

*$\varepsilon$ -constants and equivariant Arakelov-Euler  
characteristics*

Chinburg, Ted and Pappas, Georgios and Taylor,  
Martin J.

2002

MIMS EPrint: **2007.80**

Manchester Institute for Mathematical Sciences  
School of Mathematics

The University of Manchester

Reports available from: <http://eprints.maths.manchester.ac.uk/>

And by contacting: The MIMS Secretary  
School of Mathematics  
The University of Manchester  
Manchester, M13 9PL, UK

ISSN 1749-9097

# $\varepsilon$ -CONSTANTS AND EQUIVARIANT ARAKELOV–EULER CHARACTERISTICS

BY TED CHINBURG<sup>1</sup>, GEORGIOS PAPPAS<sup>2</sup> AND MARTIN J. TAYLOR<sup>3</sup>

---

*Dedicated to the memory of Ali Fröhlich (1916–2001), for his vision, inspiration and encouragement.*

ABSTRACT. – Let  $\mathcal{X} \rightarrow \mathcal{Y}$  be a tame  $G$ -cover of regular arithmetic varieties over  $\mathbf{Z}$  with  $G$  a finite group. Assuming that  $\mathcal{X}$  and  $\mathcal{Y}$  have “tame” reduction we show how to determine the  $\varepsilon$ -constant in the conjectural functional equation of the Artin–Hasse–Weil function  $L(\mathcal{X}/\mathcal{Y}, V, s)$  for  $V$  a symplectic representation of  $G$  from a suitably refined equivariant Arakelov–de Rham–Euler characteristic of  $\mathcal{X}$ . Our result may be viewed firstly as a higher dimensional version of the Cassou–Noguès–Taylor characterization of tame symplectic Artin root numbers in term of rings of integers with their trace form, and secondly as a signed equivariant version of Bloch’s conductor formula.

© 2002 Éditions scientifiques et médicales Elsevier SAS

RÉSUMÉ. – Soit  $\mathcal{X} \rightarrow \mathcal{Y}$  un revêtement modéré, de groupe fini  $G$ , de variétés arithmétiques sur  $\mathbf{Z}$ . Sous l’hypothèse que la réduction de  $\mathcal{X}$  et  $\mathcal{Y}$  est modérée, nous montrons, en utilisant un raffinement convenable de la caractéristique d’Arakelov–de Rham–Euler de  $\mathcal{X}$ , comment déterminer la constante epsilon de l’équation fonctionnelle conjecturale de la fonction  $L(\mathcal{X}/\mathcal{Y}, V, s)$  d’Artin–Hasse–Weil pour une représentation symplectique  $V$  de  $G$ . Ce résultat peut être considéré comme la version en dimension supérieure de la caractérisation des constantes symplectiques d’Artin de Cassou–Noguès et Taylor, et aussi comme une version équivariante à signes de la formule du conducteur de Bloch.

© 2002 Éditions scientifiques et médicales Elsevier SAS

## 1. Introduction

The theory of  $\varepsilon$ -constants can be traced back to Gauss’ work on quadratic Gauss sums

$$\tau(\chi_2) = \sum_{a=1}^{p-1} \chi_2(a) e^{2\pi i a/p}$$

in which  $p$  is an odd prime and  $\chi_2 : (\mathbf{Z}/p\mathbf{Z})^* \rightarrow \mathbf{C}^*$  is the (unique) quadratic multiplicative character mod  $p$ . From the equality

$$\tau(\chi_2)^2 = \chi_2(-1)p$$

---

<sup>1</sup> Supported in part by NSF grant DMS97-01411.

<sup>2</sup> Supported in part by NSF grant DMS99-70378 and by a Sloan Research Fellowship.

<sup>3</sup> EPSRC Senior Research Fellow.

one can deduce Gauss’ quadratic reciprocity law (c.f. [29], pp. 77–78). The above formula determines  $\tau(\chi_2)$  only up to  $\pm 1$ . The difficulty in determining epsilon constants without sign ambiguities appears even in these early examples, where the sign for the quadratic Gauss sum has very important consequences for the distribution of quadratic residues (c.f. [18] VIII, §6). Gauss conjectured an exact formula for  $\tau(\chi_2)$  in 1801, which he proved in 1805 (c.f. [29], Chapter IV, §3, [25], p. 73). The study of Gauss sums took on new significance with the discovery that they occur as constants in the functional equations of Dirichlet L-series. To describe a modern conjecture descending from this result, suppose  $\mathcal{X}$  is a projective and flat regular scheme over  $\text{Spec}(\mathbf{Z})$  which is an integral model of a smooth projective variety  $X$  of dimension  $d$  over  $\mathbf{Q}$ . The Hasse–Weil zeta function of  $\mathcal{X}$  is defined in a suitable half plane of convergence by the infinite product

$$\zeta(\mathcal{X}, s) = \prod_x (1 - N(x)^{-s})^{-1}$$

where  $x$  ranges over the closed points of  $\mathcal{X}$  and  $N(x)$  is the order of the residue field of  $x$ . Denote by  $L(\mathcal{X}, s)$  the zeta function with  $\Gamma$ -factors  $L(\mathcal{X}, s) = \zeta(\mathcal{X}, s)\Gamma(\mathcal{X}, s)$ . The  $L$ -function is conjectured to have an analytic continuation and to satisfy a functional equation

$$L(\mathcal{X}, s) = \varepsilon(\mathcal{X})A(\mathcal{X})^{-s}L(\mathcal{X}, d + 1 - s)$$

with  $\varepsilon(\mathcal{X})$  and  $A(\mathcal{X})$  (the “ $\varepsilon$ -constant” and the “conductor”) real numbers which, assuming certain choices, can be defined independently of any conjectures (see [15]). The formulas in [15] give expressions for both  $\varepsilon(\mathcal{X})$  and  $A(\mathcal{X})$  as products of certain rational numbers, roots of unity and generalized Gauss sums of the form

$$\tau(\chi) = \sum_{a \in R^*} \chi(a)\psi(a)$$

in which  $R$  is a finite ring and  $\chi$  (respectively  $\psi$ ) is a multiplicative (respectively additive) character of  $R$ .

The knowledge of the numbers  $\varepsilon(\mathcal{X})$  and  $A(\mathcal{X})$  is important in many arithmetic applications. In particular, the sign of  $\varepsilon(\mathcal{X})$  influences the order of the zero or pole of  $L(\mathcal{X}, s)$  at  $s = (d + 1)/2$ . Hence, at least when  $d$  is odd, this sign should determine, via the various generalizations of the Birch and Swinnerton-Dyer conjecture, the parity of the rank of a certain group of algebraic cycles on  $\mathcal{X}$ . There is considerable interest in obtaining information about  $\varepsilon(\mathcal{X})$  and  $A(\mathcal{X})$  using other global invariants of the variety  $\mathcal{X}$ . An example of a result of this sort is Bloch’s conjectural conductor formula [3]; according to this,  $A(\mathcal{X})$  is given as the degree of the localized top Chern class of the relative differentials  $\Omega^1_{\mathcal{X}/\mathbf{Z}}$ . Bloch’s formula has been proven when  $d = 1$  [3], when  $\mathcal{X} \rightarrow \text{Spec}(\mathbf{Z})$  is “tame” (see below, [13] and [1] independently), and when all the fibers of  $\mathcal{X} \rightarrow \text{Spec}(\mathbf{Z})$  are divisors with normal crossings [28]. Let us remark here that, as we can also see directly using the predicted functional equation, determining  $A(\mathcal{X})$  is equivalent to determining the square  $\varepsilon(\mathcal{X})^2$ .

In this paper we will study an “equivariant” situation. We will assume that  $\mathcal{X}$  supports an action of a finite group  $G$ ; this produces a  $G$ -cover  $\pi: \mathcal{X} \rightarrow \mathcal{Y} := \mathcal{X}/G$ . For any finite dimensional complex representation  $V$  of  $G$  with character  $\psi$  we can consider the Artin–Hasse–Weil  $L$ -function with  $\Gamma$ -factors  $L(\mathcal{Y}, \psi, s)$ . This is the  $L$ -function of a corresponding “higher dimensional Artin motive” obtained from  $\mathcal{X}$  and  $V$ . There is again a conjectural functional equation

$$L(\mathcal{Y}, \psi, s) = \varepsilon(\mathcal{Y}, \psi)A(\mathcal{Y}, \psi)^{-s}L(\mathcal{Y}, \bar{\psi}, d + 1 - s).$$

Now assume:

(T1) the action of  $G$  on  $\mathcal{X}$  is “tame” (for every point  $x$  of  $\mathcal{X}$  the order of the inertia subgroup  $I_x \subset G$  is relatively prime to the characteristic of the residue field  $k(x)$ ), and

(T2) both schemes  $\mathcal{X}$  and  $\mathcal{Y}$  are regular and “tame” (i.e they are regular and all their special fibers are divisors with strict normal crossings with multiplicities relatively prime to the characteristic).

In this paper, we will show how, under these two assumptions, we can determine the constants  $\varepsilon(\mathcal{Y}, \psi)$  for all symplectic characters  $\psi$  of  $G$  using the  $G$ -equivariant Arakelov–Euler characteristics of a suitable de Rham complex of  $\mathcal{X}$  (see below and also Theorem 8.3). Recall that a  $G$ -representation  $V$  is called symplectic when  $V$  supports a  $G$ -invariant perfect alternating bilinear form. When  $G$  is the trivial group the “smallest” symplectic representation is the direct sum  $1 \oplus 1$  of two copies of the trivial representation. The result is interesting even in this case. Indeed, then  $\mathcal{X} = \mathcal{Y}$  and the result specializes to a formula giving  $\varepsilon(\mathcal{Y}, 1 \oplus 1) = \varepsilon(\mathcal{X})^2$  in terms of an Arakelov–de Rham–Euler characteristic; one can show that this formula amounts to Bloch’s conductor formula (see [13]). In [13], Bloch’s formula is shown under the “tameness” assumption (T2). Therefore, in the current paper we obtain an equivariant generalization of this result. Let us remark that the constants  $\varepsilon(\mathcal{Y}, \psi)$  depend intrinsically on the  $l$ -adic Galois representations which appear in the étale cohomology of the arithmetic variety. The results of this paper give one of the first instances where Arakelov theory is involved in determining an interesting Galois representation invariant of this type. It is particularly striking that our results enable us to determine the sign of  $\varepsilon(\mathcal{Y}, \psi)$  for symplectic  $\psi$ .

In [8] and [12], we have shown, under the assumptions (T1-2), that a  $G$ -equivariant de Rham–Euler characteristic can be determined using  $\varepsilon$ -constants. As we will explain below, our current results can be viewed as providing a “converse” of the main theorems of these papers. The main result of [8,12], generalizes “Fröhlich’s conjecture” (shown in [36]) on the Galois module structure of the rings of integers  $\mathcal{O}_N$  in a tame extension  $N/K$  of number fields with group  $G = \text{Gal}(N/K)$  to higher dimensional schemes over  $\mathbf{Z}$ . Fröhlich’s conjecture explains how one can determine the stable isomorphism class of  $\mathcal{O}_N$  as a  $\mathbf{Z}[G]$ -module from the signs of the constants  $\varepsilon(\text{Spec}(\mathcal{O}_K), \psi)$ ,  $\psi$  symplectic. In this zero-dimensional case, the “converse” is provided by Fröhlich’s hermitian conjecture (shown by Cassou-Noguès and Taylor in [6]). Roughly speaking, this shows that the  $\mathbf{Z}[G]$ -module  $\mathcal{O}_N$ , together with the additional structure provided by the hermitian pairing of the trace form, can be used to determine the symplectic  $\varepsilon$ -constants. A main observation of the present paper is that instead of using the trace form we can construct invariants using Arakelov hermitian metrics. From our point of view, the Cassou-Noguès–Taylor result may be reformulated as follows: they show that the  $\varepsilon$ -constants of Artin  $L$ -functions for symplectic representations of  $G$  can be recovered from the isomorphism class of  $\mathcal{O}_N$  as a “metrised  $\mathbf{Z}[G]$ -module”. Here a metrised  $\mathbf{Z}[G]$ -module is a  $\mathbf{Z}[G]$ -module  $M$  together with a  $G$ -invariant metric on  $\mathbf{C} \otimes_{\mathbf{Z}} M$ ; for the ring of integers  $\mathcal{O}_N$ , the metric on  $\mathbf{C} \otimes_{\mathbf{Z}} \mathcal{O}_N$  is given by  $z \otimes a \mapsto (\sum_{\sigma} |z\sigma(a)|^2)^{1/2}$ , where the sum extends over the distinct embeddings  $\sigma: N \rightarrow \mathbf{C}$ . The theorems in this article provide generalizations of this Cassou-Noguès–Taylor result to higher dimensions.

Now let us explain in some more detail our results and methods. We first study bounded complexes of finitely generated  $\mathbf{Z}[G]$ -modules whose determinants of cohomology are endowed with certain metrics; we call such complexes *metrised* complexes. The alternating sum of the terms of the complex yields an Euler characteristic of the complex in  $G_0(\mathbf{Z}[G])$ , the Grothendieck group of finitely generated  $\mathbf{Z}[G]$ -modules; furthermore, if the complex is perfect, in the sense that all the terms of the complex are projective, then one can form a projective Euler characteristic in the finer Grothendieck group  $K_0(\mathbf{Z}[G])$  of finitely generated projective  $\mathbf{Z}[G]$ -modules. Our initial aim is to construct an arithmetic class (or “Arakelov–

Euler characteristic”) for each bounded perfect metrised complex, which will take values in a metrised version of the projective classgroup of  $\mathbf{Z}[G]$ ; we will denote this “arithmetic” classgroup by  $A(\mathbf{Z}[G])$ .

Our interest lies in arithmetic classes which are obtained as follows: Let  $\mathcal{X}$  be a regular scheme which is projective and flat over  $\text{Spec}(\mathbf{Z})$ . We suppose that  $\mathcal{X}$  supports an action by a finite group  $G$  which is tame (assumption (T1)); we choose a  $G$ -invariant Kähler metric  $h$  on the tangent bundle of the associated complex manifold  $\mathcal{X}(\mathbf{C})$ . We are then able to construct an Arakelov–Euler characteristic for any hermitian  $G$ -bundle  $(\mathcal{F}, j)$  on  $\mathcal{X}$  by endowing the equivariant determinant of cohomology of  $R\Gamma(\mathcal{X}, \mathcal{F})$  with equivariant Quillen metrics  $j_{Q,\phi}$  for each irreducible character  $\phi$  of  $G$ . This construction can then be extended to give an Arakelov–Euler characteristic in  $A(\mathbf{Z}[G])$  for a bounded complex of hermitian  $G$ -bundles. In particular, by applying this construction to a suitable complex obtained by resolving the de Rham complex of  $\mathcal{X}$ , we obtain the equivariant de Rham Arakelov–Euler characteristic

$$\mathfrak{d}(\mathcal{X}) := \chi(R\Gamma(\mathcal{X}, \lambda^\bullet \Omega_{\mathcal{X}/\mathbf{Z}}^1, \wedge^\bullet h_{\mathbf{Q}}^D) \in A(\mathbf{Z}[G])$$

(see the beginning of Section 8). Our main result shows that the de Rham Arakelov–Euler characteristic  $\mathfrak{d}(\mathcal{X}) \in A(\mathbf{Z}[G])$ , together with the arithmetic ramification class  $\text{AR}(\mathcal{X}) \in A(\mathbf{Z}[G])$  (see 8.2), completely determines all the constants  $\varepsilon(\mathcal{Y}, \psi)$ ,  $\psi$  a symplectic character of  $G$ . To explain how this is achieved, observe that by its definition (see 3.2),  $A(\mathbf{Z}[G])$  has a “Fröhlich description”; this allows us to describe elements in  $A(\mathbf{Z}[G])$  by giving suitable homomorphisms from the group of  $\bar{\mathbf{Q}}$ -valued characters  $R_G$  of  $G$ . In Section 4 we show that, by restricting to the subgroup  $R_G^s \subset R_G$  of virtual symplectic characters of  $G$ , we obtain an image of  $A(\mathbf{Z}[G])$  in the so-called tame symplectic arithmetic classgroup  $A_T^s(\mathbf{Z}[G])$ . For  $a \in A(\mathbf{Z}[G])$ , we will denote by  $a^s \in A_T^s(\mathbf{Z}[G])$  this restriction. We also show that  $A_T^s(\mathbf{Z}[G])$  contains a subgroup  $R(\mathbf{Z}[G])$ , called the group of rational classes, which supports a natural isomorphism

$$\theta : R(\mathbf{Z}[G]) \rightarrow \text{Hom}_{\text{Gal}}(R_G^s, \mathbf{Q}^\times).$$

Our main result then is (Theorem 8.3):

**THEOREM.** – *Assume (T1) and (T2). The element  $\mathfrak{d}^s(\mathcal{X})^{-1} \cdot \text{AR}^s(\mathcal{X})$  of  $A_T^s(\mathbf{Z}[G])$  is a rational class and for any symplectic character  $\psi$*

$$\varepsilon(\mathcal{Y}, \psi) = \theta(\mathfrak{d}^s(\mathcal{X})^{-1} \cdot \text{AR}^s(\mathcal{X}))(\psi).$$

This result can be thought of as a “converse” to the main Theorems of [8] and [12]; there the class of the de Rham–Euler characteristic in  $K_0(\mathbf{Z}[G])$  is shown to be determined by  $\varepsilon$ -factors. Let us point out that the arithmetic ramification class  $\text{AR}(\mathcal{X})$  is modelled on the “ramification class”  $R(\mathcal{X}/\mathcal{Y})$  of [8]. It only depends on the branch locus of the cover  $\mathcal{X} \rightarrow \mathcal{Y}$ ; under our assumptions this branch locus is contained in a finite set of fibers of  $\mathcal{Y} \rightarrow \text{Spec}(\mathbf{Z})$ . In the zero-dimensional case  $\mathcal{X} = \text{Spec}\mathcal{O}_N$ ,  $\mathcal{Y} = \text{Spec}\mathcal{O}_K$ , the arithmetic ramification class is trivial and the Theorem amounts to the result of Cassou-Noguès–Taylor as reformulated above.

This article is structured as follows: in Section 2 we define our notation and present a number of preliminary results. Then, in Section 3, we define the arithmetic classes for suitable bounded  $\mathbf{Z}[G]$ -complexes and establish a number of their basic properties. The construction of the arithmetic class is a rather delicate matter, since we wish to produce an invariant which reflects the fact that the terms in the complex are projective, whilst the metrics are only defined on the determinants of cohomology. The main point here is to show that our notion of arithmetic class is invariant under quasi-isomorphisms which preserve metrics in an appropriate sense.

The formation of arithmetic classes may also be seen to be closely related to the refined Euler characteristics with values in relative  $K$ -groups introduced by D. Burns in [4] and [5].

Our arithmetic classes take values in the arithmetic classgroup  $A(\mathbf{Z}[G])$ . This group contains a considerable amount of information, and in practice it is often convenient to work with various image groups of this classgroup; the image groups which we require are presented in Section 4.

In Section 5 we consider an arithmetic variety  $\mathcal{X}$  which carries a tame action by a finite group  $G$  and we define an arithmetic class for a hermitian  $G$ -bundle on  $\mathcal{X}$  which supports a set of metrics on the equivariant determinant of cohomology; we then carry out a number of calculations in the case when  $\mathcal{X}$  is the spectrum of a ring of integers. These will allow us to reinterpret the above mentioned results of Cassou-Noguès–Taylor.

In Section 6 we fix a choice of Kähler metric  $h$  on the tangent bundle of  $\mathcal{X}(\mathbf{C})$ . For a complex  $(\mathcal{G}^\bullet, h_\bullet)$  of hermitian  $G$ -bundles on  $\mathcal{X}$ , we use the equivariant Quillen metrics on the equivariant determinants of hypercohomology (see [2]) to construct an arithmetic class  $\chi(\mathrm{R}\Gamma\mathcal{G}^\bullet, h_{Q,\bullet})$ ; we call this class the Arakelov–Euler characteristic of  $(\mathcal{G}^\bullet, h_\bullet)$ . We then briefly detail the functorial properties of such Euler characteristics and calculate such Euler characteristics when  $\mathcal{X}$  has dimension one.

In Section 7 we introduce the logarithmic de Rham complex of  $\mathcal{X}$ ; this is an important technical tool in the proof of the main Theorem. We also obtain an interesting intermediate result. Let  $S$  denote a finite set of primes which includes those primes where  $\mathcal{X}$  has non-smooth reduction and we let  $\Omega_{\mathcal{X}/\mathbf{Z}}^1(\log \mathcal{X}_S^{\mathrm{red}} / \log S)$  denote the sheaf of degree one relative logarithmic differentials of  $\mathcal{X}$  with respect to the morphism  $(\mathcal{X}, \mathcal{X}_S^{\mathrm{red}}) \rightarrow (\mathrm{Spec}(\mathbf{Z}), S)$  of schemes with  $\log$ -structures. Assuming (T2) this sheaf is locally free. The logarithmic de Rham complex  $\Omega_{\mathcal{X}/\mathbf{Z}}^\bullet(\log \mathcal{X}_S^{\mathrm{red}} / \log S)$  is the complex

$$\mathcal{O}_{\mathcal{X}} \rightarrow \Omega_{\mathcal{X}/\mathbf{Z}}^1(\log \mathcal{X}_S^{\mathrm{red}} / \log S) \rightarrow \dots \rightarrow \Omega_{\mathcal{X}/\mathbf{Z}}^d(\log \mathcal{X}_S^{\mathrm{red}} / \log S);$$

we view each term as carrying the hermitian metric given by the corresponding exterior power of  $h^D$ . In Theorem 7.1 we completely describe the image  $c^s(\mathcal{X})$  of the logarithmic de Rham–Arakelov–Euler characteristic  $c(\mathcal{X}) = \chi(\mathrm{R}\Gamma(\mathcal{X}, \Omega_{\mathcal{X}/\mathbf{Z}}^\bullet(\log \mathcal{X}_S^{\mathrm{red}} / \log S)), \wedge^\bullet h_Q^D)$  in the tame symplectic arithmetic classgroup  $A_T^s(\mathbf{Z}[G])$  in terms of the constants  $\varepsilon_0(\mathcal{Y}, \psi)$  (variants of  $\varepsilon(\mathcal{Y}, \psi)$ , see [15] or Section 7). This is done by separating the calculation to characters of degree 0 and to characters which are multiples of the trivial character (corresponding to the  $G$ -fixed part). The case of degree 0 characters is reduced to the results of Cassou-Noguès–Taylor in the zero-dimensional case after using the moving techniques of [12] and T. Saito’s formulas for tame  $\varepsilon_0$ -constants. In particular, we obtain (a special case of Theorem 7.1):

**THEOREM.** – *Assume (T1) and (T2). The element  $c^s(\mathcal{X})$  of  $A_T^s(\mathbf{Z}[G])$  is a rational class and for any virtual symplectic character  $\psi$  of degree zero  $\varepsilon_0(\mathcal{Y}, \psi) = \theta(c^s(\mathcal{X})^{-1})(\psi)$ .*

Finally, the calculation of the  $G$ -fixed part (Theorems 7.8 and 7.9) is obtained from the fact that the analytic torsion of the de Rham complex is zero [32] using various considerations of “metrized” duality.

In Section 8, we consider the de Rham–Arakelov–Euler characteristic  $\mathfrak{d}(\mathcal{X})$  associated to the (regular) differentials of  $\mathcal{X}/\mathbf{Z}$ , and we show how this arithmetic class determines the symplectic  $\varepsilon$ -constants of  $\mathcal{X}$ . This is obtained by combining the result for the logarithmic de Rham complex of  $\mathcal{X}$  with a calculation on the fibers of  $\mathcal{X} \rightarrow \mathrm{Spec}(\mathbf{Z})$  over  $S$  as in [12].

We would like to thank the referee and the editor for helpful comments which helped us improve the presentation. The third author wishes to express his gratitude to Christophe Soulé

for helpful conversations and exchanges of letters concerning the work presented in the latter part of this article.

## 2. Preliminary results

### 2.1. Hermitian complexes

Let  $R$  denote a commutative ring which is endowed with a fixed embedding into the field of complex numbers  $\mathbf{C}$ ; in applications  $R$  will be either  $\mathbf{Z}$ ,  $\mathbf{R}$  or  $\mathbf{C}$ . We consider bounded (cochain) complexes  $K^\bullet$  of finitely generated left  $R[G]$ -modules

$$K^\bullet : \dots \rightarrow K^i \xrightarrow{d_K^i} K^{i+1} \rightarrow \dots$$

so that the boundary maps  $d_K^i$  are all  $R[G]$ -maps. Thus the  $i$ th cohomology group, denoted  $H^i = H^i(K^\bullet)$ , is an  $R[G]$ -module. Recall that the complex  $K^\bullet$  is called *perfect* if in addition all the modules  $K^i$  are  $R[G]$ -projective.

DEFINITION 2.1. – Let  $x \mapsto \bar{x}$  denote the complex conjugation automorphism of  $\mathbf{C}$ ; we extend complex conjugation to an involution of the complex group algebra  $\mathbf{C}[G]$  by the rule  $\overline{\sum a_g g} = \sum \bar{a}_g g^{-1}$ . An element  $x \in \mathbf{C}[G]$  is called *symmetric* if  $\bar{x} = x$ . A *hermitian  $R[G]$ -complex* is a pair  $(K^\bullet, k^\bullet)$  where  $K^\bullet$  is an  $R[G]$ -complex, as above, and where each  $K_{\mathbf{C}}^i = \mathbf{C} \otimes_R K^i$  is endowed with a non-degenerate positive-definite  $G$ -invariant hermitian form

$$k^i : K_{\mathbf{C}}^i \times K_{\mathbf{C}}^i \rightarrow \mathbf{C}.$$

Thus, in particular, each  $k^i$  is left  $\mathbf{C}$ -linear and  $\overline{k^i(x, y)} = k^i(y, x)$ . The metric associated to  $k^i$  is defined by  $\|x\|^i = \sqrt{k^i(x, x)}$  for  $x \in K_{\mathbf{C}}^i$ , where  $k^i(x, x) \geq 0$  because  $k^i$  is a positive definite hermitian form.

Equivalently (as per p. 164 in [17]) we may work with the  $\mathbf{C}[G]$ -valued hermitian forms

$$\hat{k}^i : K_{\mathbf{C}}^i \times K_{\mathbf{C}}^i \rightarrow \mathbf{C}[G]$$

given by the rule that for  $x, y \in K_{\mathbf{C}}^i$

$$\hat{k}^i(x, y) = \sum_{g \in G} k^i(x, gy)g.$$

Thus  $\hat{k}^i$  is  $\mathbf{C}[G]$ -left linear and is reflexive in the sense that  $\overline{\hat{k}^i(x, y)} = \hat{k}^i(y, x)$ . Conversely, given  $\hat{k}^i$ , we may of course recoup  $k^i$  by reading off the coefficient of  $1_G$  in  $\mathbf{C}[G]$ .

Example 2.2. – The module  $\mathbf{C}[G]$  carries the so-called *standard* positive  $G$ -invariant hermitian form

$$\mu : \mathbf{C}[G] \times \mathbf{C}[G] \rightarrow \mathbf{C}$$

given by the rule  $\mu(\sum x_g g, \sum y_h h) = \sum x_g \bar{y}_g$ . Then the associated  $\mathbf{C}[G]$ -valued hermitian form  $\hat{\mu}$

$$\hat{\mu} : \mathbf{C}[G] \times \mathbf{C}[G] \rightarrow \mathbf{C}[G]$$

is the so-called multiplication form  $\hat{\mu}(x, y) = x \cdot \bar{y}$ .

### 2.2. Metrised complexes

Let  $G$  again denote a finite group, let  $\widehat{G}$  denote the set of irreducible complex characters of  $G$ , and once and for all for each  $\phi \in \widehat{G}$  we let  $W = W_\phi$  denote the simple 2-sided  $\mathbf{C}[G]$ -ideal with character  $\phi(1)\bar{\phi}$ , where  $\bar{\phi}$  is the contragredient character of  $\phi$ . For a finitely generated  $\mathbf{C}[G]$ -module  $M$  we define  $M_\phi = (M \otimes_{\mathbf{C}} W)^G$ , where  $G$  acts diagonally and on the left of each term; more generally, for a bounded complex  $P^\bullet$  of finitely generated  $\mathbf{C}[G]$ -modules, we put  $H^i = H^i(P^\bullet)$  and we write

$$P_\phi^\bullet = (P^\bullet \otimes_{\mathbf{C}} W)^G \quad \text{and} \quad H_\phi^i = (H^i \otimes_{\mathbf{C}} W)^G.$$

We then construct the complex lines

$$\det(P_\phi^\bullet) = \otimes_i (\wedge^{\text{top}} P_\phi^i)^{(-1)^i} \quad \text{and} \quad \det(H_\phi^\bullet) = \otimes_i (\wedge^{\text{top}} H_\phi^i)^{(-1)^i}$$

where for a complex vector space  $V$  of dimension  $d$ ,  $\wedge^{\text{top}} V$  denotes  $\wedge^d V$  and where for a complex line  $L$  we write  $L^{-1}$  for the dual line  $\text{Hom}(L, \mathbf{C})$ . Note also that here and in the sequel for two finite dimensional vector spaces  $V_i$  of dimension  $d_i$  we normalise the standard isomorphism  $\wedge^{d_1 d_2}(V_1 \otimes V_2) \cong \wedge^{d_1 d_2}(V_2 \otimes V_1)$  by multiplying by  $(-1)^{d_1 d_2}$ , in order to avoid subsequent sign complications. We refer to the set of lines  $\det(H_\phi^\bullet)$  as the *equivariant determinants of cohomology* of  $P^\bullet$ . From Theorem 2 in [27] we have a canonical isomorphism

$$(1) \quad \xi_\phi : \det(P_\phi^\bullet) \cong \det(H_\phi^\bullet).$$

For ease of computation we use the above definition of  $P_\phi^\bullet$ ; however, alternatively one can also work with the isotypical components  $\overline{W}P^\bullet$ , as shown in the following lemma. Here and in further applications we shall often need the renormalised form  $\nu : \mathbf{C}[G] \times \mathbf{C}[G] \rightarrow \mathbf{C}$  of the hermitian form  $\mu$  of (2.2) given by

$$\nu(x, y) = |G| \cdot \mu(x, y) \quad \text{for } x, y \in \mathbf{C}[G].$$

LEMMA 2.3. – *For a  $\mathbf{C}[G]$  module  $V$  with a  $G$ -invariant metric  $\| - \|$ , the natural isomorphism*

$$\alpha : (V \otimes_{\mathbf{C}} W)^G \cong \overline{W}V$$

*given by  $\alpha(\sum_i v_i \otimes w_i) = \sum_i \bar{w}_i v_i$  is an isometry, where both terms carry the natural metrics induced by  $\nu$  and  $\| - \|$ ; that is to say  $\overline{W}V$  carries the metric given by the restriction of  $\| - \|$ , and  $(V \otimes_{\mathbf{C}} W)^G$  carries the metric given by the restriction of the tensor metric associated to  $\| - \|$  and  $\nu$  on  $V \otimes \mathbf{C}[G]$ .*

*Proof.* – Let  $\| - \|_1$  respectively  $\| - \|_2$  denote the given metric on  $(V \otimes_{\mathbf{C}} W)^G$  respectively  $\overline{W}V$ . If  $e = |G|^{-1} \cdot \sum_g \phi(g)g$  is the central idempotent associated to  $W$ , then for  $x \in \overline{W}V$ , we have  $\alpha^{-1}(x) = |G|^{-1} \cdot \sum_{g \in G} gx \otimes ge$  and so

$$\begin{aligned} \alpha^{-1}(x) &= \frac{1}{|G|^2} \sum_{g, h \in G} gx \otimes g\phi(h)h = \frac{1}{|G|^2} \sum_{g, h \in G} gh h^{-1} \phi(h)x \otimes gh \\ &= \frac{1}{|G|} \sum_{f \in G} f \bar{e}x \otimes f = \frac{1}{|G|} \sum_{f \in G} fx \otimes f. \end{aligned}$$



Thus since  $\nu(f, f) = |G|$

$$\|\alpha^{-1}(x)\|_1^2 = \frac{1}{|G|} \sum_g \|x\|_2^2 = \|x\|_2^2. \quad \square$$

DEFINITION 2.4. – Let  $R$  again denote a subring of  $\mathbf{C}$ . A *metrised  $R[G]$ -complex* is a pair  $(P^\bullet, p_\bullet)$ , where  $P^\bullet$  is a bounded complex of finitely generated (not necessarily projective)  $R[G]$ -modules and the  $p_\phi$  are a set of metrics given by positive definite hermitian forms on the complex lines  $\det(\mathbf{H}_\phi^\bullet)$ , one for each  $\phi \in \widehat{G}$ .

Example 2.5. – A hermitian complex  $(K^\bullet, k^\bullet)$  affords a metrised complex in the following way: endow  $(K^i \otimes_{\mathbf{C}} W)^G$  with the form induced by  $k^i$  on  $K^i$  and by the restriction of the standard form on  $W$ , which is given by the restriction of  $\nu$ . The alternating tensor product of the top exterior products of these forms is then a positive definite hermitian form on the complex line  $\det(K_\phi^\bullet)$  and so induces a positive definite hermitian form on the complex line  $\det(\mathbf{H}_\phi^\bullet)$  via (1).

### 3. Arithmetic classes

#### 3.1. The arithmetic classgroup

In this sub-section we shall define the arithmetic classgroup in which our arithmetic classes take their values.

The notation is that of [9] and so we recall it only briefly:  $R_G$  denotes the group of complex characters of  $G$ ;  $\overline{\mathbf{Q}}$  is the algebraic closure of  $\mathbf{Q}$  in  $\mathbf{C}$ , so that we have the inclusion map  $\overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$ . We set  $\Omega = \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ ;  $J_f$  is the group of finite ideles in  $\overline{\mathbf{Q}}$ , that is to say the direct limit of the finite idele groups of all algebraic number fields  $E$  in  $\overline{\mathbf{Q}}$ .

Let  $\widehat{\mathbf{Z}} = \prod_p \mathbf{Z}_p$  denote the ring of integral finite ideles of  $\mathbf{Z}$ . For  $x \in \widehat{\mathbf{Z}}G^\times$ , the element  $\text{Det}(x) \in \text{Hom}_\Omega(R_G, J_f)$  is defined by the rule that for a representation  $T$  with character  $\psi$

$$\text{Det}(x)(\psi) = \det(T(x));$$

the group of all such homomorphisms is denoted

$$\text{Det}(\widehat{\mathbf{Z}}G^\times) \subseteq \text{Hom}_\Omega(R_G, J_f).$$

More generally, for  $n > 1$  we can form the group  $\text{Det}(GL_n(\widehat{\mathbf{Z}}G))$ ; as each group ring  $\mathbf{Z}_p[G]$  is semi-local we have the equality  $\text{Det}(GL_n(\widehat{\mathbf{Z}}G)) = \text{Det}(\widehat{\mathbf{Z}}G^\times)$  (see 1.2.6 in [37]).

For an  $n \times n$  invertible matrix  $A$  with coefficients in  $\mathbf{C}[G]$ ,  $|\text{Det}(A)| \in \text{Hom}(R_G, \mathbf{R}_{>0})$  is defined by the rule

$$|\text{Det}(A)|(\psi) = |\text{Det}(A)(\psi)|.$$

LEMMA 3.1. – *Extending the involution  $x \mapsto \bar{x}$  on  $\mathbf{C}[G]$  to matrices over  $\mathbf{C}[G]$  by transposition, for  $\psi \in R_G$*

$$|\text{Det}(\bar{A})|(\psi) = |\text{Det}(A)|(\psi).$$

Replacing the ring  $\widehat{\mathbf{Z}}$  by  $\mathbf{Q}$ , in the same way we construct

$$\text{Det}(\mathbf{Q}[G]^\times) \subseteq \text{Hom}_\Omega(R_G, \overline{\mathbf{Q}}^\times).$$

The product of the natural maps  $\overline{\mathbf{Q}}^\times \rightarrow J_f$  and  $|\cdot|: \overline{\mathbf{Q}}^\times \rightarrow \mathbf{R}_{>0}$  yields an injection

$$\Delta: \text{Det}(\mathbf{Q}[G]^\times) \rightarrow \text{Hom}_\Omega(R_G, J_f) \times \text{Hom}(R_G, \mathbf{R}_{>0}).$$

DEFINITION 3.2. – The arithmetic classgroup  $A(\mathbf{Z}[G])$  is defined to be the quotient group

$$(2) \quad A(\mathbf{Z}[G]) = \left( \frac{\text{Hom}_\Omega(R_G, J_f) \times \text{Hom}(R_G, \mathbf{R}_{>0})}{(\text{Det}(\widehat{\mathbf{Z}}[G]^\times) \times 1) \text{Im}(\Delta)} \right).$$

Remarks. – (1) Note that, in the case when  $G = \{1\}$ ,  $A(\mathbf{Z})$  coincides with the usual Arakelov divisor class group of  $\text{Spec}(\mathbf{Z})$  (see 7.7 for further details).

(2) As indicated in the Introduction, there are two crucial differences between this arithmetic classgroup and the hermitian classgroup of Fröhlich (see II.5 in [17]): firstly, we work with positive definite complex hermitian forms; secondly, as a consequence of this, we are able to work in a uniform manner with all characters of  $G$ .

### 3.2. The arithmetic class of a complex

Let  $(P^\bullet, p_\bullet)$  be a perfect metrised  $\mathbf{Z}[G]$ -complex; that is to say  $P^\bullet$  is a bounded metrised complex all of whose terms are finitely generated projective (and therefore locally free)  $\mathbf{Z}[G]$ -modules. For each  $i$ , suppose that  $d_i$  is the rank of  $P^i$  as a  $\mathbf{Z}[G]$ -module, and choose bases  $\{a^{ij}\}$ , respectively  $\{\alpha_p^{ij}\}$  of

$$\mathbf{Q} \otimes P^i = \sum_j \mathbf{Q}[G] \cdot a^{ij}, \quad \text{respectively} \quad \mathbf{Z}_p \otimes P^i = \sum_j \mathbf{Z}_p[G] \cdot \alpha_p^{ij}$$

over  $\mathbf{Q}[G]$  respectively  $\mathbf{Z}_p[G]$ . As both  $\{a^{ij}\}$  and  $\{\alpha_p^{ij}\}$  are  $\mathbf{Q}_p[G]$ -bases of  $\mathbf{Q}_p \otimes P^i$ , we can find  $\lambda_p^i \in GL_{d_i}(\mathbf{Q}_p[G])$  such that  $(a^{ij})_j = \lambda_p^i(\alpha_p^{ij})_j$ , where  $(a^{ij})_j$  denotes the column vector with  $j$ th entry  $a^{ij}$ .

For  $a \in \mathbf{Q} \otimes P^i$  we put

$$(3) \quad r(a) = \sum ga \otimes g \in P^i \otimes \mathbf{Q}[G].$$

Note that for  $h \in G$

$$(4) \quad r(ha) = r(a)(1 \otimes h^{-1})$$

and for  $w \in W$ , the action of  $r(a)$  on  $1 \otimes w$  is defined to be

$$(5) \quad r(a)(1 \otimes w) = \sum_g ga \otimes gw \in (P^i \otimes W)^G.$$

For each  $\phi \in \widehat{G}$  we choose an orthonormal basis  $\{w_{\phi,k}\}$  of  $W = W_\phi$  with respect to the standard form  $\nu$  on  $\mathbf{C}[G]$ , then the  $\{r(a^{ij})(1 \otimes w_{\phi,k})\}$  form a  $\mathbf{C}$ -basis of  $(P^i \otimes W)^G$ . By (4) and by linearity we have that for  $\eta = \sum_{h \in G} \eta_h h \in \mathbf{Q}[G]$

$$(6) \quad r(\eta a)(1 \otimes w) = \sum_{h,g} \eta_h g h a \otimes gw = r(a)(1 \otimes \bar{\eta} w).$$

In the sequel for given  $i$  we shall write  $\bigwedge r(a^i)(1 \otimes w_\phi)$  for the wedge product

$$\bigwedge_{j,k} (r(a^{ij})(1 \otimes w_{\phi,k})) \in \det(P^i \otimes W)^G.$$

We again adopt the notation of Section 2.2 and let  $(P^\bullet, p_\bullet)$  be a perfect metrised  $\mathbf{Z}[G]$ -complex; recall from (1) that for each  $\phi \in \widehat{G}$  we have an isomorphism

$$\xi_\phi : \det(P_\phi^\bullet) \cong \det(H_\phi^\bullet).$$

DEFINITION 3.3. – With the above notation and hypotheses  $\chi(P^\bullet, p_\bullet)$ , the arithmetic class of  $(P^\bullet, p_\bullet)$ , is defined to be that class in  $A(\mathbf{Z}[G])$  represented by the homomorphism on  $R_G$  which maps each  $\phi \in \widehat{G}$  to the value in  $J_f \times \mathbf{R}_{>0}$

$$\prod_p \left( \prod_i \text{Det}(\lambda_p^i)(\phi)^{(-1)^i} \right) \times p_\phi \left( \xi_\phi \left( \otimes_i \left( \bigwedge r(a^i)(1 \otimes w_\phi) \right)^{(-1)^i} \right) \right)^{\frac{1}{\phi(1)}}.$$

In the sequel we shall refer to the first coordinate as the *finite* coordinate and the second coordinate as the *archimedean* coordinate. In order to verify that this class is well-defined, we now show that it is independent of choices:

(i) If  $\{\tilde{\alpha}_p^{ij}\}$  is a further set of  $\mathbf{Z}_p[G]$ -bases for the  $\mathbf{Z}_p \otimes P^i$ , then we can find  $z_p^i \in GL_{d_i}(\mathbf{Z}_p[G])$  such that

$$(\tilde{\alpha}_p^{ij})_j = z_p^i (\alpha_p^{ij})_j$$

and so the  $p$ -component of the finite coordinate of the homomorphism representing the class only changes by

$$\prod_i \text{Det}(z_p^i)^{(-1)^i} \in \text{Det}(\mathbf{Z}_p[G]^\times).$$

(ii) If  $\{\tilde{a}^{ij}\}$  is a further set of  $\mathbf{Q}[G]$ -bases for the  $\mathbf{Q} \otimes P^i$ , then we can find  $\eta^i \in GL_{d_i}(\mathbf{Q}[G])$  such that

$$(\tilde{a}^{ij})_j = \eta^i (a^{ij})_j.$$

Now for each pair  $i, j$ , we have the equality  $\tilde{a}^{ij} = \sum_l \eta_{jl}^i a^{il}$  and so by (6) we get

$$r(\tilde{a}^{ij})(1 \otimes w_{\phi,k}) = \sum_l r(a^{il})(1 \otimes \bar{\eta}_{jl}^i w_{\phi,k});$$

hence

$$\bigwedge r(\tilde{a}^i)(1 \otimes w_\phi) = \text{Det}(\bar{\eta}^i)(\bar{\phi})^{\phi(1)} \bigwedge r(a^i)(1 \otimes w_\phi).$$

As the  $\eta_{jl}^i$  have rational coefficients  $\text{Det}(\bar{\eta}^i)(\bar{\phi}) = \text{Det}(\eta^i)(\phi)$ , and so the homomorphism representing the class only changes by the homomorphism which maps  $\phi$  to

$$\prod_i \text{Det}(\eta^i)(\phi)^{(-1)^i} \times \prod_i |\text{Det}(\eta^i)(\phi)^{(-1)^i}|;$$

and again this comes from an element of the denominator of (2).

(iii) If  $\{\tilde{w}_{\phi,k}\}$  is a further orthonormal basis of  $W$ , then the wedge product  $\bigwedge r(a^i)(1 \otimes \tilde{w}_{\phi})$  differs from  $\bigwedge r(a^i)(1 \otimes w_{\phi})$  by a power of the determinant of a unitary base-change, which therefore has absolute value 1.

The following two properties of arithmetic classes follow readily from the definition.

LEMMA 3.4. – *Let  $(P^{\bullet}, p_{\bullet}), (Q^{\bullet}, q_{\bullet})$  be perfect metrised  $\mathbf{Z}[G]$ -complexes and endow the complex  $P^{\bullet} \oplus Q^{\bullet}$  with metrics  $p_{\phi} \otimes q_{\phi}$  on the equivariant determinants of cohomology via the identification*

$$\det(\mathbf{H}^{\bullet}(P^{\bullet} \oplus Q^{\bullet})) = \det(\mathbf{H}^{\bullet}(P^{\bullet})) \otimes \det(\mathbf{H}^{\bullet}(Q^{\bullet})).$$

Then

$$\chi(P^{\bullet} \oplus Q^{\bullet}, p_{\bullet} \otimes q_{\bullet}) = \chi(P^{\bullet}, p_{\bullet})\chi(Q^{\bullet}, q_{\bullet}).$$

*Proof.* – This follows on choosing bases for  $P^{\bullet}$  and  $Q^{\bullet}$  and then using these bases to form a basis of  $P^{\bullet} \oplus Q^{\bullet}$ .  $\square$

Recall that  $|\cdot|$  denotes the standard metric on  $\mathbf{C}$ .

LEMMA 3.5. – *If  $P^{\bullet}$  is an acyclic perfect metrised  $\mathbf{Z}[G]$ -complex and if we endow each complex line  $\det(\mathbf{H}^{\bullet}(P^{\bullet})) = \det(\{0\}) = \mathbf{C}$  with the metric  $|\cdot|$ , then  $\chi(P^{\bullet}, |\cdot|) = 1$ .*

*Proof.* – As  $P^{\bullet}$  is acyclic and its terms are projective, it is isomorphic to a complex

$$\dots \rightarrow W^{i-1} \oplus W^i \rightarrow W^i \oplus W^{i+1} \rightarrow \dots$$

where the  $W^i$  are all projective and where the boundary maps are projection to the second factor. Using bases of the  $W^i$  to form bases of the  $P^i$ , together with the standard properties of  $\det$ , we see that the products in 3.3 all telescope to 1.  $\square$

LEMMA 3.6. – *If  $p_{\bullet}$  and  $q_{\bullet}$  are two sets of metrics on the equivariant determinants of cohomology of  $P^{\bullet}$ , then for each  $\phi \in \widehat{G}$ ,  $p_{\phi} = \alpha(\phi)^{\phi(1)}q_{\phi}$  for a unique positive real number  $\alpha(\phi)$ . The class  $\chi(P^{\bullet}, p_{\bullet})\chi(P^{\bullet}, q_{\bullet})^{-1}$  in  $A(\mathbf{Z}[G])$  is represented by the homomorphism which maps each  $\phi \in \widehat{G}$  to the value  $1 \times \alpha(\phi)$ .*

*Proof.* – This follows immediately from (3.3).  $\square$

### 3.3. Invariance under quasi-isomorphism

Let  $(C^{\bullet}, c_{\bullet})$  and  $(D^{\bullet}, d_{\bullet})$  denote bounded (not necessarily perfect) metrised  $\mathbf{Z}[G]$ -complexes and suppose that there is a  $\mathbf{Z}[G]$ -cochain map  $\alpha: C^{\bullet} \rightarrow D^{\bullet}$ . Recall that  $\alpha$  is called a *quasi-isomorphism* if it induces an isomorphism on the cohomology of the complexes. Theorem 2 in [27] implies that if  $\alpha$  is a quasi-isomorphism, then it induces natural isomorphisms

$$\det(\mathbf{H}(\alpha_{\phi})) : \det(\mathbf{H}^{\bullet}(C^{\bullet}_{\phi})) \cong \det(\mathbf{H}^{\bullet}(D^{\bullet}_{\phi}))$$

so that the following square commutes:

$$\begin{array}{ccc} \det(C^{\bullet}_{\phi}) & \longrightarrow & \det(D^{\bullet}_{\phi}) \\ \downarrow & & \downarrow \\ \det(\mathbf{H}^{\bullet}(C^{\bullet}_{\phi})) & \longrightarrow & \det(\mathbf{H}^{\bullet}(D^{\bullet}_{\phi})) \end{array}$$

where the top horizontal map is  $\det(\alpha_\phi)$  and where the vertical isomorphisms are  $\xi_{C,\phi}$  and  $\xi_{D,\phi}$  of (1).

DEFINITION 3.7. – A quasi-isomorphism  $\alpha: C^\bullet \rightarrow D^\bullet$  is called a *metric quasi-isomorphism* from  $(C^\bullet, c_\bullet)$  to  $(D^\bullet, d_\bullet)$  if  $c_\phi = d_\phi \circ \det(H(\alpha_\phi))$  for each  $\phi \in \widehat{G}$ .

The following result is an immediate consequence of the definitions:

LEMMA 3.8. – *Suppose again that  $\alpha: C^\bullet \rightarrow D^\bullet$  is a quasi-isomorphic cochain map and that metrics  $d_\phi$  are given on the  $\det(H^\bullet(D^\bullet_\phi))$ . Then there is a unique set of metrics  $c_\phi$  on  $\det(H^\bullet(C^\bullet_\phi))$  such that  $\alpha: (C^\bullet, c_\bullet) \rightarrow (D^\bullet, d_\bullet)$  is a metric quasi-isomorphism; we call the metrics  $c_\bullet$  the metrics on the equivariant determinants of cohomology induced from  $d_\bullet$  via  $\alpha$ . If  $\beta: C^\bullet \rightarrow D^\bullet$  is a further quasi-isomorphic cochain map and if  $H^i(\alpha) = H^i(\beta)$  for all  $i$ , then  $\det(H(\alpha_\phi)) = \det(H(\beta_\phi))$  for all  $\phi \in \widehat{G}$  and so  $\alpha$  and  $\beta$  induce the same metrics on the equivariant determinant of cohomology of  $C^\bullet$ .*

The main result of this sub-section is

DEFINITION-THEOREM 3.9. – *With the above notation and hypotheses, let*

$$\alpha: (C^\bullet, c_\bullet) \rightarrow (D^\bullet, d_\bullet)$$

*be a metric quasi-isomorphism and suppose further that we can find perfect metrised  $\mathbf{Z}[G]$ -complexes  $(P^\bullet, p_\bullet)$ , respectively  $(Q^\bullet, q_\bullet)$  which support metric quasi-isomorphisms  $f: (P^\bullet, p_\bullet) \rightarrow (C^\bullet, c_\bullet)$ , respectively  $g: (Q^\bullet, q_\bullet) \rightarrow (D^\bullet, d_\bullet)$ . Then  $\chi(P^\bullet, p_\bullet) = \chi(Q^\bullet, q_\bullet)$ .*

*In particular: for a metrised  $\mathbf{Z}[G]$ -complex  $(C^\bullet, c_\bullet)$  with the property that  $C^\bullet$  is quasi-isomorphic to a perfect complex  $P^\bullet$ , we let  $p_\bullet$  denote the metrics on the equivariant determinant of cohomology of  $P^\bullet$  induced by  $c_\bullet$ ; then we can unambiguously define the arithmetic class of  $(C^\bullet, c_\bullet)$  to be the class  $\chi(P^\bullet, p_\bullet)$ ; this class depends only on  $(C^\bullet, c_\bullet)$  and not on the particular choice of perfect complex  $P^\bullet$ . Thus with this definition we have the equality*

$$\chi(C^\bullet, c_\bullet) = \chi(D^\bullet, d_\bullet).$$

Before proving the theorem we first need some preliminary results.

LEMMA 3.10. – *Given maps of  $\mathbf{Z}[G]$ -complexes  $M^\bullet \xrightarrow{\varphi} L^\bullet \xleftarrow{\pi} N^\bullet$  with  $\pi$  a surjective quasi-isomorphism and with  $M^\bullet$  perfect, there exists a  $\mathbf{Z}[G]$ -cochain map  $\psi: M^\bullet \rightarrow N^\bullet$  such that  $\pi \circ \psi = \varphi$ .*

*Proof.* – See VI.8.17 in [31].  $\square$

COROLLARY 3.11. – *If  $0 \rightarrow A^\bullet \xrightarrow{\alpha} B^\bullet \xrightarrow{\beta} C^\bullet \rightarrow 0$  is an exact sequence of perfect  $\mathbf{Z}[G]$ -complexes and if  $A^\bullet$  is acyclic, then there exists a cochain map  $i: C^\bullet \rightarrow B^\bullet$  which is a section of  $\beta$ .*

*Proof.* – Apply the above lemma to  $C^\bullet = C^\bullet \xleftarrow{\beta} B^\bullet$ .  $\square$

*Proof of theorem.* – First we choose an acyclic perfect complex  $L^\bullet$  and a map  $\lambda: L^\bullet \rightarrow D^\bullet$  such that  $\lambda \oplus g$  is surjective. We then endow the equivariant determinants of the cohomology of  $L^\bullet$  with the trivial metrics  $l_\bullet$  as per Lemma 3.5. Then by 3.4 and 3.5

$$\chi(L^\bullet \oplus Q^\bullet, l_\bullet q_\bullet) = \chi(Q^\bullet, q_\bullet).$$

Thus, without loss of generality, we may now assume that  $g$  is surjective.

Consider the diagram

$$P^\bullet \xrightarrow{f} C^\bullet \xrightarrow{\alpha} D^\bullet \xleftarrow{g} Q^\bullet.$$

By Lemma 3.10 we can find a  $\mathbf{Z}[G]$ -map  $\beta: P^\bullet \rightarrow Q^\bullet$  such that  $\alpha \circ f = g \circ \beta$ . As  $f, g$  and  $\alpha$  are all quasi-isomorphisms,  $\beta$  is also a quasi-isomorphism.

As previously, by adding an acyclic complex with trivial metrics  $(L^\bullet, l_\bullet)$  to  $(P^\bullet, p_\bullet)$ , setting  $P'^\bullet = L^\bullet \oplus P^\bullet$  and  $p'_\bullet = l_\bullet p_\bullet$ , we obtain a surjective quasi-isomorphism  $\beta': P'^\bullet \rightarrow Q^\bullet$  and

$$\chi(P'^\bullet, p'_\bullet) = \chi(P'^\bullet, l_\bullet p_\bullet) = \chi(P^\bullet, p_\bullet).$$

We let  $f': P'^\bullet \rightarrow C^\bullet$  denote the composition of  $f$  with the natural projection map. Then in general of course it will not be true that  $\alpha \circ f' = g \circ \beta'$ ; however, as  $L^\bullet$  is acyclic, we do know that  $\alpha \circ f'$  and  $g \circ \beta'$  agree on cohomology, i.e.  $H^i(\alpha \circ f') = H^i(g \circ \beta')$  for all  $i$ .

In order to complete the proof of Theorem 3.9, we apply Corollary 3.11 to choose a section  $\gamma: Q^\bullet \rightarrow P'^\bullet$  of  $\beta'$ . Again as per Lemma 3.5 we endow the equivariant determinants of cohomology of  $\ker \beta'$  with the trivial metric  $s_\bullet$ ; as per Lemma 3.4 we endow  $P'^\bullet$  with the metric  $\tilde{q}_\bullet$  given by  $s_\bullet \cdot \gamma_* q_\bullet$ . Then

$$\chi(P'^\bullet, \tilde{q}_\bullet) = \chi(\ker \beta', s_\bullet) \chi(\gamma Q^\bullet, \gamma_* q_\bullet) = \chi(\gamma Q^\bullet, \gamma_* q_\bullet) = \chi(Q^\bullet, q_\bullet).$$

However, as the metrics  $q_\bullet$ , on the equivariant determinants of the cohomology of  $Q^\bullet$ , are induced from  $d_\bullet$  via  $H(g)$ , the metrics  $\tilde{q}_\bullet$  are the transport to  $P'^\bullet$  of the metrics  $d_\bullet$  via  $H(g \circ \beta') = H(\alpha \circ f')$ . Thus  $p'_\bullet$  and  $\tilde{q}_\bullet$  are both transports of the  $d_\bullet$  via  $H(g \circ \beta') = H(\alpha \circ f')$ , and so by Lemma 3.8 they are equal. Therefore we have shown

$$\chi(P^\bullet, p_\bullet) = \chi(P'^\bullet, p'_\bullet) = \chi(P'^\bullet, \tilde{q}_\bullet) = \chi(Q^\bullet, q_\bullet)$$

which is the desired result.  $\square$

## 4. Arithmetic classgroups

### 4.1. Symplectic arithmetic classes

The arithmetic classgroup  $A(\mathbf{Z}[G])$  carries a great deal of information. In consequence, it is often advantageous in practice to work with various image groups. The most important of these is the *symplectic* arithmetic classgroup.

Recall that by the Hasse–Schilling norm theorem

$$(7) \quad \text{Det}(\mathbf{Q}[G]^\times) = \text{Hom}_\Omega^+(R_G, \overline{\mathbf{Q}}^\times)$$

where the right-hand expression denotes Galois equivariant homomorphisms whose values on  $R_G^s$ , the group of virtual symplectic characters, are all totally positive. By analogy with the map  $\Delta$  of Section 3.1, we again have a diagonal map

$$\Delta^s : \text{Hom}_\Omega^+(R_G^s, \overline{\mathbf{Q}}^\times) \rightarrow \text{Hom}_\Omega(R_G^s, J_f) \times \text{Hom}(R_G^s, \mathbf{R}_{>0})$$

where  $\Delta^s(f) = f \times |f| = f \times f$ .

DEFINITION 4.1. – The group of *symplectic arithmetic classes*  $A^s(\mathbf{Z}[G])$  is defined to be the quotient group

$$A^s(\mathbf{Z}[G]) = \frac{\text{Hom}_\Omega(R_G^s, J_f) \times \text{Hom}(R_G^s, \mathbf{R}_{>0})}{\text{Im } \Delta^s \cdot (\text{Det}^s(\widehat{\mathbf{Z}}[G]^\times) \times 1)}$$

where  $\text{Det}^s(\widehat{\mathbf{Z}}[G]^\times)$  denotes the restriction of  $\text{Det}(\widehat{\mathbf{Z}}[G]^\times)$  to  $R_G^s$ . In general, given a homomorphism  $f$  on  $R_G$ , we shall write  $f^s$  for the restriction of  $f$  to  $R_G^s$ . Clearly, restriction from  $R_G$  to  $R_G^s$  induces a homomorphism

$$\rho: A(\mathbf{Z}[G]) \rightarrow A^s(\mathbf{Z}[G]).$$

**4.2. Torsion classes**

Let  $K_0\text{T}(\mathbf{Z}[G])$  denote the Grothendieck group of finite, cohomologically trivial  $\mathbf{Z}[G]$ -modules and let  $K_0\text{T}(\mathbf{Z}_p[G])$  denote the Grothendieck group of finite, cohomologically trivial  $\mathbf{Z}_p[G]$ -modules. Thus the decomposition of a finite module into its  $p$ -primary parts induces the direct sum decomposition

$$K_0\text{T}(\mathbf{Z}[G]) = \bigoplus_p K_0\text{T}(\mathbf{Z}_p[G]).$$

We write  $K_0(\mathbf{F}_p[G])$  for the Grothendieck group of finitely generated projective  $\mathbf{F}_p[G]$ -modules; since each such module may be considered as a finite, cohomologically trivial  $\mathbf{Z}_p[G]$ -module, we have a natural map

$$K_0(\mathbf{F}_p[G]) \rightarrow K_0\text{T}(\mathbf{Z}_p[G]).$$

From Chapter 1, Theorem 3.3 in [37] recall that there is the Fröhlich isomorphism

$$K_0\text{T}(\mathbf{Z}[G]) \cong \frac{\text{Hom}_\Omega(R_G, J_f)}{\text{Det}(\widehat{\mathbf{Z}}[G]^\times)};$$

thus there is a natural map  $\nu: K_0\text{T}(\mathbf{Z}[G]) \rightarrow A(\mathbf{Z}[G])$ , induced by  $f \mapsto f \times 1$  for  $f \in \text{Hom}_\Omega(R_G, J_f)$ .

In order that our invariants agree with the standard invariants in Arakelov theory, our convention here is that of I.3.2 in [37]: namely, if  $M = \mathbf{Z}_p[G]/\alpha\mathbf{Z}_p[G]$  is a  $\mathbf{Z}_p$ -torsion  $\mathbf{Z}_p[G]$ -module, then the class of  $M$  in  $K_0\text{T}(\mathbf{Z}[G])$  is represented by  $\text{Det}(\alpha)$ ; this then is the inverse of the description given in 4.4 in [7]. It will be important in the sequel to keep this in mind when performing various torsion calculations in Sections 7 and 8.

**4.3. Tame arithmetic classes**

Although we shall ultimately always be interested in forming arithmetic classes over the integral group ring  $\mathbf{Z}[G]$ , in carrying out calculations it will often be advantageous to work with more general group rings, where we allow tame coefficients. With this in mind, we let  $T$  denote the maximal abelian tame extension of  $\mathbf{Q}$  in  $\overline{\mathbf{Q}}$  and we set

$$\text{Det}^s(\widehat{\mathcal{O}}_T[G]^\times) = \lim_{\overline{L}} \text{Det}^s(\widehat{\mathcal{O}}_L[G]^\times)$$

where the direct limit extends over all finite extensions  $L$  of  $\mathbf{Q}$  in  $T$  and where  $\widehat{\mathcal{O}}_L$  is the ring of integral adèles  $\widehat{\mathbf{Z}} \otimes \mathcal{O}_L$ .

In arithmetic calculations we shall often need to work with the *tame symplectic arithmetic classgroup* defined as

$$(8) \quad A_T^s(\mathbf{Z}[G]) = \frac{(\text{Det}^s(\widehat{\mathcal{O}}_T[G]^\times) \cdot \text{Hom}_\Omega(R_G^s, J_f)) \times \text{Hom}(R_G^s, \mathbf{R}_{>0})}{\text{Im}(\Delta^s) \cdot (\text{Det}^s(\widehat{\mathcal{O}}_T[G]^\times) \times 1)}.$$

Inclusion then induces a surjective homomorphism

$$\eta: A^s(\mathbf{Z}[G]) \rightarrow A_T^s(\mathbf{Z}[G]).$$

For a perfect metrised  $\mathbf{Z}[G]$ -complex  $(P^\bullet, p_\bullet)$ , we write  $\chi^s(P^\bullet, p_\bullet)$  for the image of  $\chi(P^\bullet, p_\bullet)$  in  $A_T^s(\mathbf{Z}[G])$ .

#### 4.4. Rational classes

Rational classes are ubiquitous in arithmetic applications. The subgroup of *rational symplectic arithmetic classes* is defined to be the subgroup of  $A_T^s(\mathbf{Z}[G])$  generated by  $\text{Hom}_\Omega(R_G^s, \mathbf{Q}^\times) \times 1$ , that is to say

$$R^s(\mathbf{Z}[G]) = \frac{(\text{Hom}_\Omega(R_G^s, \mathbf{Q}^\times) \cdot \text{Det}^s(\widehat{\mathcal{O}}_T[G]^\times) \times 1) \cdot \text{Im}(\Delta^s)}{(\text{Det}^s(\widehat{\mathcal{O}}_T[G]^\times) \times 1) \cdot \text{Im}(\Delta^s)}.$$

The natural map  $\overline{\mathbf{Q}}^\times \hookrightarrow J_f$  induces a map

$$\theta': \text{Im}(\Delta^s) \cdot (\text{Hom}_\Omega(R_G^s, J_f) \times 1) \rightarrow \text{Hom}_\Omega(R_G^s, J_f)$$

which is defined as follows: consider  $h \in \text{Im}(\Delta^s) \cdot (\text{Hom}_\Omega(R_G^s, J_f) \times 1)$  and let  $h_f$  respectively  $h_\infty$  denote the finite respectively archimedean component of  $h$ . Then  $h_\infty$  determines a unique element  $h'_\infty$  of  $\text{Im}(\Delta^s)$ ; we define  $\theta'(h) = h_f h'_\infty^{-1}$ . Clearly  $\theta'$  vanishes on  $\text{Im}(\Delta^s)$  and so induces a homomorphism

$$\theta: R^s(\mathbf{Z}[G]) \rightarrow \frac{\text{Hom}_\Omega(R_G^s, \mathbf{Q}^\times) \text{Det}^s(\widehat{\mathcal{O}}_T[G]^\times)}{\text{Det}^s(\widehat{\mathcal{O}}_T[G]^\times)}.$$

From [6] (see also Corollary 3 to Theorem 17 in [17]) we know that

$$(9) \quad \text{Hom}_\Omega(R_G^s, \mathbf{Q}^\times) \cap \text{Det}^s(\widehat{\mathcal{O}}_T[G]^\times) = \{1\}$$

and so by (9) we see that  $\theta$  may be written as an isomorphism

$$\theta: R^s(\mathbf{Z}[G]) \rightarrow \text{Hom}_\Omega(R_G^s, \mathbf{Q}^\times).$$

#### 4.5. Passage to degree zero

In this sub-section we describe a useful procedure for changing arithmetic classes by passage to characters of degree zero. In practice this will allow us to disregard various free classes which arise in our calculations. For an abelian group  $A$  and for  $f \in \text{Hom}(R_G, A)$ , we write



$\tilde{f} \in \text{Hom}(R_G, A)$  for the homomorphism defined by the rule  $\tilde{f}(\chi) = f(\chi - \chi(1)1_G)$ , where  $1_G$  denotes the trivial character of  $G$ . Note that for  $z \in \mathbf{Z}_p[G]^\times$

$$\widetilde{\text{Det}}(z) = \text{Det}(zd^{-1}) \quad \text{where } d = \text{Det}(z)(1_G),$$

and so  $\text{Det}(\widetilde{\mathbf{Z}_p[G]^\times}) \subset \text{Det}(\mathbf{Z}_p[G]^\times)$ ; similarly  $\widetilde{\text{Im}(\Delta)} \subset \text{Im}(\Delta)$ . Thus for a class  $c \in A(\mathbf{Z}[G])$ , represented by a homomorphism  $f$  under (2), we can unambiguously define a new class  $\tilde{c}$ , depending only on  $c$ , to be that class represented by the homomorphism  $\tilde{f}$ .

The map  $c \mapsto \tilde{c}$  can be interpreted in the following fashion in terms of  $G$ -fixed points together with the induction map

$$\text{Ind} : A(\mathbf{Z}) \rightarrow A(\mathbf{Z}[G])$$

given in terms of character maps by  $\text{Ind}(f)(\psi) = f(\text{Res}_G^{\{1\}}\psi) = f(\psi(1) \cdot 1_{\{1\}})$  for  $\psi \in R_G$ .

LEMMA 4.2. – *With the notation of Section 3.2, let  $c = \chi(P^\bullet, p_\bullet) \in A(\mathbf{Z}[G])$  and let  $c_0$  be the class in  $A(\mathbf{Z})$  of  $(P^{\bullet G}, p_1)$ , where  $P^{\bullet G}$  denotes the complex obtained from  $P^\bullet$  by taking  $G$ -fixed points and where  $p_1$  denotes the metric on the determinant of the cohomology of  $P^{\bullet G}$ , obtained by identifying  $P_{\mathbf{C}}^{\bullet G}$  with the isotypic component of the  $P_{\mathbf{C}}^\bullet$  for the trivial character of  $G$ . There is then an equality  $\tilde{c} = c \cdot \text{Ind}(c_0)^{-1}$  in  $A(\mathbf{Z}[G])$ .*

*Proof.* – Let  $\Sigma = \sum_{g \in G} g$ . As each term of  $P^\bullet$  is projective,  $P^{\bullet G} = \Sigma P^\bullet$ . We adopt the notation of Section 3.2 and assume that  $f$  is the representative character map for the class  $c = \chi(P^\bullet, p_\bullet)$  obtained by using local bases  $\{\alpha_p^{ij}\}, \{a^{ij}\}$ . Then we let  $h$  denote the representative for the class  $c_0 = \chi(P^{\bullet G}, p_1)$  obtained by using local bases  $\{\Sigma \alpha_p^{ij}\}, \{\Sigma a^{ij}\}$ . To prove the lemma it will then suffice to show that  $f(1_G) = h(1_{\{1\}})$ .

We start by considering the non-archimedean coordinates. With the notation of Section 3.2 we have  $(a^{ij})_j = \lambda_p^i(\alpha_p^{ij})_j$ , and so  $(\Sigma a^{ij})_j = e \cdot \lambda_p^i(\Sigma \alpha_p^{ij})_j$  where  $e = \Sigma/|G|$  is the idempotent associated to the trivial character of  $G$ . Since  $\det(e \lambda_p^i) = \text{Det}(\lambda_p^i)(1_G)$ , we conclude that the non-archimedean coordinates of  $f$  and  $h$  are equal.

To conclude we consider the archimedean coordinates. As  $e$  respectively  $1$  is a basis element of length 1 for the trivial isotypic component of  $\mathbf{C}[G]$  respectively  $\mathbf{C}$  with respect to  $\nu_G$  respectively  $\nu_1$  (see 2.3 for the definition of  $\nu$ ), then as in Section 3.2 we see that the archimedean coordinate of  $f(1_G)$  respectively  $h(1_{\{1\}})$  is obtained by evaluating  $p_1$  on the wedge product  $\bigwedge \alpha_G(r_G(a^{ij} \otimes e))^{(-1)^i}$  respectively  $\bigwedge \alpha_1(\Sigma a^{ij} \otimes 1)^{(-1)^i}$  (see 2.3 to recall the definition of  $\alpha$ ). Since

$$\alpha_G(r_G(a^{ij} \otimes e)) = \alpha_G(\Sigma a^{ij} \otimes e) = \Sigma a^{ij} = \alpha_1(\Sigma a^{ij} \otimes 1)$$

it follows that the archimedean coordinates of  $f(1_G)$  and  $h(1_{\{1\}})$  are also equal.  $\square$

## 5. Arithmetic applications

### 5.1. Preliminary results

Let  $\mathcal{X}$  as given in the introduction: Thus we suppose that  $\mathcal{X}$  is a projective scheme over  $\text{Spec}(\mathbf{Z})$  with structure morphism  $f : \mathcal{X} \rightarrow \text{Spec}(\mathbf{Z})$ . Suppose further that  $\mathcal{X}$  is flat over  $\text{Spec}(\mathbf{Z})$  with equidimensional fibres of dimension  $d$  and that the generic fibre of  $\mathcal{X}$  is smooth. For the sake of brevity, in the sequel we shall refer to  $\mathcal{X}$  simply as an arithmetic variety. Suppose further that  $\mathcal{X}$  is endowed with an action  $(\mathcal{X}, G)$  by a given finite group  $G$ . Since  $\mathcal{X}$  is projective, the quotient scheme  $\mathcal{Y} = \mathcal{X}/G$  is defined and we denote the quotient morphism by  $\pi : \mathcal{X} \rightarrow \mathcal{Y}$ .

Throughout this section we suppose that  $\mathcal{X}$  and  $\mathcal{Y}$  satisfy hypotheses (T1) and (T2). Since, by (T1),  $G$  acts tamely on  $\mathcal{X}$ , we note that by the valuative criterion for properness it follows that  $G$  must act freely on the generic fiber  $\mathcal{X}_{\mathbf{Q}}$  (see 1.2.4(d) in [8]). Let  $b$  denote the branch locus on  $\mathcal{Y}$  of the cover  $\mathcal{X}/\mathcal{Y}$  which is then contained in a finite set of fibers of  $\mathcal{Y} \rightarrow \text{Spec}(\mathbf{Z})$ . By hypothesis (T2) we know that the branch locus  $b$  is a Cartier divisor on  $\mathcal{Y}$  with strictly normal crossings. We now consider the construction of arithmetic classes for complexes of sheaves on  $\mathcal{X}$ . For a detailed account of the formation of Euler characteristics (without metrics) associated to a tame action, the reader is referred to [11]. Let  $\mathcal{F}^\bullet$  denote a bounded complex of coherent  $G$ - $\mathcal{X}$  sheaves. Consider a  $G$ -stable open affine cover  $\mathcal{U}$  of  $\mathcal{X}$  and take the chain complex  $C^\bullet$  which is the associated simple complex to the double complex  $C^\bullet(\mathcal{U}, \mathcal{F}^\bullet)$ . There is an isomorphism in the derived category between  $C^\bullet$  and  $\text{R}\Gamma(\mathcal{X}, \mathcal{F}^\bullet)$  which induces isomorphisms

$$\det(\mathbf{H}^\bullet(\text{R}\Gamma(\mathcal{X}, \mathcal{F}^\bullet))_\phi) \cong \det(\mathbf{H}^\bullet(C^\bullet)_\phi) \quad \text{for all } \phi \in \widehat{G},$$

and from Theorem 2.7 in [7] we know that the terms in  $C^\bullet$  may be taken to be cohomologically trivial  $G$ -modules.

LEMMA 5.1. – *For  $C^\bullet$  as above, there is a perfect  $\mathbf{Z}[G]$ -complex  $P^\bullet$  with a quasi-isomorphism  $\gamma: P^\bullet \rightarrow C^\bullet$ .*

*Proof.* – For full details we refer to the proof of Theorem 1.1 in [7]; so we shall now briefly only sketch the proof for the reader’s convenience. From Lemma III.12.3 in [23] we may construct a quasi-isomorphism  $\gamma_1: P_1^\bullet \rightarrow C^\bullet$  where the complex  $P_1^\bullet$  is a bounded complex of finitely generated  $\mathbf{Z}[G]$ -modules all of whose terms except the initial term,  $P_1^N$  say, are free  $\mathbf{Z}[G]$ -modules. Since the mapping cylinder of  $\gamma_1$  is acyclic with all terms, except possibly  $P_1^N$ , being cohomologically trivial  $\mathbf{Z}[G]$ -modules, we therefore deduce that  $P_1^N$  is a cohomologically trivial  $\mathbf{Z}[G]$ -module, and it may therefore be written as the quotient of two projective  $\mathbf{Z}[G]$ -modules; replacing  $P_1^N$  by this perfect complex of length 2 provides  $P^\bullet$  and  $\gamma$ .  $\square$

DEFINITION 5.2. – Suppose now that we are given metrics  $h_\phi$  on the  $\det(\mathbf{H}^\bullet(\text{R}\Gamma(\mathcal{X}, \mathcal{F}^\bullet))_\phi)$  for all  $\phi \in \widehat{G}$ . These metrics then induce metrics  $p_\phi$  on  $\det(\mathbf{H}^\bullet(P_\phi^\bullet))$  and by Theorem 3.9 we know that the arithmetic class  $\chi(P^\bullet, p_\bullet)$  is independent of choices; we denote this class

$$\chi(\text{R}\Gamma(\mathcal{X}, \mathcal{F}^\bullet), h_\bullet)$$

and the image of this class in the symplectic arithmetic classgroup  $A^s(\mathbf{Z}[G])$  will be denoted  $\chi^s(\text{R}\Gamma(\mathcal{X}, \mathcal{F}^\bullet), h_\bullet)$ .

The following results describe some basic properties of such arithmetic classes. The first two results follow immediately from 3.4 and 3.6.

PROPOSITION 5.3. – *Let  $\mathcal{F}^\bullet, \mathcal{G}^\bullet$  be bounded complexes of coherent  $G$ - $\mathcal{X}$  sheaves; let  $h_\bullet, g_\bullet$  be metrics on the equivariant determinants of cohomology of  $\text{R}\Gamma(\mathcal{X}, \mathcal{F}^\bullet)$ , respectively  $\text{R}\Gamma(\mathcal{X}, \mathcal{G}^\bullet)$ . Then*

$$\chi(\text{R}\Gamma(\mathcal{X}, \mathcal{F}^\bullet \oplus \mathcal{G}^\bullet), h_\bullet g_\bullet) = \chi(\text{R}\Gamma(\mathcal{X}, \mathcal{F}^\bullet), h_\bullet) \cdot \chi(\text{R}\Gamma(\mathcal{X}, \mathcal{G}^\bullet), g_\bullet).$$

PROPOSITION 5.4. – *Let  $j_\bullet$  denote a further set of metrics on the equivariant determinants of cohomology of  $\text{R}\Gamma(\mathcal{X}, \mathcal{F}^\bullet)$  and suppose that for each  $\phi \in \widehat{G}$ ,  $h_\phi = \alpha(\phi)^{\phi(1)} j_\phi$  for  $\alpha(\phi) \in \mathbf{R}_{>0}$ . Then the hermitian class*

$$\chi(\text{R}\Gamma(\mathcal{X}, \mathcal{F}^\bullet), h) \cdot \chi(\text{R}\Gamma(\mathcal{X}, \mathcal{F}^\bullet), j)^{-1}$$

is represented by the homomorphism which maps each  $\phi \in \widehat{G}$  to

$$\varphi \longmapsto 1 \times \alpha(\phi).$$

PROPOSITION 5.5. – If  $0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{H} \rightarrow 0$  is an exact sequence of coherent  $G$ - $\mathcal{X}$  sheaves with metrics  $f_\bullet, g_\bullet, h_\bullet$ , on their equivariant determinants of cohomology, with the property that  $f_\phi \otimes h_\phi = g_\phi$  under the isomorphisms

$$\det(H^\bullet(\mathcal{F})_\phi) \otimes \det(H^\bullet(\mathcal{H})_\phi) \cong \det(H^\bullet(\mathcal{G})_\phi)$$

for each  $\phi \in \widehat{G}$ , then there is an equality of arithmetic classes

$$\chi(\mathrm{R}\Gamma(\mathcal{X}, \mathcal{F}), f_\bullet) \cdot \chi(\mathrm{R}\Gamma(\mathcal{X}, \mathcal{H}), h_\bullet) = \chi(\mathrm{R}\Gamma(\mathcal{X}, \mathcal{G}), g_\bullet).$$

Proof. – Let  $\mathcal{U}$  denote a  $G$ -stable affine cover of  $\mathcal{X}$ . Then we get the associated exact sequence of Čech complexes

$$0 \rightarrow C^\bullet(\mathcal{U}, \mathcal{F}) \rightarrow C^\bullet(\mathcal{U}, \mathcal{G}) \rightarrow C^\bullet(\mathcal{U}, \mathcal{H}) \rightarrow 0.$$

For brevity we put  $C_1^\bullet = C^\bullet(\mathcal{U}, \mathcal{F})$ ,  $C_2^\bullet = C^\bullet(\mathcal{U}, \mathcal{G})$ ,  $C_3^\bullet = C^\bullet(\mathcal{U}, \mathcal{H})$ . As mentioned at the start of this section, since the  $G$ -action is tame, we can then find perfect  $\mathbf{Z}[G]$ -complexes with surjective quasi-isomorphisms

$$P_2^{\bullet'} \xrightarrow{\gamma} C_2^\bullet, \quad P_3^\bullet \rightarrow C_3^\bullet.$$

We assert that we can construct a commutative diagram in which the vertical maps are all surjective quasi-isomorphisms and in which the  $P_i^\bullet$  are perfect  $\mathbf{Z}[G]$ -complexes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & C^\bullet(\mathcal{U}, \mathcal{F}) & \longrightarrow & C^\bullet(\mathcal{U}, \mathcal{G}) & \longrightarrow & C^\bullet(\mathcal{U}, \mathcal{H}) \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & P_1^\bullet & \longrightarrow & P_2^\bullet & \longrightarrow & P_3^\bullet \longrightarrow 0 \end{array}$$

The result will then follow on taking bases for the  $P_j^i$  for  $j = 1, 3$  and using these to form bases of the  $P_2^i$ .

We now briefly sketch the construction of  $P_1^\bullet$  and  $P_2^\bullet$ . By 3.10 we can find a cochain map  $\beta$  such that the following diagram commutes:

$$\begin{array}{ccc} C_2^\bullet & \longrightarrow & C_3^\bullet \\ \uparrow \gamma & & \uparrow \\ P_2^{\bullet'} & \xrightarrow{\beta} & P_3^\bullet \end{array}$$

By adding a free acyclic complex to  $P_2^{\bullet'}$  we may assume that  $\beta$  is surjective; this then implies that  $\ker \beta$  is a perfect complex, and so the restriction of  $\gamma$  to  $\ker \beta$  provides a quasi-isomorphism to  $C_1^\bullet$ . By adding a free acyclic complex to  $\ker \beta$  we obtain a surjective quasi-isomorphism onto  $C_1^\bullet$ , and the resulting complex is denoted by  $P_1^\bullet$ .  $\square$

PROPOSITION 5.6. – Suppose that the  $G$ - $\mathcal{X}$  sheaf  $\mathcal{F}$  is fibral, that is to say it is supported over a finite set of primes  $S$  in  $\mathrm{Spec}(\mathbf{Z})$ . Then the equivariant determinants of cohomology all

identify with the trivial complex line  $\mathbf{C}$ , which we endow with the standard metric  $|\cdot|$  and

$$\chi(\mathrm{R}\Gamma(\mathcal{X}, \mathcal{F}), |\cdot|_{\bullet}) = \nu \circ f_*^T(\mathcal{F})$$

where  $f_*^T$  denotes the composition of

$$\bigoplus_{p \in S} K_0(G, \mathcal{X}_p) \xrightarrow{\oplus f_{p*}} \bigoplus_{p \in S} K_0(\mathbf{F}_p[G]) \rightarrow K_0\mathrm{T}(\mathbf{Z}[G]).$$

Here the first map is induced by the structure maps  $f_p: \mathcal{X}_p \rightarrow \mathrm{Spec}(\mathbf{F}_p)$  for  $p \in S$  (see Theorem 1.1 in [7]), and the second map is as described in Section 4.2.

*Proof.* – This follows at once from the definition of  $\chi(\mathrm{R}\Gamma(\mathcal{X}, \mathcal{F}), |\cdot|_{\bullet})$  and from Section 4.2. (Note that this, in part, justifies the choice of convention in Section 4.2.)  $\square$

### 5.2. Rings of integers

The remainder of this article is devoted to the study of images of arithmetic classes in various arithmetic situations. In this sub-section we shall consider the case where  $\mathcal{X}$  is the spectrum of a ring of integers; thus in this sub-section we consider the case  $\mathcal{X} = \mathrm{Spec}(\mathcal{O}_N)$  for a ring of integers  $\mathcal{O}_N$  of a number field  $N$  which is at most tamely ramified over a number field  $K$ , with  $N/K$  Galois and  $G = \mathrm{Gal}(N/K)$ .

Our main result here is Theorem 5.9, which is closely related to the work of Fröhlich in Chapter VI of [17] and to the proof of the Second Fröhlich Conjecture in [6].

Suppose that  $\mathfrak{a}$  is a  $G$ -stable  $\mathcal{O}_N$ -ideal and let  $\mathcal{F} = \tilde{\mathfrak{a}}$  be the associated  $G$ - $\mathcal{X}$  sheaf viewed as a complex concentrated in degree zero. As  $\mathcal{X}$  is affine

$$H^i(\mathcal{X}, \mathcal{F}) = \begin{cases} \mathfrak{a} & \text{if } i = 0, \\ \{0\} & \text{if } i > 0. \end{cases}$$

We endow  $\mathfrak{a}_{\mathbf{C}} = \mathbf{C} \otimes_{\mathbf{Z}} \mathfrak{a} = \mathbf{C} \otimes_{\mathbf{Q}} N$  with the  $G$ -invariant positive definite Hecke form  $h: \mathbf{C} \otimes_{\mathbf{Q}} N \times \mathbf{C} \otimes_{\mathbf{Q}} N \rightarrow \mathbf{C}$  which is defined by the rule

$$h(\lambda \otimes m, \nu \otimes n) = \frac{1}{|G|} \lambda \bar{\nu} \sum_{\sigma} \sigma(m) \overline{\sigma(n)}$$

where the sum extends over the embeddings  $\sigma: N \rightarrow \mathbf{C}$ . Thus, as in (2.5),  $h$  determines metrics on the  $\det((\mathbf{C} \otimes_{\mathbf{Q}} N)_{\phi})$  for  $\phi \in \hat{G}$ ; we denote this set of metrics by  $\det h_{\bullet}$ .

*Remark.* – We refer to the form  $h$  as the Hecke form since this form was introduced by Hecke in his proof of the functional equation for L-functions; see for instance 9.3 in [24].

We write  $\mu_K$  for the  $G$ -invariant positive hermitian form on  $\mathbf{C} \otimes_{\mathbf{Q}} K[G]$  given by the rule

$$\mu_K \left( \sum_g x_g g, \sum_h y_h h \right) = \frac{1}{|G|} \sum_{\rho} \sum_{g,h} \delta_{g,h} \rho(x_g) \overline{\rho(y_h)}$$

where the first right-hand sum extends over all embeddings  $\rho$  of  $K$  into  $\mathbf{C}$ . Again as per (2.5)  $\mu_K$  induces metrics  $\det \mu_{K,\phi}$  on the  $\det((\mathbf{C} \otimes_{\mathbf{Q}} K[G])_{\phi})$  for each  $\phi \in \hat{G}$ ; we denote this set of metrics by  $\det \mu_{K,\bullet}$ , or  $\det \mu_{\bullet}$  when  $K$  is clear from the context.

In the sequel, since  $\mathcal{X} = \text{Spec}(\mathcal{O}_N)$  is affine, for brevity we shall write  $\chi(\mathfrak{a}, \det h_\bullet)$  in place of  $\chi(R\Gamma(\mathcal{X}, \mathfrak{a}), \det h_\bullet)$  etc. The following result is an equivariant version of the usual discriminant-index theorem:

PROPOSITION 5.7. – *The following equality holds in  $A(\mathbf{Z}[G])$*

$$\chi(\mathcal{O}_N, \det h_\bullet) \cdot \chi(\mathfrak{a}, \det h_\bullet)^{-1} = \nu(\mathcal{O}_N/\mathfrak{a})$$

where  $\nu$  is the map on torsion classes of Section 4.2.

*Proof.* – This follows from Propositions 5.5 and 5.6 applied to the exact sequence

$$0 \rightarrow \mathfrak{a} \rightarrow \mathcal{O}_N \rightarrow \mathcal{O}_N/\mathfrak{a} \rightarrow 0. \quad \square$$

DEFINITION 5.8. – For a given prime ideal  $\mathfrak{p}$  of  $\mathcal{O}_K$ , let  $f_{\mathfrak{p}}$  denote the residue class extension degree of  $\mathfrak{p}$  in  $K/\mathbf{Q}$ , denote by  $I_{\mathfrak{p}}$  the inertia group of a chosen prime ideal of  $\mathcal{O}_N$  above  $\mathfrak{p}$ ; and let  $u_{\mathfrak{p}}$  denote the augmentation character of  $I_{\mathfrak{p}}$  (that is to say the regular character minus the trivial character). Define

$$\text{Pf}_p(\mathcal{O}_N) : R_G^s \rightarrow (-p)^{\mathbf{Z}} \text{ by the rule } \text{Pf}_p(\mathcal{O}_N)(\psi) = \prod_{\mathfrak{p}|p} (-p)^{\frac{1}{2} f_{\mathfrak{p}}(\psi, \text{Ind}_{I_{\mathfrak{p}}}^G u_{\mathfrak{p}})}$$

for  $\psi \in R_G^s$ , where  $(\cdot, \cdot)$  denotes the standard inner product on  $R_G$ . Note that

$$(\psi, \text{Ind}_{I_{\mathfrak{p}}}^G u_{\mathfrak{p}}) = (\psi|_{I_{\mathfrak{p}}}, u_{\mathfrak{p}})$$

is an even integer, since  $\psi|_{I_{\mathfrak{p}}}$  is a symplectic character of the cyclic group  $I_{\mathfrak{p}}$  and is therefore a sum of characters of the form  $\theta + \bar{\theta}$ . We then define  $\text{Pf}(\mathcal{O}_N)$  to be the idele valued function, defined on symplectic characters, which is  $\text{Pf}_p(\mathcal{O}_N)$  at primes over  $p$  and which is 1 at the archimedean primes.

Let  $\delta_K \in \text{Hom}(R_G^s, \mathbf{R}_{>0})$  be the homomorphism

$$\delta_K(\psi) = |d_K|^{\psi(1)/2}$$

where  $d_K$  is the discriminant of  $K/\mathbf{Q}$ .

The main result of this sub-section is the following description of the tame arithmetic classes  $\chi^s(\mathcal{O}_N, \det h_\bullet)$  and  $\chi^s(\mathcal{O}_K G, \det \mu_{K\bullet})$ .

THEOREM 5.9. – (a) *The class  $\chi^s(\mathcal{O}_N, \det h_\bullet)$  in  $A_T^s(\mathbf{Z}[G])$  is represented by the homomorphism  $\tilde{\varepsilon}_\infty^s(K)^{-1} \text{Pf}(\mathcal{O}_N)^{-1} \times \delta_K$ , where, for a symplectic character  $\psi$ ,  $\tilde{\varepsilon}_\infty^s(K)(\psi)$  is the archimedean epsilon factor  $\varepsilon_\infty(K, \psi - \psi(1) \cdot 1_G)$  (see Section 7);*

(b) *the class  $\chi^s(\mathcal{O}_K G, \det \mu_{K\bullet})$  in  $A_T^s(\mathbf{Z}[G])$  is represented by the homomorphism  $1 \times \delta_K$ .*

Before proceeding with the proof of the theorem, we first introduce some notation and establish some preparatory results.

For a prime number  $p$ , let  $\beta_p$  be an  $\mathcal{O}_{K,p}[G]$ -basis of  $\mathcal{O}_{N,p}$  and let  $b$  be a  $K[G]$ -basis of  $N$  (so that  $b$  is a so-called normal basis of  $N/K$ ). Recall (see I.4 of [17]) that for a character  $\psi$  of  $G$  the resolvent  $(b | \psi)$  is defined to be the value  $\text{Det}(\sum_{g \in G} g(b)g^{-1})(\psi)$ ; note that, with the notation of (3) in Section 3.2,  $(b | \psi) = \text{Det}(r(b))(\bar{\psi})$  and so by (4) we have proved the following particular instance of the Galois action formula for resolvents (cf. Theorem 20A in [16])

$$(10) \quad (g(b) | \psi) = (b | \psi) \cdot \det(\psi)(g).$$

The local resolvents  $(\alpha_p \mid \psi)$  are defined similarly (see *loc. cit.*).

Set  $\Omega_K = \text{Gal}(\overline{\mathbf{Q}}/K)$  and recall that we write  $\Omega$  for  $\Omega_{\mathbf{Q}}$ . For an  $\Omega$ -module  $A$ , let

$$\mathcal{N}_{K/\mathbf{Q}} : \text{Hom}_{\Omega_K}(R_G, A) \rightarrow \text{Hom}_{\Omega}(R_G, A)$$

denote the co-restriction map of (3.3) in II.3 of [16]; we extend the domain of this map to include resolvents, which are not in general  $\Omega_K$ -equivariant, as per (3.1) in III.3 in [16].

We now recall the  $p$ -adic absolute value function and some related constructions from [9]; for full details see (3.1) and (3.2) in *loc. cit.* Let  $L = \mathbf{Q}(\zeta_p)$ . By Lemma 3.1 *loc. cit.* we know that we can find  $\lambda \in L_p$  such that  $\lambda^{p-1} = -p$ . Let  $R_G(\overline{\mathbf{Q}}_p)$  denote the ring of  $\overline{\mathbf{Q}}_p$ -characters of  $G$  and set  $\Omega_p = \text{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ . For  $g \in \text{Hom}(R_G(\overline{\mathbf{Q}}_p), \overline{\mathbf{Q}}_p^\times)$  we shall say that  $\|g\|$  is *well-defined* if for each  $\phi \in R_G$  there is an integer  $n_\phi$  such that  $g(\phi)\lambda^{n_\phi}$  is a unit and we define  $\|g\| : R_G \rightarrow \lambda^{\mathbf{Z}}$  by the rule that  $\|g\|(\phi) = \lambda^{n_\phi}$ .

Once and for all we fix a field embedding  $h : \overline{\mathbf{Q}} \rightarrow \overline{\mathbf{Q}}_p$ . From II.2.1 in [16],  $h$  induces an isomorphism

$$h^* : \text{Hom}_{\Omega_L}(R_G, (\overline{\mathbf{Q}} \otimes \mathbf{Q}_p)^\times) \cong \text{Hom}_{\Omega_{L_p}}(R_G(\overline{\mathbf{Q}}_p), \overline{\mathbf{Q}}_p^\times).$$

For  $f \in \text{Hom}_{\Omega_L}(R_G, (\overline{\mathbf{Q}} \otimes \mathbf{Q}_p)^\times)$  define  $\|f\| = h^{*-1}(\|h^*f\|)$ ; we shall say that  $\|f\|$  is well-defined when  $\|h^*f\|$  is well-defined.

In the sequel we employ a standard abuse of notation and write  $\text{Det}(\mathcal{O}_{L_p}[G]^\times)$  for  $h^*(\text{Det}(\mathcal{O}_{L_p}[G]^\times))$ .

**THEOREM 5.10.** – For  $\psi \in R_G^s$ ,  $\text{sign}(\mathcal{N}_{K/\mathbf{Q}}(b \mid \psi)) = \varepsilon_\infty(K, \psi - \psi(1)1_G)$ .

*Proof.* – This is III.4.9 of [16].  $\square$

**PROPOSITION 5.11.** – We have  $\mathcal{N}_{K/\mathbf{Q}}(\beta_p \mid -)^s \cdot \text{Pf}_p(\mathcal{O}_N)^{-1} \in \text{Det}^s(\mathcal{O}_{T,p}[G]^\times)$  where we recall from Section 4.2 that  $T$  denotes the maximal abelian tame extension of  $\mathbf{Q}$  in  $\overline{\mathbf{Q}}$ .

*Proof.* – Let  $\tau^*$  denote the adjusted Galois Gauss sum of (3.9) in [36] (or see IV.1.7. in [16]). From the discussion following Theorem 2 in [36] we know that we can find  $z_p \in \mathbf{Z}_p[G]^\times$  such that for all  $\phi \in R_G$

$$\mathcal{N}_{K/\mathbf{Q}}(\beta_p \mid \phi) = \text{Det}(z_p)(\phi)\tau^*(\phi).$$

Recall that we have fixed a choice of field embedding  $h : \overline{\mathbf{Q}} \rightarrow \overline{\mathbf{Q}}_p$ . By Theorem 4 in [9] we know that  $\|\tau_p^*\| = \|\tilde{\varepsilon}_{0,p}\|$  is well-defined. Writing

$$\tau^* = \tau' \tau_p^* \quad \text{where } \tau' = \prod_{q \neq p} \tau_q^*$$

we get

$$\mathcal{N}_{K/\mathbf{Q}}(\beta_p \mid -) = \text{Det}(z_p)\tau' \tau_p^* = \text{Det}(z_p)\tau' \tau_p^* \|\tau_p^*\| \|\tau_p^*\|^{-1}$$

and by Theorem 4 in *loc. cit.*

$$\tau_p^* \|\tau_p^*\| \in \text{Det}(\mathcal{O}_{L_p}[G]^\times) \quad \text{and} \quad \tau' \in \text{Det}(\mathcal{O}_{T,p}[G]^\times).$$

From Theorem 7.4 in [30] we know that each value of  $\tau_p^s$  is plus or minus an integral power of  $p$ . Thus for a symplectic character  $\psi$  of  $G$ ,

$$|\tau_p^*(\psi)| = Nf_p(\psi)^{\frac{1}{2}} = \prod_{\mathfrak{p}|p} N\mathfrak{p}^{\frac{1}{2}(\psi, \text{Ind}_{I_{\mathfrak{p}}}^G u_{\mathfrak{p}})} = \pm \text{Pf}_p(\mathcal{O}_N)(\psi)$$

where  $Nf_p(\psi)$  denotes the  $p$ -part of the absolute norm of the Artin conductor of  $\psi$ . As  $\|\tau_p^{*s}\|(\psi)$  and  $\text{Pf}_p(\mathcal{O}_N)(\psi)$  are both integral powers of  $-p$ , we deduce that  $\|\tau_p^{*s}\|^{-1} = \text{Pf}_p(\mathcal{O}_N)$  as required.  $\square$

*Proof of Theorem 5.9.* – We begin by proving (a). Let  $\{x_i\}$  denote a  $\mathbf{Z}$ -basis of  $\mathcal{O}_K$ . Then  $\{x_i b\}$  respectively  $\{x_i \beta_p\}$  is a  $\mathbf{Q}[G]$ -basis respectively a  $\mathbf{Z}_p[G]$ -basis for  $N$  respectively  $\mathcal{O}_{N,p}$ . With the previous notation choose  $\lambda_p \in K_p[G]$  such that  $b = \lambda_p \beta_p$  and write  $x_i \lambda_p = \sum_j \lambda_p^{ij} x_j$  with  $\lambda_p^{ij} \in \mathbf{Q}_p[G]$ . Then

$$x_i b = x_i \lambda_p \beta_p = \sum_j \lambda_p^{ij} x_j \beta_p$$

and so the matrix  $(\lambda_p^{ij})_{ij}$  transforms the  $\mathbf{Q}_p[G]$ -basis  $\{x_i \beta_p\}$  into the basis  $\{x_i b\}$ ; therefore the finite coordinate of the representing homomorphism of the arithmetic class  $\chi(\mathcal{O}_N, \det h_{\bullet})$  is

$$\prod_p \text{Det}(\lambda_p^{ij}) = \prod_p \mathcal{N}_{K/\mathbf{Q}}(\lambda_p) = \prod_p \mathcal{N}_{K/\mathbf{Q}}(b | -) \cdot \mathcal{N}_{K/\mathbf{Q}}(\beta_p | -)^{-1}.$$

To obtain the archimedean coordinate for a chosen irreducible character  $\phi$  we have to extend our notation and choose a positive integer  $n_{\phi}$  such that  $\det(\phi)^{n_{\phi} \phi(1)}$  is trivial. We then write  $\psi = n_{\phi} \phi(1) \bar{\phi}$  and set

$$W_{\psi} = W_{\phi}^{n_{\phi}}$$

where  $W_{\phi}^{n_{\phi}}$  denotes the direct sum of  $n_{\phi}$  copies of  $W_{\phi}$ . We endow  $W_{\psi}$  with the hermitian form,  $\nu_{\psi}$  say, given by the orthogonal sum of the hermitian forms on the  $W_{\phi}$ , and we let  $\{w_{\psi,k}\}$  denote the basis of  $W_{\psi}$  derived from the bases  $\{w_{\phi,l}\}$  of  $W_{\phi}$ . We must now consider the wedge product

$$(11) \quad \bigwedge_{i,k} (x_i \cdot r(b)(1 \otimes w_{\psi,k})) = \bigwedge_i \bigwedge_k \left( \sum_g x_i g(b) \otimes g w_{\psi,k} \right) = \bigwedge_{i,k} y_i (1 \otimes w_{\psi,k})$$

where  $y_i = x_i(b | \psi)$ ; here we obtain the second equality from the fact that

$$\bigwedge_k \left( \sum_g g(b) \otimes g w_{\psi,k} \right) = \text{Det}(r(b))(\bar{\psi}) \bigwedge_k w_{\psi,k} = (b | \psi) \bigwedge_k w_{\psi,k}.$$

A priori  $(b | \psi) \in \mathbf{C} \otimes_{\mathbf{Q}} N$ ; however, because  $\det(\psi) = 1$ , by the Galois action formula (10),  $(b | \psi) \in \mathbf{C} \otimes_{\mathbf{Q}} K$ . Therefore

$$\bigwedge_i x_i(b | \psi) = N_{K/\mathbf{Q}}(b | \psi) \bigwedge_i x_i.$$

To complete the proof of (a), note first that as the  $\{x_i\}$  are fixed by  $G$  and as  $\{w_{\psi,k}\}$  is an orthonormal basis for the form  $\nu_{\psi}$ ,

$$(12) \quad h \otimes \nu_\psi(x_i \otimes w_{\psi,k}, x_j \otimes w_{\psi,l}) = \frac{1}{|G|} \sum_{\sigma} \sigma(x_i) \overline{\sigma(x_j)} \delta_{k,l} = \sum_{\rho} \rho(x_i) \overline{\rho(x_j)} \delta_{k,l}.$$

In the sequel we shall write  $(h \otimes \nu_\psi)^G$  for the restriction of  $h \otimes \nu_\psi$  from  $(\mathbf{C} \otimes_{\mathbf{Q}} N) \otimes W$  to  $((\mathbf{C} \otimes_{\mathbf{Q}} N) \otimes W)^G$ . Hence the archimedean coordinate of the representing homomorphism of  $\chi(\mathcal{O}_N, \det h_\bullet)$  at  $\phi$  is the  $n_\phi \phi(1)$ -st root of

$$\begin{aligned} & \det \left( h_\psi \left( \bigwedge_{i,k} x_i r(b) (1 \otimes w_{\psi,k}) \right) \right) \\ &= \det \left( (h \otimes \nu_\psi)^G (x_i r(b) \otimes w_{\psi,k}, x_j r(b) \otimes w_{\psi,l}) \right)^{1/2} \\ &= |\mathcal{N}_{K/\mathbf{Q}}(b \mid \psi)| \det \left( (h \otimes \nu_\psi)^G (x_i \otimes w_{\psi,k}, x_j \otimes w_{\psi,l}) \right)^{1/2} \\ &= |d_K|^{\psi(1)/2} |\mathcal{N}_{K/\mathbf{Q}}(b \mid \psi)|. \end{aligned}$$

Note that the square roots in the above right-hand terms (which are of course taken to be positive) arise since we are dealing with the metrics which are, of course, given by the square root of the corresponding positive definite hermitian forms. This then shows that the class  $\chi(\mathcal{O}_N, \det h_\bullet)$  is represented by the homomorphism which maps an irreducible character  $\phi$  to the value

$$\mathcal{N}_{K/\mathbf{Q}}(b \mid \phi) \prod_p \mathcal{N}_{K/\mathbf{Q}}(\beta_p \mid \phi)^{-1} \times |\mathcal{N}_{K/\mathbf{Q}}(b \mid \phi)| \cdot \delta_K(\phi).$$

We now consider  $\chi^s(\mathcal{O}_N, \det h_\bullet)$ . Then by Theorem 5.10 this class is represented by the homomorphism which maps a symplectic character  $\psi$  to the value

$$\mathcal{N}_{K/\mathbf{Q}}(b \mid \psi) \prod_p \mathcal{N}_{K/\mathbf{Q}}(\beta_p \mid \psi)^{-1} \times \tilde{\varepsilon}_\infty(K, \psi) \mathcal{N}_{K/\mathbf{Q}}(b \mid \psi) \delta_K(\psi).$$

Since

$$(\psi \mapsto \tilde{\varepsilon}_\infty(K, \psi) \cdot \mathcal{N}_{K/\mathbf{Q}}(b \mid \psi)) \in \text{Hom}_\Omega^+(R_G^s, \overline{\mathbf{Q}}^\times),$$

we conclude that the class is also represented by

$$\psi \mapsto \tilde{\varepsilon}_\infty(K, \psi)^{-1} \prod_p \mathcal{N}_{K/\mathbf{Q}}(\beta_p \mid \psi)^{-1} \times \delta_K(\psi)$$

and the result then follows from 5.11.

The proof for (b) is similar, but considerably easier, because we may replace  $b$  and all the  $\beta_p$  by 1 throughout in the above. Indeed, we see immediately that, with these choices, the finite coordinate is 1. Since

$$\begin{aligned} & (\mu_K \otimes \nu_\phi)^G \left( \sum_g x_i g \otimes gw_{\phi,k}, \sum_h x_j h \otimes hw_{\phi,l} \right) \\ &= \frac{1}{|G|} \sum_{\rho} \sum_{g,h} \rho(x_i) \overline{\rho(x_j)} \mu(g, h) \nu(gw_{\phi,k}, hw_{\phi,l}) \\ &= \delta_{k,l} \sum_{\rho} \rho(x_i) \overline{\rho(x_j)} \end{aligned}$$



we have

$$\det \mu_{K,\phi} \left( \bigwedge_{i,k} \sum_g x_i g \otimes gw_{\phi,k} \right) = \delta_K(\phi). \quad \square$$

### 6. Equivariant Quillen metrics

#### 6.1. Definition of arithmetic classes

In this section we again consider an arithmetic variety  $\mathcal{X}$  with fibral dimension  $d$  and  $G$ -action such that (T1) and (T2) hold. Since  $G$  acts tamely on  $\mathcal{X}$ ,  $G$  must act freely on the complexified generic fibre  $X := \mathcal{X} \times_{\mathbf{Z}} \mathbf{C}$ ; in the sequel we shall abuse terminology and identify  $X$  with the complex manifold  $\mathcal{X}(\mathbf{C})$  of its complex points. We fix a Kähler metric  $h^Y$  on  $Y$  which is invariant under complex conjugation. We denote by  $h_X = h^{TX}$  the Kähler metric on  $X$  which is the pullback of  $h^Y$ ; this is then also invariant under complex conjugation. We shall always adopt the standard normalisation of multiplying the Kähler form dual to  $h_X$  by  $(i/2\pi)^d$ ; this will ensure that the  $L^2$ -metric is then norm compatible with Serre duality (see 1.4 in [22]). We endow  $\Omega_X^n$ , the sheaf of regular  $n$ -forms on  $X$ , with the normalised metric  $|G|^{-1} \wedge^n h_X^D$ , denoted for brevity by  $\wedge^n h^D$ , to ensure that it agrees with  $\wedge^n h_Y^D$  on forms pulled back from  $Y$ .

A hermitian  $G$ -bundle on  $\mathcal{X}$  is a pair  $(\mathcal{F}, f)$ , where  $\mathcal{F}$  is a locally free  $G$ - $\mathcal{X}$  sheaf with the property that the induced holomorphic vector bundle  $\mathcal{F}_{\mathbf{C}}$  over  $X$  supports a  $G$ -invariant hermitian metric  $f$ , which is invariant under complex conjugation.

The complex lines  $\det(\mathbf{H}^\bullet(\mathbf{R}\Gamma(\mathcal{X}, \mathcal{F})))_\phi$ , for  $\phi \in \widehat{G}$ , carry metrics  $f_{L^2,\phi}$  coming from the  $L^2$ -metric of Hodge theory for the Dolbeault resolution. As per Section II in [2], the  $f_{L^2,\phi}$  can be transformed to equivariant Quillen metrics  $f_{Q,\phi}$  for  $\phi \in \widehat{G}$ . One of the main objectives of this article is the study of the arithmetic classes

$$\chi(\mathbf{R}\Gamma(\mathcal{X}, \mathcal{F}), f_{Q,\bullet}) \quad \text{in } A(\mathbf{Z}[G]).$$

More generally, we shall also consider a bounded complex  $\mathcal{G}^\bullet$  of hermitian  $G$ -bundles on  $\mathcal{X}$ , with  $g^i$  denoting the hermitian form on  $\mathcal{G}^i$ . Then the  $g^\bullet$  induce metrics  $g_{Q,\phi}^\bullet$  on the equivariant determinant of the hypercohomology of  $\mathcal{G}^\bullet$ , and so the arithmetic class

$$\chi(\mathbf{R}\Gamma(\mathcal{X}, \mathcal{G}^\bullet), g_{Q,\bullet}^\bullet) \quad \text{in } A(\mathbf{Z}[G])$$

is defined; explicitly, we may identify the equivariant determinant of  $\det(\mathbf{H}^\bullet(\mathbf{R}\Gamma(\mathcal{X}, \mathcal{G}^\bullet)))_\phi$  with the product

$$\det(\mathbf{H}^\bullet(\mathbf{R}\Gamma(\mathcal{X}, \mathcal{G}^\bullet)))_\phi = \bigotimes_i \det(\mathbf{H}^\bullet(\mathbf{R}\Gamma(\mathcal{X}, \mathcal{G}^i)))_\phi^{(-1)^i}$$

and so

$$\chi(\mathbf{R}\Gamma(\mathcal{X}, \mathcal{G}^\bullet), g_{Q,\bullet}^\bullet) = \prod_i \chi(\mathbf{R}\Gamma(\mathcal{X}, \mathcal{G}^i), g_{Q,\bullet}^i)^{(-1)^i}.$$

See Section III of [2] for details. In the sequel we shall write  $g_{Q,\bullet}$  for the metrics on the equivariant determinant of hypercohomology induced by the  $\{g_{Q,\bullet}^i\}$ .

#### 6.2. 1-dimensional subschemes

In this sub-section we place ourselves in the situation described in Section 5.1. In particular, (T1) and (T2) are satisfied. Recall that by hypothesis (T2)  $b$  is a divisor with strictly normal

crossings. Here we consider an irreducible regular connected closed horizontal sub-scheme  $\mathcal{Z}$  of  $\mathcal{X}$  of dimension one; we may therefore write  $\mathcal{Z} = \text{Spec}(\mathcal{O}_N)$  for some ring of integers  $N$ , where  $G$  acts tamely on  $N$ . As previously we put  $K = N^G$  consider  $\mathcal{F} = \mathcal{O}_{\mathcal{Z}}$  and endow  $\mathcal{F}$  with the Hecke form  $h$  of Section 5.2.

Next we recall the Pfaffian divisor from Section 2 of [9]: for each symplectic character  $\psi$  of  $G$ , the Pfaffian divisor  $\text{Pf}(\mathcal{X}, \psi)$  is a divisor on  $\mathcal{Y}$  which is supported on the branch locus  $b$ . Let  $\mathcal{W} = \pi(\mathcal{Z})$  so that  $\mathcal{W}$  is a closed sub-scheme of  $\mathcal{Y}$ . Throughout this sub-section we shall suppose that  $\mathcal{W}$  meets  $b$  transversely and at smooth points of  $b$ . As we shall see in the next section, in practice we can often reduce to this situation by means of a moving lemma – subject to certain base extensions.

From Theorem 5.9 we know that  $\chi^s(\mathcal{O}_{\mathcal{Z}}, \det h_{\bullet})$  is represented by the homomorphism  $\tilde{\varepsilon}_{\infty}(K)^{-1} \text{Pf}(\mathcal{O}_N)^{-1} \times \delta_K$ . Let  $\{b_i\}$  denote the irreducible components of  $b$ ; let  $\eta_i$  denote the generic point of an irreducible component,  $B_i$  say, of  $\pi^{-1}(b_i)$ ; let  $I_i$  denote the inertia group of  $\eta_i$  and recall that  $u_i$  denotes the augmentation character of  $I_i$ . From (2.1) in [9] we know that for  $\psi \in R_G^s$

$$(13) \quad \text{Pf}(\mathcal{X}, \psi) = \frac{1}{2} \sum_i (\psi, \text{Ind}_{I_i}^G u_i) b_i.$$

A closed point  $\mathfrak{p}$  of  $\mathcal{W}$  (above  $p$ , say) is ramified in  $\mathcal{Z}/\mathcal{W}$  if and only if it is a point of intersection of  $\mathcal{W}$  and some  $b_i$ . Since we have assumed that  $\mathcal{W}$  intersects  $b$  transversely at smooth points of  $b$ ,  $I_{\mathfrak{p}}$  is a conjugate of  $I_i$  and recall that we denote the residue class degree of the point  $\mathfrak{p}$  by  $f_{\mathfrak{p}}$ . In the sequel for such a point  $\mathfrak{p}$  we write  $n(\mathfrak{p}) = i$ . By (5.8) for  $\psi \in R_G^s$  we have

$$\text{Pf}_p(\mathcal{Z}, \psi) = \prod_{\mathfrak{p}} (-p)^{\frac{1}{2} f_{\mathfrak{p}} (\psi, \text{Ind}_{I_{n(\mathfrak{p})}}^G u_{n(\mathfrak{p})})}$$

where the product extends over all points of intersection of  $\mathcal{W}$  with the fibre of  $b$  above  $p$ . We therefore denote the right-hand expression by  $\text{deg}(\mathcal{W} \cdot \text{Pf}_p(\mathcal{X}, \psi))$ , and we let  $\text{deg}(\mathcal{W} \cdot \text{Pf}(\mathcal{X}, \psi))$  denote the finite idele whose  $p$ th component is  $\text{deg}(\mathcal{W} \cdot \text{Pf}_p(\mathcal{X}, \psi))$ . (Note that almost all  $p$ -components are 1 and that the use of  $-p$  in place of  $p$  means that of course we are using degree in a non-standard way.) Writing  $\tilde{\varepsilon}_{\infty}(\mathcal{W})$  for  $\tilde{\varepsilon}_{\infty}(K)$ , we have now shown that the class  $\chi^s(\mathcal{O}_{\mathcal{Z}}, \det h_{\bullet})$  is represented by the homomorphism

$$\tilde{\varepsilon}_{\infty}(\mathcal{W})^{-1} \cdot \text{deg}(\mathcal{W} \cdot \text{Pf}(\mathcal{X}))^{-1} \times \delta_K.$$

Since the Dolbeault complex of a point is trivial, the equivariant Quillen metrics associated to the metrics  $h_{\bullet}$  are precisely the  $\det(h_{\bullet})$  (cf. Definitions 2.1 and 2.2 in [2]). So finally we have now established the main result of this sub-section

**THEOREM 6.1.** – *The symplectic arithmetic class  $\chi^s(\mathcal{O}_{\mathcal{Z}}, \det h_{Q\bullet})$  is represented by the homomorphism*

$$\tilde{\varepsilon}_{\infty}(\mathcal{W})^{-1} \text{deg}(\mathcal{W} \cdot \text{Pf}(\mathcal{X}))^{-1} \times \delta_K.$$

### 6.3. Invariance under passage to degree zero

In this sub-section we establish a number of results concerning the independence, with respect to the choice of hermitian metric, of arithmetic classes after passage to degree zero by the method described in Section 4.5. Recall that we denote the complexified generic fibre of  $\mathcal{X}$  by  $X$ .

THEOREM 6.2. – *Suppose that  $\mathcal{F}$  is a hermitian  $G$ -bundle on  $X$  and let  $f, f'$  be two  $G$ -invariant hermitian metrics on  $\mathcal{F}$ . Then there exists a positive real number  $c$  such that for each  $\phi \in \widehat{G}$*

$$f_{Q,\phi} = c^{\phi(1)^2} f'_{Q,\phi}$$

and so

$$\tilde{\chi}(\mathrm{R}\Gamma\mathcal{F}, f_{Q\bullet}) = \tilde{\chi}(\mathrm{R}\Gamma\mathcal{F}, f'_{Q\bullet}).$$

*Proof.* – For each  $\phi \in \widehat{G}$ , let  $\beta_\phi$  be the positive real number such that  $\beta_\phi f_{Q,\phi} = f'_{Q,\phi}$ . We extend  $\beta$  to  $R_G$  by setting  $\beta(\phi) = \beta_\phi^{1/\phi(1)}$ ,  $\beta(\phi + \psi) = \beta(\phi)\beta(\psi)$  etc. In [2], Bismut considers the central function  $\sigma$  on  $G$

$$\sigma = \sum_{\phi \in \widehat{G}} 2 \log(\beta_\phi) \phi(1)^{-1} \phi;$$

the Anomaly Formula in Theorem 2.5 of [2] shows that  $\sigma(g)$  may be evaluated in terms of integrals over the fixed points of  $g$ . However, since  $G$  acts freely on  $X$ , for each  $g \in G$ ,  $g \neq 1_G$ , the sub-variety of fixed points  $X^g = \{x \in X(\mathbf{C}) \mid x^g = x\}$  is empty. Thus we immediately deduce that  $\sigma(g) = 0$  whenever  $g \neq 1_G$ . This then shows that  $\sigma$  is a scalar multiple of the regular character and the result follows.  $\square$

Next we consider the direct image of a hermitian bundle on a closed sub-scheme of a regular arithmetic variety  $\mathcal{X}$ . The formation of standard (i.e., non-hermitian) Euler characteristics respects closed immersions; however, this need not be the case for arithmetic classes, as the associated Quillen metrics may change. The precise variation in the arithmetic classes, that we wish to consider, was determined in Theorem 0.1 in [2].

We begin by considering a  $G$ -equivariant closed immersion  $i: \mathcal{Z} \rightarrow \mathcal{X}$  of an arithmetic variety  $\mathcal{Z}$  which also supports a tame action by  $G$ . Let  $\mathcal{F}$  denote a locally free  $G$ - $\mathcal{Z}$  sheaf. Since  $\mathcal{X}$  is regular, we may resolve  $i_*\mathcal{F}$  by a bounded complex  $\mathcal{G}^\bullet$  of locally free coherent  $G$ - $\mathcal{X}$  modules. We then have natural isomorphisms in the derived category of  $\mathbf{Z}[G]$ -modules

$$(14) \quad \mathrm{R}\Gamma(\mathcal{Z}, \mathcal{F}) \cong \mathrm{R}\Gamma(\mathcal{X}, i_*\mathcal{F}) \cong \mathrm{R}\Gamma(\mathcal{X}, \mathcal{G}^\bullet)$$

and hence, for each  $\phi \in \widehat{G}$ , we obtain isomorphisms

$$(15) \quad \sigma_\phi : \det H^\bullet(\mathrm{R}\Gamma(\mathcal{X}, \mathcal{G}^\bullet))_\phi \cong \det H^\bullet(\mathrm{R}\Gamma(\mathcal{Z}, \mathcal{F}))_\phi.$$

In order to describe the relevant metrics that we wish to place on these determinants of cohomology, we need some further notation. Let  $Z = \mathcal{Z}_{\mathbf{C}}$  and let  $TZ$  denote the tangent bundle of  $Z$ . We let  $h^{TZ}$  denote the restriction of  $h$  to  $TZ$ . Let  $N_{Z|X}$  denote the normal bundle to  $Z$  in  $X$  and let  $h^{N_{Z|X}}$  be the metric on  $N_{Z|X}$  induced by  $h$ . Let  $f$  denote a given  $G$ -invariant metric on  $\mathcal{F}$ ; we then endow each term  $\mathcal{G}^i$  of  $\mathcal{G}^\bullet$  with a  $G$ -invariant hermitian metric  $g^i$  in such a way that the metrics  $\{g^i\}$  satisfy Bismut’s Condition A with respect to  $h^{N_{Z|X}}$  and  $f$ .

We now wish to compare the arithmetic classes  $\chi(\mathrm{R}\Gamma(\mathcal{Z}, \mathcal{F}), f_{Q\bullet})$  and  $\chi(\mathrm{R}\Gamma(\mathcal{X}, \mathcal{G}^\bullet), g_{Q\bullet})$ .

Let  $\alpha_\phi$  be the unique positive real number such that under the isomorphism  $\sigma_\phi$  of (15)

$$\sigma_\phi^*(f_{Q,\phi}) = \alpha_\phi g_{Q,\phi}.$$

Then by Proposition 5.4 we see that the arithmetic class

$$\chi(\mathrm{R}\Gamma(\mathcal{X}, \mathcal{G}^\bullet), g_{Q\bullet}) \cdot \chi(\mathrm{R}\Gamma(\mathcal{Z}, \mathcal{F}), f_{Q\bullet})^{-1}$$

is represented by the homomorphism  $1 \times \alpha^{-1} \in \text{Hom}_{\Omega_{\mathbf{Q}}}(R_G^s, J_f) \times \text{Hom}(R_G^s, \mathbf{R}_{>0})$  which maps the character  $\phi$  to  $1 \times \alpha_{\phi}^{-1/\phi(1)}$  (so that of course  $\alpha(\phi) = \alpha_{\phi}^{1/\phi(1)}$ ).

**THEOREM 6.3.** – *With the above notation and hypotheses there is a positive real number  $b$  such that for each  $\phi \in \widehat{G}$ ,  $\alpha_{\phi} = b^{\phi(1)^2}$  and so*

$$\tilde{\chi}(\text{R}\Gamma(\mathcal{X}, \mathcal{G}^{\bullet}), g_{Q^{\bullet}}) = \tilde{\chi}(\text{R}\Gamma(\mathcal{Z}, \mathcal{F}), f_{Q^{\bullet}}).$$

*Proof.* – In Theorem 0.1 in [2] Bismut considers the central function  $\tau$

$$\tau = \sum_{\phi \in \widehat{G}} 2 \log(\alpha_{\phi}) \phi(1)^{-1} \phi$$

and shows that  $\tau(g)$  may be evaluated in terms of integrals over the fixed points of  $g$ . As in the proof of 6.2 we deduce that  $\tau(g) = 0$  whenever  $g \neq 1_G$ . This then shows that  $\tau$  is again a scalar multiple of the regular character and the result follows.  $\square$

We now interpret the above results in terms of arithmetic classes.

**PROPOSITION 6.4.** – *Let  $(\mathcal{F}_j, f_j)$  for  $j = 1, \dots, n$  and  $(\mathcal{G}_k, g_k)$  for  $k = 1, \dots, m$  be hermitian bundles on closed  $G$ -subschemes  $i_j: \mathcal{Z}_j \rightarrow \mathcal{X}$ ,  $i_k: \mathcal{W}_k \rightarrow \mathcal{X}$  such that*

$$\sum_j [i_{j*} \mathcal{F}_j] = \sum_k [i_{k*} \mathcal{G}_k] \quad \text{in } K_0(G, \mathcal{X}).$$

*Then there is an equality of classes in  $A(\mathbf{Z}[G])$*

$$\prod_j \tilde{\chi}(\text{R}\Gamma(\mathcal{Z}_j, \mathcal{F}_j), f_{j, Q^{\bullet}}) = \prod_k \tilde{\chi}(\text{R}\Gamma(\mathcal{W}_k, \mathcal{G}_k), g_{k, Q^{\bullet}}).$$

*Proof.* – We first choose resolutions by locally free  $G$ - $\mathcal{X}$  sheaves

$$\mathcal{A}_j^{\bullet} \rightarrow i_{j*} \mathcal{F}_j, \quad \mathcal{B}_k^{\bullet} \rightarrow i_{k*} \mathcal{G}_k.$$

From the definition of  $K_0(G, \mathcal{X})$ , we can find locally free  $G$ - $\mathcal{X}$  sheaves  $D_{a,b}, E_{c,d}$  and an isomorphism, which we henceforth treat as an equality,

$$\begin{aligned} & \bigoplus_b D_{2,b} \bigoplus_d E_{1,d} \bigoplus_i E_{3,d} \bigoplus_{j, a \text{ even}} \mathcal{A}_j^a \bigoplus_{k, b \text{ odd}} \mathcal{B}_k^b \\ (16) \quad & = \bigoplus_d E_{2,d} \bigoplus_b D_{1,b} \bigoplus_j D_{3,b} \bigoplus_{j, a \text{ odd}} \mathcal{A}_j^a \bigoplus_{k, b \text{ even}} \mathcal{B}_k^b \end{aligned}$$

where the  $G$ - $\mathcal{X}$  sheaves  $D_{a,b}, E_{c,d}$  fit into exact sequences

$$0 \rightarrow E_{1,d} \rightarrow E_{2,d} \rightarrow E_{3,d} \rightarrow 0$$

$$0 \rightarrow D_{1,b} \rightarrow D_{2,b} \rightarrow D_{3,b} \rightarrow 0.$$

We then endow the sheaves  $E_{3,d}$  and  $D_{3,b}$  with arbitrary  $G$ -invariant metrics  $\xi_{3,d}$  and  $\eta_{3,b}$ ; we then choose  $G$ -invariant metrics  $\xi_{1,d}, \xi_{2,d}, \eta_{1,b}, \eta_{2,b}$  on  $E_{1,d}, E_{2,d}, D_{1,b}, D_{2,b}$  satisfying

Condition A as above, so that by Theorem 6.3:

$$\tilde{\chi}(\mathrm{R}\Gamma D_{1,b}, \xi_{1,d,Q}) \cdot \tilde{\chi}(\mathrm{R}\Gamma D_{3,b}, \xi_{3,d,Q}) = \tilde{\chi}(\mathrm{R}\Gamma D_{2,b}, \xi_{2,d,Q}) \quad \text{etc.}$$

We then endow the sheaves  $\mathcal{A}_j^a, \mathcal{B}_k^b$  with  $G$ -invariant metrics  $\alpha_j^a, \beta_k^b$  satisfying Condition A, so that by Theorem 6.3

$$\tilde{\chi}(\mathrm{R}\Gamma \mathcal{A}_j^\bullet, \alpha_{j,Q^\bullet}) = \tilde{\chi}(\mathrm{R}\Gamma(i_{j*} \mathcal{F}_j), f_{j,Q^\bullet}) \quad \text{etc.}$$

The desired equality then follows from (16) and Theorem 6.2.  $\square$

### 7. Logarithmic differentials

In this section we consider the Arakelov–Euler characteristic associated to the logarithmic de Rham complex of an arithmetic variety  $\mathcal{X}$  with fibral dimension  $d$ . We begin by relating this class to an arithmetic class associated to the top Chern class of the logarithmic differentials of  $\mathcal{X}$ . After allowing for various innocuous base field extensions, we shall use the moving techniques of [12] to express this top Chern class as a difference of two horizontal 1-cycles together with a relatively innocuous fibral term. We shall then be able to use the results of Section 5.2 to show that the arithmetic class associated to the logarithmic de Rham complex of  $\mathcal{X}$  has the remarkable property of characterising symplectic  $\varepsilon_0$ -constants of  $\mathcal{X}$ .

Recall that in § 6.A we have fixed a Kähler metric  $h_X$  on the tangent bundle of  $X = \mathcal{X}(\mathbf{C})$  and metrics  $\wedge^\bullet h^D$  on  $\wedge^\bullet \Omega_X$ .

In this section we again suppose that  $\mathcal{X}$  and  $\mathcal{Y}$  satisfy hypotheses (T1) and (T2). Let  $S$  denote a finite set of prime numbers which contains all the primes which support the branch locus, together with all primes  $p$  where the fibre  $\mathcal{Y}_p$  fails to be smooth. We put  $S' = S \cup \{\infty\}$ .

Let  $\chi(\mathcal{Y}_{\mathbf{Q}}) = \chi(\mathcal{Y}(\mathbf{C}))$  denote the Euler characteristic of the generic fibre of  $\mathcal{Y}$ . Note that in all cases  $d \cdot \chi(\mathcal{Y}_{\mathbf{Q}})$  is an even integer, so that we may define  $\xi_S: R_G \rightarrow \mathbf{Q}^\times$  by the rule

$$\xi_S(\phi) = \prod_{p \in S} p^{\phi(1) \cdot d \cdot \chi(\mathcal{Y}_{\mathbf{Q}})/2}.$$

Let  $\Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\mathrm{red}} / \log S)$  denote the sheaf of degree one relative logarithmic differentials with respect to the morphism  $(\mathcal{Y}, \mathcal{Y}_S^{\mathrm{red}}) \rightarrow (\mathrm{Spec}(\mathbf{Z}), S)$  of schemes with log-structures (see [26]). Under our hypotheses (T1) and (T2)  $\Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\mathrm{red}} / \log S)$  is a locally free  $\mathcal{Y}$ -sheaf of rank  $d$ , and furthermore the cover  $\mathcal{X}/\mathcal{Y}$  is log-étale, so that

$$(17) \quad \Omega_{\mathcal{X}/\mathbf{Z}}^1(\log \mathcal{X}_S^{\mathrm{red}} / \log S) = \pi^* \Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\mathrm{red}} / \log S).$$

The main goal of this section is the study of the arithmetic class (see Section 6.1)

$$\begin{aligned} c &= \chi(\mathrm{R}\Gamma(\wedge^\bullet \Omega_{\mathcal{X}/\mathbf{Z}}^1(\log \mathcal{X}_S^{\mathrm{red}} / \log S), \wedge^\bullet h_Q^D)) \\ &= \prod_{i=0}^d \chi(\mathrm{R}\Gamma(\wedge^i \Omega_{\mathcal{X}/\mathbf{Z}}^1(\log \mathcal{X}_S^{\mathrm{red}} / \log S), \wedge^i h_Q^D))^{(-1)^i}. \end{aligned}$$

To explain our main result we need to introduce some notation on  $\varepsilon_0$ -constants. For a more detailed account see Section 4 in [9] and Section 2 and Section 5 in [8]. For a given prime number  $p$ , we choose a prime number  $l = l_p$  which is different from  $p$  and we

fix a field embedding  $\mathbf{Q}_l \rightarrow \mathbf{C}$ ; then, following the procedure of Section 8 in [15], each of the étale cohomology groups  $H_{\text{ét}}^i(\mathcal{X} \times \mathbf{Q}_p, \mathbf{Q}_l)$  for  $0 \leq i \leq 2d$ , affords a continuous complex representation of the local Weil–Deligne group. Thus, after choosing both an additive character  $\psi_p$  of  $\mathbf{Q}_p$  and a Haar measure  $dx_p$  of  $\mathbf{Q}_p$ , for each complex character  $\theta$  of  $G$  the complex number  $\varepsilon_{0,p}(\mathcal{Y}, \theta, \psi_p, dx_p, l_p)$  is defined. (For a representation  $V$  of  $G$  with character  $\theta$  this term was denoted  $\varepsilon_{p,0}(X \otimes_G V, \psi_p, dx_p, l)$  in 2.4 of [8].) Setting

$$\tilde{\varepsilon}_{0,p}(\mathcal{Y}, \theta, \psi_p, dx_p, l_p) = \varepsilon_{0,p}(\mathcal{Y}, \theta - \theta(1) \cdot 1, \psi_p, dx_p, l_p),$$

by Corollary 1 to Theorem 1 in [9] we know that when  $\theta$  is symplectic,  $\tilde{\varepsilon}_{0,p}(\mathcal{Y}, \theta, \psi_p, dx_p, l_p)$  is a non-zero rational number, which is independent of choices, and  $\theta \mapsto \tilde{\varepsilon}_{0,p}(\mathcal{Y}, \theta)$  defines an element

$$\tilde{\varepsilon}_{0,p}^s(\mathcal{Y}) \in \text{Hom}_{\Omega}(R_G^s, \mathbf{Q}^{\times}).$$

In the case where  $\mathcal{X}$  is the spectrum of a ring of integers  $\mathcal{O}_N$  of a number field  $N$  and  $K = N^G$ , we shall write  $\varepsilon_{0,p}(K)$  for  $\varepsilon_{0,p}(\mathcal{Y})$ .

Analogously, for the Archimedean prime  $\infty$  of  $\mathbf{Q}$ , Deligne provides a definition for  $\varepsilon_{\infty}(\mathcal{Y})$  and from 5.5.2 and 5.4.1 in [8] we recall that

$$\tilde{\varepsilon}_{\infty}^s(\mathcal{Y}) \in \text{Hom}_{\Omega}(R_G^s, \pm 1).$$

For  $\phi \in R_G^s$  almost all  $\tilde{\varepsilon}_{0,v}^s(\mathcal{Y}, \phi)$  are equal to 1; the global  $\tilde{\varepsilon}_0$ -constant of  $\phi$  is

$$\tilde{\varepsilon}_0^s(\mathcal{Y}, \phi) = \prod_v \tilde{\varepsilon}_{0,v}^s(\mathcal{Y}, \phi)$$

and we define

$$\varepsilon_{0,S}^s(\mathcal{Y}, \phi) = \tilde{\varepsilon}_0^s(\mathcal{Y}, \phi) \prod_{v \in S'} \varepsilon_{0,v}(\mathcal{Y}, \phi(1)).$$

The main result of this section is (we always assume that hypotheses (T1) and (T2) are satisfied):

**THEOREM 7.1.** – *The arithmetic class  $\mathfrak{c}^s$  lies in the group of rational classes  $R^s(\mathbf{Z}[G])$  and*

$$\theta(\mathfrak{c}^s) = \xi_S^s \cdot \varepsilon_{0,S}^s(\mathcal{Y})^{-1}.$$

By way of preparation for the proof of Theorem 7.1, we shall initially work with an arbitrary locally free  $\mathcal{Y}$ -sheaf  $\mathcal{E}$ ; only towards the end of the section shall we need to specialise to the case where  $\mathcal{E} = \Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}} / \log S)$ . Throughout this section we adopt the notation and hypotheses of [12]. For  $i \geq 0$  let  $c^i(\mathcal{E}) = \gamma^i(\mathcal{E} - \text{rk}(\mathcal{E}))$  which lies in  $F^i K_0(\mathcal{Y})$ , the  $i$ th component of the Grothendieck  $\gamma$ -filtration. We define  $\underline{c}^i(\mathcal{E})$  to be the class

$$\underline{c}^i(\mathcal{E}) \equiv c^i(\mathcal{E}) \text{ mod } F^{i+1} K_0(\mathcal{Y}).$$

**LEMMA 7.2.** – *Let  $\mathcal{E}$  be as above, let  $\mathcal{L}$  denote an arbitrary line bundle on  $\mathcal{Y}$  and suppose that  $n_0$  is a given negative integer. Then there exist an integer  $n_1 \leq n_0$  and integers  $l_n$  for  $n_1 \leq n \leq n_0$ , which depend only on  $\text{rk}(\mathcal{E})$ , such that for all  $i \geq 0$*

$$(18) \quad \underline{c}^i(\mathcal{E}) \equiv \sum_{n=n_1}^{n_0} l_n \underline{c}^i(\mathcal{E} \otimes \mathcal{L}^n) \text{ mod } F^{i+1} K_0(\mathcal{Y}).$$

*Proof.* – This is Lemma 5.3 in [12] with  $\mathcal{L}$  replaced by  $\mathcal{L}^{-1}$ .  $\square$

DEFINITION 7.3. – Let  $m$  be a given positive integer. We shall call a finite Galois extension  $M$  of  $\mathbf{Q}$  *harmless for  $m$* , if  $M/\mathbf{Q}$  is non-ramified at  $S$  and if the extension degree  $[M : \mathbf{Q}]$  is congruent to 1 mod  $m$ .

*Remark.* – From Lemma 9.1.2 in [8] we know that we can construct harmless for  $m$  extensions whose residue class fields over  $S$  are arbitrarily large.

If  $m$  is a positive integer and if  $M$  is harmless for  $m$ , let  $e : \text{Spec}(\mathcal{O}_M) \rightarrow \text{Spec}(\mathbf{Z})$  be the structure morphism, write  $\mathcal{Y}'$  for the base extension  $\mathcal{Y} \times_{\mathbf{Z}} \mathcal{O}_M$ , and  $\mathcal{E}'$  for the pullback of  $\mathcal{E}$  to  $\mathcal{Y}'$ .

Suppose now that an integer  $m$  is given and that  $\mathcal{E}$  has rank  $d$ ; for an integer  $n$  we put  $\mathcal{E}(n) = \mathcal{E} \otimes_{\mathcal{O}_{\mathcal{Y}}}(n)$ . From 5.1 in [12] we know that we can find a negative integer  $n_0$  with the following property: let  $n_1$  be an integer chosen as in Lemma 7.2 with respect to the negative integer  $n_0$ ; then for each  $n$ ,  $n_1 \leq n \leq n_0$ , there is an open subset  $U_n$  of  $\text{Spec}(\mathbf{Z})$ , which contains  $S$ , an extension  $M$  which is harmless for  $m$ , and a (possibly non-effective) 1-cycle  $D'_n$  on  $\mathcal{Y}'$  whose irreducible components are horizontal and meet  $b$  transversely and at points which are smooth points of both  $D'_n$  and  $b$ , such that

$$c^d(\mathcal{E}'(n))|_{U_n} = [O_{D'_n \times U_n}] + T_n \quad \text{in } K_0(\mathcal{Y}' \times U_n)$$

where  $T_n$  is supported on closed points. With the notation of Lemma 7.2, we set  $U = \bigcap_{n=n_1}^{n_0} U_n$ , so that for all  $n$ ,  $n_1 \leq n \leq n_0$  we have

$$c^d(\mathcal{E}'(n))|_U = [O_{D'_n \times U}] + T_n|_U \quad \text{in } K_0(\mathcal{Y}' \times U).$$

Pushing forward by  $e$  and using the fact that  $\mathcal{O}_M$  is free over  $\mathbf{Z}$ , we get

$$(19) \quad [M : \mathbf{Q}] \cdot c^d(\mathcal{E}(n))|_U = [O_{e_* D'_n \times U}] + e_*(T_n|_U) \quad \text{in } K_0(\mathcal{Y} \times U).$$

In the sequel we work with a chosen such extension  $M$ . We write  $\widehat{\pi^* D'_n}$  for the normalisation of  $\pi^* D'_n$  and we endow both  $e_* \pi^* D'_n$  and  $e_* \widehat{\pi^* D'_n}$  with the Hecke form  $h_n$  of Section 6.2; we denote their arithmetic classes by  $\chi(e_* \pi^* D'_n, \det h_n \bullet)$  and  $\tilde{\chi}(e_* \widehat{\pi^* D'_n}, \det h_n \bullet)$ . Proposition 5.7, together with Lemma 7.4 below, shows that the two resulting classes coincide after passage to degree zero

$$(20) \quad \tilde{\chi}(e_* \pi^* D'_n, \det h_n \bullet) = \tilde{\chi}(e_* \widehat{\pi^* D'_n}, \det h_n \bullet).$$

LEMMA 7.4. – *Suppose that  $\mathcal{F}$  is a coherent  $\mathcal{Y}$ -sheaf which is supported on a single prime  $p$  and suppose: either that  $p \notin S$ ; or that, if  $p \in S$ , then  $\mathcal{F}$  is supported over a finite number of points of  $\mathcal{Y}$ . Then  $f_{p*}(\pi^* \mathcal{F})$  is a free class in  $K_0(\mathbf{F}_p[G])$ .*

*Proof.* – Let  $h : \mathcal{Y} \rightarrow \text{Spec}(\mathbf{Z})$  denote the structure morphism of  $\mathcal{Y}$  and suppose first that  $\mathcal{F}$  is the coherent  $\mathcal{Y}$ -sheaf given by the structure sheaf of a closed point of  $\mathcal{Y}$ . As  $f_* = h_* \pi_*$  and  $\pi_* \pi^* \mathcal{F} = \mathcal{F} \otimes_{\mathcal{O}_{\mathcal{Y}}} \pi_* \mathcal{O}_{\mathcal{X}}$  in  $G_0(G, \mathcal{Y})$ , the result follows readily from the normal basis theorem.

Suppose now that  $p \notin S$ . For a  $p$ -regular element  $g \in G$ ,  $g \neq 1$ ,  $\mathcal{X}_p^g = \emptyset$ , since  $G$  acts freely away from  $S$ . Thus by the Lefschetz–Riemann–Roch theorem, we know that the Brauer trace of  $g$  on  $f_{p*}(\mathcal{F})$  is zero; hence we may conclude that  $f_{p*}(\mathcal{F})$  is a free class.  $\square$

Recall that  $\tilde{c}$  denotes the arithmetic class obtained from  $c$  by passage to degree zero, as per Section 4.5. As an intermediate step towards proving Theorem 7.1, we first show that the result holds in degree zero:

THEOREM 7.5. – *The arithmetic class  $\tilde{c}^s$  lies in the group of rational classes  $R^s(\mathbf{Z}[G])$  and*

$$\theta(\tilde{c}^s) = \tilde{\varepsilon}_0^s(\mathcal{Y})^{-1}.$$

*Proof.* – We apply the above work where we now take  $\mathcal{E} = \Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}} / \log S)$  and where we take  $n_0$  sufficiently small and negative to guarantee that  $\mathcal{E}^D(-n)$  has a regular section for all  $n \leq n_0$ . Recall that  $\pi^* \mathcal{E}$  is endowed with the metric  $h^D$ , the dual of the Kähler metric; we endow  $\pi^* \mathcal{O}_{\mathcal{Y}}(n)$  with a chosen  $G$ -invariant metric  $\nu_n$ .

By 7.2 together with 7.4 and Proposition 6.4, we know that

$$(21) \quad \tilde{\chi}(\text{R}\Gamma(\wedge^\bullet \pi^* \mathcal{E}), (\wedge^\bullet h^D)_Q) = \prod_{n=n_0}^{n_1} \tilde{\chi}(\text{R}\Gamma(\wedge^\bullet \pi^* \mathcal{E}(n)), (\wedge^\bullet h^D \otimes \nu_n)_Q)^{l_n}.$$

Let  $\mathcal{W}_n$  denote the closed one dimensional sub-scheme of  $\mathcal{Y}$  cut out by the regular section of  $\mathcal{E}^D(-n)$  and put  $\mathcal{Z}_n = \pi^* \mathcal{W}_n$ ; so that we have the Koszul quasi-isomorphism

$$\wedge^\bullet \mathcal{E}(n) \rightarrow \mathcal{O}_{\mathcal{W}_n}.$$

By 6.4 we know that

$$(22) \quad \tilde{\chi}(\text{R}\Gamma(\wedge^\bullet \pi^* \mathcal{E}(n)), (\wedge^\bullet h^D \otimes \nu_n)_Q) = \tilde{\chi}(\text{R}\Gamma \mathcal{O}_{\mathcal{Z}_n}, j_n)$$

where  $j_n$  denotes the Hecke metric on  $\mathcal{O}_{\mathcal{Z}_n}$ . Next observe that  $[\wedge^\bullet \pi^* \mathcal{E}(n)] = (-1)^d c^d(\pi^* \mathcal{E}(n))$ , and also by 7.4 we know that  $\tilde{\chi}(e_* \pi^* T_n, \pi^* | - |) = 0$ ; hence by Proposition 6.4, together with (19)–(22), we may conclude that

$$(23) \quad \begin{aligned} \prod_{n=n_1}^{n_0} \tilde{\chi}(\text{R}\Gamma \mathcal{O}_{\mathcal{Z}_n}, j_n)^{l_n [M:\mathbf{Q}]} &= \prod_{n=n_1}^{n_0} \tilde{\chi}(e_* \pi^* D'_n, \det h_{n_\bullet})^{(-1)^d l_n} \\ &= \prod_{n=n_1}^{n_0} \tilde{\chi}(e_* \widehat{\pi^* D'_n}, \det h_{n_\bullet})^{(-1)^d l_n}. \end{aligned}$$

Let  $\mathcal{C} = (-1)^d c^d(\mathcal{E})$  and consider the restriction of  $\mathcal{C}$  to an irreducible component  $b_i$  of  $b$  over  $p$ ; in this way we obtain a punctual virtual sheaf whose length we denote by  $n_i$ . If, for  $\psi \in R_G^s$ , we have  $\text{Pf}_p(\mathcal{X}, \psi) = \sum_i q_i b_i$  (see (13)), then we may define  $\deg(\mathcal{C} \cdot \text{Pf}(\mathcal{X}))(\psi) \in J_f$  to be the idele whose component at primes over  $p$  is  $(-p)^{\sum_i q_i n_i}$ . We then use this construction to define the symplectic arithmetic class  $\mathfrak{h} \in A_T^s(\mathbf{Z}[G])$  to be that class which is represented by the homomorphism

$$(\tilde{\varepsilon}_\infty^s(\mathcal{Y}) \cdot \deg(\mathcal{C} \cdot \text{Pf}(\mathcal{X}))) \times 1.$$

In fact, from Theorem 1 in [9], we know that  $\mathfrak{h}$  is a rational class and that moreover

$$(24) \quad \theta(\mathfrak{h}) = \tilde{\varepsilon}_\infty^s(\mathcal{Y}) \prod_{p < \infty} \widetilde{\varepsilon_{0,p}^s(\mathcal{Y})} = \tilde{\varepsilon}_0^s(\mathcal{Y}).$$

By Theorems 5.9 and 6.1, the left-hand arithmetic class in (23) above is represented by the character function given on characters of degree zero by



$$\begin{aligned}
 & \prod_{n=n_1}^{n_0} (\tilde{\varepsilon}_\infty^s(\widehat{\pi^* D'_n}) \deg(\widehat{\pi^* D'_n} \cdot \text{Pf}(\mathcal{X}')))^{-(-1)^d l_n} \times 1 \\
 (25) \quad & = \prod_{n=n_1}^{n_0} (\tilde{\varepsilon}_\infty^s(e_* \pi^* D'_n) \deg(e_* \pi^* D'_n \cdot \text{Pf}(\mathcal{X})))^{-(-1)^d l_n} \times 1.
 \end{aligned}$$

By (18) and (19) together with Theorem 5.5.2 in [8] we know that

$$\prod_{n=n_1}^{n_0} \tilde{\varepsilon}_\infty^s(e_* \pi^* D'_n)^{l_n} = \tilde{\varepsilon}_\infty^s(\mathcal{Y})^{[M:\mathbf{Q}]}.$$

Again by (18) and (19)

$$(26) \quad \prod_{n=n_1}^{n_0} (\deg(e_* \pi^* D'_n \cdot \text{Pf}(\mathcal{X})))^{l_n} = \deg((-1)^d \mathcal{C} \cdot \text{Pf}(\mathcal{X}))^{[M:\mathbf{Q}]}.$$

By (21)–(23) we know that

$$\tilde{\chi}(\text{R}\Gamma(\wedge^\bullet \pi^* \mathcal{E}), (\wedge^\bullet h^D)_Q)^{[M:\mathbf{Q}]} = \prod_{n=n_1}^{n_0} \tilde{\chi}(e_* \pi^* \widehat{D}'_n, \det h_{n\bullet})^{(-1)^d l_n}$$

and by the above work the right-hand class is represented by the same homomorphism as  $\mathfrak{h}^{-[M:\mathbf{Q}]}$ . Thus, by varying  $M$ , we see that

$$\tilde{\chi}(\text{R}\Gamma(\wedge^\bullet \pi^* \mathcal{E}), (\wedge^\bullet h^D)_Q) = \mathfrak{h}^{-1}$$

and so by (24)

$$\theta(\tilde{\chi}(\text{R}\Gamma(\wedge^\bullet \pi^* \mathcal{E}), (\wedge^\bullet h^D)_Q)) = \tilde{\varepsilon}_0^s(\mathcal{Y})^{-1}. \quad \square$$

Before embarking on the proof of Theorem 7.1, we first need a number of preliminary results.

LEMMA 7.6. – (a) *For a coherent  $G$ - $\mathcal{X}$  sheaf  $\mathcal{F}$  there is a quasi-isomorphism of complexes of abelian groups*

$$(\text{R}\Gamma \mathcal{F})^G \cong \text{R}\Gamma(\mathcal{F}^G).$$

(b) *If  $(\mathcal{F}, f)$  is a hermitian  $G$ -bundle on  $\mathcal{X}$ , then there is an equality in  $A(\mathbf{Z}[G])$*

$$\chi((\text{R}\Gamma \mathcal{F})^G, f_{Q,1}) = \chi(\text{R}\Gamma(\mathcal{F}^G), (f^G)_Q).$$

*Proof.* – Part (a) follows at once on expressing  $\text{R}\Gamma \mathcal{F}$  and  $\text{R}\Gamma(\mathcal{F}^G)$  in terms of Čech complexes for a given affine cover of  $\mathcal{Y}$  (which pulls back to an affine cover of  $\mathcal{X}$ , since  $\mathcal{X}/\mathcal{Y}$  is finite) and then taking  $G$ -invariants of the first complex. Part (b) is then immediate since  $f_{Q,1}$  (the Quillen metric for the trivial character) is constructed by forming the Quillen metric associated to the restriction of  $f$  to the trivial isotypical component of  $\mathcal{F}_\mathbf{C}$ , namely  $(\mathcal{F}_\mathbf{C})^G$ . See II.a in [2] for further details.  $\square$

Next we note the following elementary result from 3.2:

LEMMA 7.7. – *When  $G$  is the trivial group, then there is an isomorphism  $\gamma: A(\mathbf{Z}) \rightarrow \mathbf{R}_{>0}$ , (which coincides with the degree map on p. 162 of [34]). Furthermore, if a class  $\mathfrak{e} \in A(\mathbf{Z})$  has  $\gamma(\mathfrak{e})^2 \in \mathbf{Q}_{>0}$ , then the symplectic class  $\mathfrak{e}^s$  is a rational class and  $\theta(\mathfrak{e}^s) = \gamma(\mathfrak{e})^2$ .*

*Proof.* – For a rational finite idele  $j \in J_{\mathbf{Q},f}$  we write  $c(j)$  for the positive rational number which generates the fractional  $\mathbf{Z}$ -ideal given by the content of  $j$ . The first part of the lemma then follows from 3.2 on noting that the map from  $J_{\mathbf{Q},f} \times \mathbf{R}_{>0} \rightarrow \mathbf{R}_{>0}$  given by mapping  $(j, r) \mapsto c(j)r^{-1}$  has kernel  $(\widehat{\mathbf{Z}}^\times \times 1) \cdot \Delta(\mathbf{Q}^\times)$ . To show the second part of the lemma we first note that the class  $\mathfrak{e}$  is represented by  $1 \times \gamma(\mathfrak{e})^{-1}$ ; and that the symplectic characters of the trivial group are the even multiples of the trivial character. It therefore follows that  $\mathfrak{e}^s$  is represented by  $1 \times \gamma(\mathfrak{e})^{-2}$ , which has the same class in  $A_T^s(\mathbf{Z})$  as  $\gamma(\mathfrak{e})^2 \times 1$ .  $\square$

We denote by  $\mathcal{Y}_S^{\text{red}}$  the disjoint union of the reduced fibres of  $\mathcal{Y}$  over  $p \in S$ . Let  $\mathcal{Y}_i$ , for  $i \in \mathcal{I}$ , denote the irreducible components of  $\mathcal{Y}_S^{\text{red}}$ , so that

$$\mathcal{Y}_S^{\text{red}} = \bigcup_{i \in \mathcal{I}} \mathcal{Y}_i.$$

Let  $p_i$  denote the prime which supports  $\mathcal{Y}_i$  and let  $\chi_c(\mathcal{Y}_i^*)$  denote the  $\ell$ -adic Euler characteristic with compact supports of  $\mathcal{Y}_i^* = \mathcal{Y}_i - \bigcup_{j \neq i} \mathcal{Y}_j$ , the non-singular part of  $\mathcal{Y}_i$ .

Thanks to Theorem 7.5, in order to prove Theorem 7.1, we need only show that, with the notation of 4.2,  $\mathfrak{c}_0^s$  is a rational class and that

$$\theta(\mathfrak{c}_0^s) = \xi_S(2 \cdot 1_G) \prod_{v \in S'} \varepsilon_{0,v}(\mathcal{Y}, 2 \cdot 1_G)^{-1}.$$

Therefore, by 7.7, it will suffice to show that

$$\gamma(\mathfrak{c}_0)^2 = \xi_S(2 \cdot 1_G) \prod_{v \in S'} \varepsilon_{0,v}(\mathcal{Y}, 2 \cdot 1_G)^{-1} \in \mathbf{Q}_{>0}.$$

From (17), we know that for all non-negative  $j$

$$\left( \pi_* \bigwedge^j \Omega_{\mathcal{X}/\mathbf{Z}}^1(\log \mathcal{X}_S^{\text{red}} / \log S) \right)^G = \bigwedge^j \Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}} / \log S)$$

hence

$$\mathfrak{c}_0 = \prod_{j=0}^d \chi(\text{R}\Gamma \wedge^j \Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}} / \log S), \wedge^j h_{Q,1}^D)^{(-1)^j}$$

which we write more succinctly as

$$\mathfrak{c}_0 = \chi(\text{R}\Gamma \wedge^\bullet \Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}} / \log S), \wedge^\bullet h_{Q,1}^D).$$

We therefore see that it is enough to show the following two results (always under hypotheses (T1) and (T2)):

THEOREM 7.8. –

$$\gamma \circ \chi(\text{R}\Gamma \wedge^\bullet \Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}} / \log S), \wedge^\bullet h_{Q,1}^D)^2 = \prod_{i \in \mathcal{I}} p_i^{-(m_i-1)\chi_c(\mathcal{Y}_i^*)}$$

for any Kähler metric  $j$  on the complex tangent bundle  $TY$ .

*Remark 1.* – In fact this result can also easily be proved using the arithmetic Riemann–Roch Theorem of Gillet and Soulé; this alternative approach to the calculation of Arakelov–Euler characteristics is explored in [13]; here, however, we shall provide a direct proof, which is due to Bismut and which was shown to us by C. Soulé.

*Remark 2.* – Note that the theorem shows that the lefthand Arakelov–Euler characteristic is in fact independent of the chosen Kähler metric  $h$ . In the proof of the theorem we shall see that the fact that the metric on the determinant of cohomology is independent of choices comes down to two key-points: firstly, by a theorem of Ray–Singer the analytic torsion associated to the full de Rham complex is zero for any Kähler metric  $h$ ; secondly the  $L^2$ -metric associated to  $h$  is compatible with Serre duality, which is of course independent of choices.

THEOREM 7.9. –

$$\varepsilon_{0,S}(\mathcal{Y}, 2 \cdot 1_G) = \xi_S(2 \cdot 1_G) \prod_{i \in \mathcal{I}} p_i^{(m_i-1)\chi_c(\mathcal{Y}_i^*)} \in \mathbf{Q}_{>0}.$$

We begin by proving Theorem 7.9. For a place  $v$  of  $\mathbf{Q}$  we calculate the  $\varepsilon_{0,v}$ -constants with respect to the standard Haar measures  $dx_v$  of  $\mathbf{Z}_v$  and with respect to the Tate–Iwasawa additive character  $\psi_v$  of  $\mathbf{Q}_v$  (see [35] pp. 316–319).

We first consider the case of a finite prime  $p$ . From Theorem 2 in [33] we know that

$$\varepsilon_{0,p}(2 \cdot 1_G, \mathcal{Y}, \psi_p \circ p^{-1}, p dx_p) = \pm \prod_{p_i=p} p^{(m_i-1)\chi_c^*(\mathcal{Y}_i)}.$$

Thus by the standard transformation formulae for  $\varepsilon$ -constants (see 5.3 and 5.4 in [15])

$$\varepsilon_{0,p}(2 \cdot 1_G, \mathcal{Y}, \psi_p, dx_p) = \pm \sigma^2(p) \prod_{p_i=p} p_i^{(m_i-1)\chi_c^*(\mathcal{Y}_i)}$$

where  $\sigma$  denotes the determinant of the motive of  $\mathcal{X} \otimes_G V$  and where  $V$  denotes the trivial representation of  $G$ . From Proposition 2.2.1.a, c in [8] we know that  $\sigma^2(p) = p^{d\chi(Y_{\mathbf{Q}})}$  since  $p$  corresponds to a geometric Frobenius in [15, p. 523].

Next, we consider the archimedean prime  $v = \infty$ . From Lemma 5.1.1 in [8], we know that  $\varepsilon_{0,\infty}(2 \cdot 1_G, \mathcal{Y}, -\psi_{\infty}, dx_{\infty}) = \pm 1$ .

To show that  $\varepsilon_{0,S}(\mathcal{Y}, 2 \cdot 1_G)$  is positive, note that from (2.2) in 2.4 of [8] we know that in all cases

$$\text{sign}(\varepsilon_{0,v}(\mathcal{Y}, 2 \cdot 1_G)) = \det(\sigma)(-1_v).$$

Thus by global reciprocity  $1 = \prod_{v \in S'} \det(\sigma)(-1_v)$  and so we have indeed now shown that  $\varepsilon_{0,S}(\mathcal{Y}, 2 \cdot 1_G)$  is a positive rational number.

Prior to proving Theorem 7.8, we note that we have:

LEMMA 7.10. – *Writing  $\omega_{\mathcal{Y}/\mathbf{Z}}$  for the canonical sheaf of  $\mathcal{Y}/\mathbf{Z}$ , there is a natural isomorphism between  $\wedge^d \Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}} / \log S)$  and  $\omega_{\mathcal{Y}/\mathbf{Z}}(\mathcal{Y}_S^{\text{red}} - \mathcal{Y}_S)$ .*

*Proof.* – Recall that  $\{\mathcal{Y}_i\}_{i \in \mathcal{I}}$  denote the irreducible components of the disjoint union of the special fibres  $\mathcal{Y}_S^{\text{red}}$ . From Proposition 3.1 in [13] we know that the natural morphism

$$\omega : \Omega_{\mathcal{Y}/\mathbf{Z}} \rightarrow \Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}} / \log S)$$

has the same kernel and cokernel as the natural map

$$a: \bigoplus_{p \in S} \mathcal{O}_y/p\mathcal{O}_y \rightarrow \bigoplus_{i \in \mathcal{I}} \mathcal{O}_{\mathcal{Y}_i}.$$

The result then follows on taking determinants, since  $\omega_{\mathcal{Y}/\mathbf{Z}} \cong \det \Omega_{\mathcal{Y}/\mathbf{Z}}$ .  $\square$

*Proof of Theorem 7.8.* – For brevity we regard the isomorphic degree map  $\gamma$  of 7.7 as an identification and we again put  $\mathcal{E} = \Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}}/\log S)$ .

For  $0 \leq n \leq d$ , the Duality Theorem in III. 11 of [23] gives a quasi-isomorphism of complexes

$$\text{R}\Gamma(\text{Hom}_{\mathcal{O}_y}(\wedge^n \mathcal{E}, \omega_{\mathcal{Y}/\mathbf{Z}}[d])) \cong \text{RHom}_{\mathbf{Z}}(\text{R}\Gamma(\wedge^n \mathcal{E}), \mathbf{Z}).$$

Because we have normalised the Kähler form as in Section 6.1, the associated  $L^2$ -norm is compatible with Serre duality (see 1.4 of [22]), and so the induced isomorphisms on complex cohomology are isometries when the complex cohomology groups are endowed with their  $L^2$ -metrics. Thus we see that

$$(27) \quad \begin{aligned} \chi_{L^2}(\text{R}\Gamma(\text{Hom}_{\mathcal{O}_y}(\wedge^n \mathcal{E}, \omega_{\mathcal{Y}/\mathbf{Z}}[d]))) &= \chi_{L^2}(\text{RHom}_{\mathbf{Z}}(\text{R}\Gamma \wedge^n \mathcal{E}, \mathbf{Z})) \\ &= \chi_{L^2}(\text{R}\Gamma \wedge^n \mathcal{E})^{-1}, \end{aligned}$$

where for brevity we write  $\chi_{L^2}(\text{R}\Gamma \wedge^n \mathcal{E})$  in place of  $\chi(\text{R}\Gamma \wedge^n \mathcal{E}, \|\cdot\|_{L^2})$ .

Next we observe that by Lemma 7.10, we know that

$$\text{Hom}_{\mathcal{O}_y}(\wedge^n \mathcal{E}, \omega_{\mathcal{Y}/\mathbf{Z}}(\mathcal{Y}_p^{\text{red}} - \mathcal{Y}_p)) \cong \wedge^{d-n} \mathcal{E}.$$

Thus we obtain a quasi-isomorphism

$$\text{R}\Gamma(\wedge^{d-n} \mathcal{E}) \cong \text{R}\Gamma(\text{Hom}_{\mathcal{O}_y}(\wedge^n \mathcal{E}, \omega_{\mathcal{Y}/\mathbf{Z}}(\mathcal{Y}_S^{\text{red}} - \mathcal{Y}_S)))$$

and again the induced isomorphisms on complex cohomology are isometries with respect to their  $L^2$ -metrics. Thus we can write the number  $\chi_{L^2}(\text{R}\Gamma(\wedge^\bullet \mathcal{E}))^2$  as:

$$\prod_{n=0}^d \left[ \chi_{L^2}(\text{R}\Gamma(\wedge^n \mathcal{E}))^{(-1)^n} \cdot \chi_{L^2}(\text{R}\Gamma(\text{Hom}_{\mathcal{O}_y}(\wedge^n \mathcal{E}, \omega_{\mathcal{Y}/\mathbf{Z}}(\mathcal{Y}_S^{\text{red}} - \mathcal{Y}_S))))^{(-1)^{d-n}} \right].$$

But this latter product can be rewritten as  $\Pi_1 \cdot \Pi_2$  where  $\Pi_1$  respectively  $\Pi_2$  is the first, respectively second of the following expressions:

$$\begin{aligned} &\prod_{n=0}^d \left[ \chi_{L^2}(\text{R}\Gamma(\wedge^n \mathcal{E}))^{(-1)^n} \cdot \chi_{L^2}(\text{R}\Gamma(\text{Hom}_{\mathcal{O}_y}(\wedge^n \mathcal{E}, \omega_{\mathcal{Y}/\mathbf{Z}}[d])))^{(-1)^n} \right] \\ &\prod_{n=0}^d \left[ \chi_{L^2}(\text{R}\Gamma(\text{Hom}_{\mathcal{O}_y}(\wedge^n \mathcal{E}, \omega_{\mathcal{Y}/\mathbf{Z}})))^{-1} \right. \\ &\quad \left. \cdot \chi_{L^2}(\text{R}\Gamma(\text{Hom}_{\mathcal{O}_y}(\wedge^n \mathcal{E}, \omega_{\mathcal{Y}/\mathbf{Z}}(\mathcal{Y}_S^{\text{red}} - \mathcal{Y}_S))))^{(-1)^{d+n}} \right] \end{aligned}$$

and we note that (27) implies that  $\Pi_1 = 1$ . Hence we may conclude that  $\chi_{L^2}(\mathbb{R}\Gamma \wedge^\bullet \mathcal{E})^2$  is equal to  $\Pi_2$ . In order to evaluate  $\Pi_2$  we consider the exact sequences

$$0 \rightarrow \omega_{\mathcal{Y}/\mathbf{Z}}(\mathcal{Y}_S^{\text{red}} - \mathcal{Y}_S) \rightarrow \omega_{\mathcal{Y}/\mathbf{Z}}(\mathcal{Y}_S^{\text{red}}) \rightarrow \omega_{\mathcal{Y}/\mathbf{Z}}(\mathcal{Y}_S^{\text{red}})|_{\mathcal{Y}_S} \rightarrow 0,$$

$$0 \rightarrow \omega_{\mathcal{Y}/\mathbf{Z}} \rightarrow \omega_{\mathcal{Y}/\mathbf{Z}}(\mathcal{Y}_S^{\text{red}}) \rightarrow \omega_{\mathcal{Y}/\mathbf{Z}}(\mathcal{Y}_S^{\text{red}})|_{\mathcal{Y}_S^{\text{red}}} \rightarrow 0,$$

and we apply the exact functor  $\text{Hom}_{\mathcal{O}_Y}(\wedge^n \mathcal{E}, -)$  to get exact sequences

$$\begin{aligned} 0 \rightarrow \text{Hom}_{\mathcal{O}_Y}(\wedge^n \mathcal{E}, \omega_{\mathcal{Y}/\mathbf{Z}}(\mathcal{Y}_S^{\text{red}} - \mathcal{Y}_S)) &\rightarrow \text{Hom}_{\mathcal{O}_Y}(\wedge^n \mathcal{E}, \omega_{\mathcal{Y}/\mathbf{Z}}(\mathcal{Y}_S^{\text{red}})) \\ &\rightarrow \text{Hom}_{\mathcal{O}_Y}(\wedge^n \mathcal{E}, \omega_{\mathcal{Y}/\mathbf{Z}}(\mathcal{Y}_S^{\text{red}})|_{\mathcal{Y}_S}) \rightarrow 0 \end{aligned}$$

and

$$\begin{aligned} 0 \rightarrow \text{Hom}_{\mathcal{O}_Y}(\wedge^n \mathcal{E}, \omega_{\mathcal{Y}/\mathbf{Z}}) &\rightarrow \text{Hom}_{\mathcal{O}_Y}(\wedge^n \mathcal{E}, \omega_{\mathcal{Y}/\mathbf{Z}}(\mathcal{Y}_S^{\text{red}})) \\ &\rightarrow \text{Hom}_{\mathcal{O}_Y}(\wedge^n \mathcal{E}, \omega_{\mathcal{Y}/\mathbf{Z}}(\mathcal{Y}_S^{\text{red}})|_{\mathcal{Y}_S^{\text{red}}}) \rightarrow 0. \end{aligned}$$

Recall that  $h$  denotes the structure map  $h: \mathcal{Y} \rightarrow \text{Spec}(\mathbf{Z})$ ,  $m_i$  denotes the multiplicity of the component  $\mathcal{Y}_i$  in  $\mathcal{Y}_S$  and, as previously, for each  $i$  we let  $p_i$  denote the prime which supports  $\mathcal{Y}_i$ . For brevity we shall write  $\text{Hom}_{\mathcal{O}_Y}(\wedge^\bullet \mathcal{E}, \omega_{\mathcal{Y}/\mathbf{Z}})$  for  $\sum_n (-1)^n \text{Hom}_{\mathcal{O}_Y}(\wedge^n \mathcal{E}, \omega_{\mathcal{Y}/\mathbf{Z}})$  etc. It then follows from the above and from 5.5 and 5.6 that  $\Pi_2$  is equal to

$$\begin{aligned} \nu \circ h_{S*}(\text{Hom}_{\mathcal{O}_Y}(\wedge^\bullet \mathcal{E}, \omega_{\mathcal{Y}/\mathbf{Z}}(\mathcal{Y}_S^{\text{red}})|_{\mathcal{Y}_S^{\text{red}}}) - \text{Hom}_{\mathcal{O}_Y}(\wedge^\bullet \mathcal{E}, \omega_{\mathcal{Y}/\mathbf{Z}}(\mathcal{Y}_S^{\text{red}})|_{\mathcal{Y}_S})) \\ = \prod_{i \in \mathcal{I}} p_i^{-(m_i-1)(-1)^{d(c^d(\mathcal{E}) \cdot \mathcal{Y}_i)}}. \end{aligned}$$

However, from 3.7 in [13] (or see 5.1 in [9]), we know that

$$(-1)^d c^d(\mathcal{E}) \cdot \mathcal{Y}_i = \chi_c(\mathcal{Y}_i^*)$$

and so we have now shown

$$\chi_{L^2}(\mathbb{R}\Gamma \wedge^\bullet \mathcal{E})^2 = \prod_{i \in \mathcal{I}} p_i^{-(m_i-1)\chi_c(\mathcal{Y}_i^*)}.$$

Finally we need to allow for the fact that in the above we have used the  $L^2$ -metric instead of the given Quillen metric. From the very definition of the Quillen metric, we know that

$$\log \chi(\mathbb{R}\Gamma \wedge^n \mathcal{E}, \wedge^n j^D) = \log \chi_{L^2}(\mathbb{R}\Gamma \wedge^n \mathcal{E}) + \tau(\wedge^n \Omega_Y, \wedge^n j^D)$$

where  $\tau(\wedge^n \Omega_Y, \wedge^n j^D)$  denotes the analytic torsion associated to  $\wedge^n \Omega_Y$  with respect to the metric  $\wedge^n j^D$ . But Theorem 3.1 in [32] shows that

$$(28) \quad \sum_n (-1)^n \tau(\wedge^n \Omega_Y, \wedge^n j^D) = 0$$

and so we have now shown

$$\chi(\mathbb{R}\Gamma \wedge^\bullet \mathcal{E}, \wedge^\bullet j^D)^2 = \prod_{i \in \mathcal{I}} p_i^{-(m_i-1)\chi_c(\mathcal{Y}_i^*)}.$$

This then completes the proof of Theorem 7.8.  $\square$

Observe that Theorems 7.8 and 7.9 show that

$$\gamma \circ \chi(\mathrm{R}\Gamma(\wedge^\bullet \Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\mathrm{red}}/\log S)), \wedge^\bullet j_Q^D)^2 = \xi_S(2)\varepsilon_{0,S}(\mathcal{Y}, 2)^{-1}.$$

We conclude this section by showing that the right hand factor  $\xi_S(2)$  in the above can be removed by twisting the sheaf  $\Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\mathrm{red}}/\log S)$  by  $\mathcal{O}_{\mathcal{Y}}(-\mathcal{Y}_S)$ .

THEOREM 7.11. –

$$\gamma \circ \chi(\mathrm{R}\Gamma(\wedge^\bullet \Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\mathrm{red}}/\log S)(-\mathcal{Y}_S)), \wedge^\bullet j_Q^D)^2 = \varepsilon_{0,S}(\mathcal{Y}, 2 \cdot 1_G)^{-1}.$$

*Proof.* – Since for each  $i \geq 0$

$$\wedge^i(\Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\mathrm{red}}/\log S)(-\mathcal{Y}_S)) = \wedge^i \Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\mathrm{red}}/\log S) \otimes \mathcal{O}_{\mathcal{Y}}(-i\mathcal{Y}_S)$$

we obtain an exact sequence of complexes of sheaves

$$0 \rightarrow \wedge^\bullet(\Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\mathrm{red}}/\log S)(-\mathcal{Y}_S)) \rightarrow \wedge^\bullet \Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\mathrm{red}}/\log S) \rightarrow \mathcal{G}^\bullet \rightarrow 0$$

where for  $0 \leq i \leq d$

$$\mathcal{G}^i = \wedge^i \Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\mathrm{red}}/\log S)|_{i\mathcal{Y}_S}$$

and so by (5.5), (5.6) and the equality displayed prior to (7.11)

$$\begin{aligned} \gamma \circ \chi(\mathrm{R}\Gamma(\wedge^\bullet \Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\mathrm{red}}/\log S)(-\mathcal{Y}_S)), \wedge^\bullet j_Q^D)^2 \\ = \xi_S(2 \cdot 1_G) \cdot \varepsilon_{0,S}(\mathcal{Y}, 2 \cdot 1_G)^{-1} \cdot \chi(\nu(\mathcal{G}^\bullet))^{-2} \end{aligned}$$

and for  $0 \leq i \leq d$

$$\chi(\nu(\mathcal{G}^i))^2 = \prod_{p \in S} p^{2f_*(\mathcal{G}^i)} = \prod_{p \in S} p^{2i\chi(\Omega_{\mathcal{Y}_Q}^i)}$$

since  $\Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\mathrm{red}}/\log S)_Q = \Omega_{\mathcal{Y}_Q}^1$ . However, by Serre duality we know that

$$(-1)^{d-i} \chi(\Omega_{\mathcal{Y}_Q}^{d-i}) = (-1)^i \chi(\Omega_{\mathcal{Y}_Q}^i)$$

and so we see that

$$\sum_{i=0}^d (-1)^i i \cdot \chi(\Omega_{\mathcal{Y}_Q}^i) = \sum_{i=0}^d (-1)^{d-i} i \cdot \chi(\Omega_{\mathcal{Y}_Q}^{d-i}) = \sum_{i=0}^d (-1)^i (d-i) \cdot \chi(\Omega_{\mathcal{Y}_Q}^i)$$

hence

$$\sum_{i=0}^d (-1)^i 2i \cdot \chi(\Omega_{\mathcal{Y}_Q}^i) = d \cdot \chi(\mathcal{Y}_Q),$$

which therefore shows that

$$\prod_i \chi(\nu(\mathcal{G}^i))^{2(-1)^i} = \prod_{p \in S} p^{2i(-1)^i \chi(\Omega_{\mathcal{Y}_Q}^i)} = \prod_{p \in S} p^{d\chi(\mathcal{Y}_Q)} = \xi_S(2 \cdot 1_G)$$

as required.  $\square$

### 8. Differentials

In this the final section of the article we suppose that  $\mathcal{X}, \mathcal{Y}$  again satisfy (T1) and (T2) and we construct arithmetic classes associated to the sheaf of (regular) differentials  $\Omega^1_{\mathcal{X}/\mathbf{Z}}$ . Since  $\Omega^1_{\mathcal{X}/\mathbf{Z}}$  is not in general locally free over  $O_{\mathcal{X}}$ , we resolve it by locally free  $G$ - $\mathcal{X}$  sheaves as follows: we choose a  $G$ -equivariant embedding  $i: \mathcal{X} \rightarrow \mathcal{P}$  of  $\mathcal{X}$  into a projective bundle  $\mathcal{P}$  over  $\text{Spec}(\mathbf{Z})$ . The sheaf of differentials  $\Omega^1_{\mathcal{X}/\mathbf{Z}}$  then has a resolution by locally free  $G$ - $\mathcal{X}$  sheaves

$$(29) \quad 0 \rightarrow N^* \rightarrow P \xrightarrow{\pi} \Omega^1_{\mathcal{X}/\mathbf{Z}} \rightarrow 0$$

where  $P = i^* \Omega^1_{\mathcal{P}/\mathbf{Z}}$ , and where  $N^*$  denotes the conormal bundle associated to the regular embedding  $i$ . Let  $\mathcal{F}^\bullet$  denote the length two complex

$$\mathcal{F}^\bullet : N^* \rightarrow P$$

where the term  $P$  is deemed to have degree zero. Thus we may view  $\pi$  as inducing a quasi-isomorphism of complexes, which we abusively also denote  $\pi$ ,

$$\pi : \mathcal{F}^\bullet \rightarrow \Omega^1_{\mathcal{X}/\mathbf{Z}}.$$

Here we further abuse notation and write  $\Omega^1_{\mathcal{X}/\mathbf{Z}}$  for the complex which is  $\Omega^1_{\mathcal{X}/\mathbf{Z}}$  in degree zero and which is zero elsewhere.

For  $j \geq 0$ , recall that we have the Dold–Puppe exterior power functors  $\wedge^j$  defined on bounded complexes of locally free  $G$ - $\mathcal{X}$  sheaves and which take quasi-isomorphisms to quasi-isomorphisms. (See [14] for an account of these functors which is particularly well-suited to their use in this paper.)

We then endow the equivariant determinant of cohomology of the complex  $\wedge^j(\mathcal{F}^\bullet)$  with the metrics  $\phi_{j^\bullet}$  induced, via  $\wedge^j(\pi_{\mathbf{C}})$ , from the  $\wedge^j h^D_{Q^\bullet}$  on the determinants of cohomology of  $\Omega^j_{\mathcal{X}/\mathbf{C}}$ ; we then define arithmetic classes

$$(30) \quad \chi(\text{R}\Gamma \lambda^j \Omega^1_{\mathcal{X}/\mathbf{Z}}, \wedge^j h^D_Q) := \chi(\text{R}\Gamma \wedge^j(\mathcal{F}^\bullet), \phi_j),$$

$$(31) \quad \chi(\text{R}\Gamma \lambda^\bullet \Omega^1_{\mathcal{X}/\mathbf{Z}}, \wedge^\bullet h^D_Q) := \prod_{j=0}^d \chi(\text{R}\Gamma \lambda^j \Omega_{\mathcal{X}/\mathbf{Z}}, \wedge^j h^D_Q)^{(-1)^j}.$$

Note that here the use of the symbols  $\lambda^j$  is entirely symbolic; however, it is important to observe that the lefthand classes are independent of the chosen embedding  $i: \mathcal{X} \rightarrow \mathcal{P}$ : indeed, for a further embedding  $i'$ , with the obvious notation,  $\wedge^j(\mathcal{F}'^\bullet)$  is quasi-isomorphic to  $\wedge^j(\mathcal{F}^\bullet)$ ; furthermore their metrics on the determinant of cohomology match under the corresponding quasi-isomorphism; hence by 3.9 the arithmetic classes coincide.

The equivariant Arakelov–Euler characteristic  $\chi(\text{R}\Gamma \lambda^\bullet \Omega^1_{\mathcal{X}/\mathbf{Z}}, \wedge^\bullet h^D_Q)$  is the principal object of study in this section. Our aim here is to relate it to the epsilon constant  $\varepsilon(\mathcal{Y})$ , whose definition we now briefly recall. Let  $\mathbf{A}_{\mathbf{Q}}$  denote the ring of rational adeles; let  $\psi = \prod_v \psi_v$  denote a non-trivial additive character of  $\mathbf{A}_{\mathbf{Q}}/\mathbf{Q}$ ; let  $dx$  denote the Haar measure on  $\mathbf{A}_{\mathbf{Q}}/\mathbf{Q}$  such that  $\int_{\mathbf{A}_{\mathbf{Q}}/\mathbf{Q}} dx = 1$  and let  $dx = \prod_v dx_v$  be a factorisation of  $dx$  into local Haar measures  $dx_v$  with the property that  $\int_{\mathbf{Z}_v} dx_v = 1$  for almost all  $v$ . Recall from 3.1.1 in [8] that for  $\theta \in R_G$ ,

$$\varepsilon_v(\mathcal{Y}, \theta, \psi_v, dx_v, l_v) = \varepsilon_{0,v}(\mathcal{Y}, \theta, \psi_v, dx_v, l_v) \varepsilon(\mathcal{Y}_v, \theta).$$

Here if  $v < \infty$  then  $\varepsilon(\mathcal{Y}_v, \theta)$  is the epsilon constant associated to the special fibre  $\mathcal{Y}_v$  and if  $v = \infty$  then we take  $\varepsilon(\mathcal{Y}_v, \theta) = 1$ . We then set

$$\varepsilon(\mathcal{Y}, \theta) = \prod_v \varepsilon_v(\mathcal{Y}_v, \theta, \psi_v, dx_v, l_v).$$

Note that in this product almost all terms are 1 and moreover this product is independent of choices of additive character and Haar measure. Thus in the lefthand term we shall abuse notation and henceforth we shall not overtly mention the choices of auxiliary primes  $l_v$ .

For future reference we now need to gather together some standard results on fibral epsilon constants.

For this we require a minor variant on the notation introduced prior to Theorem 5.10. As previously, given a prime number  $p$  we fix a field embedding  $h: \overline{\mathbf{Q}} \rightarrow \overline{\mathbf{Q}}_p$ , we put  $(\overline{\mathbf{Q}})_p = \overline{\mathbf{Q}} \otimes \mathbf{Q}_p$  and we let  $J_f \rightarrow (\overline{\mathbf{Q}})_p^\times$  denote the map given by projection to the  $p$ th coordinate; given  $x \in J_f$ , we shall write  $x_p$  for the  $p$ -component of  $x$  in  $(\overline{\mathbf{Q}})_p^\times$ . Let  $|\cdot|_p: \overline{\mathbf{Q}}_p^\times \rightarrow p^{\mathbf{Q}}$  denote the  $p$ -adic absolute value which is normalised so that  $|p|_p = p^{-1}$ . We shall use the terminology of Definition 5.6 in [7] and for  $f \in \text{Hom}_{\Omega}(R_G, (\overline{\mathbf{Q}})_p^\times)$  we say that  $|f|_p$  is *well-defined* if  $|h^*(f)|_p$  takes values in  $p^{\mathbf{Z}}$ ; in this case it follows that  $|h^*(f)|_p$  respects  $\Omega_p$ -action and we then write

$$|f|_p = h^{*-1}|h^*f|_p.$$

**THEOREM 8.1.** – *For each prime number  $p$ ,  $|\varepsilon(\mathcal{Y}_p)|_p$  and  $|\varepsilon(b_p)|_p$  are well-defined. Writing  $U_p$  for the open sub-scheme  $Y_p - b_p$ , we have  $\varepsilon(\mathcal{Y}_p) = \varepsilon(U_p)\varepsilon(b_p)$  and*

$$\varepsilon(U_p)_p |\varepsilon(U_p)|_p \in \text{Det}(\mathbf{Z}_p[G]^\times)$$

and for a prime number  $q \neq p$

$$\varepsilon(U_p)_q \in \text{Det}(\mathbf{Z}_q[G]^\times).$$

*Proof.* – See [7] 5.7, 5.13 and 5.12.  $\square$

In order to make precise the fundamental relationship between  $\chi(\text{RG}\lambda^\bullet \Omega_{\mathcal{X}/\mathbf{Z}}^1, \wedge^\bullet h_{Q^\bullet}^D)$  and  $\varepsilon(\mathcal{Y})$ , we now need to introduce the *arithmetic ramification class*, which may be viewed as an arithmetic counterpart of the ramification class occurring in Theorem 1.1 in [12].

**DEFINITION 8.2.** – Let  $\text{AR}(\mathcal{X}) \in A(\mathbf{Z}[G])$  be the arithmetic class which is represented by the idele valued character function  $\beta$ , given by the rule that  $\beta$  has trivial archimedean coordinate and at a finite prime  $q$

$$\beta_q = \varepsilon(b)|\varepsilon(b_q)|_q$$

where  $b_q$  denotes the union of the components of  $b$  which are supported by  $q$ .

We are now in a position to be able to state the main result of this article:

**THEOREM 8.3.** – *Let  $\mathfrak{d}$  be the arithmetic class  $\chi(\text{RG}\lambda^\bullet \Omega_{\mathcal{X}/\mathbf{Z}}^1, \wedge^\bullet h_{Q^\bullet}^D)$ . Then  $\mathfrak{d}^s \cdot \text{AR}^s(\mathcal{X})^{-1}$  is a rational class and*

$$\theta(\mathfrak{d}^s \cdot \text{AR}^s(\mathcal{X})^{-1}) = \varepsilon^s(\mathcal{Y})^{-1}.$$

As a first step towards the proof of this theorem, we use results from [13] to show that it will suffice to establish the corresponding result after passage to degree zero:



THEOREM 8.4. – *The class  $\tilde{\mathfrak{d}}^s \cdot \widetilde{\text{AR}}^s(\mathcal{X})$  is a rational class and*

$$\theta(\tilde{\mathfrak{d}}^s \cdot \widetilde{\text{AR}}^s(\mathcal{X})^{-1}) = \tilde{\varepsilon}^s(\mathcal{Y})^{-1}.$$

We begin by showing that Theorem 8.4 implies Theorem 8.3; we then conclude the article by establishing Theorem 8.4.

Suppose then that Theorem 8.4 holds. From Section 4.5, with the notation of 4.2, we know that

$$\mathfrak{d} = \tilde{\mathfrak{d}} \cdot \text{Ind}(\mathfrak{d}_0).$$

By 5.7 in [7] we know that  $\varepsilon(b_p, 1_G)$  is  $\pm$  an integral power of  $p$ ; hence we see that  $\beta$  and  $\tilde{\beta}$  represent the same class in  $A(\mathbf{Z}[G])$  and so  $\widetilde{\text{AR}}^s(\mathcal{X}) = \text{AR}^s(\mathcal{X})$ .

THEOREM 8.5. –  *$\mathfrak{d}_0^s$  is a rational class and  $\theta(\mathfrak{d}_0^s) = \varepsilon(\mathcal{Y}, 2 \cdot 1_G)^{-1}$ .*

*Proof.* – We endow  $P$  and  $N^*$  in (29) with  $G$ -invariant hermitian metrics and denote the resulting hermitian bundles by  $\widehat{P}$  and  $\widehat{N}^*$ . We then let  $\eta_1$  denote the Bott–Chern class associated to the exact sequence (29), where  $\Omega_Y^1$  is endowed with the hermitian metric  $h^D$ , and we put  $\widehat{\Omega} = \widehat{P} - \widehat{N}^* + \eta_1$  in the arithmetic Grothendieck group  $\widehat{K}_0(\mathcal{Y})$  (see for instance II, Section 6 in [20]); we recall *loc. cit.* that  $\widehat{K}_0(\mathcal{Y})$  has a natural structure of a  $\lambda$ -ring and we write  $\hat{f}_*$  for the push forward map from  $\widehat{K}_0(\mathcal{Y})$  to  $\widehat{K}_0(\text{Spec}(\mathbf{Z}))$ . Because  $\mathcal{Y}$  is regular, we know that  $\widehat{K}_0(\mathcal{Y})$  is naturally isomorphic to  $\widehat{K}_0^1(\mathcal{Y})$  the Grothendieck group of coherent hermitian sheaves (see Lemma 13 in [21]). Thus we also have a natural map from the Grothendieck group of torsion  $\mathcal{Y}$ -sheaves supported on  $S$ , denoted  $\widehat{K}_0^S(\mathcal{Y})$ , to  $\widehat{K}_0(\mathcal{Y})$ . Recall that  $\widehat{K}_0^S(\mathcal{Y})$  is a module over the Grothendieck group of locally free  $\mathcal{Y}$ -sheaves  $\widehat{K}_0(\mathcal{Y})$ .

In Theorem 1.3 of [13], with slightly different notation, it is shown that

$$\sum_{i=0}^d \gamma \circ \chi \circ \hat{f}_*((-1)^i \lambda^i \widehat{\Omega}) = |\varepsilon(\mathcal{Y}, 1_G)|^{-1}$$

whereas the class that we now wish to study is

$$\gamma \circ \chi(\text{R}\Gamma \lambda^\bullet \Omega_{\mathcal{Y}/\mathbf{Z}}^1, \wedge^\bullet h_{Q,1}^D) = \sum_{i=0}^d \gamma \circ \chi \circ \hat{f}_*((-1)^i (\lambda^i (\widehat{P} - \widehat{N}^*) + \eta_2^{(i)})),$$

where the  $i$ th exterior power  $\wedge^i \Omega_Y = \Omega_Y^i$  carries the hermitian metric  $\wedge^i h^D$ , the terms of  $\wedge^i \mathcal{F}_C^\bullet$  carry the metrics coming from  $\widehat{P}$  and  $\widehat{N}^*$ , and where  $\eta_2^{(i)}$  is the Bott–Chern class associated to the exact sequence of hermitian bundles  $\wedge^i \mathcal{F}_C^\bullet \rightarrow \wedge^i \widehat{\Omega}_Y$ . As our first step in proving the theorem, we will show that in  $\widehat{K}_0(\mathcal{Y})$

$$(32) \quad \lambda^i (\widehat{P} - \widehat{N}^*) + \eta_2^{(i)} = \lambda^i (\widehat{P} - \widehat{N}^* + \eta_1)$$

which will then imply that

$$(33) \quad \gamma \circ \chi(\text{R}\Gamma \lambda^\bullet \Omega_{\mathcal{Y}/\mathbf{Z}}^1, \wedge^\bullet h_{Q,1}^D) = |\varepsilon(\mathcal{Y}, 1_G)|^{-1}.$$

Let us remark here that this last equality can also be shown by using Theorem 7.8 and a local calculation. From Lemma 7.10 we know that there is an exact sequence

$$0 \rightarrow K \oplus N^* \rightarrow P \rightarrow \Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}} / \log S) \rightarrow C \rightarrow 0$$

where  $K$  and  $C$  are explicitly determined torsion  $\mathcal{O}_{\mathcal{Y}}$ -modules supported on  $S$ . Thus for each  $i$ ,  $0 \leq i \leq d$ , we have an equality in  $\widehat{K}'_0(\mathcal{Y})$

$$\lambda^i(\widehat{P} - \widehat{N}^* - \widehat{K} + \eta_1) = \lambda^i(\widehat{\Omega}_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}} / \log S) - \widehat{C})$$

which we can rewrite as

$$\lambda^i(\widehat{\Omega}) + T_1^{(i)} = \lambda^i(\widehat{\Omega}_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}} / \log S)) + T_2^{(i)}$$

where  $T_1^{(i)}$  and  $T_2^{(i)}$  are the following torsion classes

$$T_1^{(i)} = \sum_{a+b=i, b>0} \lambda^a(P - N^*)\lambda^b(-K),$$

$$T_2^{(i)} = \sum_{a+b=i, b>0} \lambda^a(\widehat{\Omega}_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}} / \log S))\lambda^b(-C).$$

Next we consider the quasi-isomorphism of the Dold–Puppe exterior powers (where  $P$  and  $\Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}} / \log S)$  are both deemed to have degree zero)

$$\wedge^i(K^\bullet \oplus N^* \rightarrow P) \cong \wedge^i(\Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}} / \log S) \rightarrow C^\bullet)$$

and where  $K^\bullet$  and  $C^\bullet$  denote locally free resolutions of  $K$  and  $C$ . Hence filtering the complex  $\wedge^i(K^\bullet \oplus N^* \rightarrow P)$ , by terms  $\wedge^a(N^* \rightarrow P) \otimes \wedge^{i-a}(K^\bullet[1])$ , and filtering the complex  $\wedge^i(\Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}} / \log S) \rightarrow C^\bullet)$  by terms  $\wedge^a(\Omega_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}} / \log S)) \otimes \wedge^{i-a}(C^\bullet[-1])$  (see p. 26 in [34]), we obtain an equality in  $\widehat{K}'_0(\mathcal{Y})$

$$T_1^{(i)} + \lambda^i(\widehat{P} - \widehat{N}^*) + \eta_2^{(i)} = \lambda^i(\widehat{\Omega}_{\mathcal{Y}/\mathbf{Z}}^1(\log \mathcal{Y}_S^{\text{red}} / \log S)) + T_2^{(i)}$$

which now establishes (32).

Now each  $\varepsilon(\mathcal{Y}_p, 1_G)$  is a rational number and so by Theorem 7.9,  $\varepsilon(\mathcal{Y}, 2 \cdot 1_G) \in \mathbf{Q}_{>0}$ . Thus by Lemma 7.7 we see that if we can show

$$(34) \quad \mathfrak{d}_0 = \chi(\text{R}\Gamma \lambda^* \Omega_{\mathcal{Y}/\mathbf{Z}}^1, \wedge^\bullet h_{Q,1}^D)$$

then it will follow that  $\mathfrak{d}_0$  is a rational class and that

$$\theta(\mathfrak{d}_0^s) = \varepsilon(\mathcal{Y}, 2 \cdot 1_G)^{-1}.$$

With the notation of 7.6 above and by the very definition of  $\mathfrak{d}_0$  (see 4.2),

$$\mathfrak{d}_0 = \chi(\text{R}\Gamma \lambda^*(\mathcal{F}^\bullet)^G, \wedge^\bullet h_{Q,1}^D).$$

Thus we are now required to show that for each  $j$ ,  $0 \leq j \leq d$ , the natural map  $\wedge^j(\mathcal{F}^{\bullet G})$  to  $(\wedge^j \mathcal{F}^{\bullet})^G$  is a quasi-isomorphism. To see this it will suffice to show the result after passing to a flat neighbourhood of each closed point  $y$  of  $\mathcal{Y}$ . Writing  $\mathcal{X}' \rightarrow \mathcal{Y}'$  for the resulting base change to such a neighbourhood, we let  $x'$  respectively  $y'$  denote a closed point of  $\mathcal{X}'$  respectively  $\mathcal{Y}'$  above  $y$ . From Theorem A.1 and Lemma A.2 in [8], we know that, for a suitable choice of neighbourhood,  $\mathcal{X}'$  contains  $(G : I_{x'})$  disjoint irreducible components which are permuted transitively by  $G$  and where the component which contains  $x'$  has stabiliser  $I_{x'}$ , the inertia group of  $x'$ . If  $B_1, \dots, B_q$  are the distinct irreducible components of the inverse image  $\pi^{-1}(b)$  which contain the image of  $x'$  on  $\mathcal{X}$ , then

$$I_{x'} = I_1 \oplus \dots \oplus I_q$$

where  $I_i$  denotes the inertia group of the generic point of  $B_i$ ; moreover, each  $I_i$  carries a faithful abelian character  $\phi_i$  given by the action of  $I_i$  on the cotangent space of the generic point of  $B_i$ . To be somewhat more precise, there are integers  $n_1, \dots, n_{d+1}$  coprime to the residual characteristic of  $y$  so that, after base extension by a suitable affine flat neighbourhood  $\text{Spec}(R)$ , the connected open neighbourhood  $V$  of  $\mathcal{X}' = \mathcal{X} \times_{\mathcal{Y}} \text{Spec}(R)$  containing  $x'$  is the spectrum of

$$\frac{R[U_1, \dots, U_{d+1}]}{(U_1^{n_1} - a_1, \dots, U_{d+1}^{n_{d+1}} - a_{d+1})}.$$

Here  $a_1, \dots, a_{d+1}$  form a system of regular parameters of  $\mathcal{Y}'$ ; moreover there are integers  $m_i$  for  $1 \leq i \leq d+1$  with each  $m_i$  coprime to the residual characteristic,  $p$  say, of  $y$ , and with the property that  $a_1^{m_1} \dots a_{d+1}^{m_{d+1}} = p$ . Here, after reordering if necessary, the characters  $\phi_i$  are given by the action of  $I_i$  on  $U_i$ . It now follows that  $\Omega_{V/R}^1$  sits in an exact sequence

$$0 \rightarrow K^{\bullet} \rightarrow \Omega_{V/R}^1 \rightarrow 0$$

with

$$K^{\bullet} : \mathcal{O}_V dr \rightarrow \bigoplus_{i=1}^{d+1} \mathcal{O}_V dU_i$$

and where  $r = a_1^{m_1} \dots a_{d+1}^{m_{d+1}} - p$ . In the sequel for brevity we shall write  $K^{\bullet} = L \rightarrow E$ . Since the restriction  $\mathcal{F}^{\bullet}|_V$  is quasi-isomorphic to  $K^{\bullet}$  we are now reduced to showing that  $\wedge^m(K^{\bullet I}) \simeq (\wedge^m K^{\bullet})^I$  for all  $m \geq 0$  and for  $I = I_{x'}$ . This now follows easily since we know (see for instance Section 3 in [14]) that the complex  $\wedge^m K^{\bullet}$  is constituted entirely of terms which are tensor products of modules of the form  $\wedge^n L$  times either one or no terms of the form  $\wedge^n E$ ; the result then follows because  $L \cong \mathcal{O}_V$ , as  $I$ -modules, and because, for any non-negative  $n$ ,  $\wedge^n(E^I) \cong (\wedge^n E)^I$  (using the fact that the  $\phi_i$  come from the distinct components in a direct sum decomposition).  $\square$

*Proof of Theorem 8.4.* – We write  $\mathcal{X}_S^{\text{red}} = \bigsqcup_{p \in S} \mathcal{X}_p^{\text{red}}$ , let  $i_S^{\text{red}} : \mathcal{X}_S^{\text{red}} \rightarrow \mathcal{X}$  denote the associated closed embedding and we let  $U_S$  denote the complement of  $\mathcal{X}_S^{\text{red}}$  in  $\mathcal{X}$ .

Composing the quasi-isomorphism  $\pi : \mathcal{F}^{\bullet} \simeq \Omega_{\mathcal{X}/\mathbb{Z}}^1$  with the natural homomorphism

$$\omega : \Omega_{\mathcal{X}/\mathbb{Z}}^1 \rightarrow \Omega_{\mathcal{X}/\mathbb{Z}}^1(\log \mathcal{X}_S^{\text{red}} / \log S),$$

which is an isomorphism over  $U_S$ , we get a chain map

$$\pi' : \mathcal{F}^{\bullet} \rightarrow \Omega_{\mathcal{X}/\mathbb{Z}}^1(\log \mathcal{X}_S^{\text{red}} / \log S)$$

which is a surjective quasi-isomorphism over  $U_S$ . Hence for  $i \geq 0$  we obtain maps

$$\wedge^i \pi' : \wedge^i \mathcal{F}^\bullet \rightarrow \wedge^i \Omega_{\mathcal{X}/\mathbf{Z}}^1(\log \mathcal{X}_S^{\text{red}} / \log S)$$

which are surjective quasi-isomorphisms over  $U_S$ . Let

$$\mathcal{A}_i^\bullet = \ker(\wedge^i \pi') \quad \text{and} \quad \mathcal{B}_i^\bullet = \text{coker}(\wedge^i \pi')$$

so that  $\mathcal{B}_i^\bullet$  and the cohomology sheaves  $\mathcal{H}^j(\mathcal{A}_i^\bullet)$  of the complex  $\mathcal{A}_i^\bullet$  are all supported entirely over  $S$ . Let  $\mathfrak{I}$  denote the ideal sheaf of  $O_{\mathcal{X}}$  associated to the closed subscheme  $\mathcal{X}_S^{\text{red}}$ ; we then write  $[\mathcal{H}^j(\mathcal{A}_i^\bullet)]$  for the finite sum

$$\sum_{n \geq 0} (\mathfrak{I}^n \mathcal{H}^j(\mathcal{A}_i^\bullet) / \mathfrak{I}^{n+1} \mathcal{H}^j(\mathcal{A}_i^\bullet))$$

in  $G_0(G, \mathcal{X}_S^{\text{red}})$  and put

$$[\mathcal{H}^\bullet(\mathcal{A}_i^\bullet)] = \sum_j (-1)^j [\mathcal{H}^j(\mathcal{A}_i^\bullet)] \in G_0(G, \mathcal{X}_S^{\text{red}}).$$

We endow the equivariant determinants of cohomology of  $\mathcal{A}_i^\bullet$  and  $\mathcal{B}_i^\bullet$  with the trivial metrics  $\tau_\bullet$ . Then from 6.4 and 5.6 we know that

$$\begin{aligned} & \tilde{\chi}(R\Gamma \wedge^i \mathcal{F}^\bullet, \wedge^i h^D) \cdot \tilde{\chi}(R\Gamma \wedge^i \Omega_{\mathcal{X}/\mathbf{Z}}^1(\log \mathcal{X}_S^{\text{red}} / \log S), \wedge^i h^D)^{-1} \\ &= \tilde{\chi}(R\Gamma \mathcal{A}_i^\bullet, \tau_\bullet) \cdot \tilde{\chi}(R\Gamma \mathcal{B}_i^\bullet, \tau_\bullet)^{-1} \\ (35) \quad &= \widetilde{\nu \circ f_{S*}^{\text{red}}}([\mathcal{H}^\bullet(\mathcal{A}_i^\bullet)] - [\mathcal{H}^\bullet(\mathcal{B}_i^\bullet)]). \end{aligned}$$

Since the class  $F = (-1)^d \sum_i (-1)^i ([\mathcal{H}^\bullet(\mathcal{A}_i^\bullet)] - [\mathcal{H}^\bullet(\mathcal{B}_i^\bullet)])$  in  $G_0(G, \mathcal{X}_S^{\text{red}})$  has the property that its image in  $G_0(G, \mathcal{X}) = K_0(G, \mathcal{X})$

$$\begin{aligned} i_{S*}^{\text{red}} F &= (-1)^d \sum_i (-1)^i ([\wedge^i \mathcal{F}^\bullet] - [\wedge^i \Omega_{\mathcal{X}/\mathbf{Z}}^1(\log \mathcal{X}_S^{\text{red}} / \log S)]) \\ (36) \quad &= c^d(\Omega_{\mathcal{X}/\mathbf{Z}}) - c^d(\Omega_{\mathcal{X}/\mathbf{Z}}^1(\log \mathcal{X}_S^{\text{red}} / \log S)) \end{aligned}$$

we may take  $F = \bigoplus_{p \in S} F_p$  to be the class  $F$  in 6(a) of [12].

DEFINITION 8.6. – For each  $i \in \mathcal{I}$  we set  $\mathcal{X}_i = \pi^{-1}(\mathcal{Y}_i)$ ; thus  $\mathcal{X}_i$  is a smooth projective variety over  $\mathbf{F}_{p_i}$  of dimension  $d$  which carries a tame  $G$ -action. More generally for each non-empty subset  $\mathcal{J}$  of  $\mathcal{I}$  we define

$$\mathcal{Y}_{\mathcal{J}} = \bigcap_{j \in \mathcal{J}} \mathcal{Y}_j, \quad \mathcal{X}_{\mathcal{J}} = \bigcap_{j \in \mathcal{J}} \mathcal{X}_j$$

so that each  $\mathcal{X}_{\mathcal{J}}$  is either empty or is a smooth projective variety of dimension  $d + 1 - |\mathcal{J}|$ . Again  $\mathcal{X}_{\mathcal{J}}$  carries a tame  $G$ -action and the branch locus of the cover  $\mathcal{X}_{\mathcal{J}}/\mathcal{Y}_{\mathcal{J}}$  is a divisor with strict normal crossings. Let  $\mathcal{I}_p$  denote the subset of those  $i \in \mathcal{I}$  such that  $p_i = p$ . For  $\mathcal{J} \subset \mathcal{I}_p$  we write  $f_{\mathcal{J}}$  for the structure map  $f_{\mathcal{J}} : \mathcal{X}_{\mathcal{J}} \rightarrow \text{Spec}(\mathbf{F}_p)$  and as per 6.b in [12] we set

$$\Psi_p(\mathcal{X}_{\mathcal{J}}/\mathcal{Y}_{\mathcal{J}}) = (-1)^{d-|\mathcal{J}|} f_{\mathcal{J}*} (c^{d-|\mathcal{J}|}(\Omega_{\mathcal{X}_{\mathcal{J}}/\mathbf{F}_p})) \quad \text{in } K_0(\mathbf{F}_p[G]),$$

$$\Psi_p = \sum_{\phi \neq \mathcal{J} \subset \mathcal{I}} (-1)^{|\mathcal{J}|+1} \Psi_p(\mathcal{X}_{\mathcal{J}}/\mathcal{Y}_{\mathcal{J}})$$

and

$$\Psi = \bigoplus_{p \in S} \Psi_p \in \bigoplus_{p \in S} K_0(\mathbf{F}_p[G]).$$

**THEOREM 8.7.** – (a) *The classes  $f_{p*}F_p$  and  $(-1)^d \Psi_p$  differ by the class of a free  $\mathbf{F}_p[G]$ -module in  $K_0(\mathbf{F}_p[G])$ , and so*

$$\tilde{\nu}(f_{p*}F_p) = \tilde{\nu}(\Psi_p)^{(-1)^d}.$$

(b) *The class  $\tilde{\nu}(\Psi_p)$  is represented by the idele valued character homomorphism  $\delta_p$*

$$\delta(\theta)_v = \begin{cases} |\varepsilon_p(\mathcal{Y}_p, \bar{\theta})|_p & \text{if } v = p; \\ 1 & \text{if } v \neq p. \end{cases}$$

*Proof.* – This is the content of (A) and (B) in 6(b) of [12], but note that, as explained in Section 4.2, we adopt the opposite convention on the representative of a class in  $K_0T(\mathbf{Z}[G])$  to that used in [12].  $\square$

We are now in a position to complete the proof of Theorem 8.4. From (35), (36) and part (a) of Theorem 8.7 we know that

$$\tilde{\delta} \cdot \tilde{\epsilon}^{-1} = \tilde{\nu}(i_{S*}(F))^{(-1)^d} = \tilde{\nu}(\Psi)$$

and by part (b) of Theorem 8.7 we know that  $\tilde{\nu}(\Psi)$  is represented by the finite idele valued homomorphism on characters  $\delta = \prod_p \delta_p$ . From Theorem 7.5 we know that  $\tilde{\epsilon}$  is a rational class and that  $\theta(\tilde{\epsilon}^s) = \tilde{\epsilon}_0^s(\mathcal{Y})^{-1}$ ; it therefore follows that  $\tilde{\delta}^s$  is represented by the character function with trivial Archimedean coordinate and whose finite coordinate is

$$\tilde{\epsilon}_0^s(\mathcal{Y})^{-1} \left[ \prod_{p \in S} |\tilde{\epsilon}^s(\mathcal{Y}_p)|_p \tilde{\epsilon}^s(\mathcal{Y}_p) \tilde{\epsilon}^s(\mathcal{Y}_p)^{-1} \right].$$

Therefore, to complete the proof of Theorem 8.4, we are now reduced to showing:

**PROPOSITION 8.8.** – *The character function  $\prod_{p \in S} |\varepsilon(\mathcal{Y}_p)|_p \varepsilon(\mathcal{Y}_p)$  represents the arithmetic ramification class  $\text{AR}(\mathcal{X})$ .*

*Proof.* – For  $f, g \in \text{Hom}_{\Omega}(R_G, J_f)$  we write  $f \sim g$  if  $f$  and  $g$  represent the same class in  $A(\mathbf{Z}[G])$ . From 8.2 we need to show that

$$(37) \quad \prod_{p \in S} |\varepsilon(\mathcal{Y}_p)|_p \varepsilon(\mathcal{Y}_p) \sim \prod_{p \in S} \varepsilon(b_p) |\varepsilon(b_p)|_p.$$

With the notation of 8.1 we know that for each prime  $p \in S$ ,

$$\begin{aligned} |\varepsilon(\mathcal{Y}_p)|_p \varepsilon(\mathcal{Y}_p) &= |\varepsilon(\mathcal{Y}_p)|_p \left( \varepsilon(\mathcal{Y}_p)_p \times \prod_{q \neq p} \varepsilon(\mathcal{Y}_p)_q \right) \\ &= |\varepsilon(U_p)|_p \varepsilon(U_p)_p |\varepsilon(b_p)|_p \varepsilon(b_p)_p \times \prod_{q \neq p} \varepsilon(U_p)_q \varepsilon(b_p)_q. \end{aligned}$$

But from Theorem 8.1 we know  $|\varepsilon(U_p)|_p \varepsilon(U_p)_p \sim 1$  and  $\varepsilon(U_p)_q \sim 1$  whenever  $q \neq p$ . This then establishes (37), as required.  $\square$

## REFERENCES

- [1] ARAI K., *Conductor formula of Bloch, in tame case*, Thesis, University of Tokyo, 2000 (in Japanese).
- [2] BISMUT J.-M., Equivariant immersions and Quillen metrics, *J. Differential Geom.* **41** (1995) 53–157.
- [3] BLOCH S., Cycles on Arithmetic schemes and Euler characteristics of curves, in: *Proceedings of Symposia in Pure Math.*, Vol. **46**, Part 2, AMS, 421–450.
- [4] BURNS D., *Equivariant Tamagawa numbers and Galois module theory I*, preprint.
- [5] BURNS D., *Equivariant Tamagawa numbers and Galois module theory II*, preprint.
- [6] CASSOU-NOGUÈS PH., TAYLOR M.J., Local root numbers and hermitian Galois structure of rings of integers, *Math. Annalen*, **263** (1983) 251–261.
- [7] CHINBURG T., Galois structure of de Rham cohomology of tame covers of schemes, *Ann. Math.* **139** (1994) 443–490. Corrigendum: *Ann. Math.* **140** (1994) 251.
- [8] CHINBURG T., EREZ B., PAPPAS G., TAYLOR M.J.,  $\varepsilon$ -constants and the Galois structure of de Rham cohomology, *Ann. Math.* **146** (1997) 411–473.
- [9] CHINBURG T., EREZ B., PAPPAS G., TAYLOR M.J., On the  $\varepsilon$ -constants of arithmetic schemes, *Math. Ann.* **311** (1998) 377–395.
- [10] CHINBURG T., EREZ B., PAPPAS G., TAYLOR M.J., On the  $\varepsilon$ -constants of a variety over a finite field, *Amer. J. Math.* **119** (1997) 503–522.
- [11] CHINBURG T., EREZ B., PAPPAS G., TAYLOR M.J., Tame actions of group schemes: integrals and slices, *Duke Math. J.* **82** (2) (1996) 269–308.
- [12] CHINBURG T., PAPPAS G., TAYLOR M.J.,  $\varepsilon$ -constants and the Galois structure of de Rham cohomology II, *J. Reine Angew. Math.* **519** (2000) 201–230.
- [13] CHINBURG T., PAPPAS G., TAYLOR M.J.,  $\varepsilon$ -constants and Arakelov–Euler characteristics, *Math. Res. Lett.* **7** (2000) 433–446.
- [14] CHINBURG T., PAPPAS G., TAYLOR M.J., *Duality and hermitian Galois module structure*, to appear in the Proc. L.M.S.
- [15] DELIGNE P., Les constantes des équations fonctionnelles des fonctions  $L$ , in: *Lecture Notes in Math.*, Vol. **349**, Springer-Verlag, Heidelberg, 1974, pp. 501–597.
- [16] FRÖHLICH A., *Galois Module Structure of Algebraic Integers*, Springer Ergebnisse, Band 1, Folge 3, Springer-Verlag, Heidelberg, 1983.
- [17] FRÖHLICH A., *Classgroups and Hermitian Modules*, Progress in Mathematics, Vol. **48**, Birkhäuser, Basel, 1984.
- [18] FRÖHLICH A., TAYLOR M.J., *Algebraic Number Theory*, Cambridge Studies in Advanced Mathematics, Vol. **27**, Cambridge University Press, 1991.
- [19] FULTON W., LANG S., *Riemann–Roch Algebra*, Grundlehren, Vol. **277**, Springer-Verlag, 1985.
- [20] GILLET H., SOULÉ C., Characteristic classes for algebraic vector bundles with hermitian metrics I, II, *Ann. Math.* **131** (1990) 163–203, 205–238.
- [21] GILLET H., SOULÉ C., An arithmetic Riemann–Roch theorem, *Invent. Math.* **110** (1992) 473–543.
- [22] GILLET H., SOULÉ C., Analytic torsion and the arithmetic Todd genus. With an appendix by D. Zagier, *Topology* **30** (1) (1991) 21–54.
- [23] HARTSHORNE R., *Residues and Duality*, Lecture Notes in Math., Vol. **20**, Springer-Verlag, 1966.
- [24] HECKE E., *Mathematische Werke*, Vandenhoeck and Ruprecht, Göttingen, 1983.
- [25] IRELAND K., ROSEN M., *A Classical Introduction to Modern Number Theory*, Springer Graduate Texts in Mathematics, Vol. **84**, Springer, New York, 1982.
- [26] KATO K., Logarithmic structures of Fontaine-Illusie, in: *Proc. 1st JAMI Conf.*, Johns-Hopkins Univ. Press, 1990, pp. 191–224.
- [27] KNUSDEN F., MUMFORD D., The projectivity of moduli spaces of stable curves, *Math. Scand.* **39** (1976) 19–55.
- [28] KATO K., SAITO T., *Conductor formula of Bloch*, preprint, 2001.

- [29] LANG S., *Algebraic Number Theory*, Addison-Wesley, Reading MA, 1970.
- [30] MARTINET J., Character theory and Artin  $L$ -functions, in: Fröhlich A. (Ed.), *Algebraic Number Fields, Proc. Durham Symposium 1975*, Academic Press, London, 1977.
- [31] MILNE J.S., *Étale Cohomology*, Princeton University Press, 1980.
- [32] RAY D.B., SINGER I.M., Analytic torsion for complex manifolds, *Ann. of Math. (2)* **98** (1973) 154–177.
- [33] SAITO T.,  $\varepsilon$ -factor of a tamely ramified sheaf on a variety, *Inv. Math.* **113** (1993) 389–417.
- [34] SOULÉ C., ABRAMOVICH D., BURNOL J.-F., KRAMER J., *Lectures on Arakelov Geometry*, Cambridge Studies in Advanced Mathematics, Vol. **33**, Cambridge University Press, Cambridge, 1992.
- [35] TATE J., in: Fröhlich A. (Ed.), *Thesis in “Algebraic Number Fields”*, Academic Press, 1976.
- [36] TAYLOR M.J., On Fröhlich’s conjecture for rings of integers of tame extensions, *Invent. Math.* **63** (1981) 41–79.
- [37] TAYLOR M.J., *Classgroups of Group Rings*, LMS Lecture Notes, Vol. **91**, Cambridge University Press, Cambridge, 1984.

(Manuscript received June 16, 2000;  
accepted, after revision, September 27, 2001.)

Ted CHINBURG  
University of Pennsylvania,  
Philadelphia, PA 19104, USA  
E-mail: ted@math.upenn.edu

Georgios PAPPAS  
Michigan State University,  
E. Lansing, MI 48824, USA  
E-mail: pappas@math.msu.edu

Martin J. TAYLOR  
UMIST,  
Manchester, M60 1QD, UK  
E-mail: martin.taylor@umist.ac.uk