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On (2+1)-dimensional hydrodynamic type systems possessing pseudopotential with movable singularities

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Abstract

A certain class of integrable hydrodynamic type systems with three independent and $N \geq 2$ dependent variables is considered. We choose the existence of a pseudopotential as a criterion of integrability. It turns out that the class of integrable systems having pseudopotentials with movable singularities is described by a functional equation, which can be solved explicitly. This allows us to construct interesting examples of integrable hydrodynamic systems for arbitrary N.

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Introduction

In the papers [1, 2, 3] a general theory of integrable systems of PDEs of the form

$$\mathbf{u}_t = A(\mathbf{u})\,\mathbf{u}_x + B(\mathbf{u})\,\mathbf{u}_y, \tag{0.1}$$

where $\mathbf{u}$ is an N-component column vector, $A(\mathbf{u})$ and $B(\mathbf{u})$ are $N\times N$-matrices, was developed. The existence of sufficiently many of the hydrodynamic reductions [4, 1] has been proposed as the definition of integrability. Unfortunately, for arbitrary N it is difficult to write down explicitly the conditions for A and B, which follows from this definition. Nevertheless, for $N = 2$ the complete set of integrability conditions have been found in the paper [2]. For $N > 2$ even the verification whether a given equation is integrable is a serious task. Any attempt for classification of integrable models based directly on this definition seems to be hopeless.

To overcome the difficulties for $N > 2$ the following two observations [2, 3] are very useful. First, under some conditions of generic position, the matrix $M = (1 + tA)^{-1}(1 + kB)$ for integrable models should be diagonalizable by a point transformation $\mathbf{u} \to \Phi(\mathbf{u})$ for generic values of the parameters t and k. If the eigen-values of M are distinct, this is equivalent to the fact that the Haantjes tensor [12] of M is identically zero. This gives rise to an overdetermined system of the first order PDEs for entries of A and B. Given a system (0.1) it is not difficult to verify whether these PDEs are satisfied or not. The simplest equations from this overdetermined system also can be useful for classification of integrable models (0.1).

The second observation made in [1] is that for $N = 2$ the integrability conditions are equivalent to the existence of the scalar pseudopotential

$$\Psi_t = f(\Psi_y, \mathbf{u}), \qquad \Psi_x = g(\Psi_y, \mathbf{u}), \tag{0.2}$$

for (0.1)[1]. The scalar pseudopotential plays an important role in the theory of the universal Whitham hierarchy [5, 6, 7]. A possible importance of pseudopotentials was also noticed in [8]. The existence of pseudopotential implies a representation of (0.1) as the commutativity condition for the corresponding characteristic vector fields

$$\frac{\partial}{\partial x} - g_\xi \frac{\partial}{\partial y} + g_y \frac{\partial}{\partial \xi}, \qquad \frac{\partial}{\partial t} - f_\xi \frac{\partial}{\partial y} + f_y \frac{\partial}{\partial \xi}, \qquad \text{where} \qquad \xi = \Psi_y.$$

For recent attempts to use similar representations for integration of dispersionless PDEs see [10, 11].

In this paper we assume that the matrix $A(\mathbf{u})$ is constant and consider integrable systems of the form

$$u_{it} = \lambda_i\, u_{ix} + \sum_{1\le j\le N} b_{ij}(\mathbf{u})\, u_{jy}, \tag{0.3}$$

[1]The latter means that that the overdetermined system (0.2) for Ψ is compatible if and only if $\mathbf{u}$ is a solution of (0.1).

where $\lambda_1, \ldots, \lambda_N$ are pairwise distinct constants. Functions b_{ij} (as well as all other functions) are supposed to be locally analytic. Note that the transformation

$$u_i \to \psi_i(u_i) \tag{0.4}$$

preserves the form of the system (0.3) for arbitrary functions of one variable $\psi_i(u_i)$.

For $N = 2$ such systems were considered in [2]. Our goal is to obtain a list of the most interesting examples of integrable models (0.3) with $N > 2$. As far as we know, nobody systematically investigated such systems before us.

Example 1. Consider equation (0.3) with

$$b_{ij} = \frac{\lambda_i - \lambda_j}{u_i - u_j}\, c_j, \qquad i \neq j,$$

$$b_{ii} = -\sum_{j \neq i} b_{ij}, \tag{0.5}$$

where c_j are arbitrary constants. It is not difficult to verify that for arbitrary N this equation possesses pseudopotential (0.2), where

$$g = \sum_{i=1}^{N} c_i \log(u_i - \Psi_y), \qquad f = \sum_{i=1}^{N} c_i \lambda_i \log(u_i - \Psi_y). \tag{0.6}$$

This pseudopotential has the following structure:

$$g = \sum_{i=1}^{N} h_i(\xi, u_i), \qquad f = \sum_{i=1}^{N} \lambda_i h_i(\xi, u_i), \tag{0.7}$$

where $\xi = \Psi_y$. In Section 2 we show that (0.7) is true for any integrable equation of the form (0.3).

Notice that functions $h_i(\xi, u_i)$ in (0.6) have "moveable" singularities with respect to the variable $\xi = \Psi_y$. This means that the position of the singularity depends on $\mathbf{u}$.

In this paper we describe all equations (0.3) possessing pseudopotentials such that for any i the function $h_i(\xi, u_i)$ has a movable singularity. This leads to series of new interesting examples of integrable systems (0.3). These examples are presented in Section 1. It would be interesting to find the hydrodynamic reductions for these equations and describe the multiple waves [13, 1] in terms of these reductions.

In Sections 2-4 we deduce a functional equation describing the pseudopotentials with moveable singularities and find all it's solutions. These solutions correspond to examples of Section 1 and their degenerations.

1 Examples.

Example 2. Consider equation (0.3) given by

$$b_{ij} = c_j(\lambda_j - \lambda_i)\left(\kappa + \frac{e^{u_i - u_j}}{e^{u_i - u_j} - 1}\right), \qquad i \neq j,$$

and

$$b_{ii} = -\sum_{j \neq i} b_{ij}.$$

Here if $\kappa = -1$ or $\kappa = 0$, then c_j are arbitrary constants. For other κ the constants c_j should satisfy the following two relations

$$\sum_{i=1}^{N} c_i = 0, \qquad \sum_{i=1}^{N} \lambda_i c_i = 0. \tag{1.8}$$

It is easy to verify that for any N this equation admits a pseudopotential (0.7) with

$$h_i(\xi, u) = c_i\Big(\kappa(\xi - u) + \log(e^{\xi - u} - 1)\Big). \tag{1.9}$$

Example 3. Consider equation (0.3) with

$$b_{ij} = c_j(\lambda_j - \lambda_i)\frac{u_i - u_j + 1}{u_i - u_j} \cdot \frac{u_i}{u_j}, \qquad i \neq j,$$

$$b_{ii} = \sum_{j \neq i} c_j(\lambda_j - \lambda_i)\left(\frac{1}{u_j - u_i} + \log(u_j)\right),$$

where c_j are arbitrary constants satisfying conditions (1.8). This equation has a pseudopotential (0.7) with

$$h_i(\xi, u) = c_i\Big((\xi + 1)\log(u) - \log(u - \xi)\Big). \tag{1.10}$$

The equation from Example 1 is a particular case of the following model:

Example 4. Let

$$b_{ij} = \frac{(\lambda_i - \lambda_j)c_j P(u_i) M(u_j)}{u_i - u_j}, \qquad i \neq j,$$

$$b_{ii} = -\sum_{j \neq i} \frac{(\lambda_i - \lambda_j)c_j P(u_j) M(u_j)}{u_i - u_j} - \sum_{j \neq i} (\lambda_i - \lambda_j) c_j B(u_j),$$

where c_j are arbitrary constants, and the functions B, M are defined by quadratures from

$$B' = (k_3 x + k_2 + z_1)\, M, \qquad M' = M\frac{-k_3 x^2 + z_1 x + z_0}{P}.$$

Here

$$P(x) = k_3x^3 + k_2x^2 + k_1x + k_0$$

is an arbitrary polynomial of degree not greater than 3, z_1, z_0 are arbitrary constants.

The corresponding equation (0.3) possesses pseudopotential (0.7), where

$$h_u(\xi, u) = -\frac{M(u)}{u-\phi(\xi)} \cdot \frac{1}{M(\phi(\xi))}, \qquad h_\xi(\xi, u) = -B(u) + \frac{P(u)M(u)}{u-\phi(\xi)},$$

$$\phi' = P(\phi)\, M(\phi).$$

For any given P and z_1, z_0 the equations for $\phi(\xi)$ and $h(\xi, u)$ can be easily solved by quadratures.

Using admissible transformations

$$u_i \to \frac{au_i + b}{cu_i + d}, \qquad i = 1, \dots, N,$$

one can reduce the polynomial P to a canonical form. For example, if all three roots of P are distinct, then without loss of generality we may put $P(x) = x(x-1)$. In this case

$$M(x) = x^{s_1}(x-1)^{s_2},$$

where $s_1 = -z_0, s_2 = z_0 + z_1$, and

$$B(x) = (s_1 + s_2 + 1) \int x^{s_1}(x-1)^{s_2} dx.$$

It is not difficult to find that

$$h(\xi, u) = \frac{1}{\phi(\xi)^{s_1}(\phi(\xi)-1)^{s_2}} \int_c^u \frac{t^{s_1}(t-1)^{s_2}}{\phi(\xi)-t} dt - c^{s_1+1}(c-1)^{s_2+1} \int \frac{d\xi}{\phi(\xi)-c},$$

where $\phi' = \phi^{s_1+1}(\phi-1)^{s_2+1}$.

Other two canonical forms are $P = x$ and $P = 1$. The latter generates Example 1 if $z_1 = z_0 = 0$.

2 Pseudopotentials.

A pair of equations of the form

$$\Psi_t = f(\Psi_y, u_1, \dots, u_N), \qquad \Psi_x = g(\Psi_y, u_1, \dots, u_N), \tag{2.11}$$

with respect to unknown Ψ is called a pseudopotential for equation (0.3) if the compatibility condition $\Psi_{tx} = \Psi_{xt}$ for (2.11) is equivalent to (0.3). Differentiating (2.11), we find that this compatibility condition is given by

$$f_\xi \sum_{i=1}^N (u_i)_y\, \partial_i g + \sum_{i=1}^N (u_i)_x\, \partial_i f = g_\xi \sum_{i=1}^N (u_i)_y\, \partial_i f + \sum_{i=1}^N (u_i)_t\, \partial_i g.$$

Here and below we denote Ψ_y by ξ and $\frac{\partial}{\partial u_i}$ by ∂_i. Substituting the right hand side of (0.3) for t-derivatives and splitting with respect to x and y-derivatives, we get that for any i the following relations hold:

$$\partial_i f = \lambda_i \partial_i g, \tag{2.12}$$

$$f_\xi\, \partial_i g - g_\xi\, \partial_i f = \sum_{j=1}^{N} b_{ji} \partial_j g. \tag{2.13}$$

Since λ_i are pairwise distinct, it follows from the condition (2.12) that

$$g = \sum_{i=1}^{N} h_i(\xi, u_i),$$

and

$$f = \sum_{i=1}^{N} \lambda_i h_i(\xi, u_i) + c(\xi).$$

It is easy to see that the integration constant $c(\xi)$ can be distributed between functions h_i. Thus we have arrived at (0.7).

Substituting (0.7) into (2.13), we obtain

$$\partial_i h_i(\xi, u_i) \sum_j (\lambda_j - \lambda_i) h_{j\,\xi}(\xi, u_j) = \sum_j b_{ji}\ \partial_j h_j(\xi, u_j). \tag{2.14}$$

Remark 1. If we fix a generic value ξ_0 of the variable ξ, then it follows from (2.14) that

$$b_{ii} = \sum_j (\lambda_j - \lambda_i) \phi_j(u_j) - \sum_{j \neq i} b_{ji} \frac{s_j(u_j)}{s_i(u_i)}, \tag{2.15}$$

where $\phi_j(u_j) = h_{j\,\xi}(\xi_0, u_j)$, $s_j(u_j) = \partial_j h_j(\xi_0, u_j)$.

3 Basic functional equation.

Suppose that for any i the function $h_i(\xi, u)$ has a singularity with respect to ξ and this singularity depends on u. After a transformation of the form (0.4), we may assume that for each i the singularity of $h_i(\xi, u)$ is located on the diagonal $\xi = u$. We say that $h_i(\xi, u)$ has a singularity on the diagonal, if for fixed generic u and any $\epsilon > 0$ we have $\max\{|h_i(\xi, u)|, |\xi - u| < \epsilon\} = \infty$.

Proposition 1. *Suppose $h_i(\xi, u)$ has a singularity on the diagonal $\xi = u$ for each i. Then there exist: a function $h(\xi, u)$, functions $f_i(\xi)$ and non-zero constants c_i such that*

$$h_i(\xi, u) = c_i h(\xi, u) + f_i(\xi), \tag{3.16}$$

$$b_{ji} = (\lambda_i - \lambda_j) c_i\, \partial_i h(u_j, u_i), \qquad i \neq j, \tag{3.17}$$

$$b_{ii} = \sum_{j\neq i} (\lambda_j - \lambda_i)(c_j\, \partial_i h(u_i, u_j) + f_j'(u_i)), \tag{3.18}$$

$$h(\xi, u) = \ln(\xi - u) + \text{regular part}. \tag{3.19}$$

Moreover, the following functional equation

$$h_\xi(\xi, w)\, h_v(\xi, v) + h_v(w, v)\, h_w(\xi, w) - h_v(v, w)\, h_v(\xi, v) = \nu(\xi, v) \tag{3.20}$$

holds for some function ν.

Proof. Considering (2.14) near the diagonal $\xi = u_j$, where $j \neq i$, and comparing the singularities, we obtain

$$b_{ji} = (\lambda_j - \lambda_i)\mu_j(u_j)\, \partial_i h_i(u_j, u_i), \tag{3.21}$$

where $\mu_j(u_j) = \lim_{\xi\to u_j} \frac{h_{j\,\xi}(\xi,u_j)}{\partial_j h_j(\xi,u_j)}$. We see that the function b_{ji} depends on u_i, u_j only and has the same singularity on the diagonal as $\partial_i h_i(u_j, u_i)$. Considering (2.14) near the diagonal $u_i = u_j$, comparing the singularities and using (2.15), we obtain $\partial_j h_j(\xi, u_i)s_i(u_i) = \partial_i h_i(\xi, u_i)s_j(u_i)$ or $\partial_i h_i(\xi, u_i) = \nu_i(u_i)r(\xi, u_i)$ for some functions ν_i and r. On the other hand, consider (2.14) for N generic values $\xi_1, ..., \xi_N$ of variable ξ. For each fixed $i = 1, ..., N$ we have a system of N linear equations for $b_{i1}, ..., b_{iN}$ with matrix $Q = (q_{jk})$, where $q_{jk} = \nu_j(u_j)r(\xi_k, u_j)$. This system must have a unique solution by definition of pseudopotential and therefore $\Delta = \det Q \neq 0$. It is clear that $b_{ji} = \frac{P_{ji}}{\Delta}$, where P_{ji} is regular on each diagonal $u_k = u_l$. It is easy to prove the following

Lemma. Let $\Delta(u_1, ..., u_m)$ be the determinant of an $m \times m$ matrix Q, whose entries q_{ij} have the form $q_{ij} = g_i(u_j)$ for some functions $g_1, ..., g_m$. The function Δ is not equal to zero identically iff the functions $g_1, ..., g_m$ are linearly independent. In this case $\partial_i\Delta \neq 0$ on the diagonal $u_i = u_j$ for each $i \neq j$.

From this lemma it follows that the only singularity of b_{ij} on the diagonal can be a pole of order one. Taking into account (3.21), we obtain that near $u_j = u_i$

$$\partial_i h_i(u_j, u_i) = \frac{\alpha_i(u_j)}{u_j - u_i} + \text{regular part}$$

or, after integration,

$$h_i(u_j, u_i) = -\alpha_i(u_j) \ln(u_j - u_i) + \text{regular part}.$$

Considering the singular part of (2.14) at $\xi = u_j$, we obtain $\mu_j(u_j) = -1$ and $\alpha_i'(u_j) = 0$, i.e. $\alpha_i(u_j) = -c_i$ for some constant c_i. Comparing the singularities in (2.14) at $\xi = u_i$, we find that

$$b_{ii} = \sum_{j\neq i} (\lambda_j - \lambda_i)\, \partial_i h_j(u_i, u_j).$$

Substituting this expression for b_{ii} into (2.14), we obtain

$$\sum_{j\neq i} (\lambda_j - \lambda_i) \Big(h_{j\,\xi}(\xi, u_j)\, \partial_i h_i(\xi, u_i) + \partial_i h_i(u_j, u_i)\, \partial_j h_j(\xi, u_j) - \partial_i h_j(u_i, u_j)\, \partial_i h_i(\xi, u_i) \Big) = 0. \tag{3.22}$$

Considering the singular part of (3.22) at $u_i = u_j$, we get

$$c_j(h_i(\xi,u))_u = c_i(h_j(\xi,u))_u,$$

which gives (3.16) and (3.19). Now (3.17) and (3.18) follow from (3.16) and expressions for b_{ij} and b_{ii} already obtained. Using (3.22), we arrive at the relation

$$\sum_{j\neq i}(\lambda_j-\lambda_i)c_j\Big(h_\xi(\xi,u_j)\,\partial_i h(\xi,u_i)+\partial_i h(u_j,u_i)\partial_j h(\xi,u_j)-\partial_i h(u_i,u_j)\partial_i h(\xi,u_i)+$$
$$(f_j'(\xi)-f_j'(u_i))\,\partial_i h(\xi,u_i)\Big)=0.$$

This relation gives the equation (3.20) for some function $\nu(\xi,v)$. ■

Remark 2. If a pair $h(\xi,v)$, $\nu(\xi,v)$ is a solution of (3.20), then

$$\tilde{h}(\xi,v)=h(\xi,v)+f(\xi),\qquad \tilde{\nu}(\xi,v)=\nu(\xi,v)+(f'(\xi)-f'(v))h_v(\xi,v) \tag{3.23}$$

is also a solution of (3.20). Therefore, if $\nu(\xi,v)$ has a form $(g(\xi)-g(v))h_v(\xi,v)$ for some function g, then we can bring $\nu(\xi,v)$ to 0 adding to $h(\xi,v)$ a suitable function of ξ.

Remark 3. Without loss of generality, we can assume that $f_i(\xi)=f_j(\xi)$ for all i,j. Indeed, only the linear combinations $\sum_i f_i(\xi)$ and $\sum_i \lambda_i f_i(\xi)$ appear in (0.7). Furthermore, according to Remark 2, we may put $f_i(\xi)=0$.

Remark 4. It follows from (3.17), (3.20) that for any $i\neq j$ the function $b=b_{ij}(u_i,u_j)$ satisfies the following functional equation

$$b(w,v)b_w(x,w)-b(x,v)b_v(v,w)+b(x,w)b_w(w,v)+b(x,v)b_x(x,w)=0.$$

Remark 5. It follows from (3.17), (3.18) that the equation (0.3) defined by (3.17) and (3.18) with $f_i(\xi)=0$ can be written in the following divergent form:

$$u_{it}=\lambda_i\,u_{ix}+\sigma_{iy},$$

where

$$\sigma_i=\sum_{j\neq i}(\lambda_j-\lambda_i)c_jh(u_i,u_j).$$

Thus any such equation has at least N linearly independent hydrodynamic conservation laws.

Proposition 2. *Let $h(\xi,v)$ be a solution of* (3.20) *with* $\nu(\xi,v)=0$, *then for any nonzero constants c_i the formula*

$$g=\sum_{i=1}^{N}c_ih(\xi,u_i),\qquad f=\sum_{i=1}^{N}\lambda_ic_ih(\xi,u_i) \tag{3.24}$$

defines a pseudopotential for equation (0.3) *given by* (3.17) *and* (3.18) *with* $f_i(\xi)=0$.

Let $h(\xi,v)$ be a solution of (3.20) *with $\nu(\xi,v)\neq 0$, then* (3.24) *defines pseudopotential for equation* (0.3) *given by* (3.17) *and* (3.18) *with $f_i(\xi)=0$ iff the constants c_i satisfy the relations* (1.8).

Proof. According to the previous results, (3.24) defines a pseudopotential iff

$$\sum_{j\neq i}(\lambda_j-\lambda_i)c_j\Big(h_\xi(\xi,u_j)\,\partial_i h(\xi,u_i)+\partial_i h(u_j,u_i)\,\partial_j h(\xi,u_j)-\partial_i h(u_i,u_j)\,\partial_i h(\xi,u_i)\Big)=0.$$

Substituting (3.20) into this relation, we obtain the statement of the proposition. ■

4 Classification of solutions for the functional equation.

Proposition 3. *Let a pair $h(\xi,v)$, $\nu(\xi,v)$ be a solution of* (3.20) *with asymptotic* (3.19). *Then up to substitutions of the form* (3.23) *it belongs to the following list*:

$$h(x,v)=\kappa\,(x-v)+\log(x-v),\qquad \nu(x,v)=\kappa(\kappa+1);$$

$$h(x,v)=\kappa\,(x-v)+\log(e^{x-v}-1),\qquad \nu(x,v)=\kappa(\kappa+1),$$

where κ is an arbitrary constant;

$$h(x,v)=(x+1)\log(v)-\log(v-x),\qquad \nu(x,v)=\frac{x}{v};$$

$$h(x,v)=\int_c^v\frac{P(\phi(x))\,\phi'(t)^2}{(\phi(x)-\phi(t))\,P(\phi(t))\,\phi'(x)}dt-\int\frac{\phi'(c)}{\phi(x)-\phi(c)}dx,\qquad \nu(x,v)=0.$$

Here c is a constant and the function ϕ is defined by the following differential equation:

$$\phi''=\Big(\frac{2P'(\phi)}{3P(\phi)}+\frac{Z(\phi)}{P(\phi)}\Big)\,\phi'^{\,2},$$

where

$$P(x)=k_3x^3+k_2x^2+k_1x+k_0,\qquad Z(x)=z_1x+z_0$$

are arbitrary polynomials such that $\deg P\le 3$ *and* $\deg Z\le 1$.

Proof. According to (3.19), we have an expansion of the form

$$h(w,v)=\ln(w-v)+\sum_{i=0}^{\infty}a_i(w)(w-v)^i \tag{4.25}$$

as v tends to w. To describe the solutions of the functional equation (3.20), let us investigate a set of conditions relating the functions a_i. Using expansion (4.25) for $h(v,w)$ and $h(w,v)$ and equating the coefficients of different powers of $v-w$, we obtain an infinite sequence of PDEs for the function $h(x,v)$ and the coefficients $a_i(v)$. The simplest three of these PDEs read as follows:

$$h_{vvv} - 2h_v h_{xv} + 2a_1 h_{vv} = 0, \tag{4.26}$$

$$h_{vvvv} - 3h_v h_{xvv} + 3a_1 h_{vvv} + 6(a_1' + 2a_2)h_{vv} + 3(a_1'' + 6a_2' + 12a_3)h_v = 0, \tag{4.27}$$

$$\begin{gathered} h_{vvvvv} - 4h_v h_{xvvv} + 4a_1 h_{vvvv} + 12(a_1' + 2a_2)h_{vvv} + 12(a_1'' + 4a_2' + 6a_3)h_{vv} \\ +4(a_1''' + 6a_2'' + 12a_3')h_v = 0. \end{gathered} \tag{4.28}$$

Substituting expansion (4.25) for $h(x,v)$ to (4.26), we observe that all coefficients $a_i,\ i > 2$ can be expressed as certain differential polynomials of a_1 and a_2. For example,

$$a_3 = -\frac{1}{12}(a_1'' + 2a_1 a_1' + 4a_2').$$

This means that the function $h(x,v)$ is uniquely determined by functions $a_1(v)$ and $a_2(v)$.

The expansion of equations (4.27) and (4.28) leads to differential relations between a_1 and a_2. In particular, the simplest relation following from (4.27) has the form

$$a_1''' + 6a_1 a_1'' + 6{a_1'}^2 - 6a_1^2 a_1' + 12(a_2 a_1' + a_2' a_1) = 0.$$

If $a_1 \neq 0$, this implies

$$a_2 = \frac{C - a_1'' - 6a_1 a_1' + 2a_1^3}{12 a_1} \tag{4.29}$$

for some constant C. Eliminating a_2 with the help of (4.29), we arrive at an overdetermined system of ODEs for function a_1. If $C = 0$, we find from this system that

$$a_1^2\, a_1^{(4)} + 2a_1(3a_1^2 - a_1')\, a_1^{(3)} - 4a_1\,(a_1'')^2 + 2({a_1'}^2 - 9a_1^2 a_1' + 4a_1^4)\, a_1'' - 16 a_1^3 {a_1'}^2 = 0. \tag{4.30}$$

The general solution of this equation can be written as

$$a_1(x) = -\frac{3Z(\phi(x))}{2P(\phi(x))}\,\phi'(x), \qquad \phi'' = \Big(\frac{2P'(\phi)}{3P(\phi)} + \frac{Z(\phi)}{P(\phi)}\Big)\,{\phi'}^2,$$

where

$$P(x) = k_3 x^3 + k_2 x^2 + k_1 x + k_0, \qquad Z(x) = z_1 x + z_0$$

are arbitrary polynomials such that deg $P \le 3$ and deg $Z \le 1$. For any given P and Z the equation for ϕ can be easily integrated by quadratures. Example 4 from Section 1 after transformation $u \to \phi(u)$ describes the pseudopotential generated by such a function a_1.

Let $C \neq 0$. Then a simple analysis of the ODE system for a_1 shows that either $a_1' = 0$ or

$$4(a_1')^3 + 12a_1^2\,(a_1')^2 + 12(a_1^4 - Ca_1)\,a_1' + 4a_1^6 + 4Ca_1^3 + C^2 = 0. \tag{4.31}$$

It is easy to verify that if $a_1' = 0$, then $a_i' = 0$ for any i and therefore $h(x,v) = H(x - v)$ for some function H. Solving equation (4.26), we get

$$H(x) = c_1 + c_2 x + \log(1 - e^{-c_3 x})$$

or

$$H(x) = c_1 + c_2 x + \log(c_3 x).$$

These solutions correspond to the model of Example 2 and it's degeneration.

The left hand side of (4.31) can be decomposed into three factors. Each factor gives rise to a differential equation of the form

$$a_1' + (a_1 + k)^2 = 0, \tag{4.32}$$

where k is related to the constant C from (4.29) by $C = 2k^3$. The corresponding model is described in Example 3.

The case $a_1 = 0$ should be considered separately. It is easy to get that in this case

$$a_2'' + 36\, a_2^2 = 0. \tag{4.33}$$

It turns out that it is a particular case of the model described by (4.30). In the corresponding formulas one has to put $Z = 0$. ■

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