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Permutation groups of finite Morley rank: an Oberwolfach talk

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Permutation Groups of Finite Morley Rank

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joint work with

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Oberwolfach
7 August 2007

. . . a time to plant, and a time
to pluck up that which is planted.
Ecclesiastes 3:2

I. Even Type Theorem

II. Black Box Groups

III. Permutation groups of fMr

Groups of finite Morley rank:

- have a rank function

$$\{\text{ Definable sets in } G^n \} \xrightarrow{rk} \mathbb{N} \cup \{0\}$$

- behaves like dimension of Zariski closed sets
- Axiomatised by natural axioms

Algebraicity Conjecture:

Simple infinite groups of fMr are algebraic groups over algebraically closed fields.

Even Type Theorem

Altınel, B, Cherlin:

If a simple group G of fMr contains an infinite elementary abelian 2-group

then G is a Chevalley group over an a.c. field of char 2.

Proof: a book, to appear in AMS Monographs Series, 2007 or 2008.

Final version submitted to the publishers **yesterday**, 6 August 2007.

Corollary:

Classification of simple algebraic groups in characteristic 2

A simple algebraic group over an a.c. field of characteristic 2 is a Chevalley group

Relations with finite group theory

The proof of the Even Type Theorem uses **ideas**,

not the result of CFSG.

Three sources of inspiration:

- Original proof, 1960s–1980s
- Second generation: Gorenstein, Lyons, Solomon, Aschbacher, Smith
- Third generation: Meierfrankenfeld, Stellmacher, Stroth

It is a kind of a "homomorphic image" of CFSG:

shorter, but still a book of 560 pages.

I. Even Type Theorem

II. Black Box Groups

III. Permutation groups of fMr

B, Burdges, Cherlin:

If a simple group G of fMr contains an involution

then either

- G is a Chevalley group in characteristic 2, **or**
- G contains an abelian divisible torsion 2-subgroup ("2-torus").

One of the most surprising proofs in the theory:

uses ideas from computational finite group theory: **black box groups**.

Proof uses a trick from black box group computations:

If t is an involution, we have a partial map

$$\begin{aligned} \zeta : G &\longrightarrow C_G(t) \\ x &\mapsto \sqrt{t \cdot t^x} \cdot x^{-1}, \end{aligned}$$

if $t \cdot t^x$ is "of odd order"

In finite case: used by **Bray**.

Equivariance. If $c \in C_G(t)$ then

$$\begin{aligned}\zeta(cx) &= \sqrt{t \cdot t^{cx}} \cdot (cx)^{-1} \\ &= \sqrt{t \cdot t^{x}} \cdot (x)^{-1} c^{-1} \\ &= \zeta(x) \cdot c^{-1}.\end{aligned}$$

- The domain of ζ is a union of cosets of $C_G(t)$
- ζ is 1–1 on every coset of $C_G(t)$
- In finite case, uniform distribution of random x on G results in uniform distribution of $\zeta(x)$ on $C_G(t)$.
- In fMr case – similar calculations with "dimension" and "degree".

I. Even Type Theorem

II. Black Box Groups

III. Permutation groups of fMr

One of the major applications of CFSG in other areas of mathematics (including model theory):

via classification of 2-transitive groups.

Groups of finite Morley rank appear in model theory as "Galois" groups.

Therefore the study of **actions** of groups of finite Morley rank is relevant.

We are working in the category of **definable** actions.

Rational actions of algebraic group are definable.

A sample result

B, Cherlin:

There exists a function $f(n)$ such that
for a definably primitive permutation group (G, Ω) of fMr,

$$\mathrm{rk}\, G \leqslant f(\mathrm{rk}\, \Omega).$$

Surprisingly, the result was unknown for rational actions of algebraic groups!

Example

Let K be an a.c. field and

$$G = K \oplus \cdots \oplus K$$

be a direct-sum of n copies of the additive group of K.

The following is a faithful action of G on the affine space $\mathbb{A}^2$:

$$(a_1, \ldots, a_n) : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ y + a_1 x + \cdots + a_n x^n \end{pmatrix}$$

Example (transitive imprimitive action)

Again, let

$$A = K \oplus \cdots \oplus K$$

be a K-vector space of dimension n.

Let $T \simeq K^*$ be a torus acting on A by matrices

$$\mathrm{diag}(t, t^2, \ldots, t^n).$$

Let B be a hyperplane in "general position".

Then

$$\bigcap_{t \in T} B^t = 0$$

and the right coset action of the natural semidirect product $G = AT$ on $\Omega = G/B$ is faithful and transitive.

Key idea:

Let (G, Ω) be a definably primitive group.

If

$\mathrm{rk}\, G$ is "very big" relative to $\mathrm{rk}\, \Omega$,

then

G acts **generically** n**-transitively** on Ω,

that is, has a generic orbit on $\Omega^n = \Omega \times \cdots \times \Omega$ for some n.

Not much known about generically transitive actions of algebraic groups.

Popov:

bounds for **algebraic** generically n-transitive actions of reductive algebraic groups in characteristic 0.

Some values:

G	A_l	B_l; $l > 3$	C_l; $l > 2$	D_l; $l > 4$	E_6	E_7	E_8	F_4	G_2
$\mathrm{gtd}(G)$	$l+2$	3	3	3 if l is even 4 if l is odd	4	3	2	2	2

Next key idea: appearance of involutions

If G is generically n-transitive

then it is generically 2-transitive

and contains an **involution**, i.e. an element of order 2.

Involutions play crucial role in the proof.

Heavy guns:

- Sylow 2-subgroups are infinite (**B, Burdges, Cherlin**)
- the even type case is excluded by **Altınel, B, Cherlin**

Can assume existence of 2-tori.

Key Lemma

Let

(G, Ω) be a definably primitive permutation group of fMr,

T a definable divisible abelian subgroup of G,

T_0 its torsion subgroup, and

$O(T)$ the largest definable torsion free subgroup of T.

Then $\mathrm{rk}(T/O(T)) \leq \mathrm{rk}(\Omega)$.

Informally: tori are "rigid".

End of proof

$\mathrm{rk}(\Omega) = k$

$\mathrm{rk}(T/O(T)) \leq k$.

If G is generically n-transitive, $n \gg k$,

G has a huge Sym_n-section (factor of the setwise stabilizer of a generic n-tuple),

and huge elementary abelian 2-subgroups

which starts to contradict to everything, including

Toricity Theorem (**Burdges-Cherlin**):

every involution belongs to a 2-torus.

Explicit bounds?

Conjecture

Let $G = G^\circ$ act faithfully, definably, transitively and generically $(n+2)$-transitively on a set Ω of Morley rank n.

Then the pair (G, Ω) is equivalent to the projective general linear group $\mathrm{PGL}_{n+1}(F)$ acting on the projective space $\mathbb{P}^n(F)$ for some a.c. field F.

Conjecture

Let $H = H^\circ$ act
faithfully, definably and generically n-transitively
on an abelian group V of Morley rank n.

Then

- $V = F^n$ for some a.c. field F, and
- H is $\mathrm{GL}_n(F)$ in its natural action on F^n.

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