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We consider smooth $2n$-dimensional manifolds M, equipped with a smooth action θ of the circle S^1. Such an M determines a *normally complex S^1-manifold* (M,θ,c_ν) whenever an equivariant complex structure c_ν is chosen for the normal bundle $\nu(i)$ of some equivariant embedding $i\colon M\to\mathbb{C}^k$. In this situation, we use the standard action of S^1 on the unit sphere $S^{2m+1}\subset\mathbb{C}^{m+1}$ to define a $2(m+n)$-dimensional smooth manifold $W_m=S^{2m+1}\times_{S^1}M$, and a k-dimensional complex vector bundle $q_m\colon E_m\to\mathbb{C}P^m$, where $E_m=S^{2m+1}\times_{S^1}\mathbb{C}^k$. Then i extends to an embedding $i'\colon W_m\to E_m$, and the complex structure c_ν extends to a complex structure c' on the normal bundle $\nu(i')$.

The composition $p_m\colon W_m\xrightarrow{i'}E_m\xrightarrow{q_m}\mathbb{C}P^m$ is complex oriented in the sense of [1] and determines a complex cobordism class $\Phi_m(M,\theta,c_\nu)\in U^{-2n}(\mathbb{C}P^m)$. The standard embedding $\iota_m\colon\mathbb{C}P^m\to\mathbb{C}P^{m+1}$ acts by $\iota_m^*\Phi_{m+1}=\Phi_m$, hence the inverse sequence $(\Phi_m(M,\theta,c_\nu):m\geqslant 0)$ defines an element of $\varprojlim U^{-2n}(\mathbb{C}P^m)\simeq U^{-2n}(\mathbb{C}P^\infty)$. We write this element as $\Phi(M,\theta,c_\nu)$ and call it the *universal S^1-equivariant genus* of (M,θ,c_ν). The genus Φ was used to deduce fundamental results on Hirzebruch genera of S^1-manifolds in [2] and [3]. An analogue of Φ for a compact Lie group has been studied by several authors since the 1960s (see [4] and [1]).

The ring $U^*(\mathbb{C}P^\infty)$ is isomorphic to the algebra $\varOmega^*[[u]]$ of formal power series over $\varOmega^*=U^*(pt)$, where u is the cobordism class of the inverse sequence $(\iota_m:m\geqslant 0)$. So our initial aim is to construct stably complex manifolds $G_j(M)$ of dimension $2(n+j)$ such that $\Phi(M,\theta,c_\nu)=\sum_{j=0}^\infty[G_j(M)]\,u^j$. Clearly, $G_0=M$ with the stably complex structure c_ν.

We consider $(S^3)^j=\{(y_1,z_1;\dots;y_j,z_j)\in\mathbb{C}^{2j}:|y_i|^2+|z_i|^2=1,\ 1\leqslant i\leqslant j\}$, on which the torus T^j acts freely by $t\cdot(y_1,z_1;\dots;y_j,z_j)=(t_1y_1,t_1^{-1}z_1;\dots;t_jy_j,t_{j-1}^{-1}t_j^{-1}z_j)$ for all $t=(t_1,\dots,t_j)$. The quotient manifold $B_j=(S^3)^j/T^j$ is a j-fold iterated S^2-sphere bundle over $B_0=pt$ and admits complex line bundles $\eta_1,\dots,\eta_j$ such that $E(\eta_i)=(S^3)^j\times_{T^j}\mathbb{C}$ via the action $t\cdot z=t_i^{-1}z$ for $z\in\mathbb{C}$. For $j>0$, the isomorphism $\tau(B_j)\oplus\mathbb{C}^j\cong(\overline{\eta}_1\oplus\eta_1)\oplus\dots\oplus(\overline{\eta}_j\oplus\eta_{j-1}\eta_j)$ defines a stably complex structure c_j^∂ such that $B_j=\partial D_j$, where D_j is the associated 3-disc bundle, hence $[B_j]=0$ for $j>0$.

Theorem 1. *For every $j\geqslant 0$ the manifold $G_j(M)$ is given by the quotient $(S^3)^j\times_{T^j}M$, where T^j acts on M by $t\cdot x=\theta(t_j^{-1})\cdot x$. The stably complex structure $c(j)$ on $G_j(M)$ is induced from the fibration $M\to G_j(M)\to B_j$.*

The proof combines 1.7 in [1] with the fact (Proposition 2.2 in [5]) that the classifying maps for the bundles η_j over B_j represent a basis for $U_*(\mathbb{C}P^\infty)$ that is dual to the basis $\{u^k:k=0,1,2,\dots\}$ in $U^*(\mathbb{C}P^\infty)$.

Remark 2. Our $G_j(M^{2n})$ are clearly complex bordant to the *non-connected* manifolds $\gamma^j(M^{2n})$ of Theorem 6.3 in [6]. Therefore, we deduce that the completion map of the homotopical bordism ring $MU_*^{S^1}$ at its augmentation ideal (see Ch. 27 in [7]) restricts to Φ on geometrical classes. Also, the $G_j(M^{2n})$ are *diffeomorphic* to the manifolds $M^{2(n+j)}$ of [8], so $c(j)$ induces a stably complex structure on the latter.

We interpret M as a *tangentially stably complex S^1-manifold* (M,θ,c_τ) whenever a complex vector bundle $\xi\to M$ and an isomorphism

$$c_\tau\colon\tau(M)\oplus\mathbb{C}^{l-n}\to\xi \qquad (1)$$

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are chosen, where $\tau(M)$ is the stable tangent bundle, in such a way that the composition

$$\rho(t)\colon \xi \xrightarrow{c_\tau^{-1}} \tau(M)\oplus\mathbb{C}^{l-n} \xrightarrow{d\theta(t)\oplus I} \tau(M)\oplus\mathbb{C}^{l-n} \xrightarrow{c_\tau} \xi, \qquad t\in S^1, \tag{2}$$

is a complex transformation for any $t\in S^1$, where $d\theta(t)$ is the differential of the action by $\theta(t)$. Hence, (2) corresponds to a representation $\rho\colon S^1\to \mathrm{Hom}_{\mathbb{C}}(\xi,\xi)$. Up to natural equivalence, the structure (M,θ,c_τ) *determines* (M,θ,c_ν), and we may define $\Phi(M,\theta,c_\tau)$ to be the $\Phi(M,\theta,c_\nu)$ associated with the corresponding structure c_ν. On the other hand, c_ν may *not* determine an equivariant c_τ ([7], Ch. 28).

Let $x\in M$ be an isolated fixed point of the action θ. Then the associated representation $\rho_x\colon S^1\to GL(l,\mathbb{C})$ decomposes the fibre $\xi_x\cong\mathbb{C}^l$ as $\mathbb{C}^n\oplus\mathbb{C}^{l-n}$, where ρ_x acts (without trivial summands) as $\rho_{x,1}\oplus\cdots\oplus\rho_{x,n}$ on $\mathbb{C}^n$ and trivially on $\mathbb{C}^{l-n}$. Here $\rho_{x,j}$ is the one-dimensional representation of weight $\omega_{x,j}$, that is, it acts by $\rho_{x,j}(t)v=t^{\omega_{x,j}}v$. To each isolated fixed point x we may therefore assign a sequence $\omega_x=(\omega_{x,1},\dots,\omega_{x,n})$ of weights. Moreover, the isomorphism $c_{x,\tau}$ in (1) induces an *orientation* of the tangent space τ_x, which allows us to define $\sigma(x)$ as the sign of the determinant of the linear map

$$\tau_x(M)\xrightarrow{I\oplus 0}\tau_x(M)\oplus\mathbb{C}^{l-n}\xrightarrow{c_{\tau,x}}\xi_x\cong\mathbb{C}^n\oplus\mathbb{C}^{l-n}\xrightarrow{\pi}\mathbb{C}^n,$$

where π is the projection onto the first summand. The sign is important in toric topology [9].

Example 3. The S^1-manifold $(B_1=S^2,\theta,c_1^\partial)$, where $\theta(t)[z_1:z_2]=[tz_1:z_2]$, has fixed points $x_1=[1,0]$ and $x_2=[0,1]$ with weights $\omega_{x_1,1}=\omega_{x_2,1}=1$ and signs $\sigma(x_1)=-\sigma(x_2)=1$.

Now let $\{[q](u)\in\Omega^*[[u]]:q\in\mathbb{Z}\}$ denote the power system of the formal group $f(u,v)$ for complex cobordism (see [10]). The q-series $[q](u)=qu \mod(u^2)$ is defined uniquely by $[0](u)=0$ and $[q](u)=f(u,[q-1](u))$ for all $q\in\mathbb{Z}$.

Theorem 4. *If the action θ has a finite set X of isolated fixed points, then*

$$\Phi(M,\theta,c_\tau)=\sum_{x\in X}\sigma(x)\prod_{i=1}^{n}\frac{1}{[\omega_{x,i}](u)} \tag{3}$$

holds in $U^{-2n}(\mathbb{C}P^\infty)$.

The proof combines Proposition 3.8 in [1] with the construction of $\sigma(x)$ above.

Remark 5. If the S^1-manifold M is *almost complex*, then we may take $l=n$ in (1). Then $\sigma(x)=1$ for all $x\in X$, and (3) reduces to Krichever's formula ((2.7) in [2]).

We may reformulate Theorems 1 and 4 for effective actions of the torus T^k on any $2n$-dimensional manifold with $k\leqslant n$.

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