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# Curve Crossing for the Reflected Lévy Process at zero and infinity

Mladen Savov \*

## Abstract

Let  $R_t = \sup_{0 \leq s \leq t} X_s - X_t$  be a Lévy process reflected in its maximum. We give necessary and sufficient conditions for finiteness of passage times above power law boundaries at zero and infinity. Information, as to when the expected passage time for  $R_t$  is finite, is given.

*Keywords:* Reflected process, passage times, power law boundaries.

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# 1 Introduction

Let  $X = (X_t, t \geq 0)$ ,  $X_0 = 0$ , be a Lévy process with characteristic triplet  $(\gamma, \sigma, \Pi)$ , where  $\gamma \in \mathbb{R}$ ,  $\sigma \geq 0$  and the Lévy measure  $\Pi$  has the property  $\int_{-\infty}^{\infty} 1 \wedge x^2 \Pi(dx) < \infty$ . We use  $\bar{\Pi}(x)$  to denote the tail of the measure,  $\int_{y \geq |x|} \Pi(dy)$ , and  $\bar{\Pi}^{(+)}$  and  $\bar{\Pi}^{(-)}$  to denote correspondingly the positive and the negative tail of the Lévy measure.

A very important quantity is the characteristic exponent of  $X$ , which has the following form: For any  $\theta \in \mathbb{R}$ ,

$$\Psi(\theta) = i\gamma\theta - \frac{\sigma^2\theta^2}{2} + \int_{-\infty}^{\infty} (e^{i\theta x} - 1 - i\theta x 1_{\{|x| \leq 1\}}) \Pi(dx). \quad (1.1)$$

When  $Ee^{\lambda X_1}$  exists for  $\lambda \in (-\lambda_0, \lambda_0)$ , we can extend  $\Psi$  analytically in some neighborhood of the real line in the complex plane and refer to the Laplace exponent  $\psi$ , which relates to  $\Psi$  via the identity

$$\psi(\theta) = \ln Ee^{\theta X_1} = -\Psi(-i\theta).$$

With any Lévy process we associate the reflected process, which is defined as follows :

$$R_t = \bar{X}_t - X_t \text{ for any } t \geq 0,$$

where  $\bar{X}_t = \sup_{0 \leq s \leq t} X_s$ . We note that whenever we have the notation  $\bar{Y}_t$ , we will mean  $\bar{Y}_t = \sup_{s \in I \cap [0, t]} Y_s$ , where  $I$  is either  $\mathbb{R}_+$  or  $\mathbb{Z}_+$ .

The reflected process plays an important role in the theory of random walks and Lévy processes, and has many applications in finance, genetics etc. Thus for example the optimal time to exercise a "Russian option" is the first time the reflected process crosses a fixed level (Shepp and Shiryaev [12], [13] and Asmussen, [1]). For more discussions and basic properties of the reflected process we refer to [5].

The aim of this paper is to obtain necessary and sufficient conditions (NASC), both at 0 and  $\infty$ , for the almost sure (a.s.) finiteness of passage times of  $R_t$  out of power law regions of the form  $[0, rt^\kappa]$ , for  $r > 0$ ,  $\kappa \geq 0$ , and for the finiteness of expected values of passage times of  $R_t$  out of linear ( $\kappa = 1$ ) or parabolic ( $\kappa = 1/2$ ) regions at  $\infty$ .

Section 2 essentially extends results for random walks in [5]. Thus NASC are obtained as to whether  $\limsup_{t \rightarrow \infty} R_t/t^\kappa$  is a.s. finite or not for any

$\kappa \in \mathbb{R}_+$ . To achieve this we rely on very useful stochastic bounds discovered recently by Doney, see [7]. The section is completed by discussing the finiteness of expected values of passage times of  $R_t$ .

In section 3 new results for the passage time of  $R_t$  at 0 are obtained. The NASC are very similar to the ones at  $\infty$ . It turns out that the integrability of the Lévy measure, see Theorem 3.1 (i) and Theorem 3.2 (ii), plays the same role as the finiteness of particular moments of the Lévy process, see Theorem 2.1 (b).

The proofs are given in section 4 and section 5, while some technical results are collected in the appendix section 6.

## 2 Passage Times above power law boundaries at infinity

In [5] results about the first exit time of the reflected random walk out of power law regions are obtained. These include NASC for the first exit time to be a.s. finite and information about its expectation. In this section we extend these results to reflected Lévy processes. The main techniques in proving Theorem 2.1 and Theorem 2.2, are the stochastic bound discovered recently by Doney [7], for Theorem 2.1, and functions of  $R_t$  defining martingales on  $\mathbb{R}^+$  for Theorem 2.2. A direct approach to Theorem 2.1 is possible, but it requires tedious, long calculations. So we define for  $\kappa \geq 0$ ,  $r > 0$ :

$$\tau_\kappa(r) = \min\{t \geq 0 : R_t > r(t+1)^\kappa\}, \quad (2.1)$$

where  $t+1$  is used for  $t$  to avoid the case when  $\tau_\kappa(r) = 0$  a.s. Let  $X^+ = X_+ = \max\{X, 0\}$  and  $X^- = X_- = \max\{-X, 0\}$ . Our main result is:

**Theorem 2.1.** (a) *Suppose  $\kappa = 0$ . Then  $\tau_0(r) = \tau(r) < \infty$  a.s. for  $\forall r \geq 0$  iff  $X$  is not a positive subordinator, and if so, then in fact  $Ee^{\lambda\tau(r)} < \infty$  at least for  $\lambda$  small enough for all  $r \geq 0$ .*

(b) *Suppose  $\kappa > 0$ . We have  $\tau_\kappa(r) < \infty$  a.s. for all  $r > 0$  iff*

(i) *for  $\kappa > 1$ :  $E(X_1^-)^{1/\kappa} = \infty$ ;*

(ii) *for  $0 < \kappa \leq 1$ :  $E(X_1^-)^{1/\kappa} = \infty$  or  $\liminf_{t \rightarrow \infty} \frac{X_t}{t^\kappa} = -\infty$  a.s.*

**Remarks.** (i) Note that  $\tau_\kappa(r) < \infty$  a.s.,  $\forall r > 0$ , is equivalent to

$$\limsup_{t \rightarrow \infty} \frac{R_t}{t^\kappa} = \infty \text{ a.s.} \quad (2.2)$$

This may not seem obvious, but can be proved similarly as in Lemma 3.1. in [5]. For alternative proof, see [11].

(ii) Also note, that if the embedded random walk,  $\widehat{X} = (X_n, n > 0)$ , is considered, then the following inequality holds

$$R_n^{\widehat{X}} \leq R_n, \quad (2.3)$$

where  $R^{\widehat{X}}$  is the reflected process for  $\widehat{X}$ . In turn (2.3) entails

$$\tau_\kappa^{\widehat{X}}(r) \geq \tau_\kappa(r), \quad (2.4)$$

for any  $\kappa \geq 0$  and  $r > 0$ .

(iii) We exclude the case of positive subordinator since then  $R_t \equiv 0$ . In this case obviously  $\tau_\kappa(r) = \infty$  a.s.

(iv) For analytic conditions equivalent to  $\liminf_{t \rightarrow \infty} \frac{X_t}{t^\kappa} = -\infty$  a.s. we refer to [6].

The second result considers the expected value of the passage time of  $R_t$  above linear and square root boundaries. It will extend the corresponding result in [5].

**Theorem 2.2.** (a) Suppose  $EX_1^2 = \alpha^2 < \infty$  and  $EX_1 = 0$ . Then

(i)  $E\tau_{1/2}(\alpha r) < \infty$  for  $r < 1$ ;

(ii)  $E\tau_{1/2}(\alpha r) = \infty$  for  $r \geq 1$ .

(b) Suppose  $EX_1 < 0$ ,  $E|X_1| < \infty$  and  $E(X_1^+)^2 < \infty$ . Then

(i)  $E\tau_1(r) < \infty$  for  $r < -EX_1$ ;

(ii)  $E\tau_1(r) = \infty$  for  $r \geq -EX_1$ .

**Remarks.** (i) It seems that the well understood approach to estimate the expectation of the first exit time is via functions of  $R_t$  that define martingales on  $\mathbb{R}_+$ . This restrains us to linear and square root boundaries and it is not clear how this approach could be extended for a general boundary when

$\frac{1}{2} < \kappa < 1$ .

(ii) The restriction  $E(X_1^+)^2 < \infty$  for the linear case seems redundant but we have been unable to remove it. Generally, it seems to be difficult to obtain results for the finiteness of the expected values of the passage times when  $EX^2 = \infty$ , not only for the reflected process, but for random walks as well. For short discussion we refer to [5].

### 3 Passage Times above power law boundaries at zero

In this section we discuss passage time of the reflected process above power law boundaries at zero. The results are somewhat similar to section 2, which may be a surprise, since the behaviour at zero has no analogue in the random walk setting. To avoid notational complications we are going to study when

$$\limsup_{t \rightarrow 0} \frac{R_t}{t^\kappa} = \infty \text{ a.s.} \quad (3.1)$$

This is equivalent to

$$\tilde{\tau}_\kappa(r) = \inf\{t > 0 : R_t > rt^\kappa\} = 0 \text{ a.s. } \forall r > 0.$$

The first theorem deals with Lévy processes with bounded variation.

**Theorem 3.1.** *Let  $X$  be a Lévy process with bounded variation and drift  $d$ . Then the following statements hold*

(i) *For  $\kappa > 1$ , (3.1) holds iff*

$$\int_0^1 \bar{\Pi}^{(-)}(x^\kappa) dx = \infty \text{ or } d < 0$$

(ii) *For  $\kappa \leq 1$ , we have*

(a) *If  $\kappa < 1$ , or  $\kappa = 1$  and  $d \geq 0$ , then*

$$\lim_{t \rightarrow 0} \frac{R_t}{t^\kappa} = 0 \text{ a.s.} \quad (3.2)$$

(b) If  $\kappa = 1$  and  $d < 0$ , then

$$\lim_{t \rightarrow 0} \frac{R_t}{t} = -d \text{ a.s.} \quad (3.3)$$

Next we deal with the unbounded variation case. We have the following result :

**Theorem 3.2.** *Let  $X$  be a Lévy process with unbounded variation. Then*

(i) *If  $\kappa \geq 1$ , then (3.1) holds.*

(ii) *If  $1/2 \leq \kappa < 1$ , then (3.1) holds iff*

$$(A) \int_0^1 \bar{\Pi}^{(-)}(x^\kappa) dx = \infty \text{ or}$$

$$(B) \liminf_{t \rightarrow 0} \frac{X_t}{t^\kappa} = -\infty \text{ a.s.}$$

(iii) *If  $\kappa < 1/2$ , then*

$$\limsup_{t \rightarrow 0} \frac{R_t}{t^\kappa} < \infty \text{ a.s.} \quad (3.4)$$

**Remarks.** (i) Now it is worth mentioning the similarity between Theorem 3.1 (ii), Theorem 3.2 (ii) and Theorem 2.1 (ii). The integrability of the negative Lévy tail is directly comparable to the finiteness of  $E(X_1^-)^{1/\kappa}$ .

(ii) It needs to be mentioned, that (A) and (B) in (ii) are not equivalent. Thus, for example, for  $\kappa = 1/2$ , (A) fails, while (B) can happen, see Theorem 2.2. in [3]. Moreover, for  $\frac{1}{2} < \kappa < 1$ ,  $\int_0^1 \bar{\Pi}(x^\kappa) dx < \infty$  implies  $\liminf_{t \rightarrow 0} \frac{X_t}{t^\kappa} = 0$  a.s., see Theorem 2.1. in [3]

(iii) Analytic conditions for  $\liminf_{t \rightarrow 0} \frac{X_t}{t^\kappa} = -\infty$  a.s. can be found in [3].

## 4 Proofs for section 2

*Proof of Theorem 2.1.* We start with the proof of (a). If  $X$  is a negative subordinator, then  $R_t = -X_t$  and the statement that  $\tau(r) < \infty$  a.s. is clear from the fact that  $X_t$  drifts to  $-\infty$ . Now if we also assume, without loss of generality, that  $\bar{\Pi}^{(-)}(1) > 0$ , then obviously

$$\tau(r) < \wp(r) := \inf\{t : X \text{ has jumped } [r]+1 \text{ times with jump less than } -1\}$$

Now since  $\wp(r)$  is a sum of independent exponentially distributed random variables it has gamma distribution and hence  $Ee^{\lambda\tau(r)} \leq Ee^{\lambda\wp(r)} < \infty$  for  $\lambda$  small enough.

To show  $Ee^{\lambda\tau(r)} < \infty$ , for some  $\lambda > 0$ , for a general Lévy process, we invoke Theorem 2.1 (a) in [5] for the embedded random walk defined in remark (ii) of Theorem 2.1 and use inequality (2.4).

We shall prove the forward part of both (i) and (ii) in (b) together. Assume (2.2) holds.

Denote by  $\{\zeta_i\}_{i \geq 0}$  the stopping times defined recursively by

$$\zeta_{i+1} = \inf\{t > \zeta_i : |\Delta X_t| > 1\} \text{ and } \zeta_0 = 0.$$

Then we use Theorem 1.1 in [7] to construct a stochastic bound  $M_n$  for  $X_t$  with the following property:

$$X_t \leq M_n = \sup_{\zeta_n \leq t < \zeta_{n+1}} X_t = S_n^+ + m_0, \quad \text{for } \zeta_n \leq t < \zeta_{n+1}, \quad (4.1)$$

where  $S_n^+$  is a random walk with steps

$$Y_i = X_{\zeta_i} - \sup_{\zeta_{i-1} \leq t < \zeta_i} X_t + \sup_{\zeta_i \leq t < \zeta_{i+1}} X_t - X_{\zeta_i},$$

and  $m_0 = \sup_{t \leq \zeta_1} X_t$ . In fact  $Y_i$  can be represented in the following useful way

$$Y_i = J_i + \tilde{X}_{\zeta_i} - \sup_{\zeta_{i-1} \leq t < \zeta_i} \tilde{X}_t + \sup_{\zeta_i \leq t < \zeta_{i+1}} \tilde{X}_t - \tilde{X}_{\zeta_i} \stackrel{d}{=} J_1 + \tilde{X}_{\zeta_1}, \quad (4.2)$$

where  $J_i = \Delta X_{\zeta_i}$  and  $\tilde{X}$  is obtained from  $X$  by removing all jumps bigger in absolute value than 1. Then the Lévy measure of  $\tilde{X}$  has compact support and hence for example from [10], we have that for any  $\lambda > 0$ ,  $Ee^{\lambda\tilde{X}} < \infty$ .

Now with  $N(t) := \max\{i : \zeta_i \leq t\}$  we have the following inequality

$$R_t = \bar{X}_t - X_t \leq \bar{M}_{N(t)} - M_{N(t)} + M_{N(t)} - X_t.$$

Recall now that  $m_0 \geq 0$  a.s. and hence  $\bar{M}_{N(t)} - M_{N(t)} = \bar{S}_{N(t)}^+ - S_{N(t)}^+ = R_{N(t)}^+$ . Moreover by (4.1) and (4.2) we get  $M_{N(t)} - X_t = m_0 + \tilde{S}_{N(t)}^+ - \tilde{X}_t$ , where  $\tilde{S}_n^+ = S_n^+ - \sum_{k \leq n} J_k$ . These observations entail the following useful upper bound for  $R_t$ :

$$R_t \leq R_{N(t)}^+ + m_0 + \tilde{S}_{N(t)}^+ - \tilde{X}_t. \quad (4.3)$$

We shall show now that

$$\limsup_{t \rightarrow \infty} \frac{m_0 + \tilde{S}_{N(t)}^+ - \tilde{X}_t}{t^\kappa} = 0 \text{ a.s.} \quad (4.4)$$

Indeed note that

$$m_0 + \tilde{S}_{N(t)}^+ = \sup_{\zeta_{N(t)} \leq s < \zeta_{N(t)+1}} \tilde{X}_s,$$

gives immediately

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{m_0 + \tilde{S}_{N(t)}^+ - \tilde{X}_t}{t^\kappa} &\leq \limsup_{t \rightarrow \infty} \frac{2 \sup_{\zeta_{N(t)} \leq s < \zeta_{N(t)+1}} |\tilde{X}_s - \tilde{X}_{\zeta_{N(t)}}|}{t^\kappa} = \\ &\lim_{t \rightarrow \infty} \frac{N(t)^\kappa}{t^\kappa} \limsup_{n \rightarrow \infty} \frac{2 \sup_{\zeta_n \leq s < \zeta_{n+1}} |\tilde{X}_s - \tilde{X}_{\zeta_n}|}{n^\kappa} \leq \\ &C \limsup_{n \rightarrow \infty} \frac{2 \sup_{\zeta_n \leq s < \zeta_{n+1}} |\tilde{X}_s - \tilde{X}_{\zeta_n}|}{n^\kappa}, \end{aligned}$$

where  $\lim_{t \rightarrow \infty} \frac{N(t)^\kappa}{t^\kappa} = C > 0$  a.s. follows by the strong law of large numbers. Now set

$$V_n = \sup_{\zeta_n \leq s < \zeta_{n+1}} |\tilde{X}_s - \tilde{X}_{\zeta_n}|$$

and observe that  $V_n \stackrel{d}{=} V_0$  and  $\{V_i\}$  are mutually independent. So to get (4.4) we simply need

$$\limsup_{n \rightarrow \infty} \frac{V_n}{n^\kappa} = 0 \text{ a.s.} \quad (4.5)$$

To achieve this recall, that  $\tilde{X}_t$ , as well as  $\zeta_1$ , has finite moments of any order. This easily implies, that  $V_0$  has moments of any order and hence for any  $\varepsilon > 0$ ,

$$\sum_{n \geq 0} P(V_n > \varepsilon n^\kappa) = \sum_{n \geq 0} P(V_0 > \varepsilon n^\kappa) < \infty.$$

Then a simple application of the Borel-Cantelli lemma entails (4.5) and hence (4.4). Finally we conclude that (2.2) and (4.3) along with the strong law of large numbers give

$$\limsup_{t \rightarrow \infty} \frac{R_{N(t)}^+}{t^\kappa} = C \limsup_{n \rightarrow \infty} \frac{R_n^+}{n^\kappa} = \infty \text{ a.s.}$$

Now all we need to do is, apply Theorem 2.1. in [5] to the random walk  $S_n^+$  to deduce, that either  $E(Y_-)^{\frac{1}{\kappa}} = \infty$ , or

$$\liminf_{n \rightarrow \infty} \frac{S_n^+}{n^\kappa} = -\infty \text{ a.s. for } \kappa \in (0, 1].$$

Then the definition of  $Y$  implies,  $EX_-^{\frac{1}{\kappa}} = \infty$  in case  $EY_-^{\frac{1}{\kappa}} = \infty$ , and similarly

$$\liminf_{n \rightarrow \infty} \frac{S_n^+}{n^\kappa} = -\infty \text{ a.s.}$$

entails

$$\liminf_{t \rightarrow \infty} \frac{X_t}{t^\kappa} = -\infty \text{ a.s.}$$

The backward part of (b) is much simpler, since we can directly use  $R_t \geq -X_t$ , or invoke the embedded random walk, when  $E(X_1^-)^{1/\kappa} = \infty$ , and by Theorem 2.1. in [5] to get

$$\limsup_{n \rightarrow \infty} \frac{R_n^{\widehat{X}}}{n^\kappa} = \infty \text{ a.s.}$$

Inequality (2.3) concludes the proof.  $\square$

*Proof of Theorem 2.2.* We deduce (i) for both (a) and (b) from Theorem 2.2 in [5] and inequality (2.4). Now we concentrate on (ii). Observe that since  $EX_1 = 0$ ,  $X_t$  is a martingale, and its maximum process  $\overline{X}_t$  has bounded variation, so  $R_t$  is a semimartingale. Moreover  $EX_t^2 < \infty$  entails  $E\overline{X}_t^2 < \infty$ , which in turn gives  $ER_t^2 < \infty$ .

Next we apply Ito's formula to get

$$R_t^2 = 2 \int_0^t R_{s-} dR_s + [R]_t = [R]_t + 2 \int_0^t R_{s-} d\overline{X}_s - 2 \int_0^t R_{s-} dX_s. \quad (4.6)$$

Now by the fact that  $\overline{X}$  has bounded variation, we have

$$[R]_t = [\overline{X}]_t - 2[\overline{X}, X]_t + [X]_t = [X]_t - \sum_{s \leq t} (2\Delta\overline{X}_s \Delta X_s - \Delta\overline{X}_s^2)$$

and hence

$$R_t^2 = [X]_t - \sum_{s \leq t} (2\Delta\overline{X}_s \Delta X_s - \Delta\overline{X}_s^2) + 2 \int_0^t R_{s-} d\overline{X}_s - 2 \int_0^t R_{s-} dX_s. \quad (4.7)$$

For P a.e.  $\omega$  in  $\Omega$ ,  $\bar{X}_t(\omega) = \sum_{s \leq t} \Delta \bar{X}_s(\omega) + G(t, \omega)$ , where the function  $G(\cdot, \omega)$  is nonnegative, nondecreasing and continuous. This follows by the fact that for any given  $\omega$ ,  $\bar{X}_t(\omega)$  is a right continuous, nondecreasing and nonnegative function. Then

$$\int_0^t R_{s-} d\bar{X}_s(\omega) = \sum_{s \leq t} \Delta \bar{X}_s(\omega) R_{s-}(\omega) + \int_0^t R_{s-}(\omega) dG(s, \omega). \quad (4.8)$$

In proposition 6.1 section 6 we show that

$$\int_0^t R_{s-}(\omega) dG(s, \omega) = 0 \text{ for all } t \text{ a.s.}$$

Now using this in (4.8) and substituting (4.8) into (4.7) we derive

$$R_t^2 = [X]_t - \sum_{s \leq t} (2\Delta \bar{X}_s \Delta X_s - \Delta \bar{X}_s^2 - 2\Delta \bar{X}_s R_{s-}) - 2 \int_0^t R_{s-} dX_s \quad (4.9)$$

It remains to substitute the following identities

$$\begin{aligned} \Delta R_s &= R_s - R_{s-} = -(\Delta X_s 1_{\{\Delta X_s \leq R_{s-}\}} + R_{s-} 1_{\{\Delta X_s > R_{s-}\}}) \\ \Delta \bar{X}_s &= \Delta R_s + \Delta X_s = (\Delta X_s - R_{s-}) 1_{\{\Delta X_s > R_{s-}\}} \end{aligned}$$

in (4.9) to get

$$R_t^2 = [X]_t - \sum_{s \leq t} (\Delta X_s - R_{s-})^2 1_{\{\Delta X_s > R_{s-}\}} - 2 \int_0^t R_{s-} dX_s$$

We are ready now to finish off the proof of the theorem. We note that  $\int_0^t R_{s-} dX_s$  and  $X_t^2 - [X]_t$  are zero mean martingales. Then we apply the optional sampling theorem to the last identity to get for any  $m > 0$

$$ER_{\tau_{1/2}(\alpha r) \wedge m}^2 - \alpha^2 E\tau_{1/2}(\alpha r) \wedge m + E\left( \sum_{s \leq \tau_{1/2}(\alpha r) \wedge m} (\Delta X_s - R_{s-})^2 1_{\{\Delta X_s > R_{s-}\}} \right) = 0.$$

and hence

$$ER_{\tau_{1/2}(\alpha r) \wedge m}^2 \leq \alpha^2 E\tau_{1/2}(\alpha r) \wedge m, \quad (4.10)$$

Then if we assume  $E\tau_{1/2}(\alpha r) < \infty$ , we get by the Fatou's lemma and the definition of  $\tau(r)$

$$\liminf_{m \rightarrow \infty} (ER_{\tau_{1/2}(\alpha r) \wedge m}^2) \geq E(R_{\tau_{1/2}(\alpha r)}^2) > r^2 \alpha^2 (E\tau_{1/2}(\alpha r) + 1)$$

and by the monotone convergence theorem

$$\lim_{m \rightarrow \infty} E(\tau_{1/2}(\alpha r) \wedge m) = E(\tau_{1/2}(\alpha r)).$$

Inequality (4.10) then gives

$$(1 - r^2)\alpha^2 E\tau_{1/2}(\alpha r) > r^2\alpha^2 > 0$$

and there is an obvious contradiction when  $r \geq 1$ . So we get for  $r \geq 1$   $E\tau_{1/2}(\alpha r) = \infty$ .

For (b), (ii), without loss of generality, we assume  $EX_1 = -1$ . Also from [11] and  $E(X_1^+)^2 < \infty$  we have  $l = E\bar{X}_\infty < \infty$ . Denote for each  $q \geq 1$

$$T_q = \inf\{t > 0 : R_t > t + q\}.$$

Now assume  $ET_q < \infty$  for each  $q \geq 1$ . An easy application of the optional sampling theorem to  $X_{T_q \wedge m}$ , followed by the monotone convergence theorem and the Fatou's lemma, yields

$$ET_q + q \leq ER_{T_q} \leq \lim_{m \rightarrow \infty} ER_{T_q \wedge m} = E\bar{X}_{T_q} + ET_q \leq l + ET_q.$$

So when  $q > l$  we must have  $ET_q = \infty$ .

Next observe, that for  $q \geq 1$

$$\begin{aligned} ET_q 1_{\{T_q > 1\}} &\geq \int_0^{1-\varepsilon} E(T_q | R_1 = y, T_q > 1) P(R_1 \in dy, T_q > 1) \geq \\ &\int_0^{1-\varepsilon} E(T_{1+q-y}) P(R_1 \in dy, T_q > 1) \geq ET_{q+\varepsilon} P(R_1 \in (0, 1-\varepsilon), T_1 > 1). \end{aligned}$$

The first inequality comes from narrowing the possible values of  $R_1$ , while the second, which reads off  $E(T_q | R_1 = x, T_q > 1) \geq ET_{q-x}$ , for  $x \in (0, 1)$ , is verified using the fact that  $R_t$  is a Markov process. Now if we assume that  $X$  is not a negative drift, then  $\exists \delta > 0 : P(R_1 \in (0, 1-\varepsilon), T_1 > 1) > \delta$ . So we obtain

$$ET_q 1_{\{T_q > 1\}} \geq \delta ET_{q+\varepsilon} \geq \delta ET_{q+\varepsilon} 1_{\{T_{q+\varepsilon} > 1\}}$$

and repeating this step finitely many times we get

$$ET_q 1_{\{T_q > 1\}} > CET_{2l} 1_{\{T_{2l} > 1\}} = \infty,$$

where  $C$  is some constant. This entails  $ET_1 = \infty$ . □

## 5 Proofs of section 3

First of all, we observe, that since we study the behaviour at zero, we can always assume that the Lévy measure is carried by  $[-1, 1]$ . So with this in mind we proceed with the proof of Theorem 3.1.

Recall that since  $X$  has bounded variation we can write

$$X_t = dt + Y_t + Z_t, \quad (5.1)$$

where  $Y$  is a driftless positive subordinator and  $Z$  is a driftless negative subordinator.

To show (i), let first

$$\int_0^1 \bar{\Pi}^{(-)}(x^\kappa) dx < \infty. \quad (5.2)$$

Then applying Theorem 9, Chapter 3 in [2] to  $-Z_t$  in (5.1), we easily get

$$\lim_{t \rightarrow 0} \frac{Z_t}{t^\kappa} = 0 \text{ a.s.}$$

So if  $d \geq 0$  we have the bound

$$R_t \leq \sup_{s \leq t} (Y_s + ds) - Y_t - dt - Z_t = -Z_t$$

and (3.1) fails. If  $d < 0$  we know that  $\lim_{t \rightarrow 0} \frac{X_t}{t} = d$  a.s. and hence  $\lim_{t \rightarrow 0} \frac{X_t}{t^\kappa} = -\infty$  a.s. Then by the simple inequality  $R_t \geq -X_t$  we deduce (3.1).

Assume now (5.2) fails. Then a standard argument, see [2], page 85, Theorem 9, gives for any  $c > 1$ ,  $-\Delta X_t > ct^\kappa$  i.o., which along with  $R_t \geq -\Delta X_t 1_{\{\Delta X_t < 0\}}$  shows that (3.1) holds.

For (ii), (a), all we need to observe is that from (5.1) we have

$$R_t \leq \sup_{s \leq t} (Y_s + ds) - Y_t - dt - Z_t \leq 0 \vee -dt - Z_t,$$

and recall  $\lim_{t \rightarrow 0} \frac{Z_t}{t} = 0$  a.s.

For (ii), (b), we note that

$$\lim_{t \rightarrow 0} \frac{\sup_{s \leq t} X_s}{t} \leq \lim_{t \rightarrow 0} \frac{Y_t - Z_t}{t} = 0 \text{ a.s.}$$

and hence

$$\lim_{t \rightarrow 0} \frac{R_t}{t} = \lim_{t \rightarrow 0} \frac{\sup_{s \leq t} X_s - X_t}{t} = \lim_{t \rightarrow 0} \frac{-X_t}{t} = -d \text{ a.s.}$$

This finishes the proof of Theorem 3.1.

*Proof of Theorem 3.2.* For  $\kappa \geq 1$ , we invoke a standard result of Rogozin [9], to get  $\liminf_{t \rightarrow 0} \frac{X_t}{t} = -\infty \text{ a.s.}$

For iii) we argue using a result of Khintchine [8] :

$$\limsup_{t \rightarrow 0} \frac{X_t}{\sqrt{2t \ln |\ln t|}} = \limsup_{t \rightarrow 0} \frac{-X_t}{\sqrt{2t \ln |\ln t|}} = \sigma \text{ a.s.},$$

which shows (3.1) fails. The same way we get for  $\kappa = 1/2$  and  $\sigma > 0$ , (3.1) fails and for  $1/2 < \kappa < 1$  and  $\sigma > 0$ , (3.1) holds.

From now on we set  $\sigma = 0$  and proceed with (ii). Suppose first  $\kappa = 1/2$ . Then (A) fails. In view of Theorem 2.2. in [3] we have either

$$\liminf_{t \rightarrow \infty} \frac{X_t}{\sqrt{t}} = -\infty \text{ a.s.},$$

or (B) holds, which implies (3.1), or

$$\limsup_{t \rightarrow 0} \frac{|X_t|}{\sqrt{t}} < \infty \text{ a.s.},$$

which entails (3.1) and (B) fail simultaneously.

For  $1/2 < \kappa < 1$ , if  $\int_0^1 \bar{\Pi}(x^\kappa) dx < \infty$ , then in view of Theorem 2.1 in [3], (B) and (3.1) fail, while if  $\int_0^1 \bar{\Pi}^{(-)}(x^\kappa) dx = \infty$ , then Theorem 3.1 in [3] entails that (3.1) holds, since  $\liminf_{t \rightarrow 0} \frac{X_t}{t^\kappa} = -\infty \text{ a.s.}$

It remains to consider the case

$$\int_0^1 \bar{\Pi}^{(-)}(x^\kappa) dx < \infty = \int_0^1 \bar{\Pi}^{(+)}(x^\kappa) dx.$$

Write  $X = X^+ + X^-$  as a sum of two independent Lévy processes, where  $X^+$  is a zero mean, spectrally positive Lévy process and  $X^-$  is a zero mean,

spectrally negative Lévy process. Then since  $\int_0^1 \bar{\Pi}^{(-)}(x^\kappa) dx < \infty$ , we apply Theorem 2.1, Proposition 4.1 and 4.2 in [3], to deduce,

$$\lim_{t \rightarrow 0} \frac{\sup_{s \leq t} |X_s^-|}{t^\kappa} = 0 \text{ a.s.}$$

and thus, with  $R_t^+ = \bar{X}_t^+ - X_t^+$ ,

$$\limsup_{t \rightarrow 0} \frac{R_t}{t^\kappa} = \limsup_{t \rightarrow 0} \frac{R_t^+}{t^\kappa} \text{ a.s.}$$

So we may additionally assume, that  $X$  is a zero mean, spectrally positive Lévy process and continue with the proof. Introduce the following functions

$$V(x) := \int_0^x y^2 \Pi(dy) \tag{5.3}$$

$$W(x) := \int_0^x \int_z^1 s \Pi(ds) dz = V(x) + x \int_x^1 s \Pi(ds). \tag{5.4}$$

Then  $W(x)$  is continuous and nondecreasing and we define for any  $\lambda > 0$  :

$$J(\lambda) := \int_0^1 e^{-\lambda \frac{y^{(2\kappa-1)/(1-\kappa)}}{W(y)^\kappa/(1-\kappa)}} \frac{dy}{y} \tag{5.5}$$

and

$$\lambda_J := \inf\{\lambda > 0 \mid J(\lambda) < \infty\} \in [0, \infty]. \tag{5.6}$$

From Theorem 3.1 in [3] applied to  $-X$  we conclude that

$$\liminf_{t \rightarrow \infty} \frac{X_t}{t^\kappa} = -\infty \text{ a.s.} \Leftrightarrow \lambda_J = \infty.$$

Thus  $\lambda_J = \infty$  implies (3.1).

So also we assume, without loss of generality,  $\lambda_J < 1$ .

First of all we will often refer to Proposition 6.2 in Section 6, where important properties for the following function

$$D(x) = \inf\{z > 0 : \frac{W(z)}{z} = \frac{1}{x^{1-\kappa}}\}, \tag{5.7}$$

are obtained.

We proceed to show that (3.1) fails. First we set up some notation. We write

$X_t = X_t^b + \tilde{X}_t^b$ , where  $X^b$  is a spectrally positive Lévy process with jumps bounded by  $b$  and  $\tilde{X}^b$  is a compensated Poisson process of jumps bigger than  $b$ . Since  $X$  is spectrally positive, a handy bound for  $\tilde{X}_t^b$  now is

$$\tilde{X}_t^b = \sum_{s \leq t} \Delta \tilde{X}_s^b - t \int_b^1 x \Pi(dx) \geq -t \int_b^1 x \Pi(dx). \quad (5.8)$$

Also we have

$$\sup_{t \leq v} R_t = \sup_{t \leq v} \sup_{s \leq t} (X_s - X_t) = \sup_{t \leq v} \sup_{s \leq t} (X_s^{D(v)} - X_t^{D(v)} + \tilde{X}_s^{D(v)} - \tilde{X}_t^{D(v)}),$$

where by (5.8), (5.4) and Proposition 6.2 (b) we are immediately able to get the bound

$$\sup_{t \leq v} R_t \leq \sup_{t \leq v} \sup_{s \leq t} (X_s^{D(v)} - X_t^{D(v)}) + \frac{vW(D(v))}{D(v)} = \sup_{t \leq v} \sup_{s \leq t} (X_s^{D(v)} - X_t^{D(v)}) + v^\kappa. \quad (5.9)$$

So it will suffice to show

$$\limsup_{v \rightarrow 0} \frac{\sup_{t \leq v} X_t^{D(v)} + \sup_{t \leq v} -X_t^{D(v)}}{v^\kappa} < \infty \text{ a.s.} \quad (5.10)$$

Now we can use inequality (4.11) in [3], which holds for any zero mean Lévy process with  $\sigma = 0$ , to get using (5.3)

$$P(\sup_{t \leq v} X_t^{D(v)} > av^\kappa) \leq 2P((X_v^{D(v)} > av^\kappa - \sqrt{2vV(D(v))})) \quad (5.11)$$

$$P(\sup_{t \leq v} -X_t^{D(v)} > av^\kappa) \leq 2P(-X_v^{D(v)} > av^\kappa - \sqrt{2vV(D(v))}). \quad (5.12)$$

We invoke (5.4) followed by (b) and (d) from Proposition 6.2, to get  $\sqrt{2vV(D(v))} = o(v^\kappa)$ . This entails that for any  $\varepsilon > 0$  and  $v \leq v(\varepsilon)$  we have

$$P(\sup_{t \leq v} X_t^{D(v)} > av^\kappa) \leq 2P(X_v^{D(v)} > (a - \varepsilon)v^\kappa) \quad (5.13)$$

$$P(\sup_{t \leq v} -X_t^{D(v)} > av^\kappa) \leq 2P(-X_v^{D(v)} > (a - \varepsilon)v^\kappa). \quad (5.14)$$

Now for any  $a > e + \varepsilon$  we have by Proposition 6.3 :

$$\max\{P(\sup_{t \leq v} X_t^{D(v)} > av^\kappa), P(\sup_{t \leq v} -X_t^{D(v)} > av^\kappa)\} \leq e^{-\rho v^\kappa / D(v)},$$

where it must be clear that  $\rho = a - \varepsilon - e$ .

Now it is not difficult to finish the proof. We choose  $v_n = D^{\leftarrow}(1/2^n)$  (see Proposition 6.2 for definition) and get by the last inequality and Proposition 6.2, (c)

$$\sum_{n>0} P(\sup_{t \leq v_n} |X_t^{D(v_n)}| > av_n^\kappa) \leq 2K^{-1} \sum_{n>0} K e^{-\rho \frac{(1/2^n)^{(2\kappa-1)/(1-\kappa)}}{W(1/2^n)^\kappa/(1-\kappa)}},$$

where  $K = \ln 2$ . Then putting  $q = 2^{\frac{2\kappa-1}{1-\kappa}}$ , we get

$$\begin{aligned} \sum_{n>0} P(\sup_{t \leq v_n} |X_t^{D(v_n)}| > av_n^\kappa) &\leq K^{-1} \sum_{n>0} K e^{-q\rho \frac{(1/2^{n+1})^{(2\kappa-1)/(1-\kappa)}}{W(1/2^n)^\kappa/(1-\kappa)}} \leq \\ &K^{-1} \int_0^1 e^{-\rho q \frac{y^{(2\kappa-1)/(1-\kappa)}}{W(y)^\kappa/(1-\kappa)}} \frac{dy}{y}. \end{aligned}$$

Choose  $\rho$ , correspondingly  $a$ , such that  $q\rho = 1$  and use  $J(1) < \infty$  (recall  $\lambda_J < 1$ ) to get

$$\sum_{n>0} P(\sup_{t \leq v_n} |X_t^{D(v_n)}| > av_n^\kappa) < \infty,$$

where  $a = \frac{1}{q} + \varepsilon + e$ . Then the Borel-Cantelli lemma gives (5.10) over  $\{v_n\}$ , which in turn reads off as

$$\limsup_{n \rightarrow \infty} \frac{\sup_{s \leq v_n} R_s}{v_n^\kappa} < \infty \text{ a.s.}$$

Then for arbitrary  $s \in [v_{n+1}, v_n)$  we have

$$\begin{aligned} \frac{R_s}{s^\kappa} &\leq \frac{\sup_{t \leq v_n} R_t}{v_{n+1}} \leq \frac{\sup_{t \leq v_n} R_t}{v_n^\kappa} \frac{v_n^\kappa}{v_{n+1}^\kappa} = 2^{\kappa/(1-\kappa)} \left( \frac{W(2^{-n})}{W(2^{-n-1})} \right)^{\kappa/(1-\kappa)} \frac{\sup_{t \leq v_n} R_t}{v_n^\kappa} \leq \\ &2^{2\kappa/(1-\kappa)} \frac{\sup_{t \leq v_n} R_t}{v_n^\kappa}, \end{aligned}$$

where we have made use of the definition of  $v_n$  and the fact that  $\frac{W(x)}{x} \uparrow \infty$  as  $x \downarrow 0$ . This establishes the theorem.  $\square$

## 6 Appendix

The first technical result in this section is the following proposition.

**Proposition 6.1.** *Let  $X$  be a Lévy process. Then*

$$\int_0^t R_{s-} d\bar{X}_s = \sum_{s \leq t} \Delta \bar{X}_s(\omega) R_{s-}(\omega) \text{ for all } t \text{ a.s.}$$

*Proof.* Note that  $\int_0^t R_{s-} d\bar{X}_s$  is increasing in  $t$ , so it will be sufficient to show that for any fixed  $t$

$$\int_0^t R_{s-} d\bar{X}_s = \sum_{s \leq t} \Delta \bar{X}_s(\omega) R_{s-}(\omega) \text{ a.s.}$$

Note that  $\bar{X}$  is monotone and fix a path  $\omega$ . Write

$$\bar{X}_u(\omega) = \sum_{s \leq u} \Delta \bar{X}_s(\omega) + G(u, \omega),$$

where  $G(\cdot, \omega)$  is nondecreasing and continuous. Then  $G(\cdot, \omega)$  defines a diffuse measure on  $\mathbb{R}_+$ . Thus

$$\int_0^t R_{s-} d\bar{X}_s = \sum_{s \leq t} \Delta \bar{X}_s(\omega) R_{s-}(\omega) + \int_{\text{supp } G(\omega) \cap [0, t]} R_{s-} dG(s, \omega).$$

Clearly  $\text{supp } G(\omega)$  excludes the points in time,  $s \in \mathbb{R}^+$ , such that  $\bar{X}_s = \bar{X}_{s+h} = \bar{X}_{s-h}$  for some  $h > 0$ . Then we have  $\text{supp } G(\omega) \subseteq A \cup B \cup C \cup D$  with :

$$\begin{aligned} A &= \{s : s \text{ is an end or a start point of an excursion}\} \\ B &= \{s : \bar{X}_{s-} = \bar{X}_{s-h} < \bar{X}_s \text{ for some } h > 0\} \\ C &= \{s : \bar{X}_{s-h} = \bar{X}_s < \bar{X}_{s+\delta} \text{ for some } h > 0 \text{ and any } \delta > 0\} \\ D &= \{s : \bar{X}_{s-} = \bar{X}_s > \bar{X}_{s-h} \text{ for any } h > 0\}. \end{aligned}$$

Then it is immediate that  $A$  is a countable set, since the number of the excursion is. Also  $B$  is countable since by its definition the maximum should be attained by a jump. Finally we check that  $C$  is countable by its definition, which requires a neighborhood  $(s-h, s)$ , where no maximum is attained. Using the fact that  $G$  is diffuse we get

$$\int_{A \cap B \cap C \cap [0, t]} R_{s-} dG(s, \omega) = 0 \text{ a.s.}$$

The very definition of  $D$  implies,  $R_{s-} = 0$  on  $D$  a.s. This establishes the result.  $\square$

**Proposition 6.2.** *With  $W(x)$  as defined in (5.4) we have  $\frac{W(x)}{x}$  is decreasing and  $\lim_{x \rightarrow 0} \frac{W(x)}{x} = \infty$ . Then  $D(x)$  defined in (5.7) has the following properties:*

(a)  $D(x) \downarrow 0$  as  $x \downarrow 0$  and the function is continuous and increasing.

(b)  $\frac{W(D(x))}{D(x)} = x^{\kappa-1}$ .

(c)  $D^{\leftarrow}(x) = \left(\frac{x}{W(x)}\right)^{1/1-\kappa}$ , where  $D^{\leftarrow}(x)$  is the inverse function.

(d) Given that  $\lambda_J < \infty$  we have  $\frac{D(x)}{x^\kappa} \rightarrow 0$ .

*Proof.* The result is standard. The proof of (a), (b) and (c) is obvious. Then for (d) we refer to [11].  $\square$

**Proposition 6.3.** *For (5.13) and (5.14), and any  $a > \varepsilon + e$ , we have the following exponential bound :*

$$\max\{P(X_v^{D(v)} > (a - \varepsilon)v^\kappa), P(-X_v^{D(v)} > (a - \varepsilon)v^\kappa)\} \leq e^{-\rho v^\kappa / D(v)},$$

where  $\rho = a - \varepsilon - e$ .

*Proof.* An application of the Chebyshev inequality to (5.14) yields, for any  $\theta > 0$  :

$$P(-X_v^{D(v)} > (a - \varepsilon)v^\kappa) = P(e^{-\theta X_v^{D(v)}} > e^{\theta(a - \varepsilon)v^\kappa}) \leq e^{v \int_0^{D(v)} (e^{-\theta x} + \theta x - 1) \Pi(dx) - \theta(a - \varepsilon)v^\kappa}.$$

Then we use  $e^{-\theta x} + \theta x - 1 \leq \theta^2 x^2$  and the definition of  $V(x)$  and  $W(x)$ , to obtain further

$$P(-X_v^{D(v)} > (a - \varepsilon)v^\kappa) \leq e^{v\theta^2 V(D(v)) - \theta(a - \varepsilon)v^\kappa} \leq e^{v\theta^2 W(D(v)) - \theta(a - \varepsilon)v^\kappa}.$$

Finally we invoke (b) in Proposition 6.2, to deduce

$$P(\sup_{t \leq v} -X_t^{D(v)} > av^\kappa) \leq 2e^{v^\kappa \theta^2 D(v) - \theta(a - \varepsilon)v^\kappa},$$

where we put  $\theta = \frac{\gamma}{D(v)}$ , with  $\gamma > 0$ , to get

$$P(\sup_{t \leq v} -X_t^{D(v)} > av^\kappa) \leq 2e^{\frac{\gamma v^\kappa}{D(v)}(\gamma - (a - \varepsilon))}. \quad (6.1)$$

A further application of the Chebyshev inequality to (5.13), with  $\theta = \frac{1}{D(v)}$ , gives

$$P\left(e^{\frac{1}{D(v)}X_v^{D(v)}} > e^{\frac{1}{D(v)}(a-\varepsilon)v^\kappa}\right) \leq e^{v \int_0^{D(v)} (e^{x/D(v)} - \frac{1}{D(v)}x - 1)\Pi(dx) - \frac{1}{D(v)}(a-\varepsilon)v^\kappa}.$$

Now for  $u \leq 1$ , we have  $e^u - u - 1 \leq eu^2$ , and hence

$$P\left(e^{\frac{1}{D(v)}X_v^{D(v)}} > e^{\frac{1}{D(v)}(a-\varepsilon)v^\kappa}\right) \leq e^{ve \frac{1}{D(v)^2} \int_0^{D(v)} x^2 \Pi(dx) - \frac{1}{D(v)}(a-\varepsilon)v^\kappa}.$$

Recall the definition of  $W(x)$  to get

$$\int_0^{D(v)} x^2 \Pi(dx) = V(D(v)) \leq W(D(v)) = v^{\kappa-1}D(v)$$

and hence

$$P(\sup_{t \leq v} -X_t^{D(v)} > av^\kappa) \leq 2e^{\frac{v^\kappa}{D(v)}(e-(a-\varepsilon))}. \quad (6.2)$$

Then since  $a$  is arbitrary, we can choose  $(a - \varepsilon) = e + \rho$ , where  $\rho$  is arbitrary and then in (6.1) we set

$$\gamma(\gamma - (a - \varepsilon)) = e - (a - \varepsilon).$$

Thus we get the equation  $\gamma^2 - (e + \rho)\gamma = -\rho$ , which clearly has a positive root  $\gamma(\rho)$  and now choose  $\gamma = \gamma(\rho)$  to get the desired result.  $\square$

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## References

- [1] Asmussen, S., Avram, F. and Pistorius, M. (2004) Russian and American put options under exponential phase-type Lévy models. *Stochastic Process. Appl.* **109** 79-111.
- [2] Bertoin, J. *An Introduction to Lévy Processes*. Cambridge University Press, (1996).
- [3] Bertoin, J., Doney, R. A. and Maller, R. A. (2007) Passage of Lévy Processes across Power Law Boundaries at Small Times. *Ann. Probab.*, to appear

- [4] Chow, Y. S. and Teicher, H. ***Probability Theory: Independence, Interchangeability, Martingales.*** Springer-Verlag 1978
- [5] Doney, R. A. and Maller, R. A. (2007) Curve crossing for random walks reflected at their maximum. *Ann. Probab.*, **35**, 1351-1373.
- [6] Doney, R. A. and Maller, R. A. (2005) Passage times of random walks and Lévy processes across power law boundaries. *Probab. Theory Relat. Fields* 133, 57-70.
- [7] Doney, R. A. Stochastic bounds for Lévy processes. *Ann. Prob.*, (2004), **32**, 1545–1552.
- [8] Khintchine, A. Ya. (1939) Sur la croissance locale des processus stochastiques homogènes á accroissements indépendants. *Izv. Akad. Nauk SSSR* **3**, 487-508
- [9] Rogozin, B. A. (1968) Local behavior of processes with independent increments. *Theory Probab. Appl.* **16**, 575-595
- [10] Sato, K. (1999) *Lévy Processes and Infinitely Divisible Distributions.* Cambridge University Press, Cambridge.
- [11] Savov, M. Asymptotic behavior of Lévy processes. (2008) *Phd Thesis*, The University of Manchester, **to be submitted**
- [12] Shepp, L. A. and Shiryaev, A. N. (1993). The Russian Option: Reduced regret. *Ann. Appl. Probab.* **3** 631-640.
- [13] Shepp, L. A. and Shiryaev, A. N. (1996). A dual Russian option for selling short. *Probability Theory and Mathematical Statistics* (A. Ibragimov and A. Yu. Zaitsev, eds.) 209-218. Gordon and Breach, Amsterdam.