff* for every $\alpha \in I$,

$$(\phi_\alpha \circ \gamma)''(t) + \Gamma_\alpha((\phi_\alpha \circ \gamma)(t))[(\phi_\alpha \circ \gamma)'(t), (\phi_\alpha \circ \gamma)'(t)] = 0.$$

(ii) $\nabla_{T\gamma}h_2(T^2\gamma) = \nabla T h_2(T^2\gamma) \circ T\gamma = 0$ *iff* for every $\alpha \in I$:

$$\tau_\alpha^{-1} \circ \nabla_\alpha \circ \tilde{\tau}_\alpha(T\eta_2 \circ T\gamma(\partial_t^1)) = 0,$$

which after some tedious calculations results to

$$\widetilde{\nabla}_{T\gamma}T^2\gamma = 0 \iff (\phi_\alpha\circ\gamma)''(t) + \Gamma_\alpha((\phi_\alpha\circ\gamma)(t))[(\phi_\alpha\circ\gamma)'(t), (\phi_\alpha\circ\gamma)'(t)], \alpha \in I.$$

We have previously characterised the isomorphism classes of the vector bundle structures on T^2M, via a conjugacy condition through diffeomorphisms (see also [7]). In this respect, for two *g-conjugate* connections ∇ and ∇', $g : M \to M$ denoting a smooth map, we may state:

Corollary 5.1. *If γ is an autoparallel curve in (M,∇) then $g \circ \gamma$ is autoparallel in (M,∇').*

Proof. It suffices to prove that $\widetilde{\nabla}'_{Tg\circ\gamma}T^2g \circ \gamma = 0$, or equivalently for every $\beta \in I$,

$$(\phi_\beta \circ g \circ \gamma)''(t) + \Gamma'_\beta((\phi_\beta \circ g \circ \gamma)(t))[(\phi_\beta \circ g \circ \gamma)'(t), (\phi_\beta \circ g \circ \gamma)'(t)] = 0.$$

Let $G = \phi_\beta \circ g \circ \phi_\alpha^{-1}$ then,

$$\begin{array}{rl} & (\phi_\beta \circ g \circ \gamma)''(t) + \Gamma_\beta((\phi_\beta \circ g \circ \gamma)(t))[(\phi_\beta \circ g \circ \gamma)'(t), (\phi_\beta \circ g \circ \gamma)'(t)] \\ = & (G \circ \phi_\alpha \circ \gamma)''(t) + DG((\phi_\alpha \circ \gamma)(t))(\Gamma_a((\phi_a \circ \gamma)(t))[(\phi_a \circ \gamma)'(t), (\phi_a \circ \gamma)'(t)] - \\ & D(DG)((\phi_\alpha \circ \gamma)(t))[(\phi_a \circ \gamma)'(t), (\phi_a \circ \gamma)'(t)] \\ = & 0\end{array}$$

□

In the sequel we give three more examples of applications that clarify further our method.

5.3. **Second order differential equations on the model space.** If we consider the manifold $M = \mathbb{E}$ endowed with the differential structure induced by the global chart $(\mathbb{E}, id_{\mathbb{E}})$, and consider the canonical flat connection with trivial Christoffel symbols $\Gamma(x)(u) = 0$, for each $(x, u) \in \mathbb{E} \times \mathbb{E}$, then the second order tangent bundle $T^2\mathbb{E}$ becomes a vector bundle with (total) vector chart

$$\Phi : T^2\mathbb{E} \to \mathbb{E} \times \mathbb{E} \times \mathbb{E} : [(c, x)]_2 \longmapsto \big(x, c'(0), c''(0)\big).$$

This is the case either for a Banach or a Fréchet model space. In this way, if ξ is a second order vector field on $\mathbb{E}$ and $\theta : \mathbb{R} \to \mathbb{E}$ a corresponding integral curve, equations (2.2-2.3) of Theorem 2.2 reduce to

$$\begin{aligned} \theta'(t) &= \Phi^2_\alpha(\xi(\theta(t))), \\ \theta''(t) + \theta'(t) &= \Phi^3_\alpha(\xi(\theta(t))), \end{aligned}$$

which is an ordinary differential equation system of second order on $\mathbb{E}$. Independently the two differential equations can be solved uniquely under given initial conditions in the case of Banach spaces as well as in the Fréchet framework if we assume that the vector field ξ is a projective limit (cf. [12]). However, the coincidence of these solutions is not always achieved.

We observe that the above results ensure that the integral curves of a second order vector field induced by a constant function will themselves be constants.

5.4. **Second order differential equations on Lie groups.** Let G be a Lie group modelled on $\mathbb{E}$ endowed with the so-called *direct connection* ∇^G, that is the unique connection which is (μ, id_G)-conjugate with the canonical flat connection of the trivial bundle $(G \times \mathcal{G}, pr_1, G)$, where

$$\mu : G \times \mathcal{G} \xrightarrow{\simeq} TG : (g, h) \mapsto T_eL_g(h)$$

denotes the left parallelization of G and $\mathcal{G}$ the Lie algebra of G. If $b_a(x)$ gives the local expression of the isomorphism $T_eL_x : T_eG \to T_xG$ with respect to the chart (U_a, ϕ_a), then the Christoffel symbols of ∇^G take the form

$$\Gamma^G_a(x)(k, h) = -Db_a(x)(k, b_a^{-1}(x)(h)); \quad x \in \phi_a(U_a), \ k, h \in \mathbb{E},$$

(for a complete presentation of the notion of direct connection and the relevant proofs we refer to [27]). As a result, equations (2.2-2.3) of Theorem 2.2 take, in this case, the form

$$\begin{aligned} (\phi_\alpha \circ \theta)'(t) &= \Phi^2_\alpha(\xi(\theta(t))), \\ (\phi_\alpha \circ \theta)''(t) + (\phi_\alpha \circ \theta)'(t) &- \\ Db(x)((\phi_\alpha \circ \theta)'(t), b^{-1}(x)((\phi_\alpha \circ \theta)'(t))) &= \Phi^3_\alpha(\xi(\theta(t))), \end{aligned}$$

which is the local form of a second order system on G.

5.5. **A special case of a Lie group.** Let us consider the Lie group $G = \mathbb{R} - \{0\}$ endowed with the multiplication from the field $\mathbb{R}$. It is well known that the first order monoparametric curves here lead to the classical exponential mapping of real numbers. Let us try to find out what is happening with the second order tangent structure induced by a flat connection. In this framework, a second order vector $v = [(c,1)]_2 \in T_1^2 G$ corresponds bijectively to the pair $(c'(0), c''(0)) = (a,b) \in \mathbb{R}^2$. If we denote by ξ^2 the corresponding second order left invariant vector field, then

$$\xi^2(x) = T_1^2 L_x(v) = [(L_x \circ c, x)]_2 \equiv ((L_x \circ c)'(0), (L_x \circ c)''(0)) = (x \cdot a, x \cdot b).$$

If $\beta : \mathbb{R} \to G$ is an integral curve of ξ^2, then it should satisfy the relation $\xi^2(\beta(t)) = T_t^2 \beta(\partial t)$. However, since the basic second order vector field ∂t of $\mathbb{R}$ is realized by the real curve $\gamma(s) = t + s + \frac{s^2}{2}$, $s \in \mathbb{R}$, this would be equivalent to

$$(\beta(t) \cdot a, \beta(t) \cdot b) = [(\beta \circ \gamma, \beta(t)))]_2 \equiv ((\beta \circ \gamma)'(0), (\beta \circ \gamma)''(0)) = (\beta'(t), \beta''(t) + \beta'(t)).$$

As a result, equations (2),(3) of Section 2 reduce here to

$$\begin{aligned} \beta'(t) &= a \cdot \beta(t), \\ \beta''(t) + \beta'(t) &= b \cdot \beta(t). \end{aligned}$$

This system cannot be solved in general. More precisely, the solution is the classical exponential mapping $\beta(t) = e^{at}$, $t \in \mathbb{R}$, only in the case where $a^2 + a = b$. In other words, from all the points of the 2-dimensional $(x,y)-$plane coordinatising the second order vector space $T_1^2 G$, only the points of the curve $y = x^2 + x$ define left invariant second order vector fields that admit integral curves.

References

[1] M.C. Abbati, A. Mania, *On differential structure for projective limits of manifolds*, J. Geom. Phys. **29** (1999), no. 1-2, 35-63

[2] P.L. Antonelli and M. Anastasiei, *The Differential Geometry of Lagrangians which Generate Sprays*, Dordrecht: Kluwer, 1996.

[3] P.L. Antonelli, R. S. Ingarden and M. S. Matsumoto, *The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology*, Dordrecht: Kluwer, 1993.

[4] V.I. Bogachev, *Deterministic and stochastic differential equations in infinite dimensional spaces*, Acta Applicandae Mathematicae 40 (1999), 25-93.

[5] L. Del Riego and P. E. Parker, *Pseudoconvex and disprisoning homogeneus sprays*, Geom. Dedicata 55 (1995) no. 2, 211-220.

[6] C.T.J. Dodson and G.N. Galanis, *Second order tangent bundles of infinite dimensional manifolds*, J. Geom. Phys., 52 (2004), pp. 127-136.

[7] C.T.J. Dodson, G.N. Galanis and E. Vassiliou, *Isomorphism classes for Banach vector bundle structures of second tangents*, to appear in Math. Proc. Camb. Phil. Soc.

[8] C.T.J. Dodson and M.S. Radivoiovici, *Tangent and frame bundles of order two*, Analele stiintifice ale Universitatii "Al. I. Cuza", 28 (1982), 63-71.

[9] E. Dubinsky, *Differential equations and differential calculus in Montel spaces*, Trans. Amer. Math. Soc. 110 (1964), 1-21.

[10] G.N. Galanis, *On a type of linear differential equations in Fréchet spaces*, Annali della Scuola Normale Superiore di Pisa, 4 No. 24 (1997), 501-510.

[11] G.N. Galanis G. and E. Vassiliou, *A Floquet-Liapunov theorem in Fréchet spaces,* Annali della Scuola Normale Superiore di Pisa (4), 27 (1998), 427-436.

[12] G.N. Galanis and P.K. Palamides, *Nonlinear Differential Equations in Frechet spaces and Continuum-Cross-Sections*, Anal. St. Univ. 'Al.I. Cuza', **51** (2005), 41-54.

[13] G.N. Galanis, *Differential and Geometric Structure for the Tangent Bundle of a Projective Limit Manifold*, Rendiconti del Seminario Matematico di Padova, Vol. 112 (2004)

[14] R.S. Hamilton, *The inverse function theorem of Nash and Moser*, Bull. Amer. Math. Soc. 7 (1982), 65-221.

[15] R. Lemmert, *On Ordinary Differential Equations in Locally Convex Spaces,* Nonlinear Analysis, Theory, Methods and Applications 10 (1986), 1385-1390.

[16] J.A. Leslie, *On a differential structure for the group of diffeomorphisms*, Topology **46** (1967), 263-271.

[17] J.A. Leslie, *Some Frobenious theorems in Global Analysis*, J. Diff. Geom. **42** (1968), 279-297.

[18] P. Manoharan. Characterization for spaces of sections. *Proc. Am. Math. Soc.* 126, 4, (1998) 1205-1210.

[19] P. Manoharan. On the geometry of free loop spaces. *Int. J. Math. Math. Sci.* 30, 1, (2002) 15-23.

[20] K-H. Neeb, **Infinite Dimensional Lie Groups**, 2005 Monastir Summer School Lectures, Lecture Notes January 2006.
`http://wwwbib.mathematik.tu-darmstadt.de/Math-Net/Preprints/Listen/pp06.html`

[21] H. Omori, *On the group of diffeomorphisms on a compact manifold*, Proc. Symp. Pure Appl. Math., XV, Amer. Math. Soc. (1970), pp. 167-183.

[22] H. Omori, *Infinite-dimensional Lie groups*, Translations of Mathematical Monographs. **158**. Berlin: American Mathematical Society (1997).

[23] G. Prince, *Toward a classification of dynamical symmetries in classical mechanics*, Bull. Austral. Math. Soc., 27 (1983) no. 1, 53-71.

[24] W. Sarlet and F. Cantrijn, *Generalizations of Noether's theorem in classical mechanics*, SIAM Rev., 23 (1981), no. 4, 467-494.

[25] D.J. Saunders, *Jet fields, connections and second order differential equations.* J. Phys. A: Math. Gen. 20, (1987) 3261-3270

[26] F. Takens, *A global version of the inverse problem of the calculus of variations*, J. Dif. Geom., **14** (1979), 543-562.

[27] E. Vassiliou, *Transformations of Linear Connections II*, Period. Math. Hung., **17(1)** (1986), 1-11.

[28] J. Vilms, *Connections on tangent bundles*, J. Diff. Geom. 41 (1967), 235-243.

Department of Mathematics, Isfahan University of Technology, Isfahan, Iran

E-mail address: `m.aghasi@cc.iut.ac.ir`

School of Mathematics Manchester University, Manchester, M60 1QD, United Kingdom

E-mail address: `ctdodson@manchester.ac.uk`

Section of Mathematics, Naval Academy of Greece, Xatzikyriakion, Piraeus 185 39, Greece

E-mail address: `ggalanis@snd.edu.gr`

Department of Mathematics, Isfahan University of Technology, Isfahan, Iran

E-mail address: `a.suri@math.iut.ac.ir`