New relations in the algebra of the Baxter Q-operators

Belavin, A. A. and Odesskii, A. V. and Usmanov, R. A.

2002

MIMS EPrint: **2007.32**

ISSN 1749-9097

Theoretical and Mathematical Physics, **130**(3): 323–350 (2002)

NEW RELATIONS IN THE ALGEBRA OF THE BAXTER Q-OPERATORS

A. A. Belavin,[1] A. V. Odesskii,[1] and R. A. Usmanov[1]

We consider irreducible cyclic representations of the algebra of monodromy matrices corresponding to the R-matrix of the six-vertex model. At roots of unity, the Baxter Q-operator can be represented as a trace of a tensor product of L-operators corresponding to one of these cyclic representations, and this operator satisfies the TQ equation. We find a new algebraic structure generated by these L-operators and consequently by the Q-operators.

1. Introduction

Baxter [1] introduced the Q-operator and used it to solve the eight-vertex model. The Q-operators form a family that commutes with a family of the transfer matrices $T(u)$ provided the TQ equation is satisfied. The latter equation relates the two families to each other and is a key for solving the model.

The expressions for the chiral Potts model Boltzmann weights that are solutions of the star–triangle relation were found in [2] (also see [3], [4]). The R-matrix S of the model can be represented as a product of four such Boltzmann weights.

The algebraic structure of the Q-operators in the particular case of the six-vertex model and its relation to the R-matrix of the chiral Potts model was revealed by Bazhanov and Stroganov [5]. At the Nth root of unity (N is prime), they found the N-dimensional cyclic representation $\mathcal{L}$ of the Yang–Baxter algebra related to the R-matrix of the usual six-vertex model. The trace over the N-dimensional quantum space of a tensor product of L-operators has the properties of the Q-operator. In particular, it satisfies the TQ equation. Tarasov [6] described irreducible cyclic representations of the algebra of monodromy matrices corresponding to the R-matrix of the six-vertex model at roots of unity.

The Q-operator recently returned to the center of attention. It was shown [7]–[10] that for some models in statistical physics, the Q-operator is a quantum analogue of the Bäcklund transformation. In [5], the R-matrix S of the chiral Potts model was derived as an operator that intertwines tensor products of two cyclic representations of the algebra of monodromy matrices being multiplied first in one order and then in the reverse order.

The four factors generating the R-matrix S of the chiral Potts model are actually intertwiners that provide for special cases of some elementary isomorphisms of cyclic representations of the algebra of the L-operators and their tensor products. In this paper, we clarify the conditions under which two cyclic representations are equivalent and find the corresponding intertwiner. We also solve the same problem for two tensor products of a pair of cyclic representations. The obtained intertwiners generalize the well-known vertex weights of the chiral Potts model and satisfy a modification of the star–triangle equations.

The plan of the paper is as follows. In Sec. 2, we introduce the notion of cyclic representations of the algebra of L-operators. In Sec. 3, we discuss different versions of cyclic representations. In Sec. 4, we derive the TQ equation. In Sec. 5, we discuss some special cases of elementary isomorphisms acting on cyclic

[1] Landau Institute for Theoretical Physics, Chernogolovka, Moscow Oblast, Russia.

Translated from Teoreticheskaya i Matematicheskaya Fizika, Vol. 130, No. 3, pp. 383–413, March, 2002. Original article submitted October 9, 2001.

representations of the algebra of monodromy matrices. In Sec. 6, we find these isomorphisms in the general case. We show that the intertwiners of these elementary isomorphisms satisfy a generalized star–triangle equation of the chiral Potts model. In Sec. 7, we present some relations in the algebra of Q-operators. In Sec. 8, we discuss perspectives for possible future investigations. In the appendices, we prove some formulas used in the paper.

2. Cyclic representations of the Yang–Baxter algebra

Following [6], we define the R-matrix

$$R(u) = \begin{pmatrix} 1-u\omega & 0 & 0 & 0 \\ 0 & \omega(1-u) & u(1-\omega) & 0 \\ 0 & 1-\omega & 1-u & 0 \\ 0 & 0 & 0 & 1-u\omega \end{pmatrix}. \tag{1}$$

It is related to the algebra $U_{\mathrm{q}}(sl_2)$ [11]–[13] and can be obtained from the R-matrix of the standard six-vertex model by a simple transformation (see Sec. 3). For brevity, we let $\mathcal{M} = \operatorname{End}\mathbb{C}^2$, and then $R(u) = \mathcal{M}\otimes\mathcal{M}$.

The algebra of monodromy matrices $\mathcal{A}$ is generated by $A(u)$, $B(u)$, $C(u)$, $D(u)$, H, and H^{-1} subject to the relations

$$\begin{gathered} R(u)\overset{1}{L}(uv)\overset{2}{L}(v) = \overset{2}{L}(v)\overset{1}{L}(uv)R(u), \\ \big[\widehat{\omega}\otimes H, L(u)\big] = 0, \qquad HH^{-1} = H^{-1}H = 1, \\ L(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \in \mathcal{M}\otimes\mathcal{A}, \qquad \widehat{\omega} = \operatorname{diag}(1,\omega). \end{gathered} \tag{2}$$

The indices 1 and 2 over L label the two-dimensional space in which the corresponding L-operator is multiplied by the R-matrix. Both L-operators act in the same quantum space.

As shown in [6], the algebra $\mathcal{A}$ admits the coproduct Δ,

$$\begin{gathered} \Delta\big(L(u)\big) = L_1(u)L_2(u) \in \mathcal{M}\otimes\mathcal{A}\otimes\mathcal{A}, \\ \Delta(H) = H\otimes H. \end{gathered}$$

The subscripts 1 and 2 here label the quantum spaces in which the corresponding L-operators act. The L-operators considered as two-dimensional matrices (each matrix element is an operator in one of the two quantum spaces) are multiplied according to the standard matrix multiplication rule. A tensor product of some representations of the algebra $\mathcal{A}$ is therefore a representation of $\mathcal{A}$ itself.

We now define the quantum determinant

$$\operatorname{det}_q L(u) = D(u)A(u\omega^{-1}) - C(u)B(u\omega^{-1}). \tag{3}$$

It can be verified that $H^{-1}\operatorname{det}_q L(u)$ is a central element of the algebra $\mathcal{A}$. Hereafter, we set $\omega^N = 1$. As shown in [6], the center of the algebra $\mathcal{A}$ is enhanced in this case because the operators

$$\langle\mathcal{O}\rangle(u) = \prod_{k=0}^{N-1}\mathcal{O}(u\omega^k), \qquad \mathcal{O} = A, B, C, D,$$

become central, and we can define the matrix of central elements,

$$\langle L\rangle = \begin{pmatrix} \langle A\rangle & \langle B\rangle \\ \langle C\rangle & \langle D\rangle \end{pmatrix}.$$

It can be shown [6] that $L = L_1L_2$ satisfies the equation

$$\Delta\bigl(\langle L\rangle\bigr) = \langle L_1\rangle\langle L_2\rangle, \qquad \langle \det_q L\rangle = \det\langle L\rangle.$$

The N-dimensional [cyclic] representation π of the algebra $\mathcal{A}$ is defined by [6]

$$\begin{aligned} &L(u,p_1,p_2) = \begin{pmatrix} c_1c_2Z - b_1b_2u & -u(b_1d_2 - c_1a_2Z)X \\ X^{-1}(d_1b_2 - a_1c_2Z) & d_1d_2 - a_1a_2\omega uZ \end{pmatrix}, \\ &H_\pi = hZ, \qquad p_i = (a_i,b_i,c_i,d_i), \quad i=1,2. \end{aligned} \tag{4}$$

The action of the operators X and Z on the standard basis in $\mathbb{C}^N$ is

$$Z|k\rangle = \omega^k|k\rangle, \qquad X|k\rangle = |k+1\rangle, \quad k = 0,\dots,N-1, \quad |N\rangle \equiv |0\rangle.$$

We also have

$$\bigl\langle L(p_1,p_2)\bigr\rangle(v) = \begin{pmatrix} c_1^Nc_2^N - b_1^Nb_2^Nv & -v(b_1^Nd_2^N - c_1^Na_2^N) \\ d_1^Nb_2^N - a_1^Nc_2^N & d_1^Nd_2^N - a_1^Na_2^Nv \end{pmatrix}.$$

Although formulas (4) contain eight parameters (in addition to the spectral parameter), the N-dimensional representation depends only on six of them because the substitution

$$\begin{aligned} &a_1 \to \lambda a_1, \qquad a_2 \to \lambda^{-1}a_2, \\ &c_1 \to \lambda c_1, \qquad c_2 \to \lambda^{-1}c_2, \\ &b_1 \to b_1, \qquad b_2 \to b_2, \qquad d_1 \to d_1, \qquad d_2 \to d_2, \end{aligned}$$

where λ is an arbitrary number, does not change the operator $L(u,p_1,p_2)$. The same is true for the substitution

$$\begin{aligned} &b_1 \to \lambda b_1, \qquad b_2 \to \lambda^{-1}b_2, \\ &d_1 \to \lambda d_1, \qquad d_2 \to \lambda^{-1}d_2, \\ &a_1 \to a_1, \qquad a_2 \to a_2, \qquad c_1 \to c_1, \qquad c_2 \to c_2. \end{aligned}$$

Moreover, the projective equivalence class of the L-operators depends on only four parameters because

$$L(\lambda p_1,p_2) = \lambda L(p_1,p_2), \qquad L(p_1,\mu p_2) = \mu L(p_1,p_2),$$

where λ and μ are arbitrary numbers.

The two representations $L_1(u, p_1, p_2)L_2(u, p_3, p_4)$ and $L_2(u, p_3, p_4)L_1(u, p_1, p_2)$ are equivalent if and only if p_i, $i = 1, 2, 3, 4$, can be chosen such that they satisfy the conditions

$$\frac{a_i^N \pm b_i^N}{c_i^N \pm d_i^N} = \lambda_\pm, \tag{5}$$

where the $\lambda_\pm$ are independent of i (see Appendix B). The intertwiner given by the equation

$$\mathbf{S}(p_1, p_2, p_3, p_4)L_1(u, p_1, p_2)L_2(u, p_3, p_4) = L_1(u, p_3, p_4)L_2(u, p_1, p_2)\mathbf{S}(p_1, p_2, p_3, p_4) \tag{6}$$

can be explicitly expressed through the Boltzmann weights W_{pq} and $\overline{W}_{pq}$ of the chiral Potts model:

$$\begin{aligned} &\mathbf{S}(p_1, p_2, p_3, p_4) = \mathbf{P}F(p_1, p_4; X_1X_2^{-1})G(p_1, p_3; Z_1)G(p_2, p_4; Z_2)F(p_2, p_3; X_1X_2^{-1}), \\ &G(p, q; \omega^k) = W_{pq}(k), \qquad F(p, q; \omega^k) = \sum_{l=1}^{N} \omega^{kl}\overline{W}_{pq}(l), \end{aligned} \tag{7}$$

where $\mathbf{P}$ is the standard permutation operator interchanging the spaces 1 and 2.

3. Cyclic representations of the Yang–Baxter algebra in the Bazhanov–Stroganov form

Together with the L-operators introduced in the previous section, we can consider their version related to another choice of the R-matrix. We now consider the standard R-matrix of the ice model,

$$R_{\text{ice}}(x) = \begin{bmatrix} x\omega_1 - x^{-1}\omega_1^{-1} & 0 & 0 & 0 \\ 0 & x - x^{-1} & \omega_1 - \omega_1^{-1} & 0 \\ 0 & \omega_1 - \omega_1^{-1} & x - x^{-1} & 0 \\ 0 & 0 & 0 & x\omega_1 - x^{-1}\omega_1^{-1} \end{bmatrix} \tag{8}$$

and the corresponding relations in the Yang–Baxter algebra,

$$R_{\text{ice}}(x)\overset{1}{L}_{\text{ice}}(xy)\overset{2}{L}_{\text{ice}}(y) = \overset{2}{L}_{\text{ice}}(y)\overset{1}{L}_{\text{ice}}(xy)R_{\text{ice}}(x). \tag{9}$$

The N-dimensional representation of algebra (9) can be written as

$$L_{\text{ice}}(y, p_1, p_2) = \begin{pmatrix} y^{-1}c_1c_2Z_1 - b_1b_2yZ_1^{-1} & -(b_1d_2Z_1^{-1} - c_1a_2Z_1)X \\ \omega_1X^{-1}(d_1b_2Z_1^{-1} - a_1c_2Z_1) & y^{-1}d_1d_2Z_1^{-1} - a_1a_2\omega_1^2yZ_1 \end{pmatrix},$$

where the operators Z_1 and X satisfy the relations

$$Z_1^N = 1, \qquad X^N = 1, \qquad Z_1X = \omega_1XZ_1.$$

Substituting

$$\begin{aligned} &R_{12} \to C_1^{-1}(xy)C_2^{-1}(y)R_{12}C_2(y)C_1(xy), \\ &L_1(xy) \to C_1^{-1}(xy)L_1(xy)C_1(xy), \qquad L_2(y) \to C_2^{-1}(y)L_2(y)C_2(y), \end{aligned}$$

where

$$C(y) = \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix}$$

and the subscript of the matrix C indicates the space in which this matrix acts, we find that the equation $RLL = LLR$ remains valid. We then have

$$\mathcal{R}(x) = \begin{pmatrix} x\omega_1 - x^{-1}\omega_1^{-1} & 0 & 0 & 0 \\ 0 & x - x^{-1} & x(\omega_1 - \omega_1^{-1}) & 0 \\ 0 & x^{-1}(\omega_1 - \omega_1^{-1}) & x - x^{-1} & 0 \\ 0 & 0 & 0 & x\omega_1 - x^{-1}\omega_1^{-1} \end{pmatrix}, \tag{10}$$

and the L-operator is

$$\mathcal{L}(y, p_1, p_2) = \begin{pmatrix} y^{-1}c_1c_2Z_1 - b_1b_2yZ_1^{-1} & -y(b_1d_2Z_1^{-1} - c_1a_2Z_1)X \\ \omega_1 y^{-1}X^{-1}(d_1b_2Z_1^{-1} - a_1c_2Z_1) & y^{-1}d_1d_2Z_1^{-1} - a_1a_2\omega_1^2 yZ_1 \end{pmatrix}. \tag{11}$$

Exactly this L-operator was found in [5]. We call it the cyclic representation of the algebra of monodromy matrices in the Bazhanov–Stroganov form.

We now multiply $L(y)$ by yZ_1 and introduce the notation

$$v = y^2, \qquad Z = Z_1^2, \qquad \omega = \omega_1^2.$$

We then obtain operator (4). For the equation $RLL = LLR$ to hold, we must multiply the R-matrix as follows:

$$R(u) = -\omega_1 x K_1 \mathcal{R}(x) K_2^{-1},$$

where

$$u = x^2, \qquad K = \omega_1^{(\sigma^z - 1)/2} = \begin{pmatrix} 1 & 0 \\ 0 & \omega_1^{-1} \end{pmatrix}.$$

The subscript of the matrix K indicates the space in which this matrix acts. It is easy to see that the matrix R then coincides with (1) and the operator L coincides with (4).

We stress that hereinafter we let L denote the Tarasov cyclic representations and $\mathcal{L}$ denote the cyclic representations in the Bazhanov–Stroganov form. The matrix elements of the L-operators are denoted by $L_{i\alpha}^{j\beta}$ and $\mathcal{L}_{i\alpha}^{j\beta}$, $i, j = 0, 1$ and $\alpha, \beta = 0, \ldots, N-1$.

4. The Q-operator and the TQ equation

The transfer matrix constructed from $\mathcal{L}(u)$,

$$\mathcal{Q}(u) = \operatorname{tr}_0 \mathcal{L}_{10}(u)\mathcal{L}_{20}(u)\cdots\mathcal{L}_{n0}(u),$$

where the trace is evaluated in the N-dimensional space, has a very important property that makes it the Baxter Q-operator [5]. Namely, this matrix satisfies the TQ equation. We prove this statement.

We consider the equation

$$\mathcal{R}_{i_1i_2}^{j_1j_2}(u)\mathcal{L}_{j_1\alpha}^{k_1\beta}(uv)\mathcal{L}_{j_2\beta}^{k_2\gamma}(v) = \mathcal{L}_{i_2\alpha}^{j_2\beta}(uv)\mathcal{L}_{i_1\beta}^{j_1\gamma}(v)\mathcal{R}_{j_1j_2}^{k_1k_2}(u), \tag{12}$$

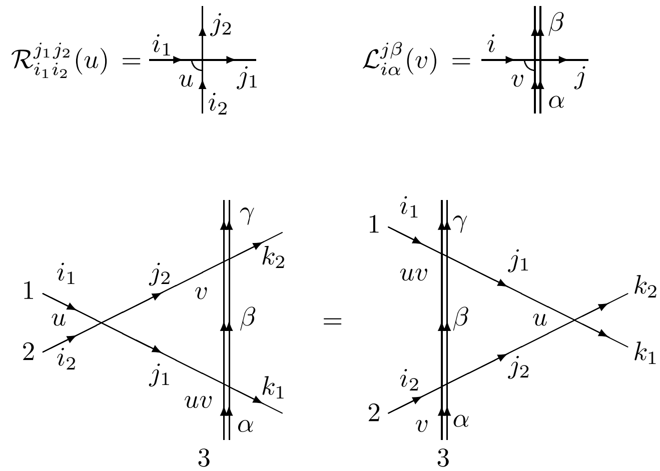

Fig. 1

which is shown graphically in Fig. 1. If the indices i_1 and k_1 are fixed, both sides of Eq. (12) are operators in the tensor product $\mathbb{C}^2 \times \mathbb{C}^N$. We act with these operators on the vector $\psi_{k_2\gamma}$, which belongs to the kernel of the operator $\mathcal{L}_{23}(v)$, that is, on the vector satisfying the equation

$$\mathcal{L}^{k_2\gamma}_{j_2\beta}(v)\psi_{k_2\gamma} = 0.$$

A kernel of $\mathcal{L}_{23}(v)$ is nontrivial only at some special values of the spectral parameter $v = v_*$: $v_*^2 = c_1 d_1/a_1 b_1$ or $v_*^2 = c_2 d_2/a_2 b_2$. It can be seen from Fig. 1 that the kernel of the operator $\mathcal{L}_{23}(v_*)$ is a subspace that is invariant with respect to the tensor product $\mathcal{L}_{13}(uv_*)\mathcal{R}_{12}(u)$. In this case, the complement of the kernel considered as a coset space is also an invariant subspace. The matrix of the operator $\mathcal{L}_{13}(uv_*)\mathcal{R}_{12}(u)$ therefore has a block-diagonal form,

$$\mathcal{L}_{13}(uv_*)\mathcal{R}_{12}(u) = \begin{pmatrix} P_1 & * \\ 0 & P_2 \end{pmatrix},$$

where all blocks are N-dimensional square matrices and stars denote the matrix elements inessential for us. We introduce the ordering of the basis vectors as follows: the first N vectors generate the kernel, and the remaining N vectors generate its complement.

Let the equations

$$P_1 = \phi_1 \mathcal{L}(uv_*\lambda), \qquad P_2 = \phi_2 \mathcal{L}(uv_*\lambda^{-1}) \tag{13}$$

hold for some values of the parameters of $\mathcal{L}(v_*)$. After multiplying n copies of the operator $\mathcal{L}_{13}(uv_*)\mathcal{R}_{12}(u)$ in the spaces 2 and 3 (see Fig. 2) and evaluating the trace, we then obtain the equation

$$\widetilde{\mathcal{Q}}(uv_*)T(u) = \phi_1^n \widetilde{\mathcal{Q}}(uv_*\lambda) + \phi_2^n \widetilde{\mathcal{Q}}(uv_*\lambda^{-1}), \tag{14}$$

where

$$\widetilde{\mathcal{Q}}(uv_*) = \operatorname{tr}_3 \mathcal{L}_{13}(uv_*)\mathcal{L}_{1'3}(uv_*)\cdots\mathcal{L}_{1^{(n)}3}$$

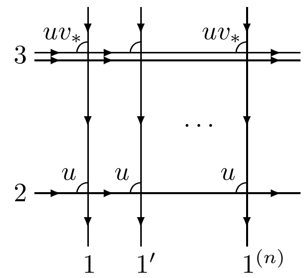

Fig. 2

and $T(u)$ is the standard transfer matrix of the ice model. Substituting $\mathcal{Q}(u) = \widetilde{\mathcal{Q}}(uv_*)$, we obtain the TQ equation.

Unfortunately, we cannot realize this scheme for the given definition of the operator $\mathcal{L}(v)$, because it is impossible to satisfy conditions (13) for any set of the parameters. But we can redefine the operator $\mathcal{L}(v)$ such that condition (13) and hence Eq. (14) do hold [5]. For this, we make p_1 depend on the spectral parameter v and p_2 remain unchanged:

$$p_1(v) = (a_1 v^{-1}, b_1, c_1, d_1 v), \qquad p_2(v) = (a_2, b_2, c_2, d_2).$$

It is shown in Appendix A that $\mathcal{Q}(u) = \mathcal{Q}\big(u, p_1(u), p_2\big)$ does satisfy the TQ equation,

$$\mathcal{Q}(u)T(u) = (u - u^{-1})^n \mathcal{Q}(u\omega) + (u\omega - u^{-1}\omega^{-1})^n \mathcal{Q}(u\omega^{-1}),$$

where $T(u)$ is the standard transfer matrix of the ice model.

5. The elementary isomorphisms: The special case

We consider two representations of the L-operator algebra: $L(u, p_1, p_2)$ and $L(u, p_2, p_1)$. Let the parameters p_1 and p_2 be such that these representations are equivalent. We introduce the operator G satisfying the equation

$$G(Z)L(u, p_1, p_2) = L(u, p_2, p_1)G(Z), \tag{15}$$

where $G(Z)$ acts in the N-dimensional space.

We now consider two tensor products of a pair of cyclic representations: $L_1(u, p_1, p_2)L_2(u, p_3, p_4)$ and $L_1(u, p_1, p_3)L_2(u, p_2, p_4)$ (note the permutation $p_2 \leftrightarrow p_3$). Let the parameters p_1, p_2, p_3, and p_4 be such that these two representations are equivalent. We introduce the intertwining operator F,

$$F(X_1 X_2^{-1})L_1(u, p_1, p_2)L_2(u, p_3, p_4) = L_1(u, p_1, p_3)L_2(u, p_2, p_4)F(X_1 X_2^{-1}), \tag{16}$$

where $F(X_1 X_2^{-1})$ acts in the N-dimensional space.

It turns out that conditions (5) certainly suffice for G and F to exist. These two operators are

$$\frac{G(p_1, p_2; \omega^k)}{G(p_1, p_2; 1)} = \prod_{j=1}^{k} \frac{d_1 b_2 - a_1 c_2 \omega^j}{b_1 d_2 - c_1 a_2 \omega^j}, \tag{17}$$

$$\frac{F(p_1, p_2; \omega^k)}{F(p_1, p_2; 1)} = \prod_{j=1}^{k} \frac{\omega a_1 d_2 - d_1 a_2 \omega^j}{c_1 b_2 - b_1 c_2 \omega^j}, \tag{18}$$

where we let $G(p_1, p_2; \omega^k)$ and $F(p_1, p_2; \omega^k)$ denote the diagonal matrix elements of the N-dimensional matrices $G(p_1, p_2; Z)$ and $F(p_1, p_2; X_1 X_2^{-1})$ in their respective eigenvalue bases (G and F cannot be brought to the diagonal form simultaneously). We calculate these two operators in a more general case in the next section, but now we merely note that formulas (17) and (18) coincide with formulas (7).

The existence of elementary isomorphisms of G and F explains the factoring of the R-matrix of the chiral Potts model. Permuting the pairs, we obtain the chain

$$(p_1, p_2)(p_3, p_4) \xrightarrow{F_1} (p_1, p_3)(p_2, p_4) \xrightarrow{G_1, G_2} (p_3, p_1)(p_4, p_2) \xrightarrow{F_2} (p_3, p_4)(p_1, p_2).$$

The factoring of the R-matrix then becomes evident.

6. The general case

6.1. The G-operator. We consider the two representations $L(u, p_1, \bar{p}_1)$ and $L(u, p_2, \bar{p}_2)$ of the L-operator algebra. We want to find the conditions for these representations to be equivalent and find the corresponding intertwiner, which is a generalization of the operator G introduced in the previous section (formula (15)). For simplicity, we let the same symbol G denote this generalized intertwiner. The two representations are equivalent if the equations

$$\begin{aligned}
&a_1^N \bar{a}_1^N = a_2^N \bar{a}_2^N, \qquad && b_1^N \bar{b}_1^N = b_2^N \bar{b}_2^N, \\
&\frac{\bar{c}_1^N \bar{d}_1^N}{\bar{a}_1^N \bar{b}_1^N} = \frac{c_2^N d_2^N}{a_2^N b_2^N}, && \frac{\bar{c}_2^N \bar{d}_2^N}{\bar{a}_2^N \bar{b}_2^N} = \frac{c_1^N d_1^N}{a_1^N b_1^N}, \\
&\frac{d_1^N \bar{d}_1^N}{a_1^N \bar{a}_1^N} = \frac{d_2^N \bar{d}_2^N}{a_2^N \bar{a}_2^N}, && d_1^N \bar{b}_1^N - a_1^N \bar{c}_1^N = d_2^N \bar{b}_2^N - a_2^N \bar{c}_2^N
\end{aligned} \tag{19}$$

hold (see Appendix C).

We consider the simplest case where we extract the Nth roots by simply erasing the letter N. As a result, we obtain the system

$$\begin{aligned}
&a_1 \bar{a}_1 = a_2 \bar{a}_2, \qquad && b_1 \bar{b}_1 = b_2 \bar{b}_2, \\
&\frac{\bar{c}_1 \bar{d}_1}{\bar{a}_1 \bar{b}_1} = \frac{c_2 d_2}{a_2 b_2}, && \frac{\bar{c}_2 \bar{d}_2}{\bar{a}_2 \bar{b}_2} = \frac{c_1 d_1}{a_1 b_1}, \\
&d_1 \bar{d}_1 = d_2 \bar{d}_2, && d_1^N \bar{b}_1^N - a_1^N \bar{c}_1^N = d_2^N \bar{b}_2^N - a_2^N \bar{c}_2^N.
\end{aligned} \tag{20}$$

We find the operator G satisfying the equation

$$G L(u, p_1, \bar{p}_1) = L(u, p_2, \bar{p}_2) G.$$

If conditions (20) are satisfied, then the operator G exists. We now prove this. Using the ansatz

$$G = G(Z),$$

we obtain the system of equations

$$G(Z) A_1 = A_2 G(Z),$$

$$G(Z) B_1 = B_2 G(Z),$$

$$G(Z) C_1 = C_2 G(Z),$$

$$G(Z) D_1 = D_2 G(Z).$$

We choose a basis $|k\rangle$, $k = 0, \dots, N-1 \pmod N$:

$$Z|k\rangle = \omega^k |k\rangle, \qquad X|k\rangle = |k+1\rangle.$$

It is clear that the matrix $G(Z)$ is diagonal in this basis. We now find its nonzero matrix elements.

The first equation of the obtained system is

$$G(Z)(c_1\bar{c}_1 Z - b_1\bar{b}_1 u) = (c_2\bar{c}_2 Z - b_2\bar{b}_2 u)G(Z).$$

Acting on the vector $|k\rangle$ with the left- and right-hand sides of this equality and comparing the coefficients at the same powers of u, we obtain the parameter restrictions

$$c_1\bar{c}_1 = c_2\bar{c}_2, \qquad b_1\bar{b}_1 = b_2\bar{b}_2. \tag{21}$$

From the fourth equation,

$$G(Z)(d_1\bar{d}_1 - a_1\bar{a}_1\omega u Z) = (d_2\bar{d}_2 - a_2\bar{a}_2\omega u Z)G(Z),$$

we similarly obtain

$$d_1\bar{d}_1 = d_2\bar{d}_2, \qquad a_1\bar{a}_1 = a_2\bar{a}_2. \tag{22}$$

The second equation is

$$G(Z)X^{-1}[d_1\bar{b}_1 - a_1\bar{c}_1 Z] = [d_2\bar{b}_2 - a_2\bar{c}_2 Z]G(Z),$$

and it implies that

$$G(\omega^{k+1}) = \frac{d_1\bar{b}_1 - a_1\bar{c}_1\omega^{k+1}}{d_2\bar{b}_2 - a_2\bar{c}_2\omega^{k+1}} G(\omega^k), \tag{23}$$

where we let $G(\omega^k)$, $k = 0, \dots, N-1$, denote the diagonal matrix elements of the matrix $G(Z)$. From the third equation,

$$G(Z)[b_1\bar{d}_1 - c_1\bar{a}_1 Z]X = [b_2\bar{d}_2 - c_2\bar{a}_2 Z]XG(Z),$$

we can easily derive

$$G(\omega^{k+1}) = \frac{b_2\bar{d}_2 - c_2\bar{a}_2\omega^{k+1}}{b_1\bar{d}_1 - c_1\bar{a}_1\omega^{k+1}} G(\omega^k) \tag{24}$$

in the same way.

Because the function $G(\omega^k)$ is single valued, we obtain

$$(d_1\bar{b}_1 - a_1\bar{c}_1\omega^{k+1})(b_1\bar{d}_1 - c_1\bar{a}_1\omega^{k+1}) = (d_2\bar{b}_2 - a_2\bar{c}_2\omega^{k+1})(b_2\bar{d}_2 - c_2\bar{a}_2\omega^{k+1}).$$

Comparing coefficients at the same powers of ω and taking (21) and (22) into account, we obtain the additional condition

$$\frac{\bar{c}_1\bar{d}_1}{\bar{a}_1\bar{b}_1} + \frac{c_1 d_1}{a_1 b_1} = \frac{\bar{c}_2\bar{d}_2}{\bar{a}_2\bar{b}_2} + \frac{c_2 d_2}{a_2 b_2}. \tag{25}$$

Further, we obtain

$$d_1^N \bar{b}_1^N - a_1^N \bar{c}_1^N = d_2^N \bar{b}_2^N - a_2^N \bar{c}_2^N \tag{26}$$

from the periodicity condition $G(\omega^{N+1}) = G(\omega)$. Using the gauge symmetries of the L-operators, we can set

$$a_1 = \bar{a}_2, \qquad b_1 = \bar{b}_2. \tag{27}$$

It then follows from (20) that (21), (22), (25), and (26) are valid. The matrix elements $G(\omega^k)$ therefore exist and are given by recursive relation (23).

We can easily rewrite $G(\omega^k)$ in terms of p_1 and $\bar{p}_1$. Substituting

$$\bar{c}_2 = \frac{c_1 \bar{c}_1}{c_2}$$

in (26) and expressing c_2 in terms of p_1 and $\bar{p}_1$,

$$c_2 = \bar{c}_1 \sqrt[N]{\frac{b_1^N \bar{d}_1^N - c_1^N \bar{a}_1^N}{d_1^N \bar{b}_1^N - a_1^N \bar{c}_1^N}} = \Lambda(p_1, \bar{p}_1)\bar{c}_1,$$

where we introduce the new function

$$\Lambda(p_1, p_2) = \sqrt[N]{\frac{b_1^N d_2^N - c_1^N a_2^N}{d_1^N b_2^N - a_1^N c_2^N}},$$

we obtain

$$\frac{G(p_1, \bar{p}_1; \omega^k)}{G(p_1, \bar{p}_1; 1)} = \Lambda(p_1, \bar{p}_1)^k \prod_{j=1}^{k} \frac{d_1 \bar{b}_1 - a_1 \bar{c}_1 \omega^j}{\bar{d}_1 b_1 - \bar{a}_1 c_1 \omega^j}.$$

We stress that G depends on only p_1 and $\bar{p}_1$.

The operator G just found generates an isomorphism of two representations $L(u, p_1, \bar{p}_1)$ and $L(u, p_2, \bar{p}_2)$ of the algebra of monodromy matrices with the parameters p_2 and $\bar{p}_2$ depending on p_1 and $\bar{p}_1$ as follows:

$$\begin{aligned}
a_2 &= \bar{a}_1, & \bar{a}_2 &= a_1, \\
b_2 &= \bar{b}_1, & \bar{b}_2 &= b_1, \\
c_2 &= \Lambda(p_1, \bar{p}_1)\bar{c}_1, & \bar{c}_2 &= \Lambda(p_1, \bar{p}_1)^{-1} c_1, \\
d_2 &= \Lambda(p_1, \bar{p}_1)^{-1}\bar{d}_1, & \bar{d}_2 &= \Lambda(p_1, \bar{p}_1) d_1.
\end{aligned} \tag{28}$$

The operator G obtained is a generalization of the operator G in (7), (15). In that special case, we must set $p_2 = \bar{p}_1$ and $\bar{p}_2 = p_1$, thus obtaining the additional constraint $\Lambda(p_1, \bar{p}_1) = 1$ on the parameters p_1 and $\bar{p}_1$.

Remark. We extract the Nth roots by simply erasing the letter N. Evidently, the general case reduces to this. A complete investigation, however, can be performed as follows. For two representations to be equivalent, their centers must coincide. But we do not compare all central elements when deriving our conditions. We must add the condition of equality of the corresponding quantum determinants to our system of equations. We have not completely investigated what this additional condition leads to. Apparently, it can be used to clarify how the Nth root can be extracted, and it strongly restricts the number of possibile variants.

6.2. The F-operator. We now consider two representations of the L-operator algebra: $L_1(u,p_1,\bar p_1)\times L_2(u,p_2,\bar p_2)$ and $L_1(u,p_3,\bar p_3)L_2(u,p_4,\bar p_4)$. We find the conditions under which these two representations are equivalent and calculate the corresponding intertwiner. The latter is a generalization of the operator F introduced in formulas (7) and (16).

The matrix of the central elements is

$$\big\langle L(u,p,\bar p)\big\rangle = \begin{pmatrix} c^N\bar c^N - b^N\bar b^N u & -u(b^N\bar d^N - c^N\bar a^N) \\ d^N\bar b^N - a^N\bar c^N & d^N\bar d^N - a^N\bar a^N u \end{pmatrix}. \tag{29}$$

The necessary condition for the equivalence of the two representations is the coincidence of their centers. Therefore,

$$\big\langle L_1(u,p_1,\bar p_1)\big\rangle\big\langle L_2(u,p_2,\bar p_2)\big\rangle = \big\langle L_1(u,p_3,\bar p_3)\big\rangle\big\langle L_2(u,p_4,\bar p_4)\big\rangle,$$

From this, we have

$$\det\big\langle L_1(u,p_1,\bar p_1)\big\rangle\det\big\langle L_2(u,p_2,\bar p_2)\big\rangle = \det\big\langle L_1(u,p_3,\bar p_3)\big\rangle\det\big\langle L_2(u,p_4,\bar p_4)\big\rangle. \tag{30}$$

We show in Appendix C that these conditions yield the following relations between the parameters (we do not consider the trivial case where $p_3=p_1$, $\bar p_3=\bar p_1$, $p_4=p_2$, and $\bar p_4=\bar p_2$):

$$\begin{aligned}
&b_1^N\bar b_1^N = b_3^N\bar b_3^N, \qquad b_2^N\bar b_2^N = b_4^N\bar b_4^N,\\
&a_1^N\bar a_1^N = a_3^N\bar a_3^N, \qquad a_2^N\bar a_2^N = a_4^N\bar a_4^N,\\
&d_1^N\bar b_1^N = d_3^N\bar b_3^N, \qquad a_2^N\bar c_2^N = a_4^N\bar c_4^N,\\
&\frac{c_2^Nd_2^N}{a_2^Nb_2^N} = \frac{\bar c_3^N\bar d_3^N}{\bar a_3^N\bar b_3^N}, \qquad \frac{c_1^Nd_1^N}{a_1^Nb_1^N} = \frac{c_3^Nd_3^N}{a_3^Nb_3^N},\\
&\frac{\bar c_2^N\bar d_2^N}{\bar a_2^N\bar b_2^N} = \frac{\bar c_4^N\bar d_4^N}{\bar a_4^N\bar b_4^N}, \qquad \frac{c_4^Nd_4^N}{a_4^Nb_4^N} = \frac{\bar c_1^N\bar d_1^N}{\bar a_1^N\bar b_1^N},\\
&b_2^N\bar c_3^N = c_2^N\bar b_1^N\frac{b_2^N\bar c_1^N - \bar a_1^Nd_2^N}{c_2^N\bar b_1^N - a_2^N\bar d_1^N},\\
&\bar a_1^Nd_4^N = \bar d_1^Na_2^N\frac{b_2^N\bar c_1^N - \bar a_1^Nd_2^N}{c_2^N\bar b_1^N - a_2^N\bar d_1^N}.
\end{aligned} \tag{31}$$

We again erase the letter N and obtain

$$\begin{aligned}
&b_1\bar b_1 = b_3\bar b_3, \qquad b_2\bar b_2 = b_4\bar b_4,\\
&a_1\bar a_1 = a_3\bar a_3, \qquad a_2\bar a_2 = a_4\bar a_4,\\
&d_1\bar b_1 = d_3\bar b_3, \qquad a_2\bar c_2 = a_4\bar c_4,\\
&\frac{c_2d_2}{a_2b_2} = \frac{\bar c_3\bar d_3}{\bar a_3\bar b_3}, \qquad \frac{c_1d_1}{a_1b_1} = \frac{c_3d_3}{a_3b_3},\\
&\frac{\bar c_2\bar d_2}{\bar a_2\bar b_2} = \frac{\bar c_4\bar d_4}{\bar a_4\bar b_4}, \qquad \frac{c_4d_4}{a_4b_4} = \frac{\bar c_1\bar d_1}{\bar a_1\bar b_1}.
\end{aligned} \tag{32}$$

Recalling the gauge symmetries of the L-operator, we observe that the substitution

$$\begin{aligned}
&a \to \lambda a, \qquad \bar{a} \to \lambda^{-1}\bar{a},\\
&c \to \lambda c, \qquad \bar{c} \to \lambda^{-1}\bar{c},\\
&b \to b, \qquad \bar{b} \to \bar{b}, \qquad d \to d, \qquad \bar{d} \to \bar{d}
\end{aligned}$$

does not change the operator $L(u,p,\bar{p})$. The same is true for the substitution

$$\begin{aligned}
&b \to \lambda b, \qquad \bar{b} \to \lambda^{-1}\bar{b},\\
&d \to \lambda d, \qquad \bar{d} \to \lambda^{-1}\bar{d},\\
&a \to a, \qquad \bar{a} \to \bar{a}, \qquad c \to c, \qquad \bar{c} \to \bar{c}.
\end{aligned}$$

Using the gauge degrees of freedom, we can set

$$b_1 = b_3, \qquad b_2 = b_4, \qquad a_1 = a_3, \qquad a_2 = a_4. \tag{33}$$

From this and from (32), we find that

$$\begin{aligned}
&\bar{b}_1 = \bar{b}_3, \qquad \bar{b}_2 = \bar{b}_4, \qquad \bar{a}_1 = \bar{a}_3, \qquad \bar{a}_2 = \bar{a}_4,\\
&d_1 = d_3, \qquad \bar{d}_2 = \bar{d}_4, \qquad c_1 = c_3, \qquad \bar{c}_2 = \bar{c}_4.
\end{aligned} \tag{34}$$

Moreover, we have the equations

$$\bar{c}_3\bar{d}_3 = c_2 d_2 \frac{\bar{a}_3\bar{b}_3}{a_2 b_2}, \qquad c_4 d_4 = \bar{c}_1\bar{d}_1 \frac{a_2 b_2}{\bar{a}_1\bar{b}_1}. \tag{35}$$

From formulas (31), we now obtain

$$\bar{c}_3^N = \frac{c_2^N \bar{b}_1^N}{b_2^N} \frac{b_2^N \bar{c}_1^N - \bar{a}_1^N d_2^N}{c_2^N \bar{b}_1^N - a_2^N \bar{d}_1^N}, \tag{36}$$

$$d_4^N = \frac{\bar{d}_1^N a_2^N}{\bar{a}_1^N} \frac{b_2^N \bar{c}_1^N - \bar{a}_1^N d_2^N}{c_2^N \bar{b}_1^N - a_2^N \bar{d}_1^N}. \tag{37}$$

We now find the matrix F intertwining the two representations in question. In particular, we prove that conditions (33)–(37) are not only necessary but also sufficient for the existence of F. We start with

$$F L_1(p_1,\bar{p}_1) L_2(p_2,\bar{p}_2) = L_1(p_3,\bar{p}_3) L_2(p_4,\bar{p}_4) F. \tag{38}$$

We now find the operator F in the form $F(X_1 X_2^{-1})$, where X_1 and X_2 are the shift matrices acting in the respective first and second N-dimensional spaces. As shown in Appendix D, F does exist and is expressed through p_1, $\bar{p}_1$, p_2, and $\bar{p}_2$,

$$\frac{F(\bar{p}_1, p_2; \omega^k)}{F(\bar{p}_1, p_2; 1)} = \Omega(\bar{p}_1, p_2)^{-k} \prod_{j=1}^{k} \frac{\bar{c}_1 b_2 - \bar{a}_1 d_2 \omega^j}{\bar{b}_1 c_2 - \bar{d}_1 a_2 \omega^j},$$

where we define the new function

$$\Omega(p_1,p_2) = \sqrt[N]{\frac{b_1^N c_2^N - d_1^N a_2^N}{c_1^N b_2^N - a_1^N d_2^N}}$$

for further convenience. We stress that F depends on only the parameters $\bar{p}_1$ and p_2. The action of F is then given by the formulas

$$\begin{aligned}
&a_3 = a_1, && \bar{a}_3 = \bar{a}_1, && a_4 = a_2, && \bar{a}_4 = \bar{a}_2,\\
&b_3 = b_1, && \bar{b}_3 = \bar{b}_1, && b_4 = b_2, && \bar{b}_4 = \bar{b}_2,\\
&c_3 = c_1, && d_3 = d_1, && \bar{c}_4 = \bar{c}_2, && \bar{d}_4 = \bar{d}_2,\\
&\bar{c}_3 = \Omega(\bar{p}_1,p_2)\frac{c_2\bar{b}_1}{b_2}, && && c_4 = \Omega(\bar{p}_1,p_2)^{-1}\frac{\bar{c}_1 b_2}{\bar{b}_1}, &&\\
&\bar{d}_3 = \Omega(\bar{p}_1,p_2)^{-1}\frac{d_2\bar{a}_1}{a_2}, && && d_4 = \Omega(\bar{p}_1,p_2)\frac{\bar{d}_1 a_2}{\bar{a}_1}. &&
\end{aligned} \tag{39}$$

It is clear that the expression obtained for F is gauge-invariant.

We recall that in addition to the gauge symmetry, the tensor product of two L-operators has another symmetry, namely, the unity can be inserted between these two operators:

$$L_1(p_3,\bar{p}_3)L_2(p_4,\bar{p}_4) = L_1(p_3,\bar{p}_3)M^{-1}ML_2(p_4,\bar{p}_4),$$

where M is an arbitrary 2×2 matrix. We set

$$M = \begin{pmatrix} \dfrac{\bar{b}_1}{b_2} & 0 \\ 0 & \dfrac{\bar{a}_1}{a_2} \end{pmatrix}$$

and apply this additional symmetry to our L-operators. As a result, we obtain the product of new L-operators, whose parameters are expressed through p_1, $\bar{p}_1$, p_2, and $\bar{p}_2$ as

$$\begin{aligned}
&a_3 = a_1, && \bar{a}_3 = a_2, && a_4 = \bar{a}_1, && \bar{a}_4 = \bar{a}_2,\\
&b_3 = b_1, && \bar{b}_3 = b_2, && b_4 = \bar{b}_1, && \bar{b}_4 = \bar{b}_2,\\
&c_3 = c_1, && d_3 = d_1, && \bar{c}_4 = \bar{c}_2, && \bar{d}_4 = \bar{d}_2,\\
&\bar{c}_3 = \Omega(\bar{p}_1,p_2)c_2, && && c_4 = \Omega(\bar{p}_1,p_2)^{-1}\bar{c}_1, &&\\
&\bar{d}_3 = \Omega(\bar{p}_1,p_2)^{-1}d_2, && && d_4 = \Omega(\bar{p}_1,p_2)\bar{d}_1. &&
\end{aligned} \tag{40}$$

The expression for $F(\omega^k)$ remains unchanged.

The case in [6] was a special case of the approach in question and can be obtained by setting

$$p_3 = p_1, \qquad \bar{p}_3 = p_2, \qquad p_4 = \bar{p}_1, \qquad \bar{p}_4 = \bar{p}_2$$

in all formulas. In particular, we can easily derive the F-operator in relation (7). For this, we must impose the additional restriction

$$\Omega(\bar{p}_1,p_2) = 1$$

on the parameters $\bar{p}_1$ and p_2.

The obtained operators G and F satisfy the relation that generalizes the star–triangle relation of the chiral Potts model [2]. Namely, we have

$$G(\tilde{q},\tilde{r};Z_1)F(\tilde{p},r;X_1X_2^{-1})G(p,q;Z_1) = \mu F(p',q';X_1X_2^{-1})G(p,r';Z_1)F(q,r;X_1X_2^{-1}), \tag{41}$$

where μ is a constant and the intertwiners depend on the parameters that are expressed in terms of p, q, and r by the formulas

$$\begin{cases} a'_r = a_r, \\ b'_r = b_r, \\ c'_r = \Omega(q,r)c_r, \\ d'_r = \Omega(q,r)^{-1}d_r, \end{cases} \qquad \begin{cases} a'_q = a_q, \\ b'_q = b_q, \\ c'_q = \Omega(q,r)^{-1}c_q, \\ d'_q = \Omega(q,r)d_q, \end{cases}$$

$$\begin{cases} a'_p = a_p, \\ b'_p = b_p, \\ c'_p = \Lambda(p,r')^{-1}c_p, \\ d'_p = \Lambda(p,r')d_p, \end{cases} \qquad \begin{cases} \tilde{a}_q = a_q, \\ \tilde{b}_q = b_q, \\ \tilde{c}_q = \Lambda(p,q)c_q, \\ \tilde{d}_q = \Lambda(p,q)^{-1}d_q, \end{cases}$$

$$\begin{cases} \tilde{a}_p = a_p, \\ \tilde{b}_p = b_p, \\ \tilde{c}_p = \Lambda(p,q)^{-1}c_p, \\ \tilde{d}_p = \Lambda(p,q)d_p, \end{cases} \qquad \begin{cases} \tilde{a}_r = a_r, \\ \tilde{b}_r = b_r, \\ \tilde{c}_r = \Omega(\tilde{p},r)c_r, \\ \tilde{d}_r = \Omega(\tilde{p},r)^{-1}d_r. \end{cases}$$

The proof of these statements is in Appendix E.

Remark. The above statement suggests a new algebraic structure related to the cyclic representations of the monodromy-matrix algebra.

We now consider a Hopf algebra with the generators $L_i^j(p_1,p_2)$, $i,j=0,1$, $p_1,p_2\in\mathbb{C}^4$. The coproduct is

$$\Delta(L_i^j) = (L_i^k)_1(L_k^j)_2.$$

The relations in the Hopf algebra are

$$G(p_1,p_2)L_i^j(p_1,p_2) = L_i^j(\tilde{p}_1,\tilde{p}_2)G(p_1,p_2),$$

$$F(\bar{p}_1,p_2)\big(L_i^k(p_1,\bar{p}_1)\big)_1\big(L_k^j(p_2,\bar{p}_2)\big)_2 = \big(L_i^k(p_1,\tilde{p}_1)\big)_1\big(L_k^j(\tilde{p}_2,\bar{p}_2)\big)_2F(\bar{p}_1,p_2),$$

$$\big(L_i^k(p_1,\bar{p}_1)\big)_1\big(L_k^j(p_2,\bar{p}_2)\big)_2 = \big(L_i^k(p_1,\bar{p}_1)\big)_1M_k^l(M^{-1})_l^m\big(L_m^j(p_2,\bar{p}_2)\big)_2,$$

where an arbitrary two-dimensional diagonal matrix is denoted by M and the parameters in the right-hand sides of the relations are expressed in terms of the parameters in the left-hand sides by formulas (28) and (40). Repeated indices imply summation.

7. The algebra of the Q-operators

In addition to the operator $\mathcal{Q}(u)$ introduced in Sec. 4 and related to the cyclic representations of the algebra of monodromy matrices in the Bazhanov–Stroganov form, we can consider the operator

$$Q(u) = \operatorname{tr}_0 L_{10}(u)L_{20}(u)\cdots L_{k0}(u),$$

where the trace is calculated in the N-dimensional space and $L_{i0}(u)$ are the cyclic representations of the monodromy-matrix algebra in the Tarasov form. The operators $Q(u)$ generate the algebra with the relations following from the properties of the operators $L(u)$,

$$Q(\lambda p_1, \bar{p}_1) = \lambda^k Q(p_1, \bar{p}_1), \tag{42}$$

$$Q(p_1, \mu\bar{p}_1) = \mu^k Q(p_1, \bar{p}_1), \tag{43}$$

$$Q(a_1, b_1, c_1, d_1, \bar{a}_1, \bar{b}_1, \bar{c}_1, \bar{d}_1) = Q(\lambda a_1, b_1, \lambda c_1, d_1, \lambda^{-1}\bar{a}_1, \bar{b}_1, \lambda^{-1}\bar{c}_1, \bar{d}_1), \tag{44}$$

$$Q(a_1, b_1, c_1, d_1, \bar{a}_1, \bar{b}_1, \bar{c}_1, \bar{d}_1) = Q(a_1, \mu b_1, c_1, \mu d_1, \bar{a}_1, \mu^{-1}\bar{b}_1, \bar{c}_1, \mu^{-1}\bar{d}_1), \tag{45}$$

$$Q(a_1, b_1, c_1, d_1, \bar{a}_1, \bar{b}_1, \bar{c}_1, \bar{d}_1) = Q(\bar{a}_1, \bar{b}_1, \Lambda\bar{c}_1, \Lambda^{-1}\bar{d}_1, a_1, b_1, \Lambda^{-1}c_1, \Lambda d_1), \tag{46}$$

$$\begin{aligned} &Q(a_1, b_1, c_1, d_1, \bar{a}_1, \bar{b}_1, \bar{c}_1, \bar{d}_1)Q(a_2, b_2, c_2, d_2, \bar{a}_2, \bar{b}_2, \bar{c}_2, \bar{d}_2) = \\ &\qquad = Q(a_1, b_1, c_1, d_1, \beta\bar{a}_1, \alpha\bar{b}_1, \alpha\bar{c}_1, \beta\bar{d}_1)Q(\beta^{-1}a_2, \alpha^{-1}b_2, \alpha^{-1}c_2, \beta^{-1}d_2, \bar{a}_2, \bar{b}_2, \bar{c}_2, \bar{d}_2), \end{aligned} \tag{47}$$

$$\begin{aligned} &Q(a_1, b_1, c_1, d_1, \bar{a}_1, \bar{b}_1, \bar{c}_1, \bar{d}_1)Q(a_2, b_2, c_2, d_2, \bar{a}_2, \bar{b}_2, \bar{c}_2, \bar{d}_2) = \\ &\qquad = Q(a_1, b_1, c_1, d_1, a_2, b_2, \Omega c_2, \Omega^{-1}d_2)Q(\bar{a}_1, \bar{b}_1, \Omega^{-1}\bar{c}_1, \Omega\bar{d}_1, \bar{a}_2, \bar{b}_2, \bar{c}_2, \bar{d}_2), \end{aligned} \tag{48}$$

where $\Lambda = \Lambda(p_1, \bar{p}_1)$, $\Omega = \Omega(\bar{p}_1, p_2)$, and α, β, λ, and μ are arbitrary numbers. The derivation of all these relations can be found in Appendix F.

8. Discussion

We note some questions relevant for future investigations.

The approach used in this paper can be applied to more general cases. In particular, it would be interesting to study the elementary isomorphisms intertwining the cyclic representations of the monodromy-matrix algebra related to the elliptic R-matrix and also to the R-matrix corresponding to the quantum algebra $U_{\mathrm{q}}(sl_n)$.

The spectrum of the transfer matrix of the six-vertex model at roots of unity is degenerate [5], [14]. Some finite-dimensional representations of the Q-operator algebra correspond to multidimensional eigensubspaces of the transfer matrix. The Q-operators act on these spaces nontrivially because these operators do not commute with each other in general. Therefore, investigating finite-dimensional representations of the Q-operator algebra given by (42)–(48) can shed light on the properties of the transfer-matrix spectrum.

It would also be interesting to clarify the relationship between the algebra of Q-operators and the $U(A_1^1)$ symmetry found in [14].[2]

Appendix A: The TQ equation

We find the kernel of the operator, $\mathcal{L}_{23}\big(v; p_1(v), p_2\big) = \tilde{\mathcal{L}}_{23}(v)$, where

$$p_1(v) = (a_1 v^{-1}, b_1, c_1, d_1 v), \qquad p_2 = (a_2, b_2, c_2, d_2).$$

We have

$$\tilde{\mathcal{L}}_{23}(v) = \begin{pmatrix} v^{-1}c_1c_2Z - b_1b_2vZ^{-1} & -v(b_1d_2Z^{-1} - c_1a_2Z)X \\ \omega v^{-1}X^{-1}(d_1b_2vZ^{-1} - a_1c_2v^{-1}Z) & d_1d_2Z^{-1} - a_1a_2\omega^2 Z \end{pmatrix}.$$

[2] After this paper was sent to the journal, we learned that isomorphisms between representations of a monodromy algebra at roots of unity were studied by Pakulyak and Sergeev for the case of the relativistic Toda chain [15].

Hereafter, we use the following basis $|\alpha\rangle$, $\alpha = 0, \dots, N-1 \pmod N$:

$$Z|\alpha\rangle = \omega^{\alpha}|\alpha\rangle, \qquad X|\alpha\rangle = |\alpha+1\rangle.$$

We consider a $2N$-dimensional vector Ψ

$$\Psi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix},$$

where Φ_1 and Φ_2 are N-dimensional vectors. We act on Ψ with the operator $\tilde{\mathcal{L}}_{23}(v)$ and equate the result to zero, thus obtaining

$$(c_1c_2Z - b_1b_2v^2Z^{-1})\Phi_1 - v^2(b_1d_2Z^{-1} - c_1a_2Z)X\Phi_2 = 0,$$

$$\omega X^{-1}v^{-1}(d_1b_2vZ^{-1} - a_1c_2v^{-1}Z)\Phi_1 + (d_1d_2Z^{-1} - a_1a_2\omega^2 Z)\Phi_2 = 0.$$

It is easy to see that this system has a solution if and only if

$$v^2 = v_*^2 = \frac{c_2d_2}{a_2b_2}.$$

The vectors generating the kernel are

$$\Psi_\alpha = \begin{pmatrix} -d_2|\alpha\rangle \\ b_2|\alpha-1\rangle \end{pmatrix} = -d_2|0,\alpha\rangle + b_2|1,\alpha-1\rangle.$$

We must now act on the vectors Ψ_α with the operator $\tilde{\mathcal{L}}_{13}(uv_*)\mathcal{R}_{12}(u)$ (it is especially useful that we can use the ordinary matrix multiplication in the two-dimensional space here). Let

$$\tilde{\mathcal{L}}_{13}(uv_*) = \begin{pmatrix} A(uv_*) & B(uv_*) \\ C(uv_*) & D(uv_*) \end{pmatrix}, \qquad \mathcal{R}_{12}(u) = \begin{pmatrix} a(u) & b(u) \\ c(u) & d(u) \end{pmatrix}.$$

Then

$$\tilde{\mathcal{L}}_{13}(uv_*)\mathcal{R}_{12}(u) = \begin{pmatrix} a(u)A(uv_*) + c(u)B(uv_*) & b(u)A(uv_*) + d(u)B(uv_*) \\ a(u)C(uv_*) + c(u)D(uv_*) & b(u)C(uv_*) + d(u)D(uv_*) \end{pmatrix}.$$

Acting on the vector Ψ_α with each of the four matrix elements, for example, we obtain

$$\begin{aligned}
(\tilde{\mathcal{L}}\mathcal{R})_0^0\Psi_\alpha &= \big[a(u)A(uv_*) + c(u)B(uv_*)\big]\big[-d_2|0,\alpha\rangle + b_2|1,\alpha-1\rangle\big] = \\
&= -d_2(u\omega - u^{-1}\omega^{-1})(v_*^{-1}u^{-1}c_1c_2\omega^{\alpha} - b_1b_2uv_*\omega^{-\alpha})|0,\alpha\rangle + \\
&\quad + \omega b_2\big\{(u-u^{-1})(v_*^{-1}u^{-1}c_1c_2\omega^{\alpha-1} - b_1b_2uv_*\omega^{1-\alpha})|1,\alpha-1\rangle + \\
&\quad + u(\omega-\omega^{-1})(-uv_*)(b_1d_2\omega^{-\alpha} - c_1a_2\omega^{\alpha})|0,\alpha\rangle\big\} = \\
&= (u-u^{-1})(v_*^{-1}u^{-1}c_1c_2\omega^{\alpha-1} - b_1b_2v_*u\omega^{1-\alpha})\Psi_\alpha.
\end{aligned}$$

Analogously,

$$(\tilde{\mathcal{L}}\mathcal{R})_1^1\Psi_\alpha = (u-u^{-1})(d_1d_2\omega^{-\alpha} - a_1a_2\omega^{2+\alpha})\Psi_\alpha,$$

$$(\tilde{\mathcal{L}}\mathcal{R})_0^1\Psi_\alpha = (u-u^{-1})(-v_*u)(b_1d_2\omega^{-\alpha-1} - c_1a_2\omega^{\alpha+1})\Psi_{\alpha+1},$$

$$(\tilde{\mathcal{L}}\mathcal{R})_1^0\Psi_\alpha = (u-u^{-1})\omega v_*^{-1}u^{-1}(d_1b_2uv_*\omega^{1-\alpha} - a_1c_2u^{-1}v_*^{-1}\omega^{\alpha-1})\Psi_{\alpha-1}.$$

It is clear that the kernel $\tilde{\mathcal{L}}_{23}(v_*)$ is actually an invariant subspace with respect to the operator $\tilde{\mathcal{L}}_{13}(uv_*)\mathcal{R}_{12}(u)$. It is easy to see that the obtained N-dimensional matrix is proportional to $\tilde{\mathcal{L}}(uv_*\omega)$,

$$\tilde{\mathcal{L}}_{13}(uv_*)\mathcal{R}_{12}(u)\big|_{\operatorname{Ker}\mathcal{L}_{23}} = (u-u^{-1})\tilde{\mathcal{L}}(uv_*\omega).$$

We now consider the vectors that belong to the complement of the kernel. If we factor by the vectors of the kernel itself, then this complement must be an invariant space with respect to the operator $\tilde{\mathcal{L}}_{13}(uv_*)\mathcal{R}_{12}(u)$. We choose the basis $\Psi'_\alpha = |0,\alpha\rangle$. Performing all calculations, we obtain $(\operatorname{mod}\Psi_\alpha)$

$$\begin{aligned}
(\tilde{\mathcal{L}}\mathcal{R})^0_0\Psi'_\alpha &= (u\omega - u^{-1}\omega^{-1})(v_*^{-1}u^{-1}c_1c_2\omega^\alpha - b_1b_2v_*u\omega^{-\alpha})\Psi'_\alpha,\\
(\tilde{\mathcal{L}}\mathcal{R})^1_1\Psi'_\alpha &= (u\omega - u^{-1}\omega^{-1})(d_1d_2\omega^{1-\alpha} - a_1a_2\omega^{1+\alpha})\Psi'_\alpha,\\
(\tilde{\mathcal{L}}\mathcal{R})^1_0\Psi'_\alpha &= (u\omega - u^{-1}\omega^{-1})(-v_*u)(b_1d_2\omega^{-\alpha} - c_1a_2\omega^\alpha)\Psi'_{\alpha+1},\\
(\tilde{\mathcal{L}}\mathcal{R})^0_1\Psi'_\alpha &= (u\omega - u^{-1}\omega^{-1})\omega v_*^{-1}u^{-1}(d_1b_2uv_*\omega^{-\alpha} - a_1c_2u^{-1}v_*^{-1}\omega^\alpha)\Psi'_{\alpha-1}.
\end{aligned}$$

As in the previous case, it is easy to see that the obtained N-dimensional matrix is proportional to $\tilde{\mathcal{L}}(uv_*\omega^{-1})$,

$$\tilde{\mathcal{L}}_{13}(uv_*)\mathcal{R}_{12}(u)\big|_{(\operatorname{Ker}\mathcal{L}_{23})^\perp} = (u\omega - u^{-1}\omega^{-1})\tilde{\mathcal{L}}(uv_*\omega^{-1}).$$

We thus find that conditions (13) are satisfied. Then,

$$\lambda = \omega, \qquad \phi_1 = u - u^{-1}, \qquad \phi_2 = u\omega - u^{-1}\omega^{-1},$$

and the TQ equation holds,

$$\mathcal{Q}(u)T(u) = (u-u^{-1})^n\mathcal{Q}(u\omega) + (u\omega - u^{-1}\omega^{-1})^n\mathcal{Q}(u\omega^{-1}),$$

where

$$\mathcal{Q}(u) = \operatorname{tr}_3\tilde{\mathcal{L}}_{13}(u)\tilde{\mathcal{L}}_{1'3}(u)\cdots\tilde{\mathcal{L}}_{1^{(n)}3}(u).$$

Appendix B: The conditions for the equivalence of representations: The Fermat curve

Let N be odd. We now prove that in the general case, the representations $L_1(u,p_1,p_2)L_2(u,p_3,p_4)$ and $L_1(u,p_3,p_4)L_2(u,p_1,p_2)$ are equivalent if and only if we can choose p_i, $i=1,2,3,4$, satisfying the conditions

$$\frac{a_i^N \pm b_i^N}{c_i^N \pm d_i^N} = \lambda_\pm,$$

which determine the Fermat curve. For convenience, we hereafter substitute

$$a_i^N \to a_i, \qquad b_i^N \to b_i, \qquad c_i^N \to c_i, \qquad d_i^N \to d_i.$$

We have

$$\big\langle L(u,p_1,p_2)\big\rangle = \begin{pmatrix} c_1c_2 - b_1b_2u & -u(b_1d_2 - c_1a_2)\\ d_1b_2 - a_1c_2 & d_1d_2 - a_1a_2u\end{pmatrix},$$

$$\big\langle L(u,p_3,p_4)\big\rangle = \begin{pmatrix} c_3c_4 - b_3b_4u & -u(b_3d_4 - c_3a_4)\\ d_3b_4 - a_3c_4 & d_3d_4 - a_3a_4u\end{pmatrix}.$$

Two representations L_π and $L_{\pi'}$ are equivalent if there exists an isomorphism P such that

$$L_{\pi'} = P L_\pi P^{-1},$$

that is, elements of L_π and $L_{\pi'}$ are in a one-to-one correspondence. The central elements $\langle L_\pi \rangle$ and $\langle L_{\pi'} \rangle$ must coincide [6]. If $\pi = \pi_1 \times \pi_2$ and $\pi' = \pi_2 \times \pi_1$, then

$$\langle L_{\pi_1} \rangle \langle L_{\pi_2} \rangle = \langle L_{\pi_2} \rangle \langle L_{\pi_1} \rangle.$$

It hence follows that

$$\big\langle L(u, p_1, p_2) \big\rangle \big\langle L(u, p_3, p_4) \big\rangle = \big\langle L(u, p_3, p_4) \big\rangle \big\langle L(u, p_1, p_2) \big\rangle.$$

Multiplying the matrices, we obtain five equations. Only three of them are independent:

$$\begin{aligned} \frac{b_1 d_2 - c_1 a_2}{d_1 b_2 - a_1 c_2} &= \frac{b_3 d_4 - c_3 a_4}{d_3 b_4 - a_3 c_4} = s, \\ \frac{a_1 a_2 - b_1 b_2}{b_1 d_2 - c_1 a_2} &= \frac{a_3 a_4 - b_3 b_4}{b_3 d_4 - c_3 a_4} = q, \\ \frac{c_1 c_2 - d_1 d_2}{b_1 d_2 - c_1 a_2} &= \frac{c_3 c_4 - d_3 d_4}{b_3 d_4 - c_3 a_4} = r, \end{aligned} \tag{B.1}$$

where s, q, and r are arbitrary constants.

We now find the constraints on p_i under which this system has solutions. We have

$$\begin{aligned} b_1 d_2 - c_1 a_2 &= s(d_1 b_2 - a_1 c_2), \\ a_1 a_2 - b_1 b_2 &= q(b_1 d_2 - c_1 a_2), \\ c_1 c_2 - d_1 d_2 &= r(b_1 d_2 - c_1 a_2). \end{aligned} \tag{B.2}$$

It turns out that (B.2) holds if p_1 and p_2 are points on a curve obtained by intersecting two planes (the projective symmetry of the operator L):

$$\begin{aligned} &\alpha_1 a_i + \beta_1 b_i + \gamma_1 c_i + \delta_1 d_i = 0, \\ &\alpha_2 a_i + \beta_2 b_i + \gamma_2 c_i + \delta_2 d_i = 0, \quad i = 1, 2. \end{aligned}$$

We find these manifolds. From the last system, we have

$$\begin{aligned} a_i &= \lambda_i c_i + \mu_i d_i, \\ b_i &= \nu_i c_i + \eta_i d_i, \quad i = 1, 2. \end{aligned} \tag{B.3}$$

We substitute (B.3) in (B.2) and obtain the system of equations for the coefficients,

$$\begin{aligned} &\eta_1 = s\eta_2, \qquad \lambda_2 = s\lambda_1, \qquad \nu_1 = \mu_2, \qquad \nu_2 = \mu_1, \\ &1 = -r\lambda_2, \qquad -1 = r\eta_1, \qquad \nu_1 = \mu_2, \\ &\lambda_1\lambda_2 - \nu_1\nu_2 = -q\lambda_2, \qquad \mu_1\mu_2 - \eta_1\eta_2 = q\eta_1, \\ &\lambda_1\mu_2 - \nu_1\eta_2 = q\nu_1 - q\mu_2, \qquad \mu_1\lambda_2 - \nu_2\eta_1 = 0. \end{aligned}$$

Solving the system, we obtain

$$\nu_1 = \mu_2 = \mu, \qquad \mu_1 = \nu_2 = \nu,$$

$$\eta_1 = \lambda_2 = \lambda, \qquad \lambda_1 = \eta_2 = \eta,$$

$$\lambda = -\frac{1}{r}, \qquad \nu\mu = (q + \eta)\lambda, \qquad s = 1.$$

Moreover, because the points p_1 and p_2 lie on the same curve, we have

$$\lambda_1 = \lambda_2 = \lambda, \qquad \mu_1 = \mu_2 = \mu, \qquad \nu_1 = \nu_2 = \nu, \qquad \eta_1 = \eta_2 = \eta.$$

Consequently,

$$\lambda = \eta, \qquad \nu = \mu.$$

As a result, we have

$$\frac{a_1 + b_1}{c_1 + d_1} = \frac{a_2 + b_2}{c_2 + d_2}, \qquad \frac{a_1 - b_1}{c_1 - d_1} = \frac{a_2 - b_2}{c_2 - d_2}.$$

Similarly, we can obtain

$$\frac{a_3 + b_3}{c_3 + d_3} = \frac{a_4 + b_4}{c_4 + d_4}, \qquad \frac{a_3 - b_3}{c_3 - d_3} = \frac{a_4 - b_4}{c_4 - d_4}.$$

Returning to the old notation

$$p_i \to p_i^N, \quad i = 1, 2, 3, 4,$$

we find that by virtue of (B.1), the points p_1, p_2, p_3, and p_4 can be related in two different ways:

$$\text{a.} \qquad \frac{a_1^N \pm b_1^N}{c_1^N \pm d_1^N} = \frac{a_2^N \pm b_2^N}{c_2^N \pm d_2^N} = \frac{a_3^N \pm b_3^N}{c_3^N \pm d_3^N} = \frac{a_4^N \pm b_4^N}{c_4^N \pm d_4^N},$$

$$\text{b.} \qquad \frac{a_1^N \pm b_1^N}{c_1^N \pm d_1^N} = \frac{a_2^N \pm b_2^N}{c_2^N \pm d_2^N} = \frac{a_3^N \mp b_3^N}{c_3^N \mp d_3^N} = \frac{a_4^N \mp b_4^N}{c_4^N \mp d_4^N}.$$

This pertains to the existence of two roots of the equation

$$\nu^2 = \mu^2 = -\frac{q}{r} + \frac{1}{r^2}.$$

But recalling the symmetries of the operator $L(u, p_1, p_2)$, we can substitute

$$b_1 \to \lambda b_1, \qquad b_2 \to \lambda^{-1} b_2, \qquad d_1 \to \lambda d_1, \qquad d_2 \to \lambda^{-1} d_2,$$

and $L(u, p_1, p_2)$ must remain unchanged. With such a substitution for $\lambda = -1$, case b is reduced to case a (we recall that N is odd).

Appendix C: The conditions for equivalence of representations in the general case

We introduce the notation

$$\phi = d^N\bar{b}^N, \qquad \psi = a^N\bar{c}^N, \qquad \beta = b^N\bar{b}^N,$$

$$\delta = a^N\bar{a}^N, \qquad \mu = \frac{c^N d^N}{a^N b^N}, \qquad \lambda = \frac{\bar{c}^N\bar{d}^N}{\bar{a}^N\bar{b}^N}.$$

Then $\langle L(u,p,\bar{p})\rangle$ can be written as

$$\langle L(u,p,\bar{p})\rangle = \begin{pmatrix} \dfrac{\mu\beta\psi}{\phi} - \beta u & -u\left(\dfrac{\lambda\beta\delta}{\psi} - \dfrac{\mu\beta\delta}{\phi}\right) \\ \phi - \psi & \dfrac{\lambda\delta\phi}{\psi} - \delta u \end{pmatrix}.$$

We consider two representations $L(u,p_1,\bar{p}_1)$ and $L(u,p_2,\bar{p}_2)$ of the L-operator algebra and find the conditions under which they are equivalent. The necessary condition for the equivalence of the two representations is a coincidence of the centers of these representations. We have

$$\langle L(u,p_1,\bar{p}_1)\rangle = \langle L(u,p_2,\bar{p}_2)\rangle. \tag{C.1}$$

By equating the coefficients of the same powers of u, we hence obtain

$$\delta_1 = \delta_2, \qquad \beta_1 = \beta_2, \qquad \phi_1 - \psi_1 = \phi_2 - \psi_2, \tag{C.2}$$

$$\frac{\lambda_1\phi_1}{\psi_1} = \frac{\lambda_2\phi_2}{\psi_2}. \tag{C.3}$$

In addition, it follows from (C.1) that

$$\det\langle L(u,p_1,\bar{p}_1)\rangle = \det\langle L(u,p_2,\bar{p}_2)\rangle.$$

In this equation, the left- and right-hand sides are second-degree polynomials in u. The roots of the left polynomial are

$$u_1 = \lambda_1, \qquad \bar{u}_1 = \mu_1,$$

and the roots of the right polynomial are

$$u_2 = \lambda_2, \qquad \bar{u}_2 = \mu_2.$$

The roots of the left- and right-hand sides must coincide. We consider the case where

$$\lambda_1 = \mu_2, \qquad \lambda_2 = \mu_1. \tag{C.4}$$

Rewriting (C.2)–(C.4) in terms of a_i, b_i, c_i, and d_i, we obtain

$$a_1^N\bar{a}_1^N = a_2^N\bar{a}_2^N, \qquad b_1^N\bar{b}_1^N = b_2^N\bar{b}_2^N,$$

$$\frac{\bar{c}_1^N\bar{d}_1^N}{\bar{a}_1^N\bar{b}_1^N} = \frac{c_2^N d_2^N}{a_2^N b_2^N}, \qquad \frac{\bar{c}_2^N\bar{d}_2^N}{\bar{a}_2^N\bar{b}_2^N} = \frac{c_1^N d_1^N}{a_1^N b_1^N},$$

$$\frac{d_1^N\bar{d}_1^N}{a_1^N\bar{a}_1^N} = \frac{d_2^N\bar{d}_2^N}{a_2^N\bar{a}_2^N},$$

$$d_1^N\bar{b}_1^N - a_1^N\bar{c}_1^N = d_2^N\bar{b}_2^N - a_2^N\bar{c}_2^N.$$

We now consider two representations $L_1(u,p_1,\bar{p}_1)L_2(u,p_2,\bar{p}_2)$ and $L_1(u,p_3,\bar{p}_3)L_2(u,p_4,\bar{p}_4)$ of the L-operator algebra. We find the conditions under which these two representations are equivalent. We have

$$\det\big\langle L_1(u,p_1,\bar{p}_1)\big\rangle \det\big\langle L_2(u,p_2,\bar{p}_2)\big\rangle = \det\big\langle L_1(u,p_3,\bar{p}_3)\big\rangle \det\big\langle L_2(u,p_4,\bar{p}_4)\big\rangle. \tag{C.5}$$

Each determinant is a second-degree polynomial in u. The roots of these polynomials are

$$u = \lambda, \qquad \bar{u} = \mu.$$

The left-hand side of (C.5) then vanishes at $u_1 = \lambda_1$, $\bar{u}_1 = \mu_1$, $u_2 = \lambda_2$, and $\bar{u}_2 = \mu_2$, and the right-hand side vanishes at $u_3 = \lambda_3$, $\bar{u}_3 = \mu_3$, $u_4 = \lambda_4$, and $\bar{u}_4 = \mu_4$. It is then clear that the left and the right roots coincide. We consider the case where

$$\mu_2 = \lambda_3, \qquad \mu_1 = \mu_3, \qquad \lambda_2 = \lambda_4, \qquad \lambda_1 = \mu_4.$$

We have

$$\big\langle L_1(u,p_1,\bar{p}_1)\big\rangle\big\langle L_2(u,p_2,\bar{p}_2)\big\rangle = \big\langle L_1(u,p_3,\bar{p}_3)\big\rangle\big\langle L_2(u,p_4,\bar{p}_4)\big\rangle, \tag{C.6}$$

and we find the roots that correspond to the separate multipliers in the left- and right-hand sides of (C.6):

$$\mu_1,\lambda_1, \qquad \mu_2,\lambda_2, \qquad \mu_1,\mu_2, \qquad \lambda_1,\lambda_2.$$

We see that the two roots λ_1 and μ_2 are interchanged.

Let $u = \lambda_2$. We act on the vector Ψ_1 (the right zero vector of the operator $\big\langle L_2(\lambda_2,p_2,\bar{p}_2)\big\rangle$) from the left with both sides of Eq. (C.6):

$$\Psi_1 = \begin{pmatrix} -\lambda_2\delta_2 \\ \psi_2 \end{pmatrix}, \qquad \big\langle L_2(\lambda_2,p_2,\bar{p}_2)\big\rangle\Psi_1 = 0.$$

This vector is also the right zero vector for $\big\langle L_2(\lambda_2,p_4,\bar{p}_4)\big\rangle$. We hence obtain the equation

$$\frac{\delta_2}{\psi_2} = \frac{\delta_4}{\psi_4}.$$

Now let $u = \mu_1$. We act on the left zero vector Ψ_2 of the operator $\big\langle L_1(\mu_1,p_1,\bar{p}_1)\big\rangle$ from the right with both sides of Eq. (C.6):

$$\Psi_2 = \begin{pmatrix} \phi_1 \\ \mu_1\beta_1 \end{pmatrix}, \qquad \Psi_2\big\langle L_1(\mu_1,p_1,\bar{p}_1)\big\rangle = 0.$$

Because this vector is the left zero vector of the operator $\big\langle L_1(\mu,p_3,\bar{p}_3)\big\rangle$, we have

$$\frac{\phi_1}{\beta_1} = \frac{\phi_3}{\beta_3}.$$

It is clear that (C.6) holds if we insert the unity $1 = MM^{-1}$ between the two factors in the right-hand side, where M is a two-dimensional matrix,

$$M = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix}.$$

Using this gauge symmetry, we can set

$$\beta_3 = \beta_1, \qquad \delta_4 = \delta_2,$$

whence it immediately follows that

$$\phi_3 = \phi_1, \qquad \psi_4 = \psi_2.$$

Multiplying the matrices in (C.6) and equating coefficients of the powers of u, it is easy to see that

$$\begin{aligned}
&\beta_4 = \beta_2, \qquad \delta_3 = \delta_1,\\
&\psi_3 = \phi_4 \frac{\mu_2 \psi_1}{\lambda_1 \phi_2},\\
&\psi_3 \beta_2 - \delta_1 \phi_4 = \psi_1 \beta_2 - \delta_1 \phi_2.
\end{aligned}$$

Two last equalities imply

$$\phi_4 = \frac{\lambda_1 \phi_2 (\beta_2 \psi_1 - \delta_1 \phi_2)}{\beta_2 \mu_2 \psi_1 - \lambda_1 \delta_1 \phi_2}, \qquad \psi_3 = \frac{\mu_2 \psi_1 (\beta_2 \psi_1 - \delta_1 \phi_2)}{\beta_2 \mu_2 \psi_1 - \lambda_1 \delta_1 \phi_2}. \tag{C.7}$$

Collecting all the obtained equations, in addition to (C.7), we have

$$\begin{aligned}
&\beta_1 = \beta_3, \qquad \beta_2 = \beta_4, \qquad \delta_1 = \delta_3, \qquad \delta_2 = \delta_4,\\
&\phi_1 = \phi_3, \qquad \psi_2 = \psi_4, \qquad \mu_2 = \lambda_3,\\
&\mu = \mu_3, \qquad \lambda_2 = \lambda_4, \qquad \lambda_1 = \mu_4.
\end{aligned}$$

Appendix D: Calculating the operator $\boldsymbol{F}$

Matrix equation (38) can be written in the form of a system of equations corresponding to the four matrix elements. We have

$$L_1(p_1, \bar{p}_1) L_2(p_2, \bar{p}_2) = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} = \begin{pmatrix} A_1 A_2 + B_1 C_2 & A_1 B_2 + B_1 D_2 \\ C_1 A_2 + D_1 C_2 & C_1 B_2 + D_1 D_2 \end{pmatrix}.$$

As a result, we obtain the system

$$\begin{aligned}
&F(X_1 X_2^{-1})(A_1 A_2 + B_1 C_2) = (A_3 a_4 + B_3 C_4) F(X_1 X_2^{-1}),\\
&F(X_1 X_2^{-1})(A_1 B_2 + B_1 D_2) = (A_3 B_4 + B_3 D_4) F(X_1 X_2^{-1}),\\
&F(X_1 X_2^{-1})(C_1 A_2 + D_1 C_2) = [C_3 a_4 + D_3 C_4] F(X_1 X_2^{-1}),\\
&F(X_1 X_2^{-1})(C_1 B_2 + D_1 D_2) = (C_3 B_4 + D_3 D_4) F(X_1 X_2^{-1}).
\end{aligned} \tag{D.1}$$

We choose a basis $|k_1, k_2\rangle$, $k_1, k_2 = 0, \ldots, N-1 \pmod{N}$,

$$\begin{aligned}
&X_1 |k_1, k_2\rangle = \omega^{k_1} |k_1, k_2\rangle, \qquad X_2 |k_1, k_2\rangle = \omega^{k_2} |k_1, k_2\rangle,\\
&Z_1 |k_1, k_2\rangle = |k_1 - 1, k_2\rangle, \qquad Z_2 |k_1, k_2\rangle = |k_1, k_2 - 1\rangle.
\end{aligned}$$

The matrix $F(X_1X_2^{-1})$ is diagonal in this basis. We now calculate its nonzero matrix elements. Substituting the expressions for A_i, B_i, C_i, and D_i, $i=1,2$, for the first equation in (D.1), for example, we have

$$F(X_1X_2^{-1})\big[(c_1\bar{c}_1Z_1-b_1\bar{b}_1u)(c_2\bar{c}_2Z_2-b_2\bar{b}_2u)-u(b_1\bar{d}_1-c_1\bar{a}_1Z_1)X_1X_2^{-1}(d_2\bar{b}_2-a_2\bar{c}_2Z_2)\big]=$$
$$=\big[(c_3\bar{c}_3Z_1-b_3\bar{b}_3u)(c_4\bar{c}_4Z_2-b_4\bar{b}_4u)-u(b_3\bar{d}_3-c_3\bar{a}_3Z_1)X_1X_2^{-1}(d_4\bar{b}_4-a_4\bar{c}_4Z_2)\big]F(X_1X_2^{-1}).$$

Opening the parenthesis and acting on the vector $|k_1,k_2\rangle$ with the left- and right-hand sides and equating coefficients of the linearly independent vectors and of the same powers of u, we obtain

$$c_1\bar{c}_1c_2\bar{c}_2=c_3\bar{c}_3c_4\bar{c}_4,\qquad b_1\bar{b}_1b_2\bar{b}_2=b_3\bar{b}_3b_4\bar{b}_4,$$
$$b_1\bar{d}_1d_2\bar{b}_2=b_3\bar{d}_3d_4\bar{b}_4,\qquad c_1\bar{a}_1a_2\bar{c}_2=c_3\bar{a}_3a_4\bar{c}_4,$$
$$F(\omega^{k+1})=\frac{b_3\bar{c}_4(\bar{d}_3a_4\omega^{k+1}-\bar{b}_3c_4)}{b_1\bar{c}_2(\bar{d}_1a_2\omega^{k+1}-\bar{b}_1c_2)}F(\omega^k)=\frac{c_1\bar{b}_2(\bar{a}_1d_2\omega^{k+1}-\bar{c}_1b_2)}{c_3\bar{b}_4(\bar{a}_3d_4\omega^{k+1}-\bar{c}_3b_4)}F(\omega^k),$$

where $F(\omega^k)$, $k=0,\dots,N-1$, are the diagonal matrix elements of the matrix F. Using the obtained restrictions on a_i, b_i, c_i, and d_i, we can reduce this system to the system

$$\bar{c}_1c_2=\bar{c}_3c_4,\qquad \bar{d}_1d_2=\bar{d}_3d_4,$$
$$F(\omega^{k+1})=\frac{\bar{d}_3a_4\omega^{k+1}-\bar{b}_3c_4}{\bar{d}_1a_2\omega^{k+1}-\bar{b}_1c_2}F(\omega^k)=\frac{\bar{a}_1d_2\omega^{k+1}-\bar{c}_1b_2}{\bar{a}_3d_4\omega^{k+1}-\bar{c}_3b_4}F(\omega^k).$$

It can be easily seen that the second equation can be derived from the first if the constraints

$$\bar{c}_3\bar{d}_3=c_2d_2\frac{\bar{a}_3\bar{b}_3}{a_2b_2},\qquad c_4d_4=\bar{c}_1\bar{d}_1\frac{a_2b_2}{\bar{a}_1\bar{b}_1}$$

are used. Because $F(\omega^k)$ is a single-valued function, we must equate the two fractions in terms of which $F(\omega^k)$ is expressed. We then obtain

$$(\bar{d}_3a_4\omega^{k+1}-\bar{b}_3c_4)(\bar{a}_3d_4\omega^{k+1}-\bar{c}_3b_4)=(\bar{d}_1a_2\omega^{k+1}-\bar{b}_1c_2)(\bar{a}_1d_2\omega^{k+1}-\bar{c}_1b_2).$$

Equating the coefficients of the same powers of ω, we obtain

$$\bar{a}_1\bar{d}_1a_2d_2=\bar{a}_3\bar{d}_3a_4d_4,\qquad \bar{b}_1\bar{c}_1b_2c_2=\bar{b}_3\bar{c}_3b_4c_4,$$
$$\bar{a}_1\bar{b}_1a_2b_2\left(\frac{\bar{c}_1\bar{d}_1}{\bar{a}_1\bar{b}_1}+\frac{c_2d_2}{a_2b_2}\right)=a_4b_4\bar{a}_3\bar{b}_3\left(\frac{\bar{c}_3\bar{d}_3}{\bar{a}_3\bar{b}_3}+\frac{c_4d_4}{a_4b_4}\right).$$

If we again take the restrictions on a_i, b_i, c_i, and d_i into account, we obtain

$$\bar{c}_1c_2=\bar{c}_3c_4, \tag{D.2}$$

that is, the same equation as before. From the obvious equation

$$F(\omega^{N+k})=F(\omega^k),$$

we also have

$$\bar{c}_1^Nb_2^N-\bar{a}_1^Nd_2^N=\bar{c}_3^Nb_4^N-\bar{a}_3^Nd_4^N. \tag{D.3}$$

We now prove that (D.2) and (D.3) follow from (33)–(37). We have

$$\psi_3 = a_3^N \bar{c}_3^N = \frac{c_2^N \bar{b}_1^N a_1^N}{b_2^N} \frac{b_2^N \bar{c}_1^N - \bar{a}_1^N d_2^N}{c_2^N \bar{b}_1^N - a_2^N \bar{d}_1^N},$$

$$\phi_4 = d_4^N \bar{b}_4^N = \frac{\bar{d}_1^N a_2^N \bar{b}_2^N}{\bar{a}_1^N} \frac{b_2^N \bar{c}_1^N - \bar{a}_1^N d_2^N}{c_2^N \bar{b}_1^N - a_2^N \bar{d}_1^N}.$$

Expressing $\bar{c}_3^N$ and d_4^N from this equation and substituting them in (D.3), we obtain an identity. Moreover, we have

$$\frac{\bar{c}_3}{d_4} = \frac{c_2}{\bar{d}_1} \frac{\bar{a}_1 \bar{b}_1}{a_2 b_2}.$$

Multiplying the last equation by

$$c_4 d_4 = \bar{c}_1 \bar{d}_1 \frac{a_2 b_2}{\bar{a}_1 \bar{b}_1},$$

we obtain (D.2). Therefore,

$$F(\omega^{k+1}) = \sqrt[N]{\frac{\bar{d}_1^N a_2^N - \bar{b}_1^N c_2^N}{\bar{a}_1^N d_2^N - \bar{c}_1^N b_2^N}} \frac{\bar{c}_1 b_2 - \bar{a}_1 d_2 \omega^{k+1}}{\bar{b}_1 c_2 - \bar{d}_1 a_2 \omega^{k+1}} F(\omega^k).$$

Appendix E: The star–triangle equation

Here, we prove the star–triangle relation

$$G(\tilde{q}, \tilde{p}; Z_1) F(\tilde{p}, r; X_1 X_2^{-1}) G(p, q; Z_1) = \mu F(p', q'; X_1 X_2^{-1}) G(p, r'; Z_1) F(q, r; X_1 X_2^{-1}) \qquad \text{(E.1)}$$

and find μ. We start with the diagram

$$\begin{array}{ccccc}
(p,q)(r,s) & \xrightarrow{F} & (p,r')(q',s) & \xrightarrow{G} & (r'',p')(q',s) \\
\big\downarrow G & & & & F \big\downarrow \\
(\tilde{q},\tilde{p})(r,s) & \xrightarrow{F} & (\tilde{q},\tilde{r})(p'',s) & \xrightarrow{G} & (r'',q'')(p'',s).
\end{array}$$

Now, if r'', q'', and p'' obtained in two different ways coincide up to gauge transformations, then we prove that Eq. (E.1) is valid. We recall the definition of the functions $\Lambda(p_1, p_2)$ and $\Omega(p_1, p_2)$:

$$\Lambda(p_1, p_2) = \sqrt[N]{\frac{b_1^N d_2^N - c_1^N a_2^N}{d_1^N b_2^N - a_1^N c_2^N}}, \qquad \Omega(p_1, p_2) = \sqrt[N]{\frac{c_1^N b_2^N - a_1^N d_2^N}{b_1^N c_2^N - d_1^N a_2^N}}.$$

We perform all calculations for the first chain of the diagram. For $(p,q)(r,s) \xrightarrow{F} (p,r')(q',s)$, we have

$$\begin{aligned}
a_r' &= a_r, & a_q' &= a_q, \\
b_r' &= b_r, & b_q' &= b_q, \\
c_r' &= \Omega(q,r) c_r, & c_q' &= \Omega(q,r)^{-1} c_q, \\
d_r' &= \Omega(q,r)^{-1} d_r, & d_q' &= \Omega(q,r) d_q.
\end{aligned}$$

For $(p,r')(q',s) \xrightarrow{G} (r'',p')(q',s)$, we have

$$\begin{aligned} a''_r &= a'_r, & a'_p &= a_p, \\ b''_r &= b'_r, & b'_p &= b_p, \\ c''_r &= \Lambda(p,r')c'_r, & c'_p &= \Lambda(p,r')^{-1}c_p, \\ d''_r &= \Lambda(p,r')^{-1}d'_r, & d'_p &= \Lambda(p,r')d_p. \end{aligned}$$

For $(r'',p')(q',s) \xrightarrow{F} (r'',q'')(p'',s)$, we have

$$\begin{aligned} a''_q &= a'_q, & a''_p &= a'_p, \\ b''_q &= b'_q, & b''_p &= b'_p, \\ c''_q &= \Omega(p',q')c'_q, & c''_p &= \Omega(p',q')^{-1}c'_p, \\ d''_q &= \Omega(p',q')^{-1}d'_q, & d''_p &= \Omega(p',q')d'_p. \end{aligned}$$

We now perform all calculations for the second chain of the diagram. For $(p,q)(r,s) \xrightarrow{G} (\tilde{q},\tilde{p})(r,s)$, we have

$$\begin{aligned} \tilde{a}_q &= a_q, & \tilde{a}_p &= a_p, \\ \tilde{b}_q &= b_q, & \tilde{b}_p &= b_p, \\ \tilde{c}_q &= \Lambda(p,q)c_q, & \tilde{c}_p &= \Lambda(p,q)^{-1}c_p, \\ \tilde{d}_q &= \Lambda(p,q)^{-1}d_q, & \tilde{d}_p &= \Lambda(p,q)d_p. \end{aligned}$$

For $(\tilde{q},\tilde{p})(r,s) \xrightarrow{F} (\tilde{q},\tilde{r})(p'',s)$, we have

$$\begin{aligned} \tilde{a}_r &= a_r, & a''_p &= \tilde{a}_p, \\ \tilde{b}_r &= b_r, & b''_p &= \tilde{b}_p, \\ \tilde{c}_r &= \Omega(\tilde{p},r)c_r, & c''_p &= \Omega(\tilde{p},r)^{-1}\tilde{c}_p, \\ \tilde{d}_r &= \Omega(\tilde{p},r)^{-1}d_r, & d''_p &= \Omega(\tilde{p},r)\tilde{d}_p. \end{aligned}$$

For $(\tilde{q},\tilde{r})(p'',s) \xrightarrow{G} (r'',q'')(p'',s)$, we have

$$\begin{aligned} a''_r &= \tilde{a}_r, & a''_q &= \tilde{a}_q, \\ b''_r &= \tilde{b}_r, & b''_q &= \tilde{b}_q, \\ c''_r &= \Lambda(\tilde{q},\tilde{r})\tilde{c}_r, & c''_q &= \Lambda(\tilde{q},\tilde{r})^{-1}\tilde{c}_q, \\ d''_r &= \Lambda(\tilde{q},\tilde{r})^{-1}\tilde{d}_r, & d''_q &= \Lambda(\tilde{q},\tilde{r})\tilde{d}_q. \end{aligned}$$

It remains to verify that the obtained L-operators actually coincide. Comparing the parameters r'', q'', and p'' obtained in the two different ways, we conclude that Eq. (41) is satisfied if

$$\begin{aligned} &\Lambda(p, r')\Omega(p', q') = \Lambda(p, q)\Omega(\tilde{p}, r), \\ &\Omega(q, r)\Lambda(p, r') = \Omega(\tilde{p}, r)\Lambda(\tilde{q}, \tilde{r}). \end{aligned} \tag{E.2}$$

It is easy to show that (E.2) is satisfied identically.

We have thus proved (E.1) for some as yet unknown μ. We now find μ^N. It is clear that

$$\mu^N = \frac{\det\ G(p,q)\det F(\tilde{p},r)\det G(\tilde{q},\tilde{r})}{\det F(q,r)\det G(p,r')\det F(p',q')}.$$

The determinants of matrices $G(p_1, p_2)$ and $F(p_1, p_2)$ can be easily evaluated. Each of these matrices can be individually reduced to the diagonal form (not simultaneously), and in each case, the diagonal matrix elements are

$$\frac{G(p_1,p_2;\omega^k)}{G(p_1,p_2;1)} = \left(\frac{b_1^N d_2^N - c_1^N a_2^N}{d_1^N b_2^N - a_1^N c_2^N}\right)^{k/N} \prod_{j=1}^{k} \frac{d_1 b_2 - a_1 c_2 \omega^j}{b_1 d_2 - c_1 a_2 \omega^j},$$

$$\frac{F(p_1,p_2;\omega^k)}{F(p_1,p_2;1)} = \left(\frac{b_1^N c_2^N - d_1^N a_2^N}{c_1^N b_2^N - a_1^N d_2^N}\right)^{k/N} \prod_{j=1}^{k} \frac{c_1 b_2 - a_1 d_2 \omega^j}{b_1 c_2 - d_1 a_2 \omega^j}.$$

We set

$$G(p_1,p_2;1) = F(p_1,p_2;1) = 1.$$

Then

$$\det G(p_1,p_2) = \left(\frac{b_1^N d_2^N - c_1^N a_2^N}{d_1^N b_2^N - a_1^N c_2^N}\right)^{(N-1)/2} \prod_{k=1}^{N-1}\prod_{j=1}^{k} \frac{d_1 b_2 - a_1 c_2 \omega^j}{b_1 d_2 - c_1 a_2 \omega^j},$$

$$\det F(p_1,p_2) = \left(\frac{b_1^N c_2^N - d_1^N a_2^N}{c_1^N b_2^N - a_1^N d_2^N}\right)^{(N-1)/2} \prod_{k=1}^{N-1}\prod_{j=1}^{k} \frac{c_1 b_2 - a_1 d_2 \omega^j}{b_1 c_2 - d_1 a_2 \omega^j}.$$

Appendix F: The relations in the Q-operator algebra

The relations in the Q-operator algebra follow from the properties of the cyclic representations of the L-operator algebra. We now prove this.

Relations (42)–(45) become evident if we recall that

$$Q(u) = \mathrm{tr}_0\, L_{10}(u)L_{20}(u)\cdots L_{n0}(u)$$

and that we have the symmetries

$$\begin{aligned} &L(\lambda p_1, \bar{p}_1) = \lambda L(p_1, \bar{p}_1), \\ &L(p_1, \mu\bar{p}_1) = \mu L(p_1, \bar{p}_1), \\ &L(a_1, b_1, c_1, d_1, \bar{a}_1, \bar{b}_1, \bar{c}_1, \bar{d}_1) = L(\lambda a_1, b_1, \lambda c_1, d_1, \lambda^{-1}\bar{a}_1, \bar{b}_1, \lambda^{-1}c_q, \bar{d}_1), \\ &L(a_1, b_1, c_1, d_1, \bar{a}_1, \bar{b}_1, \bar{c}_1, \bar{d}_1) = L(a_1, \mu b_1, c_1, \mu d_1, \bar{a}_1, \mu^{-1}\bar{b}_1, \bar{c}_1, \mu^{-1}\bar{d}_1), \end{aligned}$$

where λ and μ are arbitrary numbers. Relation (47) follows from another symmetry:

$$L_1(p_1, \bar{p}_1) L_2(p_2, s) = L_1(p_1, \bar{p}_1) M M^{-1} L_2(p_2, s),$$

or, in more detail,

$$L_1(a_1, b_1, c_1, d_1, \bar{a}_1, \bar{b}_1, \bar{c}_1, \bar{d}_1) L_2(a_2, b_2, c_2, d_2, \bar{a}_2, \bar{b}_2, \bar{c}_2, \bar{d}_2) =$$
$$= L_1(a_1, b_1, c_1, d_1, \beta\bar{a}_1, \alpha\bar{b}_1, \alpha\bar{c}_1, \beta\bar{d}_1) L_2(\beta^{-1}a_2, \alpha^{-1}b_2, \alpha^{-1}c_2, \beta^{-1}d_2, \bar{a}_2, \bar{b}_2, \bar{c}_2, \bar{d}_2),$$

where α and β are arbitrary numbers. Here,

$$M = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}.$$

Relation (46) is obtained from the equation

$$G L(a_1, b_1, c_1, d_1, \bar{a}_1, \bar{b}_1, \bar{c}_1, \bar{d}_1) G^{-1} = L(a_1, b_1, \Lambda c_1, \Lambda^{-1} d_1, \bar{a}_1, \bar{b}_1, \Lambda^{-1}\bar{c}_1, \Lambda\bar{d}_1),$$

where

$$\Lambda = \Lambda(p_1, \bar{p}_1) = \sqrt[N]{\frac{b_1^N \bar{d}_1^N - c_1^N \bar{a}_1^N}{d_1^N \bar{a}_1^N - a_1^N \bar{c}_1^N}},$$

and relation (48) is obtained from

$$F L_1(a_1, b_1, c_1, d_1, \bar{a}_1, \bar{b}_1, \bar{c}_1, \bar{d}_1) L_2(a_2, b_2, c_2, d_2, \bar{a}_2, \bar{b}_2, \bar{c}_2, \bar{d}_2) F^{-1} =$$
$$= L_1(a_1, b_1, c_1, d_1, a_2, b_2, \Omega c_2, \Omega^{-1} d_2) L_2(\bar{a}_1, \bar{b}_1, \Omega^{-1}\bar{c}_1, \Omega\bar{d}_1, \bar{a}_2, \bar{b}_2, \bar{c}_2, \bar{d}_2),$$

where

$$\Omega = \Omega(\bar{p}_1, p_2) = \sqrt[N]{\frac{\bar{c}_1^N b_2^N - \bar{a}_1^N d_2^N}{\bar{b}_1^N c_2^N - \bar{d}_1^N a_2^N}}.$$

Acknowledgments. We are grateful to B. Feigin and Yu. Stroganov for the useful discussions. One of us (A. O.) is grateful to V. Bazhanov for the discussion of the factoring property of the R-matrix of the chiral Potts model.

This work was supported by the Russian Foundation for Basic Research (Grant Nos. 01-02-16686, 99-01-01169, and 00-15-96579), the CRDF-RP1 (Grant No. 2254), and INTAS (Grant No. 00-00055).

REFERENCES

1. R. I. Baxter, *Ann. Phys.*, **70**, 193, 323 (1972); **76**, 48 (1973).
2. R. J. Baxter, J. H. H. Perk, and H. Au-Yang, *Phys. Lett. A*, **128**, 138 (1988).
3. H. Au-Yang, B. M. McCoy, J. H. H. Perk, S. Tang, and M. L. Yan, *Phys. Lett. A*, **123**, 219 (1987).
4. R. J. Baxter, V. V.Bazhanov, and J. H. H. Perk, *Int. J. Mod. Phys. B*, **4**, 803 (1990).
5. V. V. Bazhanov and Yu. G. Stroganov, *J. Stat. Phys.*, **51**, 799 (1990).
6. V. O. Tarasov, *Int. J. Mod. Phys. A*, **7**, 963 (1992).
7. V. Pasquier and M. Gaudin, *J. Phys. A*, **25**, 5243 (1992).
8. V. B. Kuznetsov and E. K. Sklyanin, *J. Phys. A*, **31**, 2241 (1998).
9. E. K. Sklyanin, "Bäcklund transformations and Baxter's Q-operator," nlin.SI/0009009 (2000).

10. G. P. Pronko, *Commun. Math. Phys.*, **212**, 687 (2000).
11. P. P. Kulish and N. Yu. Reshetikhin, *Zap. Nauch. Sem. LOMI*, **101**, 101 (1980); *J. Sov. Math.*, **23**, 2435 (1983).
12. M. Jimbo, *Lett. Math. Phys.*, **11**, 247 (1986).
13. M. Jimbo, "Topics from representations of $U_q\bigl(\{g\}\bigr)$: An introductory guide to physicists," Preprint, Faculty of Science, Kyoto University, Kyoto 606, Japan (1991).
14. T. Deguchi, K. Fabricius, and B. M. McCoy, *J. Stat. Phys.*, **102**, 701 (2001); cond-mat/9912141 (1999).
15. S. Pakuliak and S. Sergeev, "Relativistic Toda chain at root of unity II: Modified Q-operator," nlin.SI/0107062 (2001); S. Sergeev, "Exactly solvable lattice models in statistical physics [in Russian]," Doctoral dissertation, Inst. High-Energy Phys., Protvino (2001).