Effective wave propagation in a prestressed nonlinear elastic composite bar

Parnell, William J

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The University of Manchester

Effective wave propagation in a pre-stressed nonlinear elastic composite bar

W.J. Parnell

School of Mathematics, University of Manchester,

Oxford Road, Manchester, M13 9PL.

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Abstract

The problem of determining the effective incremental response of nonlinearly elastic composite materials given some initial pre-stress, is of interest in numerous application areas. In particular the case when small amplitude elastic waves pass through a pre-stressed inhomogeneous structure is of great importance. Of specific interest is how the initial finite deformation affects the microstructure and thus the subsequent response of the structure. Modelling this effect is in general extremely difficult.

In this article we consider the simplest problem of this type where the material is a one dimensional composite bar consisting of two distinct phases, periodically distributed. Neglecting lateral contractions, the initial deformation is thus piecewise homogeneous and we can therefore determine the incremental behaviour semi-analytically, given the constitutive behaviour (strain energy function) of the phases in question. We apply asymptotic homogenization theory in the deformed configuration in order to find the effective response of the deformed material in the low frequency limit where the wavelength of the propagating waves is much longer than the characteristic lengthscale of the microstructure. We close by considering the arbitrary frequency case and illustrate how the initial deformation affects the location of stop and pass bands of the material. Work is underway to confirm these results experimentally.

Keywords: Finite deformation, Composites, Homogenization, Effective wave propagation, Incremental moduli, Pass and Stop Bands

1 Introduction

This article is the first in a series which analyses the effects of an initial *finite elastic deformation* on small amplitude elastic wave propagation through composite materials. Such materials possess *microstructure* and are therefore heterogeneous, having properties which vary rapidly with respect to the macroscale (for example the size of the body). The finite deformation may be in the form of pre-stress or pre-strain but for ease of discussion we will usually refer to it here as *pre-stress*. Since the materials in question undergo large elastic deformations it is envisaged that the phases which make up the composite are of a rubber type. We assume that no plastic or viscoelastic deformation ensues and therefore the constitutive behaviour of each isotropic phase can be described by a *strain energy function* $W = W(I_1, I_2, I_3)$ where I_j are the strain invariants of the material, [2], [33]. For anisotropic phases the strain energy function has to be described as a function of a greater number of strain invariants.

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We wish to consider small amplitude linear elastic waves propagating through a deformed configuration such as the type illustrated in figure 1. In fact, for the purposes of this first article we will consider the simplest composite problem, that of the extension of a long thin composite bar made from two phases periodically distributed along the bar, see figure 2. We will neglect lateral contractions so that the problem is one dimensional. The constitutive behaviour of each phase may therefore be described by a strain energy function $W = W(\lambda)$ where λ is the principal stretch in the material. In general, elastic deformations in composite media will be inhomogeneous. In this one dimensional setting however the deformation is *piecewise* homogeneous and this simplifies the analysis somewhat.

It is well known that the propagation of elastic waves through rubber materials is strongly frequency dependent, especially when the materials in question have been pre-stressed [24], [42]. In particular the shear modulus $\mu = \mu(\omega)$ where ω is the circular frequency of propagating waves but for simplicity here we shall assume that μ is independent of frequency, so that the body is purely elastic. This is certainly a reasonable assumption in the first part of this article since we are interested in the low frequency regime where linear elastic waves propagate through the material with some effective or *homogenized* wave speed determined by the *effective incremental elastic moduli*, which themselves depend on the initial deformation. The wavelength of the propagating waves in this case is much greater than the characteristic lengthscales of the microstructure, for example in figure 1 the diameter of the inclusions and their average spacing. Therefore any frequency dependence of the material can certainly be neglected in this context as we are in the quasi-static limit [24], [42]. The properties of rubber are also highly dependent on temperature, [14], [43], and thus for the purposes of this article we assume that the initial deformations take place either isothermally or adiabatically so that thermodynamic variables can be omitted from the discussion.
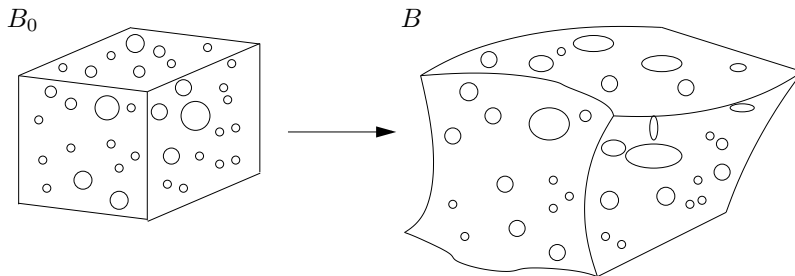

Figure 1: Figure exhibiting the general nonlinear, finite deformation of a composite material from the configuration B_0 to B. This is the most general case in which we are interested. We then consider small amplitude elastic waves propagating through the deformed configuration B in the case where there is a so-called separation of scales within the medium, i.e. when the wavelength of the propagating wave is much larger than the lengthscale defining the microstructure, in this case the characteristic lengthscale of inclusions and their average spacing.

Having understood the homogenized, low frequency regime, the second part of this article considers the arbitrary frequency case. In composite materials there exist ranges of frequencies where no propagation can occur (the so-called stop bands of the material) [44] and therefore we will consider how an initial deformation may affect these. In this analysis we again assume that material properties are independent of frequency ω although in reality this is not always the case as we discussed above. Studies of the frequency dependent (viscoelastic) case are currently underway and will be reported on in future articles.

Since its applications are numerous and varied, the effect of pre-stress on wave propagation through *homogeneous* anisotropic media has interested many workers. It has stimulated a great
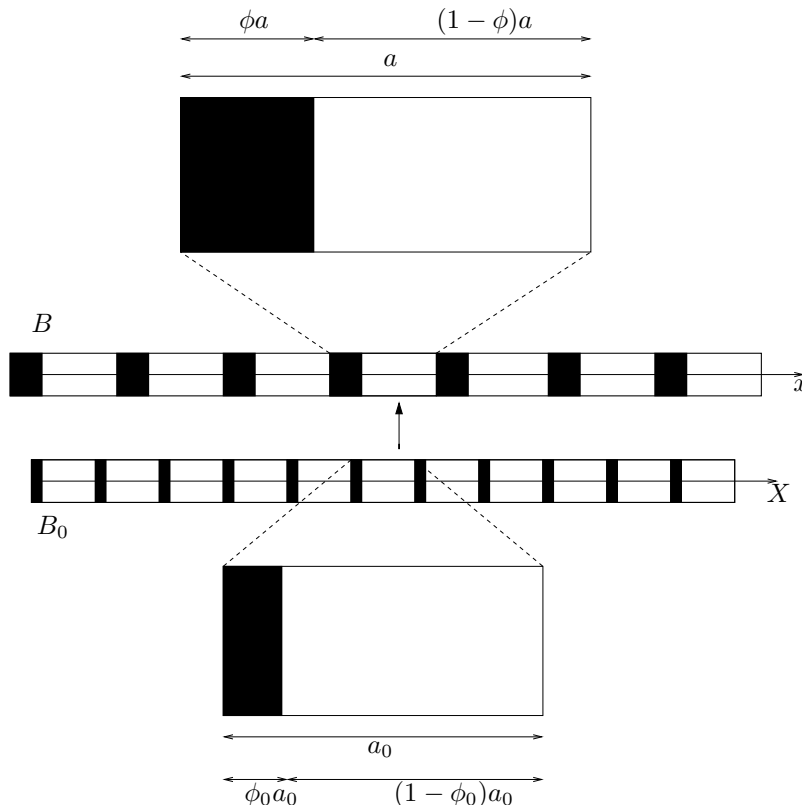
Figure 2: Figure illustrating the undeformed and deformed configurations B_0 and B respectively of the periodic composite bar considered here. Note that parameters in the undeformed configuration are labelled with a subscript 0. Volume fractions of the *inclusion* region are indicated by ϕ_0 and ϕ in configurations B_0 and B respectively.

deal of research and in the case of initial finite *elastic* deformations, (at least) two classical textbooks on the subject have been written, those by Green and Zerna [15] and Ogden [33]. The analysis of small deformations superposed on pre-stresses or pre-strains has been known ever since as *small on large*. Clearly the initial deformation must be finite to change the properties of the medium; an initial infinitesimal deformation would not affect the properties by virtue of the superposition principle for linear deformations. In a series of papers Sawyers and Rivlin [40], [41] analysed the incremental moduli of pre-stressed homogeneous media and specified restrictions on strain energy functions such that infinitesimal elastic waves can propagate in the material.

The effects of pre-stress on the propagation of surface waves on a homogeneous half-space has been studied by Fu and Devenish [10] and Destrade and Scott [5] for example. In the context of wave propagation through pre-stressed *composites*, work has mainly concentrated on composite plates consisting of a small number (i.e. two or three) of layers. Contributions to this area have come from Rogerson and Sandiford [39], Ogden and Roxburgh [34], Kaplunov and Rogerson [23], Leungvichcharoen et. al. [25], Nemat-Nasser and Amirkhizi [30] and many more (see these articles for further details). Very little work has been done however in the case of heterogeneous materials where there are *many* phases and a great deal of microstructure (so that a so-called *separation of scales* exists in the problem). This area is particularly important since in reality *all* materials are heterogeneous at some lengthscale and this affects the choice of constitutive law at

the macroscale.

Modelling infinitesimal elastic wave propagation in pre-stressed heterogeneous media plays an important role in many contexts. The earth is inherently heterogeneous and pre-stressed – therefore seismic waves propagate in a pre-stressed inhomogeneous medium [22]. Such modelling therefore has important implications in geophysical applications and the oil industry. Non-destructive testing of materials involves sending waves through structures, which due to their application are often pre-stressed. The importance of such techniques in finding cracks is of paramount importance particularly in the nuclear industry for example [16]. *Rubber* composites are used in many contexts in the automobile, aerospace and underwater vehicle industries (amongst others) and are often subjected to some form of pre-stress or pre-strain.

An extremely important application of the proposed theory is to soft tissues, muscle, tendons, etc., all of which undergo large elastic deformations continuously in the body and are often in some pre-stressed state [18]. Modelling wave propagation in such complex materials is important in, for example, imaging, locating diseased tissue, determining the acoustic properties of the lung, ultrasound techniques and many more; see Gasser et. al. [12], Holzapfel et. al. [18], Fung [11] and Fang et. al. [9] for details. In this case, the determination of the governing constitutive law (and hence strain energy function) is very important but is difficult to achieve experimentally. Since all of these biomaterials possess microstructure (and in many cases several *levels* of microstructure) they are inherently inhomogeneous and therefore modelling such media requires a theory for heterogeneous materials. The fact that these materials are alive and continuously changing provides an even greater challenge.

In many contexts, pre-strain of composites takes the form of a *plastic* deformation, particularly in metals where the material is worked until its properties are in some sense optimized. Although this article does not consider plastic deformations, strain energy functions can be used in this context as models for plastic deformation, provided the effects of hysteresis or memory are not under investigation (i.e. in the case where we do not *unload* the composite). A great deal of work in the area of incremental moduli given some plastic deformation of polycrystals has been done by Nemat-Nasser and co-workers, particularly via self consistent schemes [21], [29], [31]. Models of wave propagation in materials can also allow us to find locations of residual stresses due to some plastic deformation for example.

In the context of this work, since we are interested in the effective *incremental* moduli, we appeal to the theory of homogenization, as discussed by Bakhvalov and Panasenko [1] for example. In particular we use the method of asymptotic homogenization [38] which has proved very successful for periodic media.

The incremental moduli give a snapshot of the material behaviour after some initial finite deformation but they can also help to build a picture of the effective strain energy function of the composite material by considering a variety of deformations and finding the incremental moduli associated with each of these. This allows us to construct the effective strain energy function for the composite material and thus the theory contributes to the area of *nonlinear* homogenization. This area is not the primary focus of this article however and will be explored elsewhere at a later date. For completeness we mention that nonlinear homogenization of composites capable of large deformations has been explored by Ogden [32], Müller [28], Braides and Defranceschi [3] and Devries [6] and references therein. Layered media have been considered recently by deBotton [4]. Important contributions regarding the constitutive behaviour of filled rubbers, modelled via pseudo-elastic strain energy functions has been carried out by Ogden and Roxburgh [35] and Dorfmann and Ogden [7], [8] for example, without relation to the microstructure of the material.

We should mention that work on nonlinear homogenization for *infinitesimal* deformations has been carried out by Suquet and co-workers, [27]. This is a simpler problem than the large deformation case however since in the latter, the configuration of the microstructure changes. This is known as *microstructure evolution* or *texture evolution* in the theory of ductile metals

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(where the change is due to *plastic* deformation). Tracking the movement of each phase and how its properties are affected by the initial deformation proves very difficult [26].

In order to begin the study of effective wave propagation in nonlinear composite media which have undergone large elastic deformation, we shall consider the one dimensional composite bar problem as described above and we find the effective incremental Young's modulus of the composite semi-analytically. Having understood this initial problem, in future articles we shall consider more complex materials and thus more realistic deformations. In particular work is underway on the response of layered media, which occur frequently in many different contexts.

The effective Young's modulus of a composite bar *without* pre-stress is well understood and simple to derive [1]. The interest of this paper is therefore to determine how the initial finite deformation changes these *rest* properties of the composite medium. Key to this understanding is the microstructure evolution in the composite referred to above, which, in this one dimensional setting is relatively easy to track.

The paper proceeds as follows. In section 2 we briefly review the method of asymptotic homogenization [37], [38] as applied to this one dimensional problem in order to derive the effective properties of the bar. The one dimensional nature of the problem means that the initial deformation is piecewise homogeneous and leads only to piecewise constant *incremental* moduli and thus we can restrict attention to the homogenization of composite bars with piecewise constant elastic moduli.

In section 3 we describe how we induce the pre-stress in the composite bar, essentially by taking a unit cell of the composite and deforming each of these in a given compatible manner.

Section 4 then considers small amplitude linear elastic compressional waves as perturbations to this pre-stressed equilibrium state and we can therefore apply the well-known theory of *small on large* explained in the textbooks by Green and Zerna [15] and Ogden [33] for example. We thus obtain a linear wave equation with rapidly varying (piecewise constant) coefficients in x. Referring to the material derived in section 2 we can then state the effective homogenized wave equation for the pre-stressed composite. In order to determine the effective incremental properties we are required to state the constitutive behaviour of the material and this is done by considering alternative *strain energy functions*. Several realistic examples are chosen and their effect on the homogenized Young's modulus and density is illustrated.

In section 5 we leave the low frequency regime and consider the arbitrary frequency regime. In particular it is of interest to determine how the initial deformation affects the location of stop and pass bands in the bar – useful knowledge in practice.

Articles to follow will consider more complicated and realistic deformations in composites of higher dimensions and we will compare these results with experiments currently being performed by collaborators. Clearly the goal is to extend the methods presented here to more complicated composites including those of the laminate, fibre reinforced and particulate type and to materials involved in the applications described above. In particular as mentioned above, more general deformations of composites will certainly *not* be homogeneous and this will therefore complicate the theory substantially.

2 The effective Young's modulus of a composite bar

The composite media in which we are interested are nonlinearly elastic and capable of large (finite) deformations. Given some initial finite deformation, in many applications it is of great interest as to how the medium then responds incrementally due to small displacements and strains and thus via the linear elastic constitutive law (Hooke's law)

$$\sigma = \hat{\mathcal{A}}(x)e, \tag{2.1}$$

where σ and $e = dw/dx$ are the Cauchy stress and linearized strain associated with the (small) *incremental* deformation. The scalar function $\hat{\mathcal{A}}$ is the incremental Young's modulus and depends on the form of the initial finite deformation.

A classical application of the incremental theory is to the case of elastic waves propagating through a pre-stressed nonlinear composite medium such as the composite bar that we have envisaged above. In general since the material properties of the composite in the undeformed configuration vary with position, so does the incremental Young's modulus. However in the case of piecewise constant composite bars, the incremental modulus $\hat{\mathcal{A}}(x)$ will also be piecewise constant as we shall show below.

Let us consider the simplest case of a *two-phase* periodic composite bar which, having undergone some nonlinear deformation, then behaves incrementally as a linear elastic material via (2.1). Since we are interested in an incremental deformation from the equilibrium state, it makes sense for us to choose the reference configuration as the deformed state B and therefore let us employ a Cartesian coordinate x, parallel to the bar in this equilibrium state. The medium is periodic with a periodic cell in configuration B of length a and volume fraction $\phi_1 = \phi$ of phase 1 and $\phi_2 = 1 - \phi$ of phase 2, illustrated in figure 2. We will discuss later how the parameters in the deformed configuration B relate to those in the undeformed configuration B_0.

Assuming time harmonic motion of frequency ω and neglecting lateral contractions, the equation governing small incremental longitudinal displacements $w = w(x)$ in the medium is [13]

$$\frac{d}{dx}\left(\hat{\mathcal{A}}(x)\frac{dw}{dx}\right) + \omega^2\rho(x)w = 0 \tag{2.2}$$

where $\hat{\mathcal{A}}$ is the incremental Young's modulus and ρ is the mass density of the inhomogeneous medium. In the problem to be discussed, we will always have piecewise constant properties, so that for a two phase medium, $\hat{\mathcal{A}}$ and ρ have the form

$$\hat{\mathcal{A}}(x) = \begin{cases} \mathcal{A}_1, & x \in a[n, n+\phi], \\ \mathcal{A}_2, & x \in a[n+\phi, n+1], \end{cases} \qquad \rho(x) = \begin{cases} \rho_1, & x \in a[n, n+\phi], \\ \rho_2, & x \in a[n+\phi, n+1]. \end{cases} \tag{2.3}$$

Scaling x on a, the length of the unit cell in the deformed configuration and dividing by some characteristic measure of the Young's modulus of the material, let us say $\mathcal{A}_0$, we obtain the non-dimensionalized form

$$\frac{d}{dx}\left(\mathcal{A}(x)\frac{dw}{dx}\right) + \epsilon^2 d(x)w = 0 \tag{2.4}$$

where

$$\epsilon = ak_0, \qquad\qquad k_0^2 = \frac{\omega^2\rho_0}{\mathcal{A}_0}, \tag{2.5}$$

and where ρ_0 is some characteristic measure of mass density. Also,

$$\mathcal{A}(x) = \begin{cases} \alpha_1 = \dfrac{\mathcal{A}_1}{\mathcal{A}_0}, & x \in [n, n+\phi], \\ \alpha_2 = \dfrac{\mathcal{A}_2}{\mathcal{A}_0}, & x \in [n+\phi, n+1], \end{cases} \tag{2.6}$$

$$d(x) = \begin{cases} d_1 = \dfrac{\rho_1}{\rho_0}, & x \in [n, n+\phi], \\ d_2 = \dfrac{\rho_2}{\rho_0}, & x \in [n+\phi, n+1]. \end{cases} \tag{2.7}$$

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We will show later that the equation for incremental displacements w superposed on an initial finite deformation of a two-phase composite bar is of the form (2.4) which is precisely the reason why we are interested in homogenizing equations of this form. We do not wish to scale material properties on either of $\mathcal{A}_1$ or $\mathcal{A}_2$ since these will vary with deformation. Instead we will scale on some fixed property of the material such as an *initial* linear elastic Young's modulus before the nonlinear deformation takes place. Similarly for the density.

A condition for homogenized incremental waves to propagate in the bar is

$$\epsilon = a k_0 = \left(\frac{a \rho_0}{\mathcal{A}_0} \right) \omega \ll 1 \tag{2.8}$$

and therefore since it is clear that ϵ is a non-dimensionalized frequency, we see that this condition corresponds to us being in the low frequency regime.

Boundary conditions on the interfaces between phases are the continuity of incremental displacement and normal incremental stress:

$$[w]_-^+ = 0, \qquad\qquad [\sigma]_-^+ = 0, \qquad\qquad \text{on } x = n, n + \phi. \tag{2.9}$$

We will apply the method of asymptotic homogenization in order to obtain an effective wave equation of the form

$$\frac{\partial}{\partial x} \left(\mathcal{A}_* \frac{\partial w_*}{\partial x} \right) + d_* w_* = 0, \tag{2.10}$$

which as just discussed is applicable only in the low frequency regime, $\epsilon \ll 1$. The scalar $\mathcal{A}_*$ will then be known as the effective incremental Young's modulus, d_* is the effective density of the deformed medium and the leading order (or homogenized) displacement field w_* is (explicitly) independent of the microscale as we show shortly.

Following the scheme of asymptotic homogenization, we introduce the following independent lengthscales [1], [37]

$$x = \xi, \qquad\qquad L(\epsilon)x = z, \tag{2.11}$$

where

$$L(\epsilon) = \epsilon + L_2 \epsilon^2 + ... \tag{2.12}$$

is some asymptotic expansion in ϵ and $L_2 \in \mathbb{R}$. It turns out that the coefficient L_2 plays no part in the *leading order* homogenization analysis; it does play a part if we look for dispersive corrections to the effective incremental wavenumber, being chosen to knock out secular terms in the governing equations. We therefore find that

$$\frac{d}{dx} = \frac{\partial}{\partial \xi} + L(\epsilon) \frac{\partial}{\partial z}, \tag{2.13}$$

and we expand displacements in the asymptotic form

$$w = w_0 + \epsilon w_1 + \epsilon^2 w_2 + ... \quad . \tag{2.14}$$

Finally, since the structure is periodic on the ξ lengthscale, we impose the periodicity conditions

$$\frac{\partial^m w}{\partial \xi^m}(z, \xi) = \frac{\partial^m w}{\partial \xi^m}(z, \xi + n) \tag{2.15}$$

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where $m \in \mathbb{N}$ and $n \in \mathbb{Z}$.

The method then proceeds by substituting (2.13) and (2.14) into (2.4), (2.9) and (2.15) which results in a hierarchy of problems, one corresponding to each order in ϵ. The leading order ($O(\epsilon^0)$) problem leads us to the conclusion that w_0 is (explicitly) independent of the microscale, i.e.

$$w_0(\xi, z) = w_*(z). \tag{2.16}$$

In order to solve the $O(\epsilon)$ problem for piecewise constant $\mathcal{A}$, we choose a solution of the form

$$w_1^r(\xi, z)\Big|_n = [(\xi - \phi - n)\mathcal{B}_r + n\mathcal{C}]\frac{\partial w_*}{\partial z}(z) \tag{2.17}$$

in the nth cell (ensuring periodicity), where r refers to the rth phase ($r = 1, 2$) and

$$\mathcal{C} = \phi\mathcal{B}_1 + (1 - \phi)\mathcal{B}_2, \tag{2.18}$$

$$\mathcal{B}_r = \frac{\mathcal{A}_*}{\alpha_r} - 1, \tag{2.19}$$

$$\mathcal{A}_* = \frac{\alpha_1\alpha_2}{(1 - \phi)\alpha_1 + \phi\alpha_2}. \tag{2.20}$$

Following the asymptotic homogenization procedure as in [37], we integrate the $O(\epsilon^2)$ equation over phase 1 and then over phase 2 and sum the resulting equations. On imposing the appropriate periodicity conditions (2.15) and boundary conditions (2.9) on n and $n + \phi$, we find that the resulting homogenized wave equation is given by

$$\frac{\partial}{\partial z}\left(\mathcal{A}_*\frac{\partial w_*}{\partial z}\right) + d_*w_* = 0. \tag{2.21}$$

This is the form we were seeking, as in (2.10). We can therefore identify the effective incremental Young's modulus as $\mathcal{A}_*$ in (2.20) and the effective density in the deformed composite as

$$d_* = \phi d_1 + (1 - \phi)d_2. \tag{2.22}$$

Note that we will usually choose the scalings $\mathcal{A}_0 = \mathcal{A}_2^0$ and $\rho_0 = \rho_2^0$, i.e. the Young's modulus and density of phase two in the *undeformed* configuration. Thus for a composite bar *without* pre-stress

$$\mathcal{A}_* = \frac{\alpha_0}{(1 - \phi)\alpha_0 + \phi}, \qquad\qquad d_* = (1 - \phi) + \phi d_0, \tag{2.23}$$

where

$$\alpha_0 = \frac{\mathcal{A}_1^0}{\mathcal{A}_2^0}, \qquad\qquad d_0 = \frac{\rho_1^0}{\rho_2^0} \tag{2.24}$$

which are standard results.

3   Inducing pre-stress in the composite

Let us now consider the finite deformation of the two-phase medium and in order to do so we need to pose a constitutive law for the phases comprising the composite. As such we introduce the *strain energy function* $W = W(\lambda)$ where λ is the principal stretch associated with the one dimensional deformation [33].

In order to induce pre-stress in the *infinite* one dimensional bar, we will insist on a given deformation (and thus induced pre-strain) at the ends of the periodic cells. As such let us insist on the deformations

$$u(n) = (2n - 1)U \tag{3.1}$$

so that U is a deformation parameter and its magnitude corresponds to the amount of pre-strain induced. For example, since the cell has unit length in the *deformed* configuration, the percentage extension or compression of the bar is $2|U|/(1-2U)\%$. On imposing these displacements piecewise constant stretches are induced in the cell (independent of the cell position n and dependent only on the phase r) and the deformation can thus be described in the form

$$x = \lambda_r X + \gamma_r^n, \qquad r = 1, 2, \tag{3.2}$$

where γ_r^n is a constant in the nth cell, required for compatibility and corresponds to rigid body motion.

The important aspect of this deformation is that the value of λ_r is the same in each of the cells (for fixed r). The displacement u and Cauchy stress Σ in the deformed configuration are therefore given by

$$u(x) = x - X(x) = \frac{(\lambda_r - 1)}{\lambda_r}(x + \gamma_r^n), \qquad \Sigma(x) = \frac{\partial W}{\partial \lambda}(\lambda_r). \tag{3.3}$$

The principal stretch in the rth phase is clearly given by λ_r. Since these are piecewise constant, the equilibrium equation is therefore trivially satisfied by this deformation.

The values of λ_r and γ_r^n as functions of U are obtained by solving the (cell) problem

$$u(n) = (2n - 1)U, \qquad\qquad u(n + 1) = (2n + 1)U, \tag{3.4}$$

$$[u(n + \phi)]_-^+ = 0, \qquad\qquad [\Sigma(n + \phi)]_-^+ = \left[\frac{\partial W}{\partial \lambda}(n + \phi)\right]_-^+ = 0, \tag{3.5}$$

where the notation $[f(n)]_-^+$ indicates the jump in $f(x)$ at $x = n$.

Note that ϕ is the volume fraction of phase 1 in the deformed composite and this is related to the volume fraction of phase 1 in the *undeformed* composite, say ϕ_0 (which is the quantity which would be known in practice), by

$$\phi = \frac{\lambda_1 \phi_0}{\lambda_1 \phi_0 + \lambda_2(1 - \phi_0)}. \tag{3.6}$$

Since the problem is one dimensional, it means that the deformation is piecewise homogeneous. In higher dimensions the deformation of a composite will certainly *not* be homogeneous and will thus lead to more complicated behaviour. The main simplification brought about by considering the one dimensional problem is therefore that the incremental Young's modulus is piecewise constant and thus the effective incremental Young's modulus in the pre-stressed state can be determined simply by using the formula (2.20) as we shall now discuss.

4 Propagation of incremental elastic waves

Given the initial deformation considered above, we would like to find the effective properties of longitudinal linear elastic waves propagating along the pre-stressed bar. As such let us invoke

the small on large theory [33], denoting these incremental small displacements by w, which are governed by the equation of motion

$$\frac{\partial \sigma}{\partial x} = \rho \frac{\partial^2 w}{\partial t^2} \tag{4.1}$$

where the incremental Cauchy stress $\sigma = \sigma(x)$ is related to the incremental strain $e = dw/dx$ by

$$\sigma = \hat{\mathcal{A}}(x)e, \tag{4.2}$$

and $\hat{\mathcal{A}}$ is the piecewise constant incremental Young's modulus induced by the pre-stress described in the previous section. Following the small on large theory restricted to one dimension, this is given by[1]

$$\hat{\mathcal{A}}(x) = \lambda_r \frac{\partial^2 W(x, \lambda_r)}{\partial \lambda^2} \tag{4.3}$$

where λ_r is the stretch in the rth phase. Although $\hat{\mathcal{A}}(x)$ is piecewise constant, it is rapidly varying with respect to the macroscale (the wavelength of the envisaged propagating waves) and therefore it is of interest to homogenize this quantity in order to find the effective incremental Young's modulus of the pre-stressed composite bar and thus its response to low frequency excitation. Note again that this is not of use at higher frequencies; we analyse this regime in section 5.

Using (4.2) in (4.1), the equation of motion of time harmonic infinitesimal disturbances of frequency ω in the deformed medium is therefore

$$\frac{d}{dx}\left(\hat{\mathcal{A}}(x)\frac{dw}{dx}\right) + \omega^2 \rho(x)w = 0 \tag{4.4}$$

which is clearly of the same form as that considered in (2.2). Therefore in order to derive the effective wave equation let us consider the non-dimensional form of the equation of motion, i.e. equation (2.4):

$$\frac{d}{dx}\left(\mathcal{A}(x)\frac{dw}{dx}\right) + \epsilon^2 d(x)w = 0, \tag{4.5}$$

where

$$\mathcal{A}(x) = \begin{cases} \alpha_1 = \dfrac{1}{\mathcal{A}_0}\dfrac{d^2 W_1}{d\lambda^2}(\lambda_1), & x \in [n, n+\phi], \\ \alpha_2 = \dfrac{1}{\mathcal{A}_0}\dfrac{d^2 W_2}{d\lambda^2}(\lambda_2), & x \in [n+\phi, n+1], \end{cases} \tag{4.6}$$

and where $\mathcal{A}_0$ is the scaling used. We have denoted the strain energy functions associated with the rth phase by W_j. In order to find the effective incremental properties of the composite medium we can appeal to the theory presented in section 2. Since (4.5) has the same form as (2.4), it is homogenized in exactly the same manner, leading to a homogenized equation as in (2.21) with effective density (2.22), i.e.

$$d_* = \phi d_1 + (1-\phi)d_2 \tag{4.7}$$

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[1]We can derive this as follows. A unit length of (homogeneous) material governed by strain energy function $W(\lambda)$ is stretched to a length λ_0 and then incrementally to $\lambda_0 + \delta$. Associated with this incremental deformation is an incremental stress $\sigma = \delta\, W''(\lambda_0)$ (where $' = d/d\lambda$) and incremental strain δ/λ_0. Thus the incremental modulus is $\mathcal{A} = \sigma/e = \lambda_0 W''(\lambda_0)$.

and effective incremental elastic moduli (2.20), i.e.

$$\mathcal{A}_* = \frac{\alpha_1 \alpha_2}{(1-\phi)\alpha_1 + \phi\alpha_2} \tag{4.8}$$

where α_1 and α_2 in this case are given by (4.6). Before we can quantitatively determine these effective incremental properties we need to state the constitutive behaviour of the nonlinear elastic phases. This is achieved by discussing appropriate strain energy functions.

4.1 Specific strain energy functions

A general form of strain energy function in terms of principal *stretches* for a *three* dimensional material which is consistent with linear elasticity is

$$W = \mu \sum_m C_m W_m + \frac{1}{2}\Lambda(J-1)^2 F(J) \tag{4.9}$$

where F is a function such that $F(1) = 1$, μ and Λ are the Lamé moduli of the material for small deformations. Also, $J = \lambda_1 \lambda_2 \lambda_3$ where λ_j are the principal stretches and

$$W_m = \frac{1}{2m^2}\left(\lambda_1^{2m} + \lambda_2^{2m} + \lambda_3^{2m} - 3 - 2m\log J\right) \qquad m \neq 0, \tag{4.10}$$

$$W_0 = \lim_{m \to 0} W_m = \left(\log^2 \lambda_1 + \log^2 \lambda_2 + \log^2 \lambda_3\right) \tag{4.11}$$

and we note that $m \in \mathbb{R}$ (i.e. it does not have to be an integer). Furthermore, the coefficients C_m are restricted by the condition that

$$\sum_m C_m = 1 \tag{4.12}$$

which is necessary for consistency with linear elasticity.[2] This form can be used to obtain general expressions for compressible materials (such as those given in [20] for example). These include Varga materials [17], [19] and Hadamard materials. It may also be used for incompressible materials by taking appropriate limits.

Restricting ourselves to the one dimensional case, let us write down the appropriate analogue of (4.9), having the form

$$W = \mathcal{A}_0 \sum_m C_m W_m \tag{4.13}$$

where

$$W_m = \frac{1}{4m^2}(\lambda^{2m} - 1 - 2m\log\lambda), \qquad m \neq 0, \tag{4.14}$$

$$W_0 = \lim_{m \to 0} W_m = \frac{1}{2m}\log^2 \lambda \tag{4.15}$$

and where (4.12) is still required so that in the linear elastic limit when $\lambda = e + 1$ where $e \ll 1$ is the linear elastic strain, we obtain

$$W = \frac{1}{2}\mathcal{A}_0 e^2 + O(e^3). \tag{4.16}$$

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[2]This expression for the strain energy function was given in lectures by Professor Rodney Hill in Cambridge in 1973 (I learn from Dr. D.J. Allwright) but seems not to appear in any textbook on continuum mechanics, despite being extremely useful.

Now, let us note that if we choose

$$C_{1/2} = -1, \qquad\qquad C_1 = 2, \qquad\qquad C_m = 0 \quad \text{all other } m, \qquad (4.17)$$

we obtain

$$W = W_{\text{lin}} = \frac{1}{2}\mathcal{A}_0(\lambda - 1)^2 \qquad (4.18)$$

and therefore this strain energy function corresponds to *linear* constitutive behaviour. Let us also consider the following strain energy functions,

$$W_a = \frac{1}{4}\mathcal{A}_0(\lambda^2 - 1 - 2\log\lambda), \qquad (4.19)$$

$$W_b = \frac{1}{4}C_1\mathcal{A}_0(\lambda^2 - 1 - 2\log\lambda) + \frac{1}{16}C_2\mathcal{A}_0(\lambda^4 - 1 - 4\log\lambda), \qquad (4.20)$$

$$W_c = \frac{1}{4}\mathcal{A}_0\left(C_{-1}\left(\frac{1}{\lambda^2} - 1 + 2\log\lambda\right) + C_1\left(\lambda^2 - 1 - 2\log\lambda\right)\right), \qquad (4.21)$$

all of which give rise to nonlinear constitutive behaviour. This is illustrated in figure 3, where we plot the induced pre-stress Σ (scaled on $\mathcal{A}_0$) as a function of principal stretch in a *homogeneous* material.

Note that in rubbers, we have the property that the usual Lamé moduli behave according to $\Lambda \gg \mu$ and usually by three or four orders of magnitude. Thus, the initial Young's modulus of the material is related to the shear modulus by

$$\mathcal{A}_0 = \mu\frac{3\Lambda + 2\mu}{\Lambda + \mu} \rightarrow 3\mu, \qquad (4.22)$$
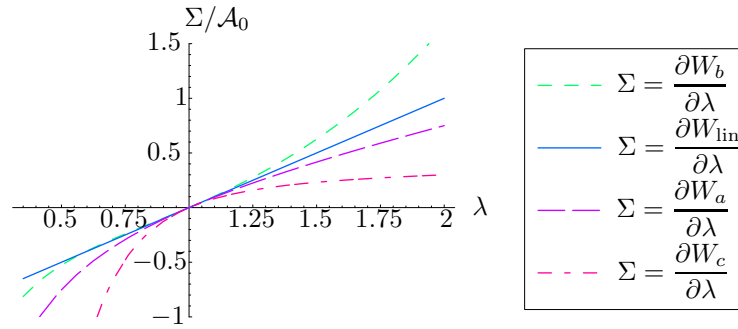
and $\mu = O(10^5)$–$O(10^6)$ Pascals.

Figure 3: Figure illustrating stress (scaled on the initial Young's modulus $\mathcal{A}_0$) as a function of stretch in a one dimensional nonlinear elastic *homogeneous* material. For W_b we use the coefficient values $C_1 = 0.2, C_2 = 0.8$ and for W_c we use $C_{-1} = 0.2, C_1 = 0.8$. The function W_b in particular has the characteristic nonlinear elastic properties of rubber for large deformation.

4.2 Homogenized density

We have shown in (4.7) that the effective density of the composite is given by

$$d_* = \phi d_1 + (1 - \phi)d_2 \qquad (4.23)$$

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where $d_j = \rho_j/\rho_0$ and ρ_0 is the chosen scaling. Since it is the *initial* material parameters in the undeformed configuration which are known in practice, we are interested in determining the functional dependence of the effective density on these properties. As such let us choose $\rho_0 = \rho_2^0$, the density of phase 2 in its undeformed state and therefore (4.23) is

$$d_* = \frac{\rho_2}{\rho_2^0}(1 - \phi) + \frac{\rho_1}{\rho_2^0}\phi. \tag{4.24}$$

Given densities ρ_1^0 and ρ_2^0 in the undeformed configuration, the densities in the *deformed* configuration are clearly given by

$$\rho_1 = \frac{1}{\lambda_1}\rho_1^0, \qquad\qquad \rho_2 = \frac{1}{\lambda_2}\rho_2^0. \tag{4.25}$$

Volume fractions in the deformed configuration are related to those in the rest state by (3.6) and thus the effective density as a function of initial parameters and the principal stretches is given by

$$d_*(\phi_0, d_0, \lambda_1, \lambda_2) = \frac{(1 - \phi_0) + \phi_0 d_0}{\lambda_1 \phi_0 + \lambda_2(1 - \phi_0)} \tag{4.26}$$

where $d_0 = \rho_1^0/\rho_2^0$ was defined in (2.24). This expression is clearly correct since $(\lambda_1\phi_0 + \lambda_2(1 - \phi_0))$ is the stretch of the material due to the initial deformation and $((1 - \phi_0) + \phi_0 d_0)$ is the effective density in the initial configuration.

In figures 4 and 5 we plot the effective density for fixed U and ϕ_0 respectively with parameters $d_0 = 10$ and $\alpha_0 = 10$ (α_0 was defined in (2.24)) and for two alternative strain energy functions (W_a and W_b given above). In figure 4 we see the linear dependence on ϕ_0 when there is no deformation ($U = 0$). Compression and extension increases or reduces the effective density relative to the material's rest configuration as we would expect. Note from figure 5 that very high compression leads to a rapid increase in effective density.
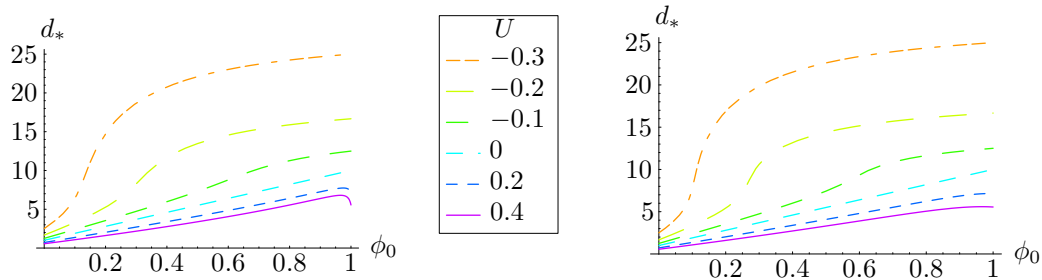

Figure 4: Figure showing the effective density of the composite bar as a function of initial volume fraction ϕ_0 of phase 1 material, given specific displacement parameters U. We take the parameter set $d_0 = 10$, $\alpha_0 = 10$. In the figure on the left each phase behaves according to the strain energy function W_a whereas on the right, they behave according to W_b with $C_1 = 0.2, C_2 = 0.8$. Note the (expected) linear behaviour when $U = 0$, i.e. no initial deformation. As should be anticipated, extension of the bar reduces its density whereas compression leads to a denser material. Dependence on the volume fraction is, however, no longer linear.
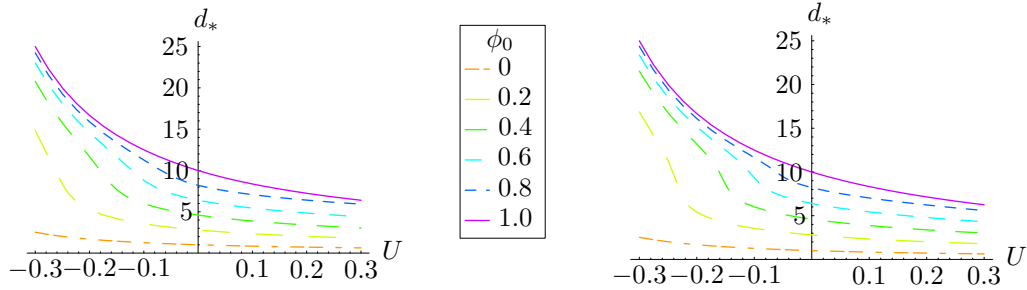
Figure 5: Figure showing the effective density of the composite bar as a function of the displacement parameter U, for specific initial volume fractions ϕ_0. We take the parameter set $d_0 = 10$, $\alpha_0 = 10$. In the figure on the left each phase behaves according to the strain energy function W_a whereas on the right, they behave according to W_b with $C_1 = 0.2, C_2 = 0.8$.

4.3 Homogenized incremental elastic moduli

We shall plot the effective incremental Young's modulus $\mathcal{A}_*$ as a function of ϕ_0 and U for a variety of strain energy functions and for specific parameters d_0 and α_0.

In figures 6 and 7 we exhibit the dependence of $\mathcal{A}_*$ on ϕ_0 and U respectively when both phases of the material behave according to the strain energy function W_a. Note that $\mathcal{A}_*$ is a non-monotonic function of U and small compressions can lead to fairly substantial increases in $\mathcal{A}_*$ whereas extensions do not significantly affect the effective modulus.

In figure 8 we can see how a change in the strain energy function significantly alters the elastic response of the composite bar compared with that in figure 6 for example. Specifically for compressions, $\mathcal{A}_*$ increases significantly for small changes in the inclusion volume fraction. As we move into the extension region however, $\mathcal{A}_*$ increases gradually as a function of ϕ_0. We observe this behaviour more clearly in figure 9 where we plot curves for fixed ϕ_0. The non-monotonic behaviour for compression is noted. This is attributed completely to the change in the type of strain energy function.

The combined effect of phase 1 behaving according to W_a and phase 2 behaving according to W_b is seen in figures 10 and 11. We note in particular the interesting behaviour of $\mathcal{A}_*$ as a function of U for fixed ϕ_0. For small (finite) compressions the effective Young's modulus can vary rapidly meaning that extremely varied material behaviour can be achieved by pre-stressing the material.
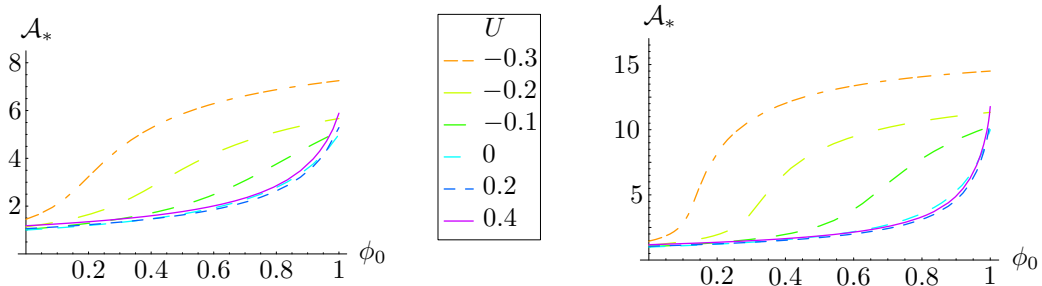

Figure 6: Figure illustrating the dependence of $\mathcal{A}_*$ on ϕ_0 for fixed U. Both phases behave according to W_a and we choose the parameters $\alpha_0 = 5$ (left) and $\alpha_0 = 10$ (right).
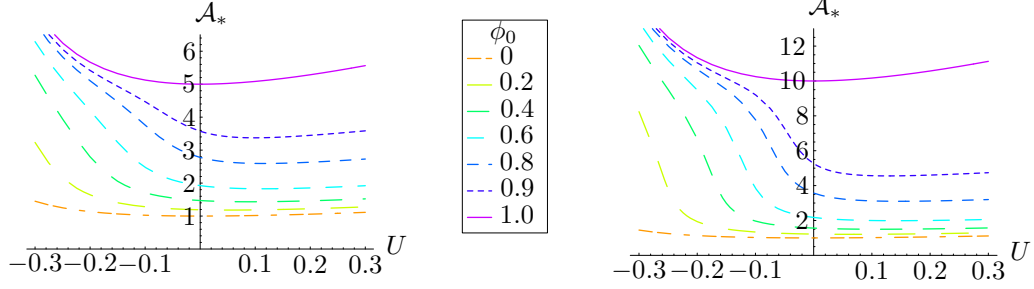
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Figure 7: Figure illustrating the dependence of $\mathcal{A}_*$ on U for fixed ϕ_0. Both phases behave according to W_a and we choose the parameters $\alpha_0 = 5$ (left) and $\alpha_0 = 10$ (right).
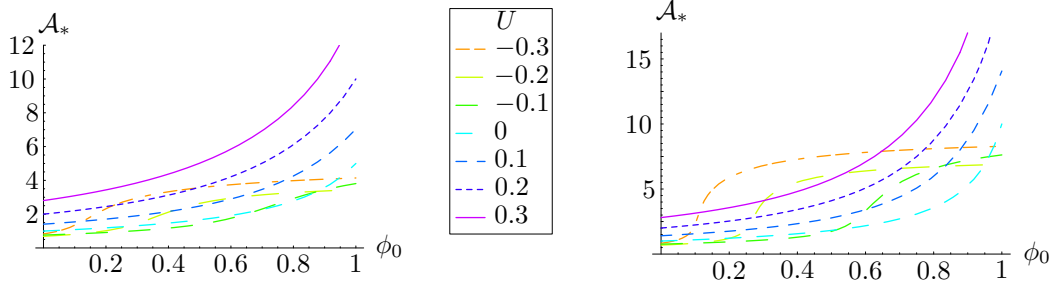

Figure 8: Figure illustrating the dependence of $\mathcal{A}_*$ on ϕ_0 for fixed U. Both phases behave according to W_b with $C_1 = 0.2$ and $C_2 = 0.8$ and we choose the parameters $\alpha_0 = 5$ (left) and $\alpha_0 = 10$ (right).
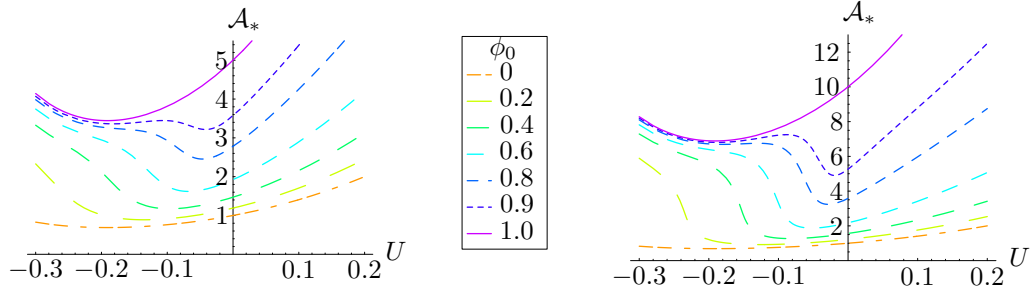

Figure 9: Figure illustrating the dependence of $\mathcal{A}_*$ on U for fixed ϕ_0. Both phases behave according to W_b with $C_1 = 0.2$ and $C_2 = 0.8$ and we choose the parameters $\alpha_0 = 5$ (left) and $\alpha_0 = 10$ (right).

4.4 Effective incremental wavespeeds

Since the effective incremental wavespeed is given by

$$c_* = \sqrt{\frac{\mathcal{A}_*}{d_*}} \tag{4.27}$$

we can see how the initial deformation affects wavespeeds by using the results of the previous two subsections and we plot the dependence of c_* on U in figures 12-14. Note in particular that
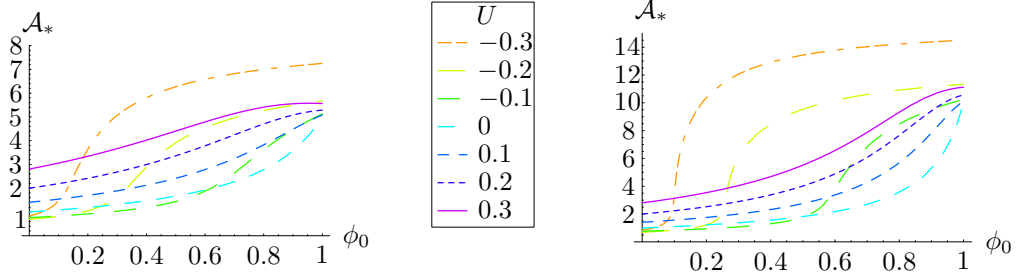
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Figure 10: Figure illustrating the dependence of $\mathcal{A}_*$ on ϕ_0 for fixed U. Phase 2 behaves according to W_b with $C_1 = 0.2$ and $C_2 = 0.8$ whereas phases 1 behaves according to W_a. We choose the parameters $\alpha_0 = 5$ (left) and $\alpha_0 = 10$ (right).
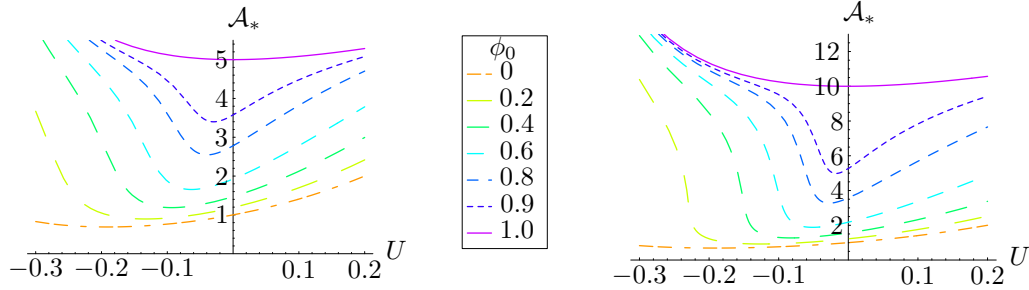

Figure 11: Figure illustrating the dependence of $\mathcal{A}_*$ on U for fixed ϕ_0. Phase 2 behaves according to W_b with $C_1 = 0.2$ and $C_2 = 0.8$ whereas phases 1 behaves according to W_a. We choose the parameters $\alpha_0 = 5$ (left) and $\alpha_0 = 10$ (right).

for the left hand pictures the $\phi_0 = 0$ and $\phi_0 = 1$ curves coincide since $d_0 = \alpha_0$ except in the case where the phases are described by different strain energy functions (figure 14). The figures on the right hand side show complex dependence of the effective wavespeed on U and ϕ_0. Note in particular the transition from the $\phi_0 = 0$ to $\phi_0 = 1$ in the figure on the right of figure 14. We note that it is *compressive* deformations which have significant effects on the incremental wavespeed.

It is easy to recognize how such material behaviour could be exploited in practice by imposing some initial deformation before sending elastic waves through the material.

5 Stop and pass bands

Thus far we have considered only low frequency behaviour where we are able to use the theory of homogenization. For higher frequencies, waves do *not* always propagate with an effective wave number. Stop bands exist which are frequency regimes where no wave propagation occurs. We shall analyse how the initial deformation affects the location of these stop bands. In the nth periodic cell, we may look for solutions z_n and w_n in the inclusion and host regions respectively,
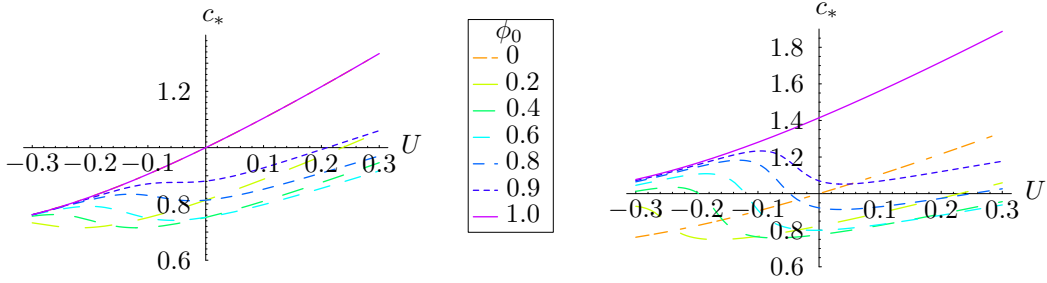
Figure 12: Figure illustrating the dependence of the incremental wavespeed c_* on U for fixed ϕ_0 when phases behave according to W_a in the parameter regime $d_0 = 5$ and $\alpha_0 = 5$ (left) and $\alpha_0 = 10$ (right). Note on the left that since $d_0 = \alpha_0$ and both phases behave according to the same *type* of strain energy function, the wavespeed for $\phi_0 = 1$ is the same as for $\phi_0 = 1$.
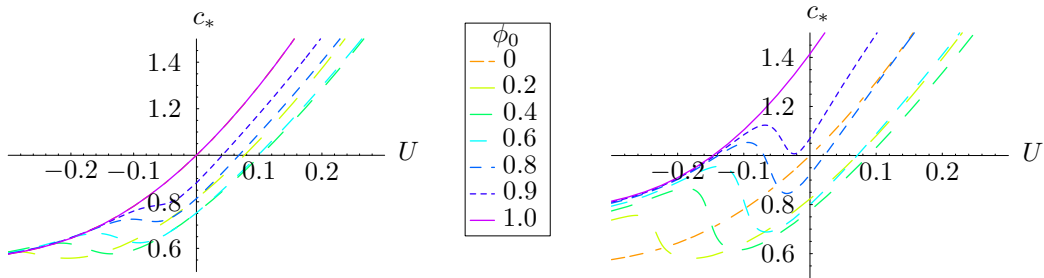

Figure 13: Figure illustrating the dependence of the incremental wavespeed c_* on U for fixed ϕ_0 when phases behave according to W_b with $C_1 = 0.2$ and $C_2 = 0.8$ in the parameter regime $d_0 = 5$ and $\alpha_0 = 5$ (left) and $\alpha_0 = 10$ (right). Note again on the left that since $d_0 = \alpha_0$ and both phases behave according to the same *type* of strain energy function, the wavespeed for $\phi_0 = 1$ is the same as for $\phi_0 = 1$.
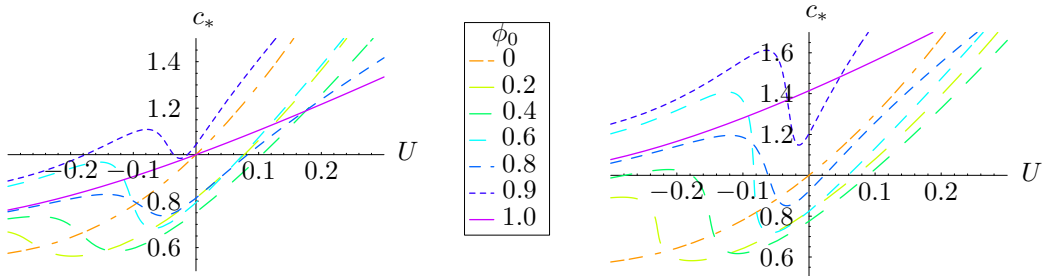

Figure 14: Figure illustrating the dependence of the incremental wavespeed c_* on U for fixed ϕ_0 when phase 2 behaves according to W_b with $C_1 = 0.2$ and $C_2 = 0.8$ and phase 1 behaves according to W_a. We consider the parameter regime $d_0 = 5$ and $\alpha_0 = 5$ (left) and $\alpha_0 = 10$ (right). In this case since each phase behaves according to a different strain energy function its dependence on U for $\phi_0 = 0$ and $\phi_0 = 1$ is different when $d_0 = \alpha_0$. Note the distinct non-monotonic behaviour in compression.

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equation (2.4) being written as

$$\frac{d^2 z_n}{dx^2} + \epsilon^2 \beta_1^2 z_n = 0, \qquad\qquad n \le x \le n + \phi, \qquad\qquad (5.1)$$

$$\frac{d^2 w_n}{dx^2} + \epsilon^2 \beta_2^2 w_n = 0, \qquad\qquad n + \phi \le x \le n + 1, \qquad\qquad (5.2)$$

where

$$\beta_1^2 = \frac{d_1}{\alpha_1} \qquad\qquad\qquad \beta_2^2 = \frac{d_2}{\alpha_2}. \qquad\qquad (5.3)$$

Continuity conditions on the interfaces at n and $n + \phi$ are

$$w_{n-1}(n) = z_n(n), \qquad\qquad w_n(n + \phi) = z_n(n + \phi), \qquad\qquad (5.4)$$

$$\frac{dw_{n-1}(n)}{dx} = \alpha \frac{dz_n(n)}{dx}, \qquad\qquad \frac{dw_n(n + \phi)}{dx} = \alpha \frac{dz_n(n + \phi)}{dx}, \qquad\qquad (5.5)$$

where $\alpha = \alpha_1/\alpha_2$.

Let us pose a travelling wave solution ansatz in the nth cell of the form [36]

$$z_n = e^{in\epsilon_*}(Ce^{i\epsilon\beta_1(x-n)} + De^{-i\epsilon\beta_1(x-n)}), \qquad\qquad (5.6)$$

$$w_n = e^{in\epsilon_*}(Ae^{i\epsilon\beta_2(x-n)} + Be^{-i\epsilon\beta_2(x-n)}), \qquad\qquad (5.7)$$

where the multiplicative factor $e^{in\epsilon_*}$ is a modulating term so that ϵ_* acts as an effective wavenumber. The remaining term accounts for the periodicity of the bar. The boundary conditions then provide a homogeneous system for the coefficients A, B, C and D and for a non-trivial solution we find that we require

$$\cos\epsilon_* = \frac{1}{4\alpha\beta}\left((1 + \alpha\beta)^2 \cos[\epsilon(\beta\phi + (1 - \phi))] - (1 - \alpha\beta)^2 \cos[\epsilon(\beta\phi - (1 - \phi))]\right), \qquad (5.8)$$

where $\beta^2 = \beta_1^2/\beta_2^2 = d/\alpha, d = d_1/d_2$. It can be checked that for low frequency waves when $\epsilon, \epsilon_* \ll 1$ we obtain

$$\epsilon_*^2 = \epsilon^2 \frac{d_*}{\mathcal{A}_*} + O(\epsilon^4) \qquad\qquad (5.9)$$

as expected and thus $k_*^2 = \epsilon_*^2/\epsilon^2 = d_*/\mathcal{A}_*$ corresponds to the homogenized wavenumber.

In figures 15-17 we plot the real and imaginary parts of ϵ_* as a function of the nondimensional frequency ϵ in order to assess how the initial deformation affects the location of stop and pass bands. Note that here we have made the assumption that μ is not frequency dependent. In reality this is not the case but the present work provides a first approximation to this more general case.

Non-zero values of $\Im(\epsilon_*)$ correspond to stop band regions where no wave propagation occurs. Note that the first pass band from $\epsilon = 0$ to the occurance of the first stop band is the homogenization regime. In each figure we consider an illustrative case with the parameter set $\phi_0 = 0.4, d_0 = 10, \alpha_0 = 20$, for the case of no deformation ($U = 0$), an extension ($U = 0.2$) and a compression ($U = -0.2$). In figure 15 we assume both phases behave according to the same *form* of strain energy function W_a but with different material parameters so that we retain inhomogeneity. Similarly in figure 16 where the phases behave according to W_b. In figure 17 we assess what happens when the materials behave according to different strain energy functions W_a and W_b. We see in each case how these affect the location of the stop bands. In particular

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we note that both extension *and* compression reduces the width of the first stop band but it is noted that compression leads to many more occurrences of stop bands over the frequency range considered.

The choice of strain energy function also considerably affects the subsequent locations of stop bands as can be seen by comparing each of the figures and noting that when one of the phases behaves according to W_b (figures 16 and 17) the second narrow stop band is almost completely eradicated.
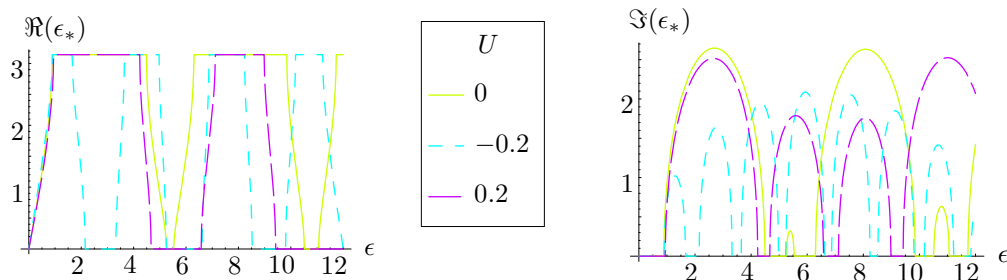

Figure 15: Figure showing the real and imaginary parts of ϵ_* as a function of ϵ for a material where both phases behave according to the same strain energy function W_a. We still retain inhomogeneity however by choosing parameters $d_0 = 10$ and $\alpha_0 = 20$ and the initial volume fraction of the inclusion phase as $\phi_0 = 0.4$. Compression *and* extension reduces the width of the first stop band but it is noted that compression leads to many more occurrences of stop bands over the frequency range considered.
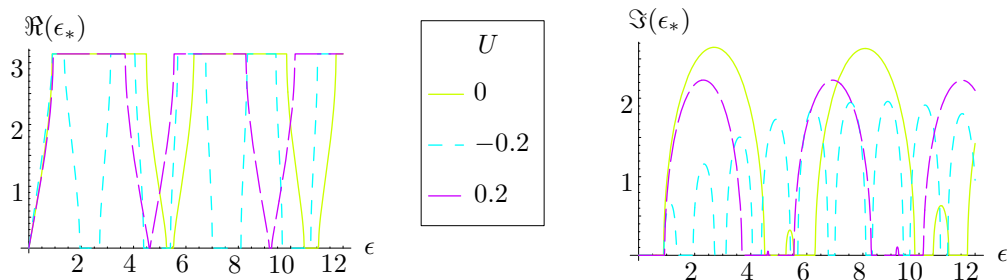

Figure 16: Figure showing the real and imaginary parts of ϵ_* as a function of ϵ for a material where both phases behave according to the same strain energy function W_b with $C_1 = 0.2$ and $C_2 = 0.8$. We retain inhomogeneity however by choosing parameters $d_0 = 10$, $\alpha_0 = 20$ and an initial volume fraction of the inclusion phase as $\phi_0 = 0.4$. Similar qualitative behaviour to figure 16 is noted, as regards how extension and compression affect the stop and pass bands. However it is noted that the second narrow stop band is almost eradicated in extension.

6   Conclusions

We have discussed how the effective incremental properties of pre-stressed composites may be determined by employing asymptotic homogenization theory in the deformed configuration. Specifically we determined the effective density and effective Young's modulus (and thus the effective wavespeed) in the separation of scales regime where the wavelength of propagating waves is much
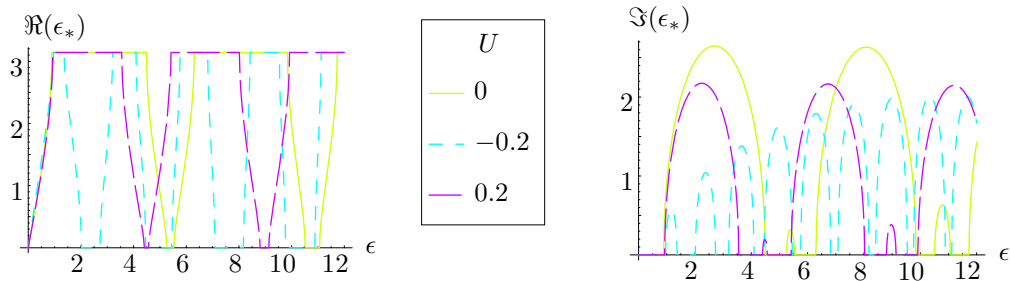
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Figure 17: Figure showing the real and imaginary parts of ϵ_* as a function of ϵ for a material where phase 2 behaves according to W_b with $C_1 = 0.2$ and $C_2 = 0.8$ and phase 1 behaves according to W_a. Parameters chosen are $d_0 = 10$, $\alpha_0 = 20$ and an initial volume fraction of the inclusion phase $\phi_0 = 0.4$. The second stop band is narrowed but not to the extent of that in figure 16.

larger than the characteristic lengthscale of the microstructure. Furthermore we discussed how the stop and pass bands of the periodic material are affected by this initial finite deformation.

We considered a one dimensional problem for ease of exposition and this meant that the effective properties could be determined semi-analytically (given the phase constitutive behaviour). Experiments are currently being performed by collaborators and will be reported on shortly. Although this was a fairly simple problem, it tells us a great deal about the behaviour of pre-stressed composites in general and particularly on how effective properties depend very strongly on the choice of strain energy function of the constituent phases, especially in the compression regime. The case of laminated composites is currently under study.

We note that we have made no mention of instabilities or buckling under pre-stress. As a first approximation in the present paper we have assumed that the material is always stable under the imposed pre-stress. Further analysis in this area is underway (perhaps more importantly in higher dimensional composites where buckling in particular can be of great interest) and our experiments will provide us with further information in this area.

Incremental moduli can themselves help to build a picture of the effective strain energy function of the material and thus could contribute to the theory of nonlinear elastic homogenization. Filled rubbers are of course renowned for their inelastic behaviour (e.g. the Mullins effect [8], [35]) due to microstructural effects. Such inelastic behaviour has thus far been incorporated via pseudo-elastic strain energy functions using damage parameters for example. It is felt that it would be of interest to attempt to link this damage parameter with specific microstructural detail in the composite medium such as interface effects between phases in order to understand its physical significance.

References

[1] N. BAKHVALOV AND G. PANASENKO. *Homogenization: Averaging Processes in Periodic Media.* Kluwer, 1989.

[2] M.C. BOYCE AND E.M. ARRUDA. Constitutive models of rubber elasticity: A review. *Rubber Chem. and Tech.*, **73**(3):504–522, 2000.

[3] A. Braides and A. Defranceschi. *Homogenization of Multiple Integrals.* Oxford University Press, 1998.

[4] G. deBotton. Transversely isotropic sequentially laminated composites in finite elasticity. *J. Mech. Phys. Sol.*, **53**:1334–1361, 2005.

[5] M Destrade and N.H. Scott. Surface waves in a deformed isotropic hyperelastic material subject to an isotropic internal contraint. *Wave Motion*, **40**(4):347–357, 2004.

[6] F. Devries. Homogenization of elastomer matrix composites: method and validation. *Composites B*, **29**:753–762, 1998.

[7] A. Dorfmann and R.W. Ogden. A pseudo-elastic model for loading, partial unloading and reloading of particle reinforced rubber. *Int. J. Sol. Struct.*, **40**:2699–2714, 2003.

[8] A. Dorfmann and R.W. Ogden. A constitutive model for the Mullins effect with permanent set in particle-reinforced rubber. *Int. J. Sol. Struct.*, **41**:1855–1878, 2004.

[9] M. Fang, R.P. Gilbert, and X. Xuming. Deriving the effective ultrasound equations for soft tissue interrogation. *Comp. Math. with Applic.*, **49**:1069–1080, 2005.

[10] Y. Fu and B. Devenish. Effects of pre-stresses on the propagation of nonlinear surface waves in an incompressible elastic half-space. *Q. J. Mech. Appl. Math.*, **49**(1):65–80, 1995.

[11] Y.C. Fung. *Biomechanics: Mechanical Properties of Living Tissue.* Springer-Verlag, New York, 2nd edition, 1993.

[12] T.C. Gasser, G.A Holzapfel, and R.W. Ogden. Hyperelastic modelling of arterial layers with distributed collagen fibre orientations. *J. Roy. Soc. Interface*, **3**:15–35, 2006.

[13] K.F. Graff. *Wave Motion in Elastic Solids.* Dover, 1991.

[14] A.E. Green and Adkins. *Large Elastic Deformations.* Clarendon Press, Oxford, 1960.

[15] A.E. Green and W. Zerna. *Theoretical Elasticity.* Oxford University Press, 1968.

[16] G. Haiat, P. Calmon, and F. Lasserre. A technique for the positioning of cracks detected by ultrasound through austenitic cladding. *Mat. Eval.*, **63**(11):1115–1121, 2005.

[17] D.M. Haughton. Inflation of thick-walled compressible elastic spherical shells. *IMA J. Appl. Math.*, **39**:259–272, 1987.

[18] G.A Holzapfel, T.C. Gasser, and R.W. Ogden. A new constitutive framework for arterial wall mechanics and a comparative study of material models. *J. Elasticity*, **61**:1–48, 2000.

[19] C.O. Horgan. Void nucleation and growth for compressible nonlinear elastic materials: an example. *Int. J. Sol. Struct.*, **29**:279–291, 1992.

[20] C.O. Horgan. Equilibrium solutions for compressible nonlinearly elastic materials. In Y.B. Fu and R.W. Ogden, editors, *Nonlinear elasticity. Theory and applications*, **283** of *LMS lecture note series*, chapter 4, pages 135–159. Cambridge University Press, 2001.

[21] T. Iwakuma and S. Nemat-Nasser. Finite elastic-plastic deformation of polycrystalline metals. *Proc. R. Soc. Lond. A*, **394**:87–119, 1984.

[22] Y.Y. KAGAN. Incremental stress and earthquakes. *Geophys. J. Int.*, **117**(2):345–364, 1994.

[23] J.D. KAPLUNOV AND G.A. ROGERSON. An asymptotically consistent model for long-wave high frequency motion in a pre-stressed elastic plate. *Math. Mech. Sol.*, **7**:581–606, 2002.

[24] B-K. KIM, S.K. YOUN, AND W-S. LEE. A constitutive model and FEA of rubber under small oscillatory load superimposed on large static deformation. *Arch. Appl. Mech.*, **73**:781–798, 2004.

[25] S. LEUNGVICHCHAROEN, A.C. WIJEYEWICKREMA, AND T. YAMAMOTO. Anti-symmetric waves in pre-stressed imperfectly bonded incompressible elastic layered composites. *Int. J. Sol. Struct.*, **41**:6873–6894, 2004.

[26] O. LOPEZ-PAMIES AND P. PONTE CASTANEDA. On the overall behaviour, microstructure evolution and macroscopic stability in reinforced rubbers at large deformations: I - Theory. *J. Mech. Phys. Sol.*, **54**(4):807–830, 2006.

[27] H. MOULINEC AND P. SUQUET. A numerical method for computing the overall response of nonlinear composites with complex microstructure. *Comput. Methods Appl. Mech. Eng.*, **157**:69–94, 1998.

[28] S. MULLER. Homogenization of nonconvex integral functionals and cellular elastic materials. *Arch. Rat. Anal. Mech.*, **99**:69–94, 1987.

[29] S. NEMAT-NASSER. *Plasticity: A Treatise on Finite Deformation of Heterogeneous Inelastic Materials.* Cambridge University Press, 2004.

[30] S. NEMAT-NASSER AND A.V. AMIRKHIZI. Finite-amplitude shear waves in pre-stressed thin elastomers. *Wave Motion*, **43**:20–28, 2005.

[31] S. NEMAT-NASSER AND M. OBATA. Rate dependent, finite elasto-plastic deformation of polycrystals. *Proc. R. Soc. Lond. A*, **407**:343–375, 1986.

[32] R.W. OGDEN. On the overall moduli of non-linear elastic composite materials. *J. Mech. Phys. Solids*, **22**:541–553, 1974.

[33] R.W. OGDEN. *Nonlinear Elastic Deformations.* Dover, New York, 1997.

[34] R.W. OGDEN AND D.G. ROXBURGH. The effect of pre-stress on the vibration and stability of elastic plates. *Int. J. Eng. Sci.*, **31**:1611–1639, 1993.

[35] R.W. OGDEN AND D.G. ROXBURGH. A pseudo-elastic model for the Mullins effect in filled rubber. *Proc. Roy. Soc. A*, **455**:2861–2878, 1999.

[36] W.J. PARNELL. *Homogenization techniques for wave propagation in composite materials.* PhD thesis, University of Manchester, 2004.

[37] W.J. PARNELL. Coupled thermoelasticity in a composite half-space. *J. Eng. Math.*, **56**:1–21, 2006.

[38] W.J. PARNELL AND I.D. ABRAHAMS. Dynamic homogenization in periodic fibre reinforced media. Quasi-static limit for SH waves. *Wave Motion*, **43**(6):474–498, 2006.

[39] G.A. ROGERSON AND K.J. SANDIFORD. The effect of finite primary deformations on harmonic waves in layered elastic media. *Int. J. Sol. Struct.*, **37**:2059–2087, 2000.

22

[40] K.N. Sawyers and R.S. Rivlin. On the speed of propagation of waves in a deformed elastic material. *J. Appl. Math. Phys. (ZAMP)*, **28**:1045–1057, 1977.

[41] K.N. Sawyers and R.S. Rivlin. The incremental shear modulus in a compressible isotropic elastic material. *J. Appl. Math. Phys. (ZAMP)*, **35**:1–11, 1984.

[42] J.C. Simo. On a fully three-dimensional finite strain viscoelastic damage model: formulation and computational aspects. *Comp. Meth. Appl. Mech. Eng.*, **60**:153–173, 1987.

[43] L.R.G. Treloar. *The Physics of Rubber Elasticity*. Oxford University Press, 3rd edition, 1975.

[44] V.V. Zalipaev, C.G. Movchan, A.B. Poulton, and R.C. McPhedran. Elastic waves and homogenization in oblique periodic structures. *Proc. R. Soc. Lond. A*, **458**:1887–1912, 2002.