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Symplectic Group Actions and Covering Spaces

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Abstract

We show that for symplectic group actions (not necessarily Hamiltonian), the reduced space defined using the cylinder-valued momentum map can also be achieved by passing to any Hamiltonian cover and then performing standard (Meyer-Marsden-Weinstein) reduction. At the same time we give a classification of all Hamiltonian coverings of a given symplectic group action. The main properties of the lifting of a group action to a cover are studied.

Keywords: lifted group action, symplectic reduction, universal covering, Hamiltonian holonomy, momentum map

Introduction

There are many instances of symplectic group actions which are not Hamiltonian—ie, for which there is no momentum map. This can occur both in applications [10] as well as in fundamental studies of symplectic geometry [1, 2, 5]. In such cases it is possible to define a "cylinder valued momentum map" [3], and then for example to perform symplectic reduction with respect to this map [13, 14]. An alternative approach is to pass to the universal cover, on which the action is always Hamiltonian, and then to perform ordinary symplectic reduction there. The principal purpose of this study is to relate the two procedures. In short we show that after appropriate projection the two reduced spaces so-constructed are the same.

In more detail, suppose a connected Lie group G acts on a connected manifold M, and let N be a covering of M. Then it may not be possible to lift the action of G, but there is a natural lift to universal covers giving an action of $\widetilde{G}$ on $\widetilde{M}$. This can then be used to define an action of $\widetilde{G}$ on the given cover N. This general construction must be well-known, but we were unable to find it in the literature, and consequently have established the main results about these lifted actions in the first section. For example, since N can be written as a quotient of $\widetilde{M}$ by a subgroup of the group of deck transformations, we use this to determine exactly which subgroup of $\widetilde{G}$ acts trivially on N. We also determine the relation between isotropy subgroups of the G action on M and the lifted action on N, and we show that the action on M is proper, then so is the lifted action on N.

In Section 2 we consider the case where M is a symplectic manifold, and G acts symplectically on M. We consider the covers of M for which the action is Hamiltonian and which form the category of Hamiltonian covers of M. The "largest" Hamiltonian cover of M is of course its universal cover $\widetilde{M}$; we give an explicit expression for its momentum map (Proposition 2.3) and we use it to define a subgroup of the fundamental group of M whose corresponding set of subgroups classifies the Hamiltonian covers (Corollary 2.7). There is also a *"smallest"* such cover, denoted $\widehat{M}$ and which was first introduced in [12], where it is called the *universal covered space* of M; we give here a different interpretation of it.

In Section 3, we consider the cylinder valued momentum map of [3] (where it is defined in a different manner, and called the *"moment réduit"*). In Theorem 3.4 we see that reduction can be carried out in two equivalent ways. One can either reduce M with respect to the cylinder valued momentum map or, alternatively, one can lift the action to the universal covering $\widetilde{M}$ (or on any other Hamiltonian cover) and then carry out (standard) symplectic reduction on it using its momentum map. The result is that the natural projection of this reduced space (inherited from the covering projection) yields the original reduced space; that is, both reduction schemes are equivalent.

1 Lifting group actions to covering spaces

1.1 The category of covering spaces

We begin by recalling a few facts about covering spaces. Many of the details can be found in any introductory book on Algebraic Topology, for example Hatcher [6]. Let (M,z_0) be a connected manifold with a chosen base point z_0, and let $q_M : (\widetilde{M},\tilde{z}_0) \to (M,z_0)$ be the universal covering. We realize the universal cover as the set of homotopy classes of paths in M with base point z_0. For definiteness, we take the base point in $\widetilde{M}$ to be the homotopy class $\tilde{z}_0$ of the trivial loop at z_0. Throughout, 'homotopic paths' will mean homotopy with fixed end-points, and all paths will be parametrized by $t \in [0,1]$.

Any cover $p_N : (N,y_0) \to (M,z_0)$ has the same universal cover $(\widetilde{M},\tilde{z}_0)$ as (M,z_0), and the covering map $q_N : (\widetilde{M},\tilde{z}_0) \to (N,y_0)$ can be constructed as follows: Let $\tilde{z} \in \widetilde{M}$ and let $z(t)$ be a representative path in M, so $z(0) = z_0$. By the path lifting property of the covering map p_N, $z(t)$ can be lifted uniquely to a path $y(t)$ in (N,y_0). Then $q_N(\tilde{z}) = y(1)$.

Let $\mathfrak{C}$ be the category of all covers of (M,z_0). The morphisms are the covering maps. Since any element $(N,y_0) \in \mathfrak{C}$ also shares $\widetilde{M}$ as universal cover, it sits in a diagram,

$$(\widetilde{M},\tilde{z}_0) \xrightarrow{q_N} (N,y_0) \xrightarrow{p_N} (M,z_0).$$

Note that with this notation for the covering maps, the map $\widetilde{M} \to M$ can be written both as q_M and as $p_{\widetilde{M}}$.

It is well-known that this category is isomorphic to the category of subgroups of the fundamental group $\pi_1(M,z_0)$ of M, where the morphisms are the inclusion homomorphisms of subgroups. The isomorphism is defined as follows. Let $p_N : (N,y_0) \to (M,z_0)$ be a cover. Then $\Gamma_N := p_{N*}(\pi_1(N,y_0))$ is the required subgroup of $\Gamma := \pi_1(M,z_0)$. Γ_N consists of the homotopy classes of closed paths in (M,z_0) whose lift to (N,y_0) is also closed, and the number of sheets of the covering p_N is equal to the index $\Gamma : \Gamma_N$. Note that since $\widetilde{M}$ is simply connected, $\Gamma_{\widetilde{M}}$ is trivial.

The inverse of this isomorphism can be defined using deck transformations. Let $\Gamma = \pi_1(M,z_0)$. Then Γ is the fibre of q_M over z_0, and it acts on $\widetilde{M}$ by deck transformations defined via the homotopy product: if $\gamma \in \Gamma$ and $\tilde{z} \in \widetilde{M}$ then $\gamma * \tilde{z}$ gives the action of γ on $\tilde{z}$. Then given $\Gamma_1 < \Gamma$, define $N = \widetilde{M}/\Gamma_1$, and put $y_0 = \Gamma_1\tilde{z}_0$. Then from the long exact sequence of homotopy, it follows that $\pi_1(N,y_0) \simeq \Gamma_1$. Furthermore, if $\Gamma_1 < \Gamma_2 < \Gamma$ then there is a well-defined morphism (covering map) $p : N_1 \to N_2$, where $N_j = \widetilde{M}/\Gamma_j$, obtained from noting that any Γ_1-orbit is contained in a unique Γ_2-orbit, so we put $p(\Gamma_1\tilde{z}) = \Gamma_2\tilde{z}$.

Let (N_1,y_1) be a cover of (M,z_0) with group Γ_1, and let $\Gamma_2 = \gamma\Gamma_1\gamma^{-1}$ be a subgroup conjugate to Γ_1 (where $\gamma \in \Gamma$). Then $N_2 = \widetilde{M}/\Gamma_2$ is diffeomorphic to N_1, but the base point is now $y_2 = \Gamma_2\tilde{z}_0$. The diffeomorphism is simply induced from the diffeomorphism $\tilde{z} \mapsto \gamma \cdot \tilde{z}$ of $\widetilde{M}$, which does not in general map y_1 to y_2.

If $\Gamma_1 \lhd \Gamma$ (normal subgroup), then the cover (N,y_1) is said to be a *normal cover*.

Let us emphasize here that we view $\Gamma = \pi_1(M,z_0)$ both as a group acting on $\widetilde{M}$ by deck transformations, and as a discrete subset of $\widetilde{M}$—the fibre over z_0. In particular, for $\gamma \in \Gamma$,

$$\gamma * \tilde{z}_0 = \gamma \tag{1.1}$$

In other words, $\tilde{z}_0$ is the identity element in Γ.

1.2 Lifting the group action

Now let G be a connected Lie group acting on the connected manifold M, and let $p_N : (N,y_0) \to (M,z_0)$ be a covering. To define the lifted action on N, we first describe the lift to $\widetilde{M}$ and then show it induces an action on N, using the covering $q_N : \widetilde{M} \to N$.

The action of G on M does not in general lift to an action of G on $\widetilde{M}$ but of the universal cover $\widetilde{G}$, which is also defined using homotopy classes of paths, with base point the identity element e. The covering map is denoted $q_G : \widetilde{G} \to G$. So if $\widetilde{g}$ is represented by a path $g(t)$ then $q_G(\widetilde{g}) = g(1)$. The product structure in $\widetilde{G}$ is given by pointwise multiplication of paths: if $\widetilde{g}_1$ is represented by a path $g_1(t)$ and $\widetilde{g}_2$ by $g_2(t)$, then $\widetilde{g}_1\widetilde{g}_2$ is represented by the path $t \mapsto g_1(t)g_2(t)$.

Definition 1.1 Let $\widetilde{g} \in \widetilde{G}$ be represented by a path $g(t)$ (with $g(0) = e$), and $\tilde{z} \in \widetilde{M}$ be represented by a path $z(t)$ (with $z(0) = z_0$). Then we define $\widetilde{g} \cdot \tilde{z}$ to be $\tilde{y} \in \widetilde{M}$, where $\tilde{y}$ is the homotopy class represented by the path $t \mapsto g(t) \cdot z(t)$. It is readily checked that the homotopy class of this path depends only on the homotopy classes $\widetilde{g}$ and $\tilde{z}$.

With this definition for the action of $\widetilde{G}$ on $\widetilde{M}$, it is clear that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{G} \times \widetilde{M} & \longrightarrow & \widetilde{M} \times \widetilde{M} \\ \downarrow & & \downarrow \\ G \times M & \longrightarrow & M \times M \end{array} \tag{1.2}$$

where the vertical arrows are $q_G \times q_M$ and $q_M \times q_M$ respectively, and the horizontal arrows are the group actions. In particular,

$$\tilde{y} = \widetilde{g} \cdot \tilde{z} \quad \Longrightarrow \quad y = g \cdot z \tag{1.3}$$

where for $\tilde{z} \in \widetilde{M}$ we denote its projection to M by z, and similarly with elements of $\widetilde{G}$. Note for future reference that it follows immediately from (1.3) that the isotropy subgroups satisfy

$$\widetilde{g} \in \widetilde{G}_{\tilde{z}} \quad \Longrightarrow \quad g \in G_z. \tag{1.4}$$

Remark 1.2 A second approach to defining the action of $\widetilde{G}$ on $\widetilde{M}$ is as follows. The action of G gives rise to an 'action' of the Lie algebra $\mathfrak{g}$. That is, to each $\xi \in \mathfrak{g}$ there is associated a vector field ξ_M on M; these are the so-called generating vector fields of the G-action. Let $N \to M$ be any covering. The covering map is a local diffeomorphism, so the vector fields ξ_M can be lifted to vector fields ξ_N on N. Because this covering map is a local diffeomorphism, this gives rise to an 'action' of $\mathfrak{g}$ on N. Now $\mathfrak{g}$ is the Lie algebra of a unique simply connected Lie group $\widetilde{G}$. To see that the vector fields on N are complete, so defining an action of $\widetilde{G}$, one needs to compare the local actions on M and N. It is not hard to see that the two definitions of actions of $\widetilde{G}$ are equivalent.

Lemma 1.3 *Let $g(t)$ be a path in G with $g(0) = e$, and $z(t)$ a path in M with $z(0) = z_0$ and $z(1) = z_1$. Then the following three homotopy classes coincide:*

$$g(t) \cdot z(t), \quad [g(t) \cdot z_0] * [g(1) \cdot z(t)], \quad z(t) * [g(t) \cdot z_1],$$

where $*$ is the homotopy product of paths.

Proof. Denote the three curves by $a(t), b(t)$ and $c(t)$ respectively. So for example,

$$c(t) = \begin{cases} z(2t) & \text{if } t \in [0, \frac{1}{2}] \\ g(2t-1) \cdot z_1 & \text{if } t \in [\frac{1}{2}, 1] \end{cases}.$$

A homotopy between a and b can be given by

$$A(t,s) = \begin{cases} g((1+s)t) \cdot z((1-s^2)t) & \text{if } t \leq \frac{1}{1+s} \\ g(1) \cdot z((1+s)t - s) & \text{if } t \geq \frac{1}{1+s} \end{cases}.$$

Then, $A(t,0) = a(t)$ and $A(t,1) = b(t)$. It is readily checked that $A(t,s)$ is continuous. A similar homotopy can be defined between a and c. $\square$

Recall that $\Gamma := \pi_1(M, z_0)$ acts on $\widetilde{M}$ by deck transformations; that is, given $\gamma \in \Gamma$ and $\tilde{z} \in \widetilde{M}$ then $\gamma \cdot \tilde{z} := \gamma * \tilde{z}$. This action is transitive on fibres of the covering map q_M. Furthermore, the fibre $q_M^{-1}(z_0)$ is the Γ-orbit of the constant loop $\tilde{z}_0$ which we identify with Γ, see equation (1.1).

Proposition 1.4 *The action of $\widetilde{G}$ on $\widetilde{M}$ commutes with the deck transformations. Furthermore, for each $\widetilde{g} \in \pi_1(G,e)$ the homotopy class $g(t)\cdot z_0$ lies in the centre of $\pi_1(M,z_0)$.*

Proof. Let $\widetilde{g} \in \widetilde{G}$, $\delta \in \Gamma$ and $\tilde{z} \in \widetilde{M}$ with $q_M(\tilde{z}) = y \in M$. We want to show that $\widetilde{g}\cdot(\delta\cdot\tilde{z}) = \delta\cdot(\widetilde{g}\cdot\tilde{z})$. By Lemma 1.3 (applied with $\gamma = \delta * \tilde{z}$), we have

$$\widetilde{g}\cdot(\delta\cdot\tilde{z}) = [\delta*\tilde{z}]*[\widetilde{g}\cdot y],$$

while again by Lemma 1.3 (now with $\gamma = \tilde{z}$),

$$\delta\cdot(\widetilde{g}\cdot\tilde{z}) = \delta*[\tilde{z}*(\widetilde{g}\cdot y)].$$

The result follows from the associativity of the homotopy product of paths.

Now let $\widetilde{g} \in \pi_1(G,e)$ and $\delta \in \Gamma$. We want to show that $[\widetilde{g}\cdot\tilde{z}_0]*\delta = \delta*[\widetilde{g}\cdot\tilde{z}_0]$, where $\tilde{z}_0$ is the constant loop at x. By Lemma 1.3, $\delta*[\widetilde{g}\cdot\tilde{z}_0] = \widetilde{g}\cdot\delta = [\widetilde{g}\cdot\tilde{z}_0]*\delta$ (since $g(1) = e$), as required. ☐

As a particular example, this leads to the following well-known result

Corollary 1.5 $\pi_1(G,e)$ *lies in the centre of $\widetilde{G}$. So the following is a central extension:*

$$1 \to \pi_1(G,e) \to \widetilde{G} \xrightarrow{q_G} G \to 1. \tag{1.5}$$

Proof. This follows by applying the proposition to the left action of $\widetilde{G}$ on itself. ☐

Now we are in a position to define the action of $\widetilde{G}$ on an arbitrary cover (N,y_0) of (M,z_0). As in §1.1, let $\Gamma_N = p_{N*}(\pi_1(N,y_0)) < \Gamma$. So, $N \simeq \widetilde{M}/\Gamma_N$. That is, a point in N is a Γ_N-orbit of points in $\widetilde{M}$.

Definition 1.6 The $\widetilde{G}$-action on N is defined simply by

$$\widetilde{g}\cdot\Gamma_N\tilde{z} := \Gamma_N(\widetilde{g}\cdot\tilde{z}).$$

This is well-defined as the actions of $\widetilde{G}$ and Γ commute, by Proposition 1.4. It is clear too that the analogues of (1.2), (1.3), and (1.4) hold with N in place of $\widetilde{M}$.

Proposition 1.7 *Let $p_N : (N,y_0) \to (M,z_0)$ be a covering map. The $\widetilde{G}$-orbits on N are the connected components of the inverse images under p_N of the orbits on M. More precisely, if $y \in p_N^{-1}(z) \subset N$ then $\widetilde{G}\cdot y$ is the connected component of $p_N^{-1}(G\cdot z)$ containing y. In particular if the G-orbits in M are closed, so too are the $\widetilde{G}$-orbits in N.*

Proof. Let $Z \subset M$ be any submanifold. Then $Z' := p_N^{-1}(Z)$ is a submanifold of N and the projection $p_N|_{Z'} : Z' \to Z$ is a covering, and if Z is closed so too is Z'. Moreover, if Z is G-invariant (hence $\widetilde{G}$-invariant), then by the equivariance of p_N so is Z', and if Z is a single orbit, then Z' is a discrete union of orbits: discrete because p_N is a covering. Since $\widetilde{G}$ is connected, the orbits are the connected components of Z'. ☐

1.3 The kernel of the lifted action

Let $\widetilde{g} \in \pi_1(G,e)$ be represented by a path $g(t)$, with $g(1) = e$. The path $g(t)\cdot z_0$ is an element of $\pi_1(M,z_0)$. Moreover, homotopic loops in G give rise to homotopic loops in M, so this induces a well-defined homomorphism

$$a_{z_0} : \pi_1(G,e) \to \pi_1(M,z_0).$$

In short, if $\widetilde{g} \in \pi_1(G,e)$ and $\tilde{z}_0$ is the trivial homotopy class at z_0, then $a_{z_0}(\widetilde{g}) = \widetilde{g}\cdot\tilde{z}_0$. It follows from Proposition 1.4 that $\mathrm{image}(a_{z_0}) \subset \mathcal{Z}(\pi_1(M,z_0))$ (the centre of π_1).

Note that, if z_0 is a fixed point for the G-action then a_{z_0} is trivial (that is, $\forall \widetilde{g} \in \widetilde{G}$, $a_{z_0}(\widetilde{g}) = \tilde{z}_0$, the trivial loop in $\pi_1(M,z_0)$).

Proposition 1.8 **(i)** *$K := \ker a_{z_0}$ is independent of the choice of z_0, and acts trivially on $\widetilde{M}$ and hence on every cover of M.*

(ii) *If (N,y_0) is a cover of (M,z_0), with associated subgroup Γ_N of $\pi_1(M,z_0)$, then $K_N := a_{z_0}^{-1}(\Gamma_N)$ is independent of the choice of base point y_0 in N, and acts trivially on N.*

(iii) *If G acts effectively on M then $G_N := \widetilde{G}/K_N$ acts effectively on N.*

Note that since the domain of a_{z_0} is $\pi_1(G,e)$ which is in the centre of $\widetilde{G}$, it follows that K_N is a normal subgroup of $\widetilde{G}$. And with the notation of the proposition, $K = K_{\widetilde{M}}$ since $\Gamma_{\widetilde{M}}$ is trivial. We will write

$$G' := \widetilde{G}/K \tag{1.6}$$

for the group acting on $\widetilde{M}$.

A particular case is where the action of G on M has a fixed point. If z_0 is such a fixed point then $a_{z_0} = 0$ and $K = \pi_1(G,e)$ so the action on M lifts to an action of G on $\widetilde{M}$, and hence on any other cover N. More generally this is true if any (and hence every) G-orbit in M is contractible in M, since in that case too a_{z_0} is trivial.

Proof. **(i)** Let η be any path from z_0 to z_0'. Then η induces an isomorphism $\eta_* : \pi_1(M,z_0) \to \pi_1(M,z_0')$, by $\eta_*(\gamma) = \eta^{-1} * \gamma * \eta$. With this notation, $a_{z_0'} = \eta_* \circ a_{z_0}$ so both have the same kernel. That K acts trivially on $\widetilde{M}$ follows from the definition of a_{z_0}: let $\tilde{z} \in \widetilde{M}$ and $\widetilde{g} \in K$, then $\widetilde{g}\cdot\tilde{z} = \widetilde{g}\cdot(\tilde{z}_0 * \tilde{z}) = a_{z_0}(\widetilde{g}) * \tilde{z} = \tilde{z}$ (using Lemma 1.3).
(ii) The argument is similar to part (i). Let $y_0' \in N$, let $z_0' = p_N(y_0') \in M$ and let η be any path from y_0 to y_0', with $\overline{\eta}$ its projection to M. The result follows from the fact that the following diagram commutes:

$$\begin{tikzcd}
 & \pi_1(G,e) \arrow[dl, "a_{z_0}"'] \arrow[dr, "a_{z_0'}"] & \\
\pi_1(M,z_0) \arrow[rr, "\overline{\eta}_*"] & & \pi_1(M,z_0') \\
\pi_1(N,y_0) \arrow[u, "p_{N*}"] \arrow[rr, "\eta_*"] & & \pi_1(N,y_0') \arrow[u, "p'_{N*}"']
\end{tikzcd}$$

Writing $N = \widetilde{M}/\Gamma_N$, if $\widetilde{g} \in a_{z_0}^{-1}(\Gamma_N)$ then $\widetilde{g} \in K\Gamma_N$ and, $\widetilde{g}\Gamma_N\tilde{z} \subset \Gamma_N K\tilde{z} = \Gamma_N\tilde{z}$ so $\widetilde{g}$ acts trivially (using Proposition 1.4 and part (i)).
(iii) Suppose $\widetilde{g} \in \widetilde{G}$ acts trivially on N, so for all $y \in N$, $\widetilde{g}\cdot y = y$. Projecting to M, this implies that $g(1)\cdot z = z$ (for all $z \in M$) so $g(1) \in \cap_{z\in M} G_z = \{e\}$. Thus $\widetilde{g} \in \pi_1(G,e)$.

To prove the statement, we first consider the case $N = \widetilde{M}$. If $\widetilde{g} \notin K$ then $a_{z_0}(\widetilde{g}) \neq \tilde{z}_0 \in \pi_1(M,z_0)$. Since $\pi_1(M,z_0)$ acts effectively (by deck transformations) on the fibre $q_M^{-1}(z_0) \simeq \pi_1(M,z_0) \subset \widetilde{M}$ it follows that $a_{z_0}(\widetilde{g})$ acts non-trivially, which is in contradiction with the assumption that $\widetilde{g}$ acts trivially.

Now suppose $\widetilde{g} \in \widetilde{G}$ acts trivially on N. We have $\widetilde{g}\Gamma_N\tilde{z}_0 = \Gamma_N\tilde{z}_0$, so that $\widetilde{g} \in \Gamma_N K = a_{z_0}^{-1}(\Gamma_N)$ as required. $\square$

1.4 Isotropy subgroups

Fix $y_0 \in N$ and let $\widetilde{g} \in \widetilde{G}_{y_0}$, the isotropy subgroup at y_0 for the $\widetilde{G}$ action on N. It follows that $q_G(\widetilde{g}) \in G_{z_0}$, where $z_0 = p_N(y_0)$, since $\widetilde{g} \cdot y = y \Rightarrow g \cdot z = z$. Consequently, $\widetilde{G}_{y_0}$ is a subgroup of $\Lambda_{z_0} := q_G^{-1}(G_{z_0})$. Restricting the exact sequence (1.5), we have

$$1 \to \pi_1(G,e) \to \Lambda_{z_0} \xrightarrow{q_G} G_{z_0} \to 1. \tag{1.7}$$

The group Λ_{z_0} consists of those homotopy classes of paths $g(t)$ with $g(0) = e$ and $g(1) \in G_{z_0}$. It follows that $g(t) \cdot z_0$ is a closed loop, so determines a well-defined element of $\pi_1(M,z_0)$. That is, the homomorphism a_{z_0} described above extends naturally to a homomorphism

$$\bar{a}_{z_0} : \Lambda_{z_0} \to \pi_1(M,z_0).$$

In contrast to a_{z_0}, this homomorphism *does* depend on z_0. Let L_{z_0} be the kernel of this homomorphism (which obviously contains K), and $L_{(N,y_0)} := \bar{a}_{z_0}^{-1}(\Gamma_N)$ (which contains K_N).

Recall that $G_N := \widetilde{G}/K_N$ from Proposition 1.8.

Proposition 1.9 *The isotropy subgroups for the lifted actions are as follows:*

(i) *at $\tilde{z}_0$ for the $\widetilde{G}$-action on $\widetilde{M}$ it is $\widetilde{G}_{\tilde{z}_0} = L_{z_0}$ and for G' it is $G'_{\tilde{z}_0} = L_{z_0}/K$*

(ii) *at y_0 for the $\widetilde{G}$-action on N it is $\widetilde{G}_{y_0} \simeq L_{(N,y_0)}$ and consequently, $(G_N)_{y_0} \simeq L_{(N,y_0)}/K_N$.*

Proof. We just prove (ii) as (i) is a special case. Let $\widetilde{g} \in \widetilde{G}$ be represented by a path $g(t)$. Then $\widetilde{g} \cdot y_0 = y_0$ implies $g(1) \in G_{z_0}$; that is, $\widetilde{g} \in \Lambda_{z_0}$. Using $y_0 = \Gamma_N \tilde{z}_0$, we have $\widetilde{g} \cdot \Gamma_N \tilde{z}_0 = \Gamma_N \tilde{z}_0$ and this is equivalent to $\widetilde{g} \cdot \tilde{z}_0 \in \Gamma_N \tilde{z}_0 = \Gamma_N$ (as in (1.1)); that is, $\bar{a}_{z_0}(\widetilde{g}) \in \Gamma$, so we are done. □

Corollary 1.10 *If the G-action on M is free, then so is the G_N-action on N.*

Proof. Since G_{z_0} is trivial, we have $\Lambda_{z_0} = \pi_1(G,e)$ and hence $\bar{a}_{z_0} = a_{z_0}$ and thus $L_{(N,y_0)} = K_N$, so $(G_N)_{y_0}$ is trivial. □

To identify the isotropy subgroups L_{z_0}/K or $L_{(N,y_0)}/K_N$ with subgroups of the isotropy subgroup G_{z_0} we define a homomorphism

$$\begin{array}{rcccl} \psi_{z_0} & : & G_{z_0} & \longrightarrow & \operatorname{coker}(a_{z_0}) \\ & & g & \longmapsto & \widetilde{g} \cdot z_0 \mod \operatorname{image}(a_{z_0}) \end{array} \tag{1.8}$$

where $\widetilde{g}$ is any lift of g. We take *right* cosets, so $g \bmod H = Hg$

The homomorphism ψ_{z_0} is well defined, for given any two lifts $\widetilde{g}_1$ and $\widetilde{g}_2$ of $g \in G_z$, define $\widetilde{g}_0 \in \pi_1(G,e)$ to be the homotopy product of the path $g_1(t)$ and the reverse path of $g_2(t)$ (which goes from g to e):

$$g_0(t) = \begin{cases} g_1(2t) & \text{for } t \in [0,\frac{1}{2}] \\ g_2(2-2t) & \text{for } t \in [\frac{1}{2},1] \end{cases} .$$

Then $\widetilde{g}_1 \cdot \tilde{z}_0 = (\widetilde{g}_0 \cdot \tilde{z}_0) * (\widetilde{g}_2 \cdot \tilde{z}_0) \in \operatorname{image}(a_{z_0}).(\widetilde{g}_2 \cdot \tilde{z}_0)$, as required.

The homomorphism $\bar{a}_{z_0}$ induces a morphism between two short exact sequences, the lower two rows of the following commutative diagram:

$$
\begin{tikzcd}
K \arrow[d] & L_{z_0} \arrow[d] & G_{z_0} \arrow[d] \\
\pi_1(G,e) \arrow[r] \arrow[d, "a_{z_0}"] & \Lambda_{z_0} \arrow[r, "p_G"] \arrow[d, "\bar{a}_{z_0}"] & G_{z_0} \arrow[d] \\
\pi_1(M,z_0) \arrow[r, "="] & \pi_1(M,z_0) \arrow[r] & 1
\end{tikzcd}
$$

where the first row consists of the kernels of the vertical homomorphisms.

Proposition 1.11 (i). *There is an exact sequence*

$$0 \to K \to L_{z_0} \to G_{z_0} \xrightarrow{\psi_{z_0}} \operatorname{coker}(a_{z_0}) \to \operatorname{coker}(\bar{a}_{z_0}) \to 0 \tag{1.9}$$

where the homomorphism $\psi_{z_0} : G_{z_0} \to \operatorname{coker}(a_{z_0})$ *is defined above (1.8). Consequently,*

(ii). $G'_{\tilde{z}_0} = \ker \psi_{z_0}$*, which is a subgroup of* G_{z_0}

(iii). $(G_N)_{y_0}$ *is isomorphic to* $\psi_{z_0}^{-1}(\Gamma_N \mod \operatorname{image}(a_{z_0}))$.

Since $\operatorname{image}(\bar{a}_{z_0})$ is not in general normal in $\pi_1(M,z_0)$, $\operatorname{coker}(\bar{a}_{z_0})$ here is just the set of right cosets of $\operatorname{image}(\bar{a}_{z_0})$ in $\pi_1(M,z_0)$; and exactness at $\operatorname{coker}(\bar{a}_{z_0})$ means only that the map $\operatorname{coker}(a_{z_0}) \to \operatorname{coker}(\bar{a}_{z_0})$ is surjective (which is obvious as $\bar{a}_{z_0}$ is an extension of a_{z_0}). The first part of the proposition would be an instance of the snake lemma, but for the fact that the groups here are not all abelian.

Proof. (i) We show exactness at each 'node' in turn (the proof is just that of the snake lemma):

• Exactness at K is clear, for the homomorphism is the restriction of $\pi_1(G,e) \to \Lambda_{z_0}$ which is injective.

• Suppose $\ell \in L_{z_0}$ is such that $p_G(\ell) = e \in G_{z_0}$. Then $\ell \in \pi_1(G,e)$ (by definition). And since $\ell \in L_{z_0}$ it follows that $a_{z_0}(\ell) = \bar{a}_{z_0}(\ell) = 1$. So $\ell \in K$, as required for exactness at L_{z_0}.

• Now suppose $g \in G_{z_0}$ is such that $\psi_{z_0}(g) = 1$, i.e., such that $\widetilde{g} \cdot z \in \operatorname{image}(a_{z_0})$ for some $\widetilde{g} \in p_G^{-1}(g) \subset \Lambda_{z_0}$. Then $\exists \widetilde{h} \in \pi_1(G,e)$ such that $\widetilde{g} \cdot z = \widetilde{h} \cdot z$. Now consider $\sigma = \widetilde{h}^{-1}\widetilde{g} \in G$. Clearly, $p_G(\sigma) = p_G(\widetilde{h}^{-1})p_G(\widetilde{g}) = p_G(\widetilde{g}) = g$ so $\sigma \in \Lambda_{z_0}$. Moreover, $\bar{a}_{z_0}(\sigma) = \bar{a}_{z_0}(\widetilde{h})^{-1}\bar{a}_{z_0}(\widetilde{g}) = 1 \in \pi_1(M,z_0)$. That is, $\sigma \in L_{z_0}$ and $p_G(\sigma) = g$, so $g \in p_G(L_{z_0})$ as required.

• Exactness at $\operatorname{coker}(a_{z_0})$. Write $j : \operatorname{coker}(a_{z_0}) \to \operatorname{coker}(\bar{a}_{z_0})$. Let $\gamma \in \ker(j) \subset \pi_1(M,z_0)$. Then $\gamma \in \operatorname{image}(\bar{a}_{z_0})$, so $\exists \widetilde{g} \in \Lambda_{z_0}$ such that $\gamma = \bar{a}_{z_0}(\widetilde{g})$. Then $p_G(\widetilde{g}) = g \in G_{z_0}$, and $\psi_{z_0}(g) = \gamma$ as required.

• Exactness at $\operatorname{coker}(\bar{a}_{z_0})$. As already stated above, this is just the fact that $\bar{a}_{z_0}$ is an extension of a_{z_0}.

(ii) $\ker \psi_{z_0} = \operatorname{image}[L_{z_0} \to G_{z_0}] \simeq L_{z_0}/K$ which is $(G')_{\tilde{z}_0}$ by Proposition 1.9.

(iii) If we replace $\pi_1(M,z_0)$ by $\Delta := \pi_1(M,z_0)/\Gamma_N$ in the bottom row of the diagram above, then $a'_{z_0} : \pi_1(G,e) \to \Delta$ has kernel equal to $K_N = a_{z_0}^{-}1(\Gamma_N)$ and $\bar{a}'_{z_0} : \Lambda_{z_0} \to \Delta$ has kernel equal to $L_{(N,y_0)}$. The proof follows now in the same way as the proof of (ii). $\square$

Notice firstly that the connected component of the identity $G^o_{z_0}$ of G_{z_0} is contained in $\ker \psi_{z_0}$. To see this it is enough to take $\widetilde{g}$ to be a path contained entirely in $G^o_{z_0}$. Secondly, notice that

$$\operatorname{image}(\psi_{z_0}) \simeq \frac{\operatorname{image}(\bar{a}_{z_0})}{\operatorname{image}(a_{z_0})}$$

so that for a given isotropy subgroup G_{z_0}, the larger the difference between the images of a_{z_0} and $\bar{a}_{z_0}$, the smaller the isotropy subgroup $G'_{\tilde{z}}$.

Remark 1.12 The argument in the second part of the proof of Proposition 1.4 can only be applied to elements of $\operatorname{image}(\bar{a}_{z_0})$ if $\delta \in F := \operatorname{Fix}(G_{z_0}, M)$, so that $g(1) \cdot \delta = \delta$. This means that $\operatorname{image}(\bar{a}_{z_0})$ merely centralizes the image of $\pi_1(F,z_0)$ in $\pi_1(M,z_0)$.

Theorem 1.13 *Let N be a cover of M, and suppose the G-action on M is effective and proper. Then the G_N-action on N is also proper.*

Proof. Since G acts properly on M there is a G-invariant Riemannian metric on M. This metric can be lifted by the covering map to one on N. Since the covering map is equivariant, it follows that the lifted metric is also G_N-invariant.

To show that action is proper, we need to show that the action map $\Phi_N : G_N \times N \to N \times N$ is closed and has compact fibres. The fibre $\Phi_N^{-1}(x,y) = \{(g,y) \in G_N \times N \mid g \cdot x = y\}$. If this is non-empty, and $h \cdot x = y$ then $\Phi_N^{-1}(x,y) \simeq h(G_N)_x$, which is compact since the G-action is proper, using Proposition 1.11.

To see that the action map is closed, consider a sequence (g_i, x_i) in $G_N \times N$ for which $(g_i \cdot x_i, x_i)$ converges to (y,z). Then of course $x_i \to z$. We claim that $g_i \cdot z \to y$. This is because,

$$d(g_i \cdot z, y) \leq d(g_i \cdot z, g_i \cdot x_i) + d(g_i \cdot x_i, y) = d(z, x_i) + d(g_i \cdot x_i, y),$$

where d is the G_N-invariant metric on N defined above. Both terms on the right tend to 0 so that $d(g_i \cdot z, y) \to 0$ as required.

Now, by Proposition 1.7 the G_N-orbits in N are closed and hence there is an $g \in G_N$ with $y = g \cdot z$. That is, $g_i \cdot z \to g \cdot z$. Consequently, $g_i(G_N)_z \to g(G_N)_z$ in $G_N/(G_N)_z$. By taking a slice to the proper $(G_N)_z$-action on G, this can be rewritten as $g_i h_i \to g$ in G_N, for some sequence $h_i \in (G_N)_z$. Since $(G_N)_z$ is compact, (h_i) has a convergent subsequence, $h_{i_k} \to h$. Then $g_{i_k} \to gh^{-1}$. It follows therefore that $(g_{i_k}, x_{i_k}) \to (gh^{-1}, z)$ and $\Phi_N(gh^{-1}, z) = (y,z)$. $\square$

Remark 1.14 There is an alternative argument for proving this theorem as follows. Any invariant (Riemannian) metric on M lifts to an invariant metric on N. By a standard result, the group $I(N)$ of isometries of N acts properly on N (see [15, problem 26, p.31] and [4, p.106], although neither give a detailed proof). Since the action of G_N is by isometries, it follows from the monomorphism $A : G_N \to I(N)$ that the action of G_N is proper. The argument we give is more direct, using the covering structure of the action.

2 Hamiltonian coverings

For the remainder of the paper, we assume the manifold M is endowed with a symplectic form ω and the Lie group G acts by symplectomorphisms. Notice that any cover $p_N : N \to M$ of M is also symplectic with form $\omega_N := p_N^* \omega$ and that, moreover, the lifted action of $\widetilde{G}$ (or G_N) on N is also symplectic. It follows that the category of all symplectic coverings of (M, ω) coincides with the category of all coverings of M. Furthermore, the deck transformations on $\widetilde{M}$ are also symplectic.

Symplectic Lie group actions are linked at a very fundamental level with the existence of *momentum maps*. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}^*$ its dual. We recall that a momentum map $\mathbf{J} : M \to \mathfrak{g}^*$ for the symplectic G-action on (M, ω) is defined by the condition that its components $\mathbf{J}^\xi := \langle \mathbf{J}, \xi \rangle$, $\xi \in \mathfrak{g}$, are Hamiltonian functions for the infinitesimal generator vector fields $\xi_M(m) := \frac{d}{dt}\big|_{t=0} \exp t\xi \cdot m$. The existence of a momentum map for the action is by no means guaranteed; however, it could be that the lifted action to a cover has this feature. For example, if the cover is simply connected (as is $\widetilde{M}$), the action necessarily has a momentum map associated. This remark leads us to the following definitions.

Definition 2.1 Let (M, z_0, ω) be a connected pointed symplectic manifold endowed with an action of the connected Lie group G. We say that the smooth covering $p_N : (N, y_0) \to (M, z_0)$ of (M, z_0) is a *Hamiltonian covering* of (M, z_0, ω) if N is connected and the lifted action of $\widetilde{G}$ (or G_N) on (N, ω_N) has a momentum map $\mathbf{J}_N : N \to \mathfrak{g}^*$ associated.

If the G-action on M is already Hamiltonian, then every cover is naturally a Hamiltonian cover, so the interesting case is where the symplectic action on M is not Hamiltonian.

The connectedness hypothesis on N assumed in the previous definition implies that any two momentum maps of the G_N-action on N differ by a constant element in $\mathfrak{g}^*$. We will assume that $\mathbf{J}_N$ is chosen so that $\mathbf{J}_N(y_0) = 0$. (This choice should perhaps be denoted $\mathbf{J}_{(N,y_0)}$, but we will refrain from the temptation!)

Definition 2.2 Let (M, z_0, ω) be a connected pointed symplectic manifold and G a Lie group acting symplectically thereon. Let $\mathfrak{H}$ be the category whose objects $\mathrm{Ob}(\mathfrak{H})$ are the pairs

$$(p_N : (N, y_0, \omega_N) \to (M, z_0, \omega),\ \mathbf{J}_N),$$

where p_N a Hamiltonian covering of (M, z_0, ω) and $\mathbf{J}_N : N \to \mathfrak{g}^*$ is the momentum map for the lifted $\widetilde{G}$- (or G_N-) action on N satisfying $\mathbf{J}_N(y_0) = 0$, and whose morphisms $\mathrm{Mor}(\mathfrak{H})$ are the smooth maps $p : (N_1, y_1, \omega_1) \to (N_2, y_2, \omega_2)$ that satisfy the following properties:

(i) p is a symplectic covering map

(ii) p is $\widetilde{G}$–equivariant

(iii) the following diagram commutes:

$$\begin{tikzcd} & \mathfrak{g}^* & \\ (N_1, y_1) \arrow[ur, "\mathbf{J}_{N_1}"] \arrow[rr, "p"] \arrow[dr, "p_{N_1}"'] & & (N_2, y_2) \arrow[ul, "\mathbf{J}_{N_2}"'] \arrow[dl, "p_{N_2}"] \\ & (M, z_0) & \end{tikzcd}$$

We will refer to $\mathfrak{H}$ as the category of *Hamiltonian coverings* of (M, z_0, ω).

It should be clear that the ingredients ω_N and $\mathbf{J}_N$ are both uniquely determined by $p_N : (N, y_0) \to (M, z_0)$ (given the symplectic form on M), so $\mathfrak{H}$ is in fact a (full) subcategory of the category of all coverings of (M, z_0).

The category of the Hamiltonian coverings of a symplectic manifold acted upon symplectically by a Lie algebra was studied in [12]. We will now use the developments in Section 1 to recover those results in the context of group actions. The study that we carry out in the following paragraphs sheds light on the *universal covered space* introduced in [12] and additionally will be of much use in Section 3 where we will spell out in detail the interplay between Hamiltonian coverings and symplectic reduction.

2.1 The momentum map on the universal cover

We now start by giving an expression for the momentum map associated to the $\widetilde{G}$-action on the universal cover $\widetilde{M}$ of M. As far as this momentum map is concerned, it does not matter if we consider the $\widetilde{G}$ or the G' action (see (1.6)) since both have the same Lie algebra and the momentum map depends only on the infinitesimal part of the action. Recall that the *Chu map* $\Psi : M \to Z^2(\mathfrak{g})$ is defined by

$$\Psi(z)(\xi, \eta) := \omega(z)\left(\xi_M(z), \eta_M(z)\right). \tag{2.1}$$

for $\xi, \eta \in \mathfrak{g}$.

Proposition 2.3 *Let (M, ω) be a connected symplectic manifold acted upon symplectically by the connected Lie group G. Then, the $\widetilde{G}$-action on $(\widetilde{M}, \widetilde{\omega} := q_M^* \omega)$ has a momentum map associated $\mathbf{J} : \widetilde{M} \to \mathfrak{g}^*$*

that can be expressed as follows: realize $\widetilde{M}$ as the set of homotopy classes of paths in M with base point z_0. Let $\widetilde{x} \in \widetilde{M}$ and $x(t)$ an element in the homotopy class $\widetilde{x}$. Then, for any $\xi \in \mathfrak{g}$

$$\langle \mathbf{J}(\widetilde{x}),\, \xi \rangle = \int_{[0,1]} x^*(\mathbf{i}_{\xi_M}\omega) = \int_0^1 \omega(x(t))\big(\xi_M(x(t)), \dot{x}(t)\big)\; \mathrm{d}t. \tag{2.2}$$

If $\widetilde{x} \in \pi_1(M, z_0)$ and $\widetilde{y} \in \widetilde{M}$ then $\widetilde{x} * \widetilde{y} \in \widetilde{M}$ and

$$\mathbf{J}(\widetilde{x} * \widetilde{y}) = \mathbf{J}(\widetilde{x}) + \mathbf{J}(\widetilde{y}). \tag{2.3}$$

The non-equivariance cocycle $\sigma_{\mathbf{J}} : \widetilde{G} \to \mathfrak{g}^*$ of $\mathbf{J}$ is given by

$$\langle \sigma_{\mathbf{J}}(\widetilde{g}), \xi \rangle = \int_0^1 \Psi(z_0)(\xi_t,\, \eta_t)\; \mathrm{d}t, \tag{2.4}$$

for any $\xi \in \mathfrak{g}$, $\widetilde{g} \in \widetilde{G}$, and $g(t)$ a curve in the homotopy class of $\widetilde{g}$, where $\xi_t = \mathrm{Ad}_{g(t)^{-1}}\xi$ and $\eta_t = \left(T_e L_{g(t)}\right)^{-1} \dot{g}(t)$, and Ψ is the Chu map defined in (2.1) above.

Momentum maps are only defined up to a constant; the one in (2.2) is normalized to vanish on the trivial homotopy class $\widetilde{z}_0$ at z_0. The expression (2.2) is very similar to the one in [9] for the momentum map of the action of a group G on the fundamental groupoid of a symplectic G-manifold.

Proof. Let $\alpha := \mathbf{i}_{\xi_M}\omega$. Since this 1-form on M is closed, it follows that $\int x^*\alpha$ depends only on the homotopy class (indeed homology class) of x; that is, $\mathbf{J}(\tilde{x})$ is well-defined by (2.2).

To show that that $\mathbf{J}$ is a momentum map for the $\widetilde{G}$-action on $\widetilde{M}$, we use the Poincaré Lemma on the closed form α. Cover the image of $x(t)$ in M by contractible well-chained open sets (open in M), $U_1, \ldots, U_n$, with $x(0) = z_0 \in U_1$ and $x(1) \in U_n$. We can enumerate these sets consecutively along the curve $x(t)$, and let $z_j = x(t_j) \in U_j \cap U_{j+1}$ lie on the curve, and $z_0 = x(0)$ (as always) and $z_n = x(1)$.

On each U_j we can write $\alpha = \mathrm{d}\phi_j$ for some function ϕ_j (in fact a local momentum for ξ_M). Then on $U_i \cap U_j$, $\mu_{i,j} := \phi_i - \phi_j$ is constant.

Now, with $I = [0,1]$ and $I_j = [t_j, t_{j+1}]$ we have

$$\int_I x^*\alpha = \sum_j \int_{I_j} x^*\, \mathrm{d}\phi_j = \sum_j (\phi_j(z_{j+1}) - \phi_j(z_j)) = \phi_n(z_n) - \phi_1(z_0) - \sum_{j=1}^{n-1} \mu_{j+1,j}\,. \tag{2.5}$$

The covering map $q_M : \widetilde{M} \to M$, $\tilde{x} \mapsto x(1)$ identifies the tangent space $T_{\tilde{x}}\widetilde{M}$ with $T_{x(1)}M$. Let $\widetilde{v} \in T_{\tilde{x}}\widetilde{M}$ arbitrary and $v = T_{\tilde{x}}q_M(\widetilde{v})$. Thus, differentiating (2.5) at $\tilde{x}$ in the direction $\widetilde{v} \in T_{\tilde{x}}\widetilde{M}$ gives

$$\mathrm{d}\left(\int x^*\alpha\right)(\widetilde{v}) = \mathrm{d}\phi_n(x(1))(v) = \alpha(x(1))(v) = \omega(\xi_M, v) = \widetilde{\omega}(\xi_{\widetilde{M}}, \widetilde{v}),$$

as required.

The identity (2.3) follows from a straightforward verification.

We conclude by computing the non-equivariance cocycle $\sigma_{\mathbf{J}}$. By definition, for any $\widetilde{g} \in \widetilde{G}$ and $\xi \in \mathfrak{g}$

$$\sigma_{\mathbf{J}}(\widetilde{g}) = \mathbf{J}(\widetilde{g} \cdot \widetilde{x}) - \mathrm{Ad}^*_{\widetilde{g}^{-1}}\mathbf{J}(\widetilde{x}),$$

for any $\widetilde{x} \in \widetilde{M}$. Take $\widetilde{x} = \widetilde{z}_0$ and use (2.2). The formula for $\sigma_{\mathbf{J}}$ then follows by recalling that $\mathbf{J}(\widetilde{z}_0) = 0$ and that the G-action on M is symplectic. $\square$

Remark 2.4 If the Chu map vanishes at one point, then clearly $\mathbf{J}$ is coadjoint-equivariant. This happens if there is an isotropic orbit in M (and hence in $\widetilde{M}$).

2.2 The Hamiltonian holonomy and Hamiltonian coverings

Definition 2.5 Let (M, z_0, ω) be a connected pointed symplectic manifold with symplectic action of the connected Lie group G. Let $\mathbf{J} : \widetilde{M} \to \mathfrak{g}^*$ be the momentum map defined in Proposition 2.3. The *Hamiltonian holonomy* $\mathcal{H}$ of the G-action on (M, ω) is defined as $\mathcal{H} = \mathbf{J}(\Gamma)$, and for an arbitrary symplectic cover $p_N : N \to M$, the homonomy group is $\mathcal{H}_N := \mathbf{J}(\Gamma_N)$, where $\Gamma = \pi_1(M, z_0)$ and $\Gamma_N = (p_N)_*\big(\pi_1(N, y_0)\big)$ (as in §1).

Proposition 2.6 *The symplectic cover $p_N : (N, y_0) \to (M, z_0)$ is Hamiltonian if and only if $\mathcal{H}_N = 0$.*

Proof. If the $\widetilde{G}$-action on N is Hamiltonian, then the momentum map is well-defined. This means that if γ is any closed loop in N, then $\mathbf{J}(\overline{\gamma}) = 0$, where $\overline{\gamma} \in \pi_1(M, z_0)$ is the image under $(p_N)_*$ of the homotopy class of γ. Conversely, if $\mathcal{H}_N = 0$ then the map $\mathbf{J} : \widetilde{M} \to \mathfrak{g}^*$ descends to a map $\mathbf{J}_N : \widetilde{M}/\Gamma_N \to \mathfrak{g}^*$, and as described in §1, $N \simeq \widetilde{M}/\Gamma_N$. □

Let us emphasize that if $p_N : (N, y_0) \to (M, z_0)$ is a Hamiltonian cover, then the momentum map $\mathbf{J}_N : N \to \mathfrak{g}^*$ is defined uniquely by the following diagram.

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\mathbf{J}} & \mathfrak{g}^* \\ {\scriptstyle q_N}\big\downarrow & & \big\downarrow{\scriptstyle =} \\ N & \xrightarrow{\mathbf{J}_N} & \mathfrak{g}^* \end{array} \tag{2.6}$$

As we pointed out in Section 1, the subgroups of the fundamental group $\Gamma = \pi_1(M, z_0)$ classify the covers of M. In a similar vein, the following result shows that the subgroups of the subgroup Γ_0 of Γ play the same rôle with respect to the Hamiltonian covers of the symplectic G-manifold (M, ω).

Define,

$$\Gamma_0 := \mathbf{J}^{-1}(0) \cap q_M^{-1}(z_0) \subset \pi_1(M, z_0). \tag{2.7}$$

Corollary 2.7 *The symplectic cover $p_N : (N, y_0) \to (M, z_0)$ is Hamiltonian if and only if $\Gamma_N < \Gamma_0$. Consequently, $\mathfrak{H}$ is isomorphic to the category of subgroups of Γ_0.*

Recall that the category $\mathfrak{S}(\Gamma)$ of subgroups of a group Γ is the category whose objects are the subgroups, and whose morphisms are the inclusions of one subgroup into another. We have therefore shown that $\mathfrak{H} \simeq \mathfrak{S}(\Gamma_0)$. Explicitly, the isomorphism is given by

$$\begin{array}{rcl} \mathfrak{H} & \longrightarrow & \mathfrak{S}(\Gamma_0) \\ \big(p_N : (N, y_0) \to (M, z_0), \mathbf{J}_N\big) & \longmapsto & \Gamma_N = (p_N)_*(\pi_1(N, y_0)). \end{array} \tag{2.8}$$

2.3 The universal Hamiltonian covering and covered spaces

As it was shown in the previous section, the Hamiltonian coverings of a symplectic G-manifold (M, ω) are characterized by the subgroups of $\Gamma_0 := \mathbf{J}^{-1}(0) \cap \pi_1(M, z_0)$.

The covering associated to the smallest possible subgroup, that is, the trivial group, is obviously the simply connected universal covering $\widetilde{M}$ of M. It is easy to check that this object satisfies in the category $\mathfrak{H}$ of Hamiltonian coverings, the same universality property that it satisfies in the general category of covering spaces, that is, $(p_{\widetilde{M}} : \widetilde{M} \to M, \mathbf{J}) \in \mathrm{Ob}(\mathfrak{H})$ and for any other Hamiltonian covering $(p_N : N \to M, \mathbf{J}_N)$ of (M, ω) there exists a morphism $q_N : (\widetilde{M}, \widetilde{\omega}) \to (N, \omega_N)$ in $\mathrm{Mor}(\mathfrak{H})$. Moreover, any other element in $\mathrm{Ob}(\mathfrak{H})$ that has this universality property is isomorphic to $(p_{\widetilde{M}} : \widetilde{M} \to M, \mathbf{J})$ (we have suppressed the dependence on base points $z_0, y_0, \tilde{z}_0$ in this discussion; if they are included the morphisms become unique—see Remark 2.9 below).

A major difference between the general category of covering spaces and the category of Hamiltonian coverings arises when we look at the covering associated to the biggest possible subgroup of Γ_0, that is, Γ_0 itself. Unlike the situation found for general coverings, where the biggest possible subgroup that one considers is the fundamental group Γ and it is associated to the trivial (identity) covering, the covering associated to Γ_0 is non-trivial (unless M is already Hamiltonian) and has an interesting universality property that is “dual” to the one exhibited by the universal covering. This special object in $\mathfrak{H}$ was first investigated in the context of Lie algebra actions in [12] where it is defined as the holonomy bundle of a $\mathfrak{g}^*$-valued connection. We return to that approach below, but first give a definition of this space more in keeping with the topological approach used so far in this paper.

Define $\widehat{M} := \widetilde{M}/\Gamma_0$. Because of the following result, $\widehat{M}$ is called the *universal covered space* of (M,ω). Recall from §1.1 that a covering $N \to M$ is said to be normal if Γ_N is a normal subgroup of Γ. Since Γ_0 is the kernel of a homomorphism $\Gamma \to \mathcal{H}$, it follows that $\widehat{M}$ is a normal covering of M. By Proposition 1.8, the group $\widehat{G} := \widetilde{G}/a_{z_0}^{-1}(\Gamma_0)$ acts effectively on $\widehat{M}$.

Proposition 2.8 *$\widehat{M}$ is a Hamiltonian normal covering of M with the universal property that for any given Hamiltonian covering $p_N : N \to M$ of M there is a Hamiltonian covering $\hat{p}_N : N \to \widehat{M}$.*

Proof. Since we have shown that $\mathfrak{H} \simeq \mathfrak{S}(\Gamma_0)$, this property of $\widehat{M}$ in $\mathfrak{H}$ follows from the corresponding property of Γ_0 in $\mathfrak{S}(\Gamma_0)$; namely that for every subgroup Γ_1 of Γ_0 there is an inclusion $\Gamma_1 \hookrightarrow \Gamma_0$. $\square$

Remark 2.9 $(\widetilde{M},\tilde{z}_0)$ and $(\widehat{M},\hat{z}_0)$ are initial and final objects in the category of Hamiltonian covers of (M,z_0) with base points; this of course corresponds to the fact that 1 and Γ_0 are initial and final objects in the category $\mathfrak{S}(\Gamma_0)$.

2.4 The connection in $M \times \mathfrak{g}^*$ and a model for the universal covered space

The universal covered space $\widehat{M}$ was introduced in [12] (though there it is denoted $\widetilde{M}$) using a connection in $M \times \mathfrak{g}^*$ proposed in [3]. Here we briefly review that definition, and show that it is equivalent to the one given above.

Let (M,ω) be a connected paracompact symplectic manifold and let G be a connected Lie group that acts symplectically on M. Take the Cartesian product $M \times \mathfrak{g}^*$ and let $\pi : M \times \mathfrak{g}^* \to M$ be the projection onto M. Consider π as the bundle map of the trivial principal fiber bundle $(M \times \mathfrak{g}^*, M, \pi, \mathfrak{g}^*)$ that has $(\mathfrak{g}^*,+)$ as Abelian structure group. The group $(\mathfrak{g}^*,+)$ acts on $M \times \mathfrak{g}^*$ by $\nu \cdot (z,\mu) := (z,\mu-\nu)$. Let $\alpha \in \Omega^1(M \times \mathfrak{g}^*;\mathfrak{g}^*)$ be the connection one-form defined by

$$\langle \alpha(z,\mu)(v_z,\nu),\xi\rangle := (\mathbf{i}_{\xi_M}\omega)(z)(v_z) - \langle \nu,\xi\rangle, \tag{2.9}$$

where $(z,\mu) \in M \times \mathfrak{g}^*$, $(v_z,\nu) \in T_zM \times \mathfrak{g}^*$, $\langle\cdot,\cdot\rangle$ denotes the natural pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$, and ξ_M is the infinitesimal generator vector field associated to $\xi \in \mathfrak{g}$.

The connection α is flat. For $(z_0,0) \in M \times \mathfrak{g}^*$, let $\widehat{M}' := (M \times \mathfrak{g}^*)(z_0,0)$ be the holonomy bundle through $(z_0,0)$ and let $\mathcal{H}(z_0,0)$ be the holonomy group of α with reference point $(z_0,0)$ (which is an Abelian zero dimensional Lie subgroup of $\mathfrak{g}^*$ by the flatness of α); in other words, $\widehat{M}'$ is the maximal integral leaf of the horizontal distribution associated to α that contains the point $(z_0,0)$ and it is hence endowed with a natural initial submanifold structure with respect to $M \times \mathfrak{g}^*$. See for example Kobayashi and Nomizu [7] for standard definitions and properties of flat connections and holonomy bundles.

The principal bundle $(\widehat{M}', M, \hat{p}, \mathcal{H}) := (\widehat{M}', M, \pi|_{(M\times\mathfrak{g}^*)(z_0,0)}, \mathcal{H}(z_0,0))$ is a reduction of the principal bundle $(M \times \mathfrak{g}^*, M, \pi, \mathfrak{g}^*)$. A straightforward verification shows that $\mathcal{H}(z_0,0)$ coincides with the Hamiltonian holonomy $\mathcal{H}$ introduced in Definition 2.5. In this sense, the momentum map $\mathbf{J} : \widetilde{M} \to \mathfrak{g}^*$ establishes a relationship between the deck transformation groups of the universal covering of M and of the holonomy bundle $\widehat{p} : \widehat{M}' \to M$. Moreover, the holonomy bundle $\widehat{M}'$ can be expressed using $\mathbf{J}$ as

$$\widehat{M}' = \{(q_M(\tilde{x}), \mathbf{J}(\tilde{x})) \mid \tilde{x} \in \widetilde{M}\}. \tag{2.10}$$

This expression allows one to easily check that $(\widehat{M}', M, \hat{p}, \mathcal{H})$ is actually a Hamiltonian covering of M with the symplectic form $\widehat{\omega}' := \widehat{p}^*\omega$. The $G_{\widehat{M}'}$-action on $\widehat{M}'$ is symplectic and is induced by the $\widetilde{G}$-action on $\widehat{M}'$ given by

$$\widetilde{g} \cdot (x, \mu) = (g \cdot x, \mathbf{J}(\widetilde{g} \cdot \widetilde{x})) = (g \cdot x, \sigma_{\mathbf{J}}(\widetilde{g}) + \mathrm{Ad}^*_{g^{-1}} \mathbf{J}(\widetilde{x})), \tag{2.11}$$

where $(x, \mu) \in \widehat{M}'$, $g = p_{\widetilde{G}}(\widetilde{g})$, and $\widetilde{x}$ is such that $p_{\widetilde{M}}(\widetilde{x}) = x$, and $\mathbf{J}(\widetilde{x}) = \mu$. The $G_{\widehat{M}'}$-action on $\widehat{M}'$ has a momentum map $\widehat{\mathbf{J}} : \widehat{M}' \to \mathfrak{g}^*$ given by $\widehat{\mathbf{J}}(x, \mu) = \mu$.

Proposition 2.10 *The universal covered space $\widehat{M} = \widetilde{M}/\Gamma_0$ is diffeomorphic to $\widehat{M}'$.*

Proof. The required diffeomorphism is implemented by the map

$$\begin{array}{cccc} \Theta: & \widetilde{M}/\Gamma_0 & \longrightarrow & \widehat{M}' \\ & [\widetilde{x}] & \longmapsto & (x(1), \mathbf{J}(\widetilde{x})). \end{array}$$

This map is well defined since by (2.3), the smooth map $\theta : \widetilde{M} \longrightarrow \widehat{M}'$ given by $\widetilde{x} \longmapsto (x(1), \mathbf{J}(\widetilde{x}))$ is Γ_0 invariant and hence it drops to the smooth map Θ. The map θ is an immersion since for any $v_{\widetilde{x}} \in T_{\widetilde{x}}\widetilde{M}$ such that $0 = T_{\widetilde{x}}\theta \cdot v_{\widetilde{x}} = \left(T_{\widetilde{x}} p_{\widetilde{M}} \cdot v_{\widetilde{x}}, T_{\widetilde{x}}\mathbf{J} \cdot v_{\widetilde{x}}\right)$, we have that $T_{\widetilde{x}} p_{\widetilde{M}} \cdot v_{\widetilde{x}} = 0$ and hence $v_{\widetilde{x}} = 0$, necessarily. Given that Γ_0 is a discrete group, the projection $\widetilde{M} \to \widetilde{M}/\Gamma_0$ is a local diffeomorphism and hence Θ is also an immersion. Addiditonally, by (2.10), the map Θ is also surjective. We conclude by showing that Θ is injective. Let $\widetilde{x}, \widetilde{y} \in \widetilde{M}$ be such that $\Theta([\widetilde{x}]) = \Theta([\widetilde{y}])$. This implies that

$$x(1) = y(1) \quad \text{and that} \quad \mathbf{J}(\widetilde{x}) = \mathbf{J}(\widetilde{y}). \tag{2.12}$$

The first equality in (2.12) implies that $\widetilde{x} * \widetilde{\overline{y}} \in \pi_1(M, z_0)$, where $\widetilde{\overline{y}}$ is the homotopy class associated to the reverse path $\overline{y}$ of y. Moreover, by the second equality in (2.12), it is easy to check that $\mathbf{J}(\widetilde{x} * \widetilde{\overline{y}}) = 0$, and hence $\widetilde{x} * \widetilde{\overline{y}} \in \Gamma_0$. Since $(\widetilde{x} * \widetilde{\overline{y}}) * \widetilde{y} = \widetilde{x}$ we can conclude that $[\widetilde{x}] = [\widetilde{y}]$, as required. Consequently, Θ being a smooth bijective immersion, it is necessarily a diffeomorphism. A straightforward verification shows that $\Theta \in \mathrm{Mor}(\mathfrak{H})$, which concludes the proof. ☐

3 Symplectic reduction and Hamiltonian coverings

Symplectic reduction is a well studied process that prescribes how to construct symplectic quotients out of the orbit spaces associated to the symplectic symmetries of a given symplectic manifold. Even though it is known how to carry this out for fully general symplectic actions [13], the implementation of this procedure is particularly convenient in the presence of a standard momentum map, that is, when the Hamiltonian holonomy is trivial (this is the so called symplectic or Marsden-Weinstein reduction [8]). Unlike the situation encountered in the general case with a non-trivial Hamiltonian holonomy, the existence of a standard momentum map implies the existence of a unique canonical symplectic reduced space. In the light of this remark the notion of Hamiltonian covering appears as an interesting and useful object for reduction. More specifically, one may ask whether, given a symplectic action on a symplectic manifold with non-trivial holonomy and with respect to which we want to reduce, we could lift the action to a Hamiltonian covering, perform reduction there with respect to a standard momentum map, and then project down the resulting space. How would this compare with the potentially complicated reduction in the original manifold? The main result in this section shows that indeed both processes yield exactly the same result.

Before we start with the presentation of this result we emphasize some points related to the actions introduced in Section 1. Let M be a manifold acted upon by the connected Lie group G and $p_N : N \to M$ a covering. In this section we will be interested in orbit spaces obtained out of the G-action on M and of the $\widetilde{G}$ and G_N-actions on N. Since by Proposition 1.8 the subgroup $K_N \subset \widetilde{G}$ acts trivially on N, the orbit spaces associated to the $\widetilde{G}$ and G_N-actions on N coincide. Another point is that since $p_{\widetilde{G}} : \widetilde{G} \to G$ is a covering map then the derivative $T_{\widetilde{e}} p_{\widetilde{G}} : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ is a Lie algebra isomorphism and hence, for any $\widetilde{g} \in \widetilde{G}$ and $\widetilde{\xi} \in \widetilde{\mathfrak{g}}$ such that $p_{\widetilde{G}}(\widetilde{g}) = g$ and $T_{\tilde{e}}p_{\widetilde{G}}(\widetilde{\xi}) = \xi$

$$T_{\tilde{e}}p_{\widetilde{G}}\left(\mathrm{Ad}_{\widetilde{g}}\widetilde{\xi}\right) = \mathrm{Ad}_g\xi. \tag{3.1}$$

In the pages that follow we will tacitly identify $\widetilde{\mathfrak{g}}$ with $\mathfrak{g}$. Moreover, since we will make no distinction between $\widetilde{\xi}$ and ξ, we will sometimes write (3.1) as $\mathrm{Ad}_{\widetilde{g}}\widetilde{\xi} = \mathrm{Ad}_g\xi$. The same applies to the covering $p_{G_N} : G_N := \widetilde{G}/K_N \to G$ and to the corresponding Lie algebra isomorphism $T_e p_{G_N} : \mathfrak{g}_N \to \mathfrak{g}$. It follows that the isotropy subgroups for the coadjoint action satisfy

$$\widetilde{G}_\mu = p_{\widetilde{G}}^{-1}(G_\mu),$$

and similarly for $(G_N)_\mu$.

3.1 The cylinder valued momentum map

Recall the definition of the holonomy of a symplectic action of G on M given in Definition 2.5: namely, $\mathcal{H} = \mathbf{J}(\Gamma)$, where as always, $\Gamma = \pi_1(M, z_0)$. Using this definition, equation (2.3) can be expressed by saying that $\mathbf{J}$ is equivariant with respect to Γ acting as deck transformations on $\widetilde{M}$ and as translations by elements of $\mathcal{H}$ on $\mathfrak{g}^*$. It follows that $\mathbf{J}$ descends to another map with values in $\mathfrak{g}^*/\mathcal{H}$. However, in general this is a difficult object to use as $\mathcal{H}$ is not necessarily a *closed* subgroup of $\mathfrak{g}^*$. To circumvent this, we proceed as follows.

Let $\overline{\mathcal{H}}$ be the closure of $\mathcal{H}$ in $\mathfrak{g}^*$. Since $\overline{\mathcal{H}}$ is a closed subgroup of $(\mathfrak{g}^*, +)$, the quotient $C := \mathfrak{g}^*/\overline{\mathcal{H}}$ is a cylinder (that is, it is isomorphic to the Abelian Lie group $\mathbb{R}^a \times \mathbb{T}^b$ for some $a, b \in \mathbb{N}$). Let $\pi_C : \mathfrak{g}^* \to \mathfrak{g}^*/\overline{\mathcal{H}}$ be the projection. Define $\mathbf{K} : M \to C$ to be the map that makes the following diagram commutative:

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\mathbf{J}} & \mathfrak{g}^* \\ {\scriptstyle q_M}\downarrow & & \downarrow{\scriptstyle \pi_C} \\ M & \xrightarrow{\mathbf{K}} & C = \mathfrak{g}^*/\overline{\mathcal{H}} \end{array} \tag{3.2}$$

In other words, $\mathbf{K}$ is defined by $\mathbf{K}(z) = \pi_C(\mathbf{J}(\tilde{z}))$, where $\tilde{z} \in \widetilde{M}$ is any path with endpoint z. We will refer to $\mathbf{K} : M \to \mathfrak{g}^*/\overline{\mathcal{H}}$ as a *cylinder valued momentum map* associated to the symplectic G-action on (M, ω). This object was introduced in [3] in a slightly different manner under the name of *"moment réduit"*.

Any other choice of Hamiltonian cover in place of $\widetilde{M}$ would render the same Hamiltonian holonomy group $\mathcal{H}$ and the same cylinder valued momentum map. If one chose a different base point $z_1 \in M$ in place of z_0 the holonomy group would remain the same, but the cylinder valued momentum map would possibly differ from $\mathbf{K}$ by a constant in $\mathfrak{g}^*/\overline{\mathcal{H}}$.

Elementary properties. The cylinder valued momentum map is a strict generalization of the standard (Kostant-Souriau) momentum map since the G-action has a standard momentum map if and only if the holonomy group $\mathcal{H}$ is trivial. In such a case the cylinder valued momentum map is a standard momentum map. The cylinder valued momentum map satisfies Noether's Theorem; that is, for any G-invariant function $h \in C^\infty(M)^G$, the flow F_t of its associated Hamiltonian vector field X_h satisfies the identity $\mathbf{K} \circ F_t = \mathbf{K}|_{\mathrm{Dom}(F_t)}$. Additionally, using the diagram (3.2) and identifying T_zM and $T_{\tilde{z}}\widetilde{M}$ via $T_{\tilde{z}}q_M$, one has that for any $v_z \in T_zM$, $T_z\mathbf{K}(v_z) = T_\mu\pi_C(\nu)$, where $\mu = \mathbf{J}(\tilde{z}) \in \mathfrak{g}^*$ and $\nu = T_{\tilde{z}}\mathbf{J}(v) \in \mathfrak{g}^*$.

Consequently, $T_z\mathbf{K}(v_z) = 0$ is equivalent to $T_{\tilde{z}}\mathbf{J}(v_z) \in \mathrm{Lie}(\overline{\mathcal{H}}) \subset \overline{\mathcal{H}}$, or equivalently $\mathbf{i}_{v_z}\omega \in \mathrm{Lie}(\overline{\mathcal{H}})$, so that

$$\ker T_z\mathbf{K} = \left[\left(\mathrm{Lie}(\overline{\mathcal{H}})\right)^\circ \cdot z\right]^\omega.$$

Here $\mathrm{Lie}(\overline{\mathcal{H}}) \subset \mathfrak{g}^*$ is the Lie algebra of $\overline{\mathcal{H}}$, and $\mathrm{Lie}(\overline{\mathcal{H}})^\circ$ its annihilator in $\mathfrak{g}$, and the upper index ω denotes the ω-orthogonal complement of the set in question. The notation $\mathfrak{k} \cdot m$ for any subspace $\mathfrak{k} \subset \mathfrak{g}$ has the usual meaning: namely the vector subspace of T_zM formed by evaluating all infinitesimal generators η_M at the point $z \in M$ for all $\eta \in \mathfrak{k}$. Furthermore, $\operatorname{range}(T_z\mathbf{K}) = T_\mu\pi_C((\mathfrak{g}_z)^\circ)$ (the Bifurcation Lemma).

Equivariance properties of the cylinder valued momentum map. There is a G-action on $\mathfrak{g}^*/\overline{\mathcal{H}}$ with respect to which the cylinder valued momentum map is G-equivariant. This action is constructed by noticing first that since G is connected it follows (see [13]) that the Hamiltonian holonomy $\mathcal{H}$ is pointwise fixed by the coadjoint action, that is, $\mathrm{Ad}^*_{g^{-1}}h = h$, for any $g \in G$ and any $h \in \mathcal{H}$. Hence, the coadjoint action on $\mathfrak{g}^*$ descends to a well defined action $\mathcal{A}d^*$ on $\mathfrak{g}^*/\overline{\mathcal{H}}$ defined so that for any $g \in G$, $\mathcal{A}d^*_{g^{-1}} \circ \pi_C = \pi_C \circ \mathrm{Ad}^*_{g^{-1}}$. With this in mind, we define $\overline{\sigma}_{\mathbf{K}} : G \times M \to \mathfrak{g}^*/\overline{\mathcal{H}}$ by

$$\overline{\sigma}_{\mathbf{K}}(g,z) := \mathbf{K}(g\cdot z) - \mathcal{A}d^*_{g^{-1}}\mathbf{K}(z).$$

Since M is connected by hypothesis, it can be shown that $\overline{\sigma}_{\mathbf{K}}$ does not depend on the point $z \in M$ and hence it defines a map $\sigma_{\mathbf{K}} : G \to \mathfrak{g}^*/\overline{\mathcal{H}}$ which is a group valued one-cocycle: for any $g,h \in G$, it satisfies the equality $\sigma_{\mathbf{K}}(gh) = \sigma_{\mathbf{K}}(g) + \mathcal{A}d^*_{g^{-1}}\sigma_{\mathbf{K}}(h)$. This guarantees that the map

$$\begin{array}{rccc} \Theta: & G\times \mathfrak{g}^*/\overline{\mathcal{H}} & \longrightarrow & \mathfrak{g}^*/\overline{\mathcal{H}} \\ & (g,\pi_C(\mu)) & \longmapsto & \mathcal{A}d^*_{g^{-1}}(\pi_C(\mu)) + \sigma_{\mathbf{K}}(g), \end{array}$$

defines a G-action on $\mathfrak{g}^*/\overline{\mathcal{H}}$ with respect to which the cylinder valued momentum map $\mathbf{K}$ is G-equivariant; that is, for any $g \in G$, $z \in M$, we have

$$\mathbf{K}(g\cdot z) = \Theta_g(\mathbf{K}(z)).$$

We will refer to $\sigma_{\mathbf{K}} : G \to \mathfrak{g}^*/\overline{\mathcal{H}}$ as the *non-equivariance one-cocycle* of the cylinder valued momentum map $\mathbf{K} : M \to \mathfrak{g}^*/\overline{\mathcal{H}}$ and to Θ as the *affine G-action* on $\mathfrak{g}^*/\overline{\mathcal{H}}$ induced by $\sigma_{\mathbf{K}}$. The infinitesimal generators of the affine G-action on $\mathfrak{g}^*/\overline{\mathcal{H}}$ are given by the expression

$$\xi_{\mathfrak{g}^*/\overline{\mathcal{H}}}(\pi_C(\mu)) = -T_\mu\pi_C(\Psi(z)(\xi,\cdot)), \tag{3.3}$$

for any $\xi \in \mathfrak{g}$, where $\mathbf{K}(z) = \pi_C(\mu)$, and $\Psi : M \to Z^2(\mathfrak{g})$ is the Chu map defined in (2.1).

The non-equivariance cocycles $\sigma_{\mathbf{J}} : \widetilde{G} \to \mathfrak{g}^*$ and $\sigma_{\mathbf{K}} : G \to \mathfrak{g}^*/\overline{\mathcal{H}}$ are related by

$$\pi_C \circ \sigma_{\mathbf{J}} = \sigma_{\mathbf{K}} \circ p_{\widetilde{G}}. \tag{3.4}$$

Proposition 3.1 *If the action of G has an isotropic orbit then the cylinder valued momentum map for this action can be chosen coadjoint equivariant.*

Proof. This follows from Remark 2.4. Let $z_0 \in M$ be a point in the isotropic orbit and construct a universal covering $\widetilde{M}$ of M by taking homotopies of curves with a fixed endpoint starting at z_0. Let $\mathbf{J} : \widetilde{M} \to \mathfrak{g}^*$ be the momentum map for the $\widetilde{G}$-action on $\widetilde{M}$ introduced in Proposition 2.3. Since the G-orbit containing z_0 is isotropic, the integrand in (2.4) is identically zero and hence $\sigma_{\mathbf{J}} = 0$ (see Remark 2.4). Therefore by (3.4) the non-equivariance cocycle $\sigma_{\mathbf{K}}$ satisfies $\sigma_{\mathbf{K}} \circ p_{\widetilde{G}} = \pi_C \circ \sigma_{\mathbf{J}} = 0$. □

Remark 3.2 For any Hamiltonian covering $p_N : N \to M$ of (M,ω) there exists a momentum map $\mathbf{J}_N : N \to \mathfrak{g}^*$ for the $\widetilde{G}$ (and also G_N) action on N such that $\mathbf{J}_N \circ q_N = \mathbf{J}$ and $\sigma_{\mathbf{J}_N} = \sigma_{\mathbf{J}}$. Consequently, $\pi_C \circ \mathbf{J}_N = \mathbf{K} \circ p_N$. Here the map $q_N : \widetilde{M} \to N$ is the $\widetilde{G}$-equivariant covering such that $p_N \circ q_N = p_{\widetilde{M}}$.

3.2 Reduction

The following result establishes a crucial relationship between the deck transformations group of $q_M : \widetilde{M} \to M$, that is, $\Gamma := \pi_1(M, z_0)$ and the deck transformations group of $\widehat{p} : \widehat{M} \to M$, that is $\mathcal{H}$.

Proposition 3.3 *Let G be a connected Lie group acting symplectically on the symplectic manifold (M, z_0, ω) with Hamiltonian holonomy $\mathcal{H}$ and let $\mathbf{J} : \widetilde{M} \to M$ be the momentum map for the lifted action on $(\widetilde{M}, \tilde{z}_0)$ defined in Proposition 2.3. Then, for any $\mu \in \mathfrak{g}^*$*

$$q_M(\mathbf{J}^{-1}(\mu + \mathcal{H})) = q_M(\mathbf{J}^{-1}(\mu)), \tag{3.5}$$

with $q_M : \widetilde{M} \to M$ the projection. More generally, for any Hamiltonian covering $p_N : (N, y_0) \to (M, z_0)$ of (M, z_0, ω), let $\mathbf{J}_N : N \to \mathfrak{g}^*$ be the momentum map for the G_N-action on N that satisfies the properties mentioned in Remark 3.2. Then, for any $\mu \in \mathfrak{g}^*$

$$p_N(\mathbf{J}_N^{-1}(\mu + \mathcal{H})) = p_N(\mathbf{J}_N^{-1}(\mu)). \tag{3.6}$$

Proof. Since $q_M(\mathbf{J}^{-1}(\mu + \mathcal{H})) = \bigcup_{\nu \in \mathcal{H}} q_M(\mathbf{J}^{-1}(\mu + \nu))$, it suffices to show that

$$q_M(\mathbf{J}^{-1}(\mu + \nu)) = q_M(\mathbf{J}^{-1}(\mu)), \text{ for any } \nu \in \mathcal{H}. \tag{3.7}$$

Let $\tilde{z} \in \mathbf{J}^{-1}(\mu)$, and let $\gamma \in \Gamma$ be such that $\mathbf{J}(\gamma) = \nu$. Then the deck transformation $\tilde{z} \mapsto \gamma * \tilde{z}$ on $\widetilde{M}$ provides a diffeomorphism of $\widetilde{M}$ which maps $\mathbf{J}^{-1}(\mu)$ to $\mathbf{J}^{-1}(\mu + \nu)$, since $\mathbf{J}(\gamma * \tilde{z}) = \mathbf{J}(\gamma) + \mathbf{J}(\tilde{z})$ (Proposition 2.3). However, $q_M(\gamma * \tilde{z}) = \tilde{z}$ so that indeed, $q_M(\mathbf{J}^{-1}(\mu)) = q_M(\mathbf{J}^{-1}(\mu + \nu))$ for any $\nu \in \mathcal{H}$.

In order to prove (3.6) let $q_N : \widetilde{M} \to N$ be the $\widetilde{G}$-equivariant covering such that $p_N \circ q_N = p_{\widetilde{M}}$. This equality and the surjectivity of q_N imply that for any set $A \subset N$, $p_N(A) = p_{\widetilde{M}} \circ q_N^{-1}(A)$. Now, the relations $\mathbf{J}_N \circ q_N = \mathbf{J}$ and (3.5) imply that $p_{\widetilde{M}}\left(q_N^{-1}(\mathbf{J}_N^{-1}(\mu + \mathcal{H}))\right) = p_{\widetilde{M}}\left(q_N^{-1}(\mathbf{J}_N^{-1}(\mu))\right)$ and hence $p_N(\mathbf{J}_N^{-1}(\mu + \mathcal{H})) = p_N(\mathbf{J}_N^{-1}(\mu))$, as required. □

The final result shows that reduction behaves well with respect to the lifting of the action to any Hamiltonian cover. More explicitly, we show that in order to carry out reduction one can either stay in the original manifold and use the in general cylinder valued momentum map or one can lift the action to a Hamiltonian cover, perform ordinary Marsden-Weinstein reduction there and then project the resulting quotient. The two strategies yield the same result. Notice that unless the Hamiltonian holonomy of the action $\mathcal{H}$ is closed in $\mathfrak{g}^*$, the reduced spaces obtained via the cylinder valued momentum map are in general not symplectic but Poisson manifolds [13].

Theorem 3.4 *Let G be a Lie group acting symplectically on the symplectic manifold (M, ω) and $p_N : (N, y_0) \to (M, z_0)$ a Hamiltonian covering. Let $\mathbf{J}_N : N \to \mathfrak{g}^*$ be the momentum map for the $\widetilde{G}$ (or G_N) action on N that satisfies the properties mentioned in Remark 3.2. Then, for any $\mu \in \mathfrak{g}^*$ and for $[\mu] = \pi_C(\mu) \in \mathfrak{g}^*/\overline{\mathcal{H}}$, the covering map $p_N : N \to M$ induces a natural projection $p_\mu : \mathbf{J}_N^{-1}(\mu)/(G_N)_\mu = \mathbf{J}_N^{-1}(\mu)/\widetilde{G}_\mu \to \mathbf{K}^{-1}([\mu])/G_{[\mu]}$. The subgroups $(G_N)_\mu$ and $G_{[\mu]}$ are the isotropies of μ and $[\mu]$ with respect to the affine actions of G_N and G on $\mathfrak{g}^*$ and $\mathfrak{g}^*/\overline{\mathcal{H}}$ using the non-equivariance cocycles of $\mathbf{J}_N$ and $\mathbf{K}$, respectively.*

If the G-action on M is free and proper then both $\mathbf{J}_N^{-1}(\mu)/(G_N)_\mu$ and $\mathbf{K}^{-1}([\mu])/G_{[\mu]}$ are Poisson manifolds and p_μ is a Poisson surjective submersion. If, additionally, the Hamiltonian holonomy $\mathcal{H}$ is closed in $\mathfrak{g}^*$, then $\mathbf{J}_N^{-1}(\mu)/(G_N)_\mu$ and $\mathbf{K}^{-1}([\mu])/G_{[\mu]}$ are symplectic manifolds and p_μ is a symplectic covering map.

Proof. If $y \in \mathbf{J}_N^{-1}(\mu)$ then $p_N(y) \in \mathbf{K}^{-1}([\mu])$, since by hypothesis $\mathbf{J}_N$ and $\mathbf{K}$ were chosen such that $\pi_C \circ \mathbf{J}_N = \mathbf{K} \circ p_N$. Let $\overline{p}_\mu : \mathbf{J}_N^{-1}(\mu) \to \mathbf{K}^{-1}([\mu])$ be the restriction of p_N to $\mathbf{J}_N^{-1}(\mu)$. Let $\widetilde{g} \in \widetilde{G}_\mu$ and $g = p_{\widetilde{G}}(\widetilde{g})$. Given that

$$\begin{aligned} g \cdot [\mu] := \mathcal{A}d^*_{g^{-1}}\pi_C(\mu) + \sigma_{\mathbf{K}}(g) &= \pi_C(\mathrm{Ad}^*_{g^{-1}}\mu) + \pi_C(\sigma_{\mathbf{J}_N}(\widetilde{g})) \\ &= \pi_C(\mathrm{Ad}^*_{\widetilde{g}^{-1}}\mu + \sigma_{\mathbf{J}_N}(\widetilde{g})) = \pi_C(\widetilde{g} \cdot \mu) = \pi_C(\mu) = [\mu], \end{aligned}$$

we conclude that $g \in G_{[\mu]}$. This remark, as well as the $\widetilde{G}$-equivariance of q_M imply the $\widetilde{G}_\mu$-equivariance of $\overline{p}_\mu$, which allows us to drop this map onto $p_\mu : \mathbf{J}_N^{-1}(\mu)/(G_N)_\mu = \mathbf{J}_N^{-1}(\mu)/\widetilde{G}_\mu \to \mathbf{K}^{-1}([\mu])/G_{[\mu]}$. It remains to be shown that p_μ is surjective. We will prove that by showing that $\overline{p}_\mu$ is surjective. First of all notice that

$$\mathbf{K}^{-1}([\mu]) = p_N(\mathbf{J}_N^{-1}(\mu + \mathcal{H})). \tag{3.8}$$

Indeed, let $z \in \mathbf{K}^{-1}([\mu])$ and $\tilde{z} \in N$ such that $z = p_N(\tilde{z})$. Then since $\pi_C(\mu) = \mathbf{K}(p_N(\tilde{z})) = \pi_C(\mathbf{J}_N(\tilde{z}))$ there exists $\nu \in \mathcal{H}$ such that $\mathbf{J}_N(\tilde{z}) = \mu + \nu$ and hence $z \in p_N(\mathbf{J}_N^{-1}(\mu + \nu))$. Conversely, if $z \in p_N(\mathbf{J}_N^{-1}(\mu + \mathcal{H}))$ then there exists $\tilde{z} \in \mathbf{J}_N^{-1}(\mu + \nu)$ for some $\nu \in \mathcal{H}$ such that $z = p_N(\tilde{z})$. Consequently, $\mathbf{K}(z) = \mathbf{K}(p_N(\tilde{z})) = \pi_C(\mathbf{J}_N(\tilde{z})) = \pi_C(\mu + \nu) = \pi_C(\mu)$. Now, by Proposition 3.3

$$\mathbf{K}^{-1}([\mu]) = p_N(\mathbf{J}_N^{-1}(\mu + \mathcal{H})) = p_N(\mathbf{J}_N^{-1}(\mu)) = \overline{p}_\mu(\mathbf{J}_N^{-1}(\mu)),$$

as required.

The last statement in the theorem is a consequence of the fact that by Corollary 1.10 and Theorem 1.13, if the G-action on M is free and proper, then so is the G_N-action on N and hence both $\mathbf{J}_N^{-1}(\mu)/(G_N)_\mu$ and $\mathbf{K}^{-1}([\mu])/G_{[\mu]}$ are regular quotient manifolds. The results in [13] guarantee that these two spaces are endowed with natural projected Poisson structures. A straightforward diagram chasing shows that p_μ preserves these two Poisson structures. Additionally, if $\mathcal{H}$ is closed in $\mathfrak{g}^*$, the results in [13] guarantee that $\mathbf{K}^{-1}([\mu])/G_{[\mu]}$ is a symplectic manifold of the same dimension as $\mathbf{J}_N^{-1}(\mu)/(G_N)_\mu$. It is easy to check that in that situation the map p_μ is a symplectic covering map. $\square$

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