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2007

MIMS EPrint: **2008.4**

Manchester Institute for Mathematical Sciences
School of Mathematics

The University of Manchester

ISSN 1749-9097

ON SYMMETRIC INVARIANTS OF CENTRALISERS IN REDUCTIVE LIE ALGEBRAS

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ABSTRACT. Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra of rank l over an algebraically closed field of characteristic 0. Let e be a nilpotent element of $\mathfrak{g}$ and let $\mathfrak{g}_e$ be the centraliser of e in $\mathfrak{g}$. In this paper we study the algebra $\mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e}$ of symmetric invariants of $\mathfrak{g}_e$. We prove that if $\mathfrak{g}$ is of type **A** or **C**, then $\mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e}$ is always a graded polynomial algebra in l variables, and we show that this continues to hold for *some* nilpotent elements in the Lie algebras of other types. In type **A** we prove that the invariant algebra $\mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e}$ is freely generated by a regular sequence in $\mathcal{S}(\mathfrak{g}_e)$ and describe the tangent cone at e to the nilpotent variety of $\mathfrak{g}$.

To Ernest Borisovich Vinberg on occasion of his 70th birthday

CONTENTS

Introduction		1
1.	Some general results	6
2.	Slodowy slices and symmetric invariants of centralisers	11
3.	Regular linear functions on centralisers	16
4.	Degrees of basic invariants	24
5.	The null-cones in type A	41
6.	Miscellany	44
7.	Appendix	47
References		48

INTRODUCTION

0.1. Let $\mathfrak{g}$ be a finite-dimensional reductive Lie algebra of rank l over an algebraically closed field $\mathbb{K}$ of characteristic zero, and let G be the adjoint group of $\mathfrak{g}$. Let $\mathcal{N}(\mathfrak{g})$ denote the nilpotent cone of $\mathfrak{g}$, i.e., the set of all nilpotent elements of $\mathfrak{g}$. Given $x \in \mathfrak{g}$ we denote by $\mathfrak{g}_x$ and G_x the centraliser of x in $\mathfrak{g}$ and G, respectively. It is well-known that $\mathfrak{g}_x = \mathrm{Lie}\ G_x = \mathrm{Lie}\ G_x^\circ$ (here and in what follows H° stands for the connected component of an algebraic group H).

D.P. and O.Y were supported in part by RFBI Grant 05-01-00988.

Inspired by a conversation with J. Brundan at the Oberwolfach meeting on enveloping algebras in March 2005, the second author put forward the following conjecture:

Conjecture 0.1. *For any $x \in \mathfrak{g}$ the invariant algebra $\mathcal{S}(\mathfrak{g}_x)^{\mathfrak{g}_x}$ is a graded polynomial algebra in l variables.*

In order to prove (or disprove) Conjecture 0.1 it suffices to consider the case where $\mathfrak{g}$ is simple and $x \in \mathcal{N}(\mathfrak{g})$. The conjecture is known to hold for some $x \in \mathcal{N}(\mathfrak{g})$. For example, when $x = 0$, it is an immediate consequence of the Chevalley Restriction Theorem. At the other extreme, when $x \in \mathcal{N}(\mathfrak{g})$ is regular, the centraliser $\mathfrak{g}_x$ is abelian of dimension l and we have $\mathcal{S}(\mathfrak{g}_x)^{\mathfrak{g}_x} = \mathcal{S}(\mathfrak{g}_x) \cong \mathbb{K}[X_1, \ldots, X_l]$ with $\deg X_i = 1$ for all i.

Conjecture 0.1 is closely related to an earlier conjecture of A. Elashvili (initiated by a question of A. Bolsinov). Recall that the *index* of a finite-dimensional Lie algebra $\mathfrak{s}$ over $\mathbb{K}$, denoted $\operatorname{ind}\mathfrak{s}$, is defined as the minimal dimension of the stabilisers of linear functions on $\mathfrak{s}$. In other words, $\operatorname{ind}\mathfrak{s} = \min\{\dim \mathfrak{s}^f \mid f \in \mathfrak{s}^*\}$ where $\mathfrak{s}^f = \{x \in \mathfrak{s} \mid f([x, \mathfrak{s}]) = 0\}$. Elashvili's conjecture states that

$$\operatorname{ind}\mathfrak{g}_x = l = \operatorname{rk}\mathfrak{g} \qquad (\forall\, x \in \mathfrak{g}).$$

According to Vinberg's inequality, $\operatorname{ind}\mathfrak{g}_x \geqslant l$ for all $x \in \mathfrak{g}$ (see [17, 1.6 & 1.7], but the equality is *much* harder to establish.

During the last decade Elashvili's conjecture drew attention of several Lie theorists. Similar to Conjecture 0.1 it reduces to the case in which $\mathfrak{g}$ is simple and $x \in \mathcal{N}(\mathfrak{g})$. For the spherical nilpotent orbits, Elashvili's conjecture was proved in [17] and [18] by the first author. For $\mathfrak{g}$ classical, Elashvili's conjecture was recently proved in [29] by the third author. In 2004, J.-Y. Charbonnel published a case-free proof of Elashvili's conjecture applicable to all simple Lie algebras; see [4]. Unfortunately, the argument in [4] has a gap in the final part of the proof, which was pointed out by L. Rybnikov. At present we are unable to close this gap. Answering a question of Elashvili, W. de Graaf used a computer programme to verify the conjecture for all nilpotent elements in the Lie algebra of type $\mathbf{E}_8$ (unpublished). Since there are many nilpotent orbits in the Lie algebras of exceptional types, it is difficult to present the results of such computations in a concise way.

To summarise, Elashvili's conjecture holds for the Lie algebras of type $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{G}_2$ and remains a challenging open problem for the Lie algebras of type $\mathbf{E}$ and $\mathbf{F}_4$. We feel that it would be very important to find a conceptual proof of Elashvili's conjecture applicable to all finite-dimensional simple Lie algebras.

0.2. The main goal of this paper is to prove Conjecture 0.1 for all nilpotent elements in the Lie algebras of type $\mathbf{A}$ and $\mathbf{C}$. Our methods also work for some nilpotent elements in the Lie algebras of type $\mathbf{B}$ and $\mathbf{D}$ and for a few nilpotent orbits in the exceptional Lie algebras.

From now on, we fix a nonregular element $e \in \mathcal{N}(\mathfrak{g}) \setminus \{0\}$ and include it into an $\mathfrak{sl}_2$-triple (e, h, f) of $\mathfrak{g}$. Let $(\,\cdot\,,\,\cdot\,)$ denote the the scalar multiple of the Killing form of $\mathfrak{g}$ such that $(e, f) = 1$, and put $\chi = (e, \cdot\,)$. The map κ from $\mathfrak{g}$ to $\mathfrak{g}^*$ which takes x to $(x, \cdot\,)$ extends uniquely to a G-equivariant isomorphism between the symmetric algebra $\mathcal{S}(\mathfrak{g})$ and the

coordinate algebra $\mathbb{K}[\mathfrak{g}]$ of $\mathfrak{g}$. This isomorphism of graded algebras will be denoted by the same letter κ and referred to as a *Killing isomorphism*. The G-equivariance of $(\,\cdot\,,\,\cdot\,)$ implies that $\mathfrak{g}_e = [e,\mathfrak{g}]^{\perp}$. On the other hand, $\mathfrak{g} = [e,\mathfrak{g}] \oplus \mathfrak{g}_f$ by the $\mathfrak{sl}_2$-theory. It follows that the Killing isomorphism κ induces an algebra isomorphism

$$\kappa_e : \mathcal{S}(\mathfrak{g}_e) \xrightarrow{\sim} \mathbb{K}[\mathfrak{g}_f], \quad x \mapsto (x,\,\cdot\,)_{|\mathfrak{g}_f} \qquad (\forall\, x \in \mathfrak{g}_e).$$

The coordinate algebra $K[\mathfrak{g}_f]$ carries a natural $\mathbb{Z}$-grading in which the linear forms on $\mathfrak{g}_f$ have degree 1. Each nonzero $\varphi \in \mathbb{K}[\mathfrak{g}_f]$ is expressed uniquely as

$$\varphi = \varphi_k + \text{ terms of higher degree},$$

where φ_k is a nonzero homogeneous element of degree $k = k(\varphi)$. We say that φ_k is the *initial term* of φ, written $\varphi_k = \mathrm{in}(\varphi)$. For $\varphi = 0$ we set $\mathrm{in}(\varphi) = 0$.

Let $\mathcal{S}_e$ denote the *Slodowy slice* $e + \mathfrak{g}_f$ at e through the adjoint orbit $G \cdot e$. The translation map $x \mapsto e + x$ induces an isomorphism of affine varieties $\tau : \mathfrak{g}_e \xrightarrow{\sim} \mathcal{S}_e$. The comorphism τ^* maps the coordinate algebra $\mathbb{K}[\mathcal{S}_e]$ isomorphically onto $\mathbb{K}[\mathfrak{g}_f]$.

Let F be a homogeneous element in $\mathcal{S}(\mathfrak{g})$. Then $\kappa(F) \in \mathbb{K}[\mathfrak{g}]$ and $\kappa(F)_{|\,\mathcal{S}_e} \in \mathbb{K}[\mathcal{S}_e]$. The above discussion shows that $\tau^*(\kappa(F)_{|\,\mathcal{S}_e}) \in \mathbb{K}[\mathfrak{g}_f]$ and $\kappa_e^{-1}\big(\mathrm{in}(\tau^*(\kappa(F)_{|\,\mathcal{S}_e}))\big) \in \mathcal{S}(\mathfrak{g}_e)$. We now put

$${}^eF := \kappa_e^{-1}\big(\mathrm{in}(\tau^*(\kappa(F)_{|\,\mathcal{S}_e}))\big).$$

Thus, to each homogeneous $F \in \mathcal{S}(\mathfrak{g})$ we assign a homogeneous element ${}^eF \in \mathcal{S}(\mathfrak{g}_e)$. Roughly speaking, ${}^eF$ is the initial component of $F_{|\kappa(\mathcal{S}_e)}$.

Proposition 0.1. *If F is a homogeneous element of $\mathcal{S}(\mathfrak{g})^G$, then ${}^eF \in \mathcal{S}(\mathfrak{g}_e)^{G_e}$.*

We give two proofs of Proposition 0.1. The first proof relies in a crucial way on some properties of the quantisation of the coordinate algebra $\mathbb{K}[\mathcal{S}_e]$ introduced in [19] (see also [11]). The second (elementary) proof is given in the Appendix.

0.3. Of particular interest are those homogeneous generating sets $\{F_1,\dots,F_l\} \subset \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ for which the resulting systems ${}^eF_1,\dots,{}^eF_l$ are algebraically independent. In Section 2 we show that if Elashvili's conjecture holds for $\mathfrak{g}_e$, then for any homogeneous system of basic invariants $F_1,\dots,F_l$ in $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ we have the inequality

$$\textstyle\sum_{i=1}^{l} \deg {}^eF_i \leqslant (\dim \mathfrak{g}_e + \mathrm{rk}\,\mathfrak{g})/2. \tag{1}$$

Furthermore, ${}^eF_1,\dots,{}^eF_l$ are algebraically independent in $\mathcal{S}(\mathfrak{g}_e)$ if and only if the equality holds in (1), that is $\sum_{i=1}^{l} \deg {}^eF_i = (\dim \mathfrak{g}_e + \mathrm{rk}\,\mathfrak{g})/2$. If this happens, we say that the system $F_1,\dots,F_l$ is *good* for e.

Given a linear function γ on $\mathfrak{g}_e$ we denote by $\mathfrak{g}_e^{\gamma}$ the stabiliser of γ in $\mathfrak{g}_e$ and set

$$(\mathfrak{g}_e^*)_{\mathrm{sing}} := \{\gamma \in \mathfrak{g}_e^* \mid \dim \mathfrak{g}_e^{\gamma} > \mathrm{ind}\,\mathfrak{g}_e\}.$$

The complement $\mathfrak{g}_e^* \setminus (\mathfrak{g}_e^*)_{\mathrm{sing}}$ consists of all *regular* linear functions of $\mathfrak{g}_e$. We prove in Section 2 that if Elashvili's conjecture holds for $\mathfrak{g}_e$, then for any good generating set $\{F_1,\dots,F_l\} \subset \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ the differentials $d_\gamma({}^eF_1),\dots,d_\gamma({}^eF_l)$ are linearly independent at $\gamma \in \mathfrak{g}_e^*$ if and only if γ is regular in $\mathfrak{g}_e^*$. When $e = 0$, this is a classical result of Lie Theory often

referred to as *Kostant's differential criterion for regularity* (note that any homogeneous generating system in $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ is good for $e = 0$ and Elashvili's conjecture is true in this case). When e is regular nilpotent, the statement follows from another theorem of Kostant saying that the restriction of the adjoint quotient map to the Slodowy slice $\mathcal{S}_e$ is an isomorphism of algebraic varieties. Beyond these two extreme cases our result seems to be new. It should be stressed, however, that if $\mathfrak{g}$ is not of type **A** or **C**, then there may exist nilpotent elements in $\mathfrak{g}$ which do not admit good generating systems in $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$. One such element in $\mathfrak{g} = \mathfrak{so}_{12}$ is exhibited in Example 4.1. Quite surprisingly, the root vectors in Lie algebras of type $\mathbf{E}_8$ provide yet another example of this kind.

0.4. Our proof of the above results relies on some geometric properties of Poisson algebras of Slodowy slices (established in [19] and [11]) and a theorem of Odesskii-Rubtsov [15] on polynomial Poisson algebras with a regular structure of symplectic leaves. All necessary background on polynomial Poisson algebras is assembled in Section 1.

Let $\mathcal{A} = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial algebra in n variables. For $g_1, \ldots, g_m \in \mathcal{A}$, we denote by $\mathcal{J}(g_1, \ldots, g_m)$ the Jacobian locus of $g_1, \ldots, g_m$, i.e., the set of all $\xi \in \operatorname{Specm} \mathcal{A}$ for which the differentials $d_\xi g_1, \ldots, d_\xi g_m$ are linearly dependent. Suppose $\mathcal{A}$ is a Poisson algebra and let $\pi \in \operatorname{Hom}_{\mathcal{A}}(\Omega^2(\mathcal{A}), \mathcal{A})$ be the corresponding Poisson bivector. Let $Z(\mathcal{A})$ denote the Poisson centre of $\mathcal{A}$. The defect of the skew-symmetric matrix $\big(\{x_i, x_j\}\big)_{1 \leqslant i,j \leqslant n}$ with entries in $\mathcal{A}$ is called the *index* of $\mathcal{A}$ and denoted $\operatorname{ind} \mathcal{A}$. It is well-known (and easily seen) that $\operatorname{tr.deg}_{\mathbb{K}} Z(\mathcal{A}) \leqslant \operatorname{ind} \mathcal{A}$. We denote by $\operatorname{Sing} \pi$ the set of all $\xi \in \operatorname{Specm} \mathcal{A}$ for which $\operatorname{rk} \pi(\xi) < n - \operatorname{ind} \mathcal{A}$. A subset $\{Q_1, \ldots Q_l\} \subset Z(\mathcal{A})$ is said to be *admissible* if $l = \operatorname{ind} \mathcal{A}$ and the Jacobian locus $\mathcal{J}(Q_1, \ldots, Q_l)$ has codimension $\geqslant 2$ in $\mathbb{A}^n$. We say that $(\mathcal{A}, \pi)$ is a *quasi-regular* Poisson algebra if $Z(\mathcal{A})$ contains an admissible subset and $\operatorname{Sing} \pi$ has codimension $\geqslant 2$ in $\operatorname{Specm} \mathcal{A}$.

Assume now that $\mathcal{A} = \mathbb{K}[x_1, \ldots, x_n]$ is graded and each x_i is homogeneous of positive degree. Let $f_1, \ldots, f_s$ be a collection of homogeneous elements in $\mathcal{A}$ such that the Jacobian locus $\mathcal{J}(f_1, \ldots, f_s)$ has codimension $\geqslant 2$ in $\operatorname{Specm} \mathcal{A}$, and denote by R the subalgebra of $\mathcal{A}$ generated by $f_1, \ldots, f_s$. Inspired by Skryabin's result [24, Theorem 5.4] on modular invariants of finite group schemes we prove that if an element $\tilde{f} \in \mathcal{A}$ is algebraic over R, then necessarily $\tilde{f} \in R$. This has the following consequence:

Theorem 0.2. *Let $\mathcal{A} = \mathbb{K}[x_1, \ldots, x_n]$ be a quasi-regular Poisson algebra of index l and suppose that $\mathcal{A} = \bigoplus_{k \geqslant 0} \mathcal{A}(k)$ is graded in such a way that $x_i \in \mathcal{A}(r_i)$ for some $r_i > 0$, where $1 \leqslant i \leqslant n$. Suppose further that $Z(\mathcal{A})$ contains an admissible set $\{Q_1, \ldots, Q_l\}$ consisting of homogeneous elements of $\mathcal{A}$. Then $Z(\mathcal{A}) = \mathbb{K}[Q_1, \ldots, Q_l]$.*

0.5. In this paper, we mostly apply Theorem 0.2 to the pair $(\mathcal{A}, \pi) = \big(\mathcal{S}(\mathfrak{g}_e), \pi_e^{PL}\big)$ where π_e^{PL} is the Poisson bivector of $\mathcal{S}(\mathfrak{g}_e)$ induced by the Lie bracket of $\mathfrak{g}_e$. In this situation $Z(\mathcal{A}) = \mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e}$. (One noteworthy application of Theorem 0.2 to quantisations of Slodowy slices can be found in Remark 2.1.) Of course, before applying Theorem 0.2 to the pair $\big(\mathcal{S}(\mathfrak{g}_e), \pi_e^{PL}\big)$ we have to make sure that our nilpotent element qualifies. That is to say, we must check that e admits a good generating system $F_1, \ldots, F_l$, that Elashvili's conjecture

holds for $\mathfrak{g}_e$, and that $\mathcal{J}({}^eF_1,\dots,{}^eF_l) = (\mathfrak{g}_e^*)_{\rm sing}$ has codimension $\geqslant 2$ in $\mathfrak{g}_e^*$. Our main result is the following:

Theorem 0.3. *Suppose e admits a good generating system $F_1,\dots,F_l$ in $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ and assume further that Elashvili's conjecture holds for $\mathfrak{g}_e$ and $(\mathfrak{g}_e^*)_{\rm sing}$ has codimension $\geqslant 2$ in $\mathfrak{g}_e^*$. Then $\mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e} = \mathcal{S}(\mathfrak{g}_e)^{G_e}$ is a polynomial algebra in ${}^eF_1,\dots,{}^eF_l$.*

Suppose $\mathfrak{g}$ is of type $\mathbf{A}_n$ of $\mathbf{C}_n$, where $n \geqslant 2$, and let $e \in \mathcal{N}(\mathfrak{g})$. By [29], Elashvili's conjecture holds for $\mathfrak{g}_e$. In Section 3 we show that the singular locus $(\mathfrak{g}_e^*)_{\rm sing}$ has codimension $\geqslant 2$ in $\mathfrak{g}_e^*$, whilst our results in Section 4 imply that in types $\mathbf{A}$ and $\mathbf{C}$ the invariant algebra $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ contains a homogeneous generating set which is good for *all* nilpotent elements in $\mathfrak{g}$ (this is no longer true in types $\mathbf{B}$ and $\mathbf{D}$). Applying Theorem 0.3 we are able to conclude that Conjecture 0.1 holds for all nilpotent elements in $\mathfrak{g}$.

Apart from the the above-mentioned results, we show in Sections 3 and 4 that the conditions of Theorem 0.3 are satisfied for some nilpotent elements in Lie algebras of types $\mathbf{B}$ and $\mathbf{D}$. Subsection 3.9 illustrates the behavior of the simple Lie algebras $\mathfrak{g}$ other than $\mathfrak{sl}_n$ and $\mathfrak{sp}_{2n}$ by producing a nilpotent element $e \in \mathfrak{g}$ for which $(\mathfrak{g}_e^*)_{\rm sing}$ has codimension 1 in $\mathfrak{g}_e^*$.

0.6. In Section 5 we study the null-cone $\mathcal{N}(e)$ of $\mathfrak{g}_e^*$, that is the subvariety of $\mathfrak{g}_e^*$ consisting of all linear functions ξ such that $\varphi(\xi) = 0$ for all $\varphi \in \mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e}$ with $\varphi(0) = 0$. Here we have to assume that $\mathfrak{g} = \mathfrak{gl}_n$. Working with the good generating set $\{F_1,\dots,F_n\} \subset \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ mentioned in (0.5) we show that the zero locus $\mathcal{N}(e)$ of the ideal $({}^eF_1,\dots,{}^eF_n)$ has codimension n in $\mathfrak{g}_e^*$ and hence ${}^eF_1,\dots,{}^eF_n$ is a regular sequence in $\mathcal{S}(\mathfrak{g}_e)$. As a consequence, we describe the tangent cone at e to the variety of all nilpotent $n \times n$ matrices over $\mathbb{K}$; see Corollary 5.5. Although the variety $\mathcal{N}(e)$ is irreducible in some interesting cases, in general it has many irreducible components. The problem of describing the irreducible components of $\mathcal{N}(e)$ for $\mathfrak{g} = \mathfrak{gl}_n$ is wide open.

0.7. Let $\tilde{e} \in \mathcal{O}_{\min}$, where $\mathcal{O}_{\min}$ is the minimal (nonzero) nilpotent orbit in $\mathfrak{g}$. The element $\tilde{e}$ is G-conjugate to a highest root vector in $\mathfrak{g}$. Recall that outside type $\mathbf{A}$ the orbit $\mathcal{O}_{\min}$ is *rigid*, i.e., cannot be obtained by Lusztig–Spaltenstein induction form a nilpotent orbit in a Levi subalgebra of $\mathfrak{g}$. We put Conjecture 0.1 to the test by investigating the invariant algebra $\mathcal{S}(\mathfrak{g}_{\tilde{e}})^{\mathfrak{g}_{\tilde{e}}}$. Here our result is as follows:

Theorem 0.4. *Suppose $\operatorname{rk}\mathfrak{g} \geqslant 2$. Then the singular locus $(\mathfrak{g}_{\tilde{e}}^*)_{\rm sing}$ has codimension $\geqslant 2$ in $\mathfrak{g}_{\tilde{e}}^*$. If $\mathfrak{g}$ is not of type $\mathbf{E}_8$, then $\tilde{e}$ admits a good generating system in $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ and the invariant algebra $\mathcal{S}(\mathfrak{g}_{\tilde{e}})^{\mathfrak{g}_{\tilde{e}}}$ is isomorphic to a graded polynomial algebra in $\operatorname{rk}\mathfrak{g}$ variables. The degrees of basic invariants of $\mathcal{S}(\mathfrak{g}_{\tilde{e}})^{\mathfrak{g}_{\tilde{e}}}$ are given in the Table 1.*

In order to prove Theorem 0.4 for Lie algebras of types $\mathbf{E}_7$ we have to use the explicit system of basic invariants for the Weyl group of type $\mathbf{E}_7$ constructed in [14]. In type $\mathbf{E}_8$, we reduce Conjecture 0.1 for $\mathfrak{g}_{\tilde{e}}$ to a specific problem on polynomial invariants of the Weyl group of type $\mathbf{E}_7$; see Theorem 4.14. In principle, this problem can be tackled by computational methods.

Type of $\mathfrak{g}$	Degrees of basic invariants
$\mathbf{A}_n$, $n \geqslant 1$	$1, 2, \ldots, n$
$\mathbf{B}_n$, $n \geqslant 3$	$1, 3, 4, \ldots, 2n-2$
$\mathbf{C}_n$, $n \geqslant 2$	$1, 3, \ldots, 2n-1$
$\mathbf{D}_n$, $n \geqslant 4$	$1, 3, 4, \ldots, 2n-4, n-1$
$\mathbf{E}_6$	$1, 4, 4, 6, 7, 9$
$\mathbf{E}_7$	$1, 4, 6, 8, 9, 11, 14$
$\mathbf{F}_4$	$1, 4, 6, 9$
$\mathbf{G}_2$	$1, 4$

TABLE 1

We adopt the Vinberg–Onishchik numbering of simple roots and fundamental weights in simple Lie algebras; see [27, Tables]. The i-th fundamental weight is denoted by ϖ_i.

1. SOME GENERAL RESULTS

1.1. Our goal in this section is twofold: to prove an extended characteristic-zero version of Skryabin's theorem [24] on invariants of finite group schemes and to obtain a slight generalisation of a result of Odesskii–Rubtsov [15] on polynomial Poisson algebras. We first recall some basics on the classical duality between differential forms and polyvector fields.

Let $\mathbb{A}^n = \mathbb{A}^n_{\mathbb{K}}$ be the n-dimensional affine space with the algebra of regular functions $\mathcal{A} = \mathbb{K}[x_1, \ldots, x_n]$. Let W denote the derivation algebra of $\mathcal{A}$. This is a free $\mathcal{A}$-module with basis consisting of partial derivatives $\partial_1, \ldots, \partial_n$ with respect to $x_1, \ldots, x_n$. Let $\Omega^1 = \operatorname{Hom}_{\mathcal{A}}(W, \mathcal{A})$ and let $\Omega = \bigoplus_{k=0}^n \Omega^k$ be the exterior $\mathcal{A}$-algebra on Ω^1. The exterior differential $d\colon \mathcal{A} \to \Omega^1$, $(df)(D) = D(f)$, extends uniquely up to a zero-square graded derivation of the $\mathcal{A}$-algebra Ω. We identify Ω^0 with $\mathcal{A}$ and regard Ω^1 as the $\mathcal{A}$-module of global sections on the cotangent bundle $T^*\mathbb{A}^n$. Note that Ω^k is a free $\mathcal{A}$-module with basis $\{dx_{i_1} \wedge \ldots \wedge dx_{i_k} \mid 1 \leqslant i_1 < \cdots < i_k \leqslant n\}$.

We view the exterior powers $\Omega^k = \bigwedge^k_{\mathcal{A}} \Omega^1$ and $\bigwedge^k_{\mathcal{A}} W$ as dual $\mathcal{A}$-modules by using the nondegenerate $\mathcal{A}$-pairing

$$\langle \alpha_1 \wedge \ldots \wedge \alpha_k, D_1 \wedge \ldots \wedge D_k \rangle = \det\big(\alpha_i(D_j)\big).$$

For $\eta \in \Omega^k$, set $\eta(D_1 \wedge \ldots \wedge D_p) := \langle \eta, D_1 \wedge \ldots \wedge D_k \rangle$. For $D \in \bigwedge^k_{\mathcal{A}} W$, set $D(\alpha_1 \wedge \ldots \wedge \alpha_k) := \langle \alpha_1 \wedge \ldots \wedge \alpha_k, D \rangle$. Then for $D \in \bigwedge^p_{\mathcal{A}} W = (\Omega^p)^*$ and $D' \in \bigwedge^q_{\mathcal{A}} W = (\Omega^q)^*$ we have

$$\begin{aligned}(D \wedge D')(\alpha_1 \wedge \ldots \wedge \alpha_{p+q}) &= \langle \alpha_1 \wedge \ldots \wedge \alpha_{p+q}, D \wedge D' \rangle \\ &= \sum (\operatorname{sgn} \sigma) D(\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(p)}) D'(\alpha_{\sigma(p+1)}, \ldots, \alpha_{\sigma(p+q)}),\end{aligned}$$

where the summation runs over the set of all permutations σ of $\{1, \ldots, p+q\}$ which are increasing on $\{1, \ldots, p\}$ and $\{p+1, \ldots, p+q\}$.

For $X \in \bigwedge_{\mathcal{A}} W$ and $\xi \in \mathbb{A}^n$, the specialisation X_ξ is a well-defined element of the exterior algebra $\bigwedge T_\xi(\mathbb{A}_n)$ on the tangent space $T_\xi(\mathbb{A}^n)$. For $X \in W$, the *left interior product* i_X is the unique $\mathcal{A}$-linear endomorphism of degree -1 on Ω such that

$$i_X(\eta)(D_1 \wedge \cdots \wedge D_k) = \eta(X \wedge D_1 \ldots \wedge D_k) \qquad \big(\forall\, \eta \in \Omega^{k+1}\big).$$

For $\omega \in \Omega^1$, the *right interior product* j_ω is the unique $\mathcal{A}$-linear endomorphism of degree -1 on $\bigwedge_{\mathcal{A}} W$ such that

$$j_\omega(D)(\alpha_1 \wedge \cdots \wedge \alpha_k) = D(\alpha_1 \wedge \ldots \wedge \alpha_k \wedge \omega) \qquad \big(\forall\, D \in \textstyle\bigwedge_{\mathcal{A}}^{k+1} W\big).$$

Using the above discussion it is easy to observe that the endomorphisms i_X and j_ω are graded derivations (a.k.a. super-derivations) of Ω and $\bigwedge_{\mathcal{A}} W$, respectively. More generally, given $X \in \bigwedge_{\mathcal{A}}^p W$ and $\omega \in \Omega^p$ one defines the *right interior product* i_X and the *left interior product* j_ω to be the unique endomorphisms of degree $-p$ on Ω and $\bigwedge_{\mathcal{A}} W$, respectively, such that

$$\langle i_X(\eta), D\rangle = \langle \eta, X \wedge D\rangle \quad \text{and} \quad \langle \eta, j_\omega(D)\rangle = \langle \eta \wedge \omega, D\rangle \qquad \big(\forall\, D \in \textstyle\bigwedge_{\mathcal{A}}^p W,\ \forall\, \eta \in \Omega\big).$$

The mappings $X \mapsto i_X$ and $\omega \mapsto j_\omega$ then give rise to $\mathcal{A}$-algebra homomorphisms $i\colon \bigwedge_{\mathcal{A}} W \to \mathrm{End}(\Omega)$ and $j\colon \Omega \to \mathrm{End}\big(\bigwedge_{\mathcal{A}} W\big)$. In other words, we have $i_X \circ i_Y = i_{X\wedge Y}$ and $j_\alpha \circ j_\beta = j_{\alpha\wedge\beta}$ for all $X, Y \in \bigwedge_{\mathcal{A}} W$ and all $\alpha, \beta \in \Omega$. Finally, $i_X(\omega) = j_\omega(X) = \langle \omega, X\rangle$ whenever $X \in \bigwedge_{\mathcal{A}}^p W$ and $\omega \in \Omega^p$.

The top components Ω^n and $\bigwedge_{\mathcal{A}}^n W$ are free modules of rank 1 over $\mathcal{A}$ generated by $dx_1 \wedge \ldots \wedge dx_n$ and $\partial_1 \wedge \ldots \wedge \partial_n$, respectively. The mappings $X \mapsto i_X(dx_1 \wedge \ldots \wedge dx_n)$ and $\omega \mapsto j_\omega(\partial_1 \wedge \ldots \wedge \partial_n)$ induce canonical $\mathcal{A}$-module isomorphisms $\bigwedge_{\mathcal{A}}^p W \cong \Omega^{n-p}$ and $\Omega^p \cong \bigwedge_{\mathcal{A}}^{n-p} W$.

1.2. For $g_1, \ldots, g_m \in \mathcal{A}$, the *Jacobian locus* $\mathcal{J}(g_1, \ldots, g_m)$ consists of all $\xi \in \mathbb{A}^n$ for which the differentials $d_\xi g_1, \ldots, d_\xi g_m$ are linearly dependent. The set $\mathcal{J}(g_1, \ldots, g_m)$ is Zariski closed in $\mathbb{A}^n$ and it coincides with $\mathbb{A}^n$ if and only if $g_1, \ldots, g_m$ are algebraically dependent. Our interpretation of Skryabin's result [24, Theorem 5.4] will be based on the following general result which is of independent interest:

Theorem 1.1. *Suppose $\mathcal{A} = \mathbb{K}[x_1, \ldots, x_n]$ is graded in such a way that each x_i is homogeneous of positive degree. Let R be the subalgebra of $\mathcal{A}$ generated by homogeneous elements $f_1, \ldots, f_s$ and assume further that $\mathcal{J}(f_1, \ldots, f_s)$ has codimension $\geqslant 2$ in $\mathbb{A}^n$. Then R is algebraically closed in $\mathcal{A}$. In other words, if $\tilde{f} \in \mathcal{A}$ is algebraic over R, then $\tilde{f} \in R$.*

Proof. For $t \in \mathbb{K}^\times$, we denote by $\rho(t)$ the automorphism of $\mathcal{A}$ such that $\rho(t)\cdot f = t^k f$ for all $f \in \mathcal{A}(k)$, where $\mathcal{A}(k)$ is the k-th graded component of $\mathcal{A}$. Let $\mathcal{Q}(R)$ be the field of fractions of R, a subfield of $\mathbb{K}(x_1, \ldots, x_n)$, and denote by $\tilde{R}$ the algebraic closure of R in $\mathcal{A}$. Since $\tilde{R}$ is nothing but the intersection of $\mathcal{A}$ with the algebraic closure of $\mathcal{Q}(R)$ in $\mathbb{K}(x_1, \ldots, x_n)$, it is a subalgebra of $\mathcal{A}$. Since all f_i are homogeneous, the subalgebra R is $\rho(\mathbb{K}^\times)$-stable. But then so is $\tilde{R}$. As a consequence, $\tilde{R}$ is a homogeneous subalgebra of $\mathcal{A}$. Thus, in order to prove the theorem it suffices to show that if a *homogeneous* element $\tilde{f} \in \mathcal{A}$ is algebraic over R, then $\tilde{f} \in R$.

We shall argue by induction on the degree of $\tilde{f}$. So assume that the statement holds for all homogeneous elements of degree less than $\deg \tilde{f}$ (when $\deg \tilde{f} = 1$, this is a valid assumption).

(a) The grading of $\mathcal{A}$ induces that on the $\mathbb{K}$-algebra Ω where we impose that $\deg dx_i = \deg x_i$. Note that $a \in \mathcal{A}$ is algebraic over R if and only if $da \wedge df_1 \wedge \ldots \wedge df_s = 0$ in Ω. Since $\mathcal{J}(g_1, \ldots, g_m)$ consists of all $\xi \in \mathbb{A}^n$ for which $d_\xi g_1 \wedge \ldots \wedge d_\xi g_m = 0$, our assumption on $f_1, \ldots, f_s$ implies that for every subset $\{i_1, \ldots, i_k\}$ of $\{1, \ldots, s\}$ the locus $\mathcal{J}(f_{i_1}, \ldots, f_{i_k})$ has codimension $\geqslant 2$ in $\mathbb{A}^n$. From this it follows that passing to smaller subsets of $\{f_1, \ldots, f_s\}$ and renumbering if necessary we can reduce our proof to the situation where for each i the polynomials $\{f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_s, \tilde{f}\}$ are algebraically independent. So let us assume from now that this is the case, and put

$$T := df_1 \wedge \ldots \wedge df_s, \qquad T_i := df_1 \wedge \ldots \wedge df_{i-1} \wedge d\tilde{f} \wedge df_{i+1} \wedge \ldots \wedge df_s \qquad (1 \leqslant i \leqslant s).$$

By our assumption, T and the T_i are *nonzero* homogeneous elements of Ω.

(b) If $\xi \notin \mathcal{J}(f_1, \ldots, f_s)$, then $d_\xi f_1, \ldots, d_\xi f_s$ are linearly independent and $d_\xi \tilde{f}$ is a linear combination of $d_\xi f_1, \ldots, d_\xi f_s$. It follows that the specialisation of T_i at ξ is a scalar multiple of $d_\xi f_1 \wedge \ldots \wedge d_\xi f_s$. As Ω is a free $\mathcal{A}$-module, this yields that T and T_i are linearly dependent as elements of the $\mathbb{K}(x_1, \ldots, x_n)$-vector space $\mathbb{K}(x_1, \ldots, x_n) \otimes_{\mathcal{A}} \Omega$. Combined with our discussion in part (a) this implies that $a_i T_i = b_i T$ for some nonzero coprime $a_i, b_i \in \mathcal{A}$. As $\mathcal{J}(f_1, \ldots, f_m)$ has codimension $\geqslant 2$ in $\mathbb{A}^n$, the function a_i must be constant. Thus, $T_i = p_i T$ where p_i is a nonzero homogeneous element of the graded algebra $\mathcal{A}$.

(c) Since $d^2 = 0$, we have $dp_i \wedge T = d(p_i T) = dT_i = 0$. Our remarks in part (a) now show that all p_i are algebraic over R. Let

$$F = S_k(X_1, \ldots, X_s)Y^k + S_{k-1}(X_1, \ldots, X_s)Y^{k-1} + \cdots + S_0(X_1, \ldots, X_s)$$

be a nonzero polynomial in $\mathbb{K}[X_1, \ldots, X_s, Y]$ of minimal possible degree in Y such that $F(f_1, \ldots, f_s, \tilde{f}) = 0$. Assume further that S_k has minimal possible total degree in $\mathbb{K}[X_1, \ldots, X_s]$ and that all $S_i(f_1, \ldots, f_s)$ are homogeneous in the graded algebra $\mathcal{A}$. Applying the exterior differential we get $0 = dF(f_1, \ldots, f_s, \tilde{f}) = \tilde{\psi} d\tilde{f} + \sum \psi_i df_i$ where

$$\begin{aligned} \tilde{\psi} &= k\tilde{f}^{k-1} S_k(f_1, \ldots, f_s) + \cdots + S_1(f_1, \ldots, f_s), \\ \psi_i &= \tilde{f}^k \frac{\partial S_k}{\partial X_i}(f_1, \ldots, f_s) + \tilde{f}^{k-1} \frac{\partial S_{k-1}}{\partial X_i}(f_1, \ldots, f_s) + \cdots + \frac{\partial S_0}{\partial X_i}(f_1, \ldots, f_s) \qquad (1 \leqslant i \leqslant m). \end{aligned}$$

As $\tilde{\psi} \neq 0$ by our choice of F, we have $d\tilde{f} = -\sum(\psi_i/\tilde{\psi}) df_i$. This forces $T_i = -(\psi_i/\tilde{\psi})T$ for all i. Then $\psi_i = -p_i \tilde{\psi}$ by our concluding remark in part (b).

(d) Part (b) also shows that each p_i is homogeneous with $\deg p_i = \deg \tilde{f} - \deg f_i < \deg \tilde{f}$. Since all p_i are algebraic over $\mathcal{A}$ by part (c), our induction hypothesis implies that $p_i \in R$ for all i. We now look again at the formulae displayed in part (c), this time keeping in mind that $\psi_i + p_i \tilde{\psi} = 0$ and $p_i \in \mathbb{K}[f_1, \ldots, f_s]$.

If at least one of the partial derivatives $\partial S_k / \partial X_i$ was nonzero, we would have a nontrivial polynomial relation for $\tilde{f}, f_1, \ldots, f_s$ with a smaller total degree of S_k. Due to our

choice of F this is impossible, however. So S_k is a nonzero constant, and there will be no harm in assuming that $S_k = 1$. Note that each equality $\psi_i + p_i\tilde{\psi} = 0$ now induces a polynomial relation for $\tilde{f}, f_1, \ldots, f_s$ of degree $\leqslant k-1$ in Y. Since such a relation is trivial by our choice of F, the coefficient $(\partial S_{k-1}/\partial X_i)(f_1, \ldots, f_s) + kp_i$ of $\tilde{f}^{k-1}$ in the relation has to be zero. In view of our remarks in part (c) we thus obtain

$$dS_{k-1}(f_1, \ldots, f_s) = -\sum kp_i df_i = -kd\tilde{f}.$$

Then $\tilde{f} = -S_{k-1}/k + \lambda$ for some $\lambda \in \mathbb{K}$, which shows that $\tilde{f} \in R$. $\square$

1.3. Now suppose that $\mathcal{A}$ possesses a Poisson structure $\{\ ,\ \}\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ and let π denote the corresponding *Poisson bivector*, the element of $\mathrm{Hom}_{\mathcal{A}}(\Omega^2, \mathcal{A})$ satisfying $\pi(df \wedge dg) = \{f, g\}$ for all $f, g \in \mathcal{A}$. In view of the duality described in (1.1) we may assume that $\pi \in \bigwedge^2_{\mathcal{A}} W$, that is

$$\langle df \wedge dg, \pi \rangle = \{f, g\} \qquad (\forall\, f, g \in \mathcal{A}).$$

Let $\mathrm{rk}\,\pi(\xi)$ denote the rank of the skew-symmetric matrix $\big(\{x_i, x_j\}\big)_{1 \leqslant i,j \leqslant n}$ at $\xi \in \mathbb{A}^n$. The *index* of the Poisson algebra $\mathcal{A}$, denoted $\mathrm{ind}\,\mathcal{A}$, is defined as

$$\mathrm{ind}\,\mathcal{A} := n - \max_{\xi \in \mathbb{A}^n} \mathrm{rk}\,\pi(\xi).$$

Let $Z(\mathcal{A})$ denote the Poisson centre of $\mathcal{A}$ and put $\mathrm{Sing}\,\pi := \{\xi \in \mathbb{A}^n \mid \mathrm{rk}\,\pi(\xi) < n - \mathrm{ind}\,\mathcal{A}\}$. Clearly, $\mathrm{Sing}\,\pi$ is a proper Zariski closed subset of $\mathbb{A}^n$. Note that $\langle df \wedge dg, \pi \rangle = 0$ for all $f \in Z(\mathcal{A})$ and all $g \in \mathcal{A}$. Hence the linear subspace $\{d_\xi f \mid f \in Z(\mathcal{A})\}$ lies in the kernel of $\pi(\xi)$ and we have

$$\mathrm{tr.\,deg}_{\mathbb{K}}\, Z(\mathcal{A}) \leqslant \mathrm{ind}\,\mathcal{A}.$$

We say that a subset $\{Q_1, \ldots Q_l\} \subset Z(\mathcal{A})$ is *admissible* if $l = \mathrm{ind}\,\mathcal{A}$ and the locus $\mathcal{J}(Q_1, \ldots, Q_l)$ has codimension $\geqslant 2$ in $\mathbb{A}^n$. It is clear from the definition that any admissible subset of $Z(\mathcal{A})$ is algebraically independent.

Definition 1.1. We call a Poisson algebra $(\mathcal{A}, \pi)$ *quasi-regular* if the Poisson centre of $\mathcal{A}$ contains an admissible subset and $\mathrm{Sing}\,\pi$ has codimension $\geqslant 2$ in $\mathbb{A}^n$.

Given $k \in \mathbb{N}$ we set

$$\pi^k := \underbrace{\pi \wedge \pi \wedge \cdots \wedge \pi}_{k \text{ factors}},$$

an element of $\bigwedge^{2k}_{\mathcal{A}} W$. The following is a slight modification of [15, Theorem 3.1].

Theorem 1.2. *Let $\mathcal{A} = \mathbb{K}[x_1, \ldots, x_n]$ be a quasi-regular Poisson algebra of index l and let $\{Q_1, \ldots, Q_l\} \subset Z(\mathcal{A})$ be an admissible set in $Z(\mathcal{A})$. Then*

$$\pi^{(n-l)/2} = \lambda j_{dQ_1 \wedge \ldots \wedge dQ_l}(\partial_1 \wedge \cdots \wedge \partial_n)$$

for some nonzero $\lambda \in \mathbb{K}$.

Proof. Set $w := j_{dQ_1\wedge\ldots\wedge dQ_l}(\partial_1 \wedge \cdots \wedge \partial_n)$, an element of $\bigwedge_{\mathcal{A}}^{n-l} W$. Since $j\colon\ \Omega \to \mathrm{End}(\bigwedge_{\mathcal{A}} W)$ is an exterior algebra homomorphism, it must be that

$$j_{dQ_i}(w) = j_{dQ_i\wedge dQ_1\wedge\ldots\wedge dQ_l}(\partial_1 \wedge \ldots \wedge \partial_n) = 0 \qquad (1 \leqslant i \leqslant l).$$

Since $Q_i \in Z(\mathcal{A})$, we also have

$$\langle df, j_{dQ_i}(\pi)\rangle = \langle df \wedge dQ_i, \pi\rangle = \{f, Q_i\} = 0 \qquad (\forall f \in \mathcal{A}).$$

Hence $j_{dQ_i}(\pi) = 0$. Since j_{dQ_i} is a graded derivation of $\bigwedge_{\mathcal{A}} W$, it follows that $j_{dQ_i}(\pi^k) = 0$ for all $k \in \mathbb{N}$. As a consequence, $j_{dQ_i}\big(\pi^{(n-l)/2}\big) = j_{dQ_i}(w) = 0$ for all $i \leqslant l$. As $l = \operatorname{ind}\mathcal{A}$, we have $\pi^{(n-l)/2} \neq 0$.

Given $\xi \in \mathbb{A}^n$ put $V_\xi := \bigcap_{i=1}^{l}\{v \in T_\xi(\mathbb{A}^n) \mid j_{d_\xi Q_i}(v) = 0\}$. Suppose $\xi \notin \mathcal{J}(Q_1, \ldots, Q_l)$. Then $d_\xi Q_1 \wedge \ldots \wedge d_\xi Q_l \neq 0$ and $\dim V_\xi = n - l$. Since the exterior algebra $\bigwedge T_\xi(\mathbb{A}^n)$ is a free module over its subalgebra $\bigwedge V_\xi$, it is straightforward to see that $\bigcap_{i=1}^{l} \operatorname{Ker} j_{d_\xi Q_i} = \bigwedge V_\xi$. As $\dim \bigwedge^{n-l} V_\xi = 1$, our earlier remarks now imply that $\pi^{(n-l)/2}$ and w are linearly dependent as elements of the vector space $\mathbb{K}(x_1, \ldots, x_n) \otimes_{\mathcal{A}} \big(\bigwedge_{\mathcal{A}} W\big)$.

Since $d_\xi Q_1 \wedge \ldots \wedge d_\xi Q_l \neq 0$, the above argument also shows that $w \neq 0$. It follows that there exist nonzero coprime $f_1, f_2 \in \mathcal{A}$ such that $f_1\pi^{(n-l)/2} = f_2 w$. As the set $\{Q_1, \ldots, Q_l\}$ is admissible, the function f_1 must be constant. As $\operatorname{Sing}\pi$ has codimension $\geqslant 2$ in $\mathbb{A}^n$, the function f_2 must be constant as well. Therefore, $\pi^{(n-l)/2} = \lambda w$ for some nonzero $\lambda \in \mathbb{K}$, as stated. □

1.4. Next we are going to apply Theorem 1.1 to determine the Poisson centre of certain graded quasi-regular Poisson algebras.

Corollary 1.3. *Let $\mathcal{A} = \mathbb{K}[x_1, \ldots, x_n]$ be a quasi-regular Poisson algebra of index l and suppose that $\mathcal{A} = \bigoplus_{k\geqslant 0} \mathcal{A}(k)$ is graded in such a way that $x_i \in \mathcal{A}(r_i)$ for some $r_i > 0$, where $1 \leqslant i \leqslant n$. Suppose further that $Z(\mathcal{A})$ contains an admissible set $\{Q_1, \ldots, Q_l\}$ consisting of homogeneous elements of $\mathcal{A}$. Then $Z(\mathcal{A}) = \mathbb{K}[Q_1, \ldots, Q_l]$.*

Proof. By our assumption, $R := \mathbb{K}[Q_1, \ldots, Q_l]$ is a graded subalgebra of $\mathcal{A}$ contained in $Z(\mathcal{A})$. Let z be an arbitrary element of $Z(\mathcal{A})$. We need to show that $z \in R$. Our discussion in (1.3) shows that

$$l = \mathrm{tr.\,deg}_{\mathbb{K}}\, \mathbb{K}(Q_1, \ldots, Q_l) \leqslant \mathrm{tr.\,deg}_{\mathbb{K}}\, Z(\mathcal{A}) \leqslant \operatorname{ind}\mathcal{A} = l,$$

implying that z is algebraic over R. Since $\mathcal{J}(Q_1, \ldots, Q_l)$ has codimension $\geqslant 2$ in $\mathbb{A}^n$, we can apply Theorem 1.1 to complete the proof. □

1.5. Let $A = \bigoplus_{k\geqslant 0} A_k$ be a commutative graded domain over a field F. Given $a \in A$ we denote by $\tilde{a}$ the initial (lowest) component of a. Given an F-subalgebra R of A we let $\tilde{R}$ denote the F-span of all $\tilde{r}$ with $r \in R$. Clearly, $\tilde{R}$ is a graded F-subalgebra of A.

Proposition 1.4. *Let $A = \bigoplus_{k\geqslant 0} A_k$ be an affine graded domain over a field F and suppose that $A_0 = F$. Then for any F-subalgebra R of A we have $\mathrm{tr.\,deg}_F\, \tilde{R} = \mathrm{tr.\,deg}_F\, R$.*

Proof. Since the fields of fractions of R and $\tilde{R}$ are isomorphic to subfields of the field of fractions of A, both $\operatorname{tr.deg}_F R$ and $\operatorname{tr.deg}_F \tilde{R}$ are finite. It follows from [30, Ch. II, § 12, Corollary 2] that the field of fractions of $\tilde{R}$ contains a transcendence basis consisting of homogeneous elements of $\tilde{R}$. From this it is immediate that $\operatorname{tr.deg}_F \tilde{R} \leqslant \operatorname{tr.deg}_F R$.

Put $m := \operatorname{tr.deg}_F \tilde{R}$ and assume for a contradiction that $m < \operatorname{tr.deg}_F R$. As every algebraically independent subset of R is contained in a transcendence basis of R, our earlier remarks then show that there exist algebraically independent elements $a_1, \dots, a_{m+1} \in R$ such that $\operatorname{tr.deg}_F F(\tilde{a}_1, \dots, \tilde{a}_{m+1}) = m$. Let $J \subset F[X_1, \dots, X_{m+1}]$ be the ideal of all polynomial relations between $\tilde{a}_1, \dots, \tilde{a}_{m+1}$. Since $F[\tilde{a}_1, \dots, \tilde{a}_{m+1}] \subset A$ is a domain of Krull dimension m, one observes easily that J is a prime ideal of codimension 1 in the polynomial algebra $F[X_1, \dots, X_{m+1}]$. As a consequence, J is generated by one polynomial of positive degree, say H.

Let $\tilde{R}_0 \subseteq \tilde{R}$ denote the subalgebra of initial components of $R_0 := F[a_1, \dots, a_{m+1}]$. We claim that $\tilde{R}_0$ is generated by the $\tilde{a}_i$'s and the initial component $\tilde{h}$ of $H(a_1, \dots, a_{m+1})$. To prove the claim we let $f(a_1, \dots, a_{m+1})$ be an arbitrary element of R_0. If $\tilde{f} := f(\tilde{a}_1, \dots, \tilde{a}_{m+1})$ is not zero, then $\tilde{f}$ is the initial component of $f(a_1, \dots, a_{m+1})$. If $\tilde{f} = 0$, then $f \in I$ implying that $f = f_0 H$ for some polynomial f_0 of smaller degree. Since A is a domain, the initial component of $f(a_1, \dots, a_{m+1})$ is nothing but $\tilde{f}_0 \tilde{h}$, where $\tilde{f}_0$ is the initial component of $f_0(a_1, \dots, a_{m+1})$. Since $\deg f_0 < \deg f$, our claim follows by induction on the degree of $f \in F[X_1, \dots, X_{m+1}]$. As a result, the algebra $\tilde{R}_0$ is finitely generated over F.

Next we note that the grading of A induces a descending filtration $\mathcal{F} = (I_k)_{k \geqslant 0}$ of R_0, where $I_k = R_0 \cap \bigoplus_{i \geqslant k} A_i$ for all k. Furthermore, $\tilde{R}_0 \cong \operatorname{gr}_{\mathcal{F}} R_0$, the corresponding graded algebra. Consequently, the algebra $\operatorname{gr}_{\mathcal{F}} R_0$ is Noetherian. Since $A_0 = F$, we now apply [2, Theorem 4.4.6(b)] to deduce that $R_0 \cong F[X_1, \dots, X_{m+1}]$ and $\operatorname{gr}_{\mathcal{F}} R_0 \cong \tilde{R}_0$ have the same Krull dimension. However, $\dim R_0 = m + 1$ whilst $\dim \tilde{R}_0 = \operatorname{tr.deg}_F \tilde{R}_0 = m$. By contradiction, the result follows. □

2. Slodowy slices and symmetric invariants of centralisers

2.1. Let $\chi = (e, \cdot\,)$ and $r = \dim \mathfrak{g}_e$. The action of $\operatorname{ad} h$ gives $\mathfrak{g}$ a graded Lie algebra structure, $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}(i)$, where $\mathfrak{g}(i) = \{x \in \mathfrak{g} \mid [h, x] = ix\}$. It is well-known that $\mathfrak{g}_e$ is a graded Lie subalgebra of the parabolic subalgebra $\mathfrak{p} := \bigoplus_{i \geqslant 0} \mathfrak{g}(i)$ of $\mathfrak{g}$, that is $\mathfrak{g}_e = \bigoplus_{\geqslant 0} \mathfrak{g}_i(i)$ where $\mathfrak{g}_e(i) = \mathfrak{g}_e \cap \mathfrak{g}(i)$. Choose a $\mathbb{K}$-basis $x_1, \dots, x_m$ of $\mathfrak{p}$ with $x_i \in \mathfrak{g}(n_i)$ for some $n_i \in \mathbb{Z}_+$, such that $x_1, \dots, x_r$ is a basis of $\mathfrak{g}_e$ and $x_i \in [f, \mathfrak{g}]$ for all $i \geqslant r + 1$. Such a basis exists because $\mathfrak{g} = \mathfrak{g}_e \oplus [\mathfrak{g}, f]$ an $\mathfrak{p}$ contains $\mathfrak{g}_e$.

Define a skew-symmetric bilinear form $\langle \cdot, \cdot \rangle$ on the subspace $\mathfrak{g}(-1)$ by setting $\langle x, y \rangle = (e, [x, y])$ for all $x, y \in \mathfrak{g}(-1)$. As $\mathfrak{g}_e \subset \mathfrak{p}$, this form is nondegenerate. Choose a basis $z_1, \dots, z_s, z_{s+1}, \dots, z_{2s}$ of $\mathfrak{g}(-1)$ such that

$$\langle z_{i+s}, z_j \rangle = \delta_{ij}, \qquad \langle z_i, z_j \rangle = \langle z_{i+s}, z_{j+s} \rangle = 0 \qquad (1 \leqslant i, j \leqslant r)$$

and denote by $\mathfrak{g}(-1)^0$ the linear span of $z_{s+1}, \dots, z_{2s}$. Let $\mathfrak{m} = \mathfrak{g}(-1)^0 \oplus \sum_{i \leqslant -2} \mathfrak{g}(i)$, a nilpotent Lie subalgebra of dimension $(\dim G \cdot e)/2$ in $\mathfrak{g}$.

Given a Lie algebra $\mathfrak{s}$ over $\mathbb{K}$ denote by $U(\mathfrak{s})$ the universal enveloping algebra of $\mathfrak{s}$. As χ vanishes on the derived subalgebra of $\mathfrak{m}$, the ideal N_χ of $U(\mathfrak{m})$ generated by all $x-\chi(x)$ with $x \in \mathfrak{m}$ has codimension 1 in $U(\mathfrak{m})$. Let $\mathbb{K}_\chi = U(\mathfrak{m})/N_\chi$, a one-dimensional $U(\mathfrak{m})$-module, and denote by 1_χ the image of 1 in $\mathbb{K}_\chi$. Set

$$Q_\chi = U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} \mathbb{K}_\chi \quad \text{and} \quad H_\chi = \mathrm{End}_{\mathfrak{g}}(Q_\chi)^{\mathrm{op}}.$$

According to [19] and [11] the associative algebra H_χ is a noncommutative filtered deformation of the coordinate algebra $\mathbb{K}[\mathcal{S}_e]$ endowed with its Slodowy grading [25, 7.4].

2.2. Given $(\mathbf{a},\mathbf{b}) \in \mathbb{Z}_+^m \times \mathbb{Z}_+^s$ we set $x^{\mathbf{a}} z^{\mathbf{b}} = x_1^{a_1} \cdots x_m^{a_m} z_1^{b_1} \cdots z_s^{m_s}$, an element of $U(\mathfrak{g})$. By the PBW theorem, the monomials $x^{\mathbf{a}} z^{\mathbf{b}} \otimes 1_\chi$, where $(\mathbf{a},\mathbf{b}) \in \mathbb{Z}_+^m \times \mathbb{Z}_+^s$, form a $\mathbb{K}$-basis of the induced $U(\mathfrak{g})$-module Q_χ. For $k \in \mathbb{Z}_+$ we denote by Q_χ^k the $\mathbb{K}$-span of all $x^{\mathbf{a}} z^{\mathbf{b}} \otimes 1_\chi$ with

$$|(\mathbf{a},\mathbf{b})|_e := \sum_{i=1}^m a_i(n_i+2) + \sum_{i=1}^s b_i \leqslant k.$$

Any element $h \in H_\chi$ is uniquely determined by its effect on the canonical generator 1_χ. We let H_χ^k denote the subspace of H_χ spanned by all $h \in H_\chi$ with $h(1_\chi) \in Q_\chi^k$. Then $H_\chi = \bigcup_{k \geqslant 0} H_\chi^k$ and $H_\chi^i \cdot H_\chi^j \subseteq H_\chi^{i+j}$ for all $i,j \in \mathbb{Z}_+$; see [19] or [11]. The increasing filtration $\{H_\chi^i \mid i \in \mathbb{Z}_+\}$ of the associative algebra H_χ is often referred to as the *Kazhdan filtration* of H_χ. The corresponding graded algebra $\mathrm{gr}\, H_\chi$ is commutative. The elements x from $Q_\chi^k \setminus Q_\chi^{k-1}$ and $H_\chi^k \setminus H_\chi^{k-1}$ are said to have *Kazhdan degree* k, written $\deg_e(x) = k$.

According to [19, Theorem 4.6] the algebra H_χ has a distinguished generating set $\Theta_1, \ldots, \Theta_r$ such that

$$\Theta_k(1_\chi) = \Big(x_k + \sum_{1 \leqslant |(\mathbf{i},\mathbf{j})|_e \leqslant n_k+2} \lambda_{\mathbf{i},\mathbf{j}}^k \, x^{\mathbf{i}} z^{\mathbf{j}}\Big) \otimes 1_\chi, \qquad 1 \leqslant k \leqslant r,$$

where $\lambda_{\mathbf{i},\mathbf{j}}^k \in \mathbb{K}$ and $\lambda_{\mathbf{i},\mathbf{j}}^k = 0$ if either $|(\mathbf{i},\mathbf{j})|_e = n_k+2$ and $|\mathbf{i}|+|\mathbf{j}| = 1$ or $\mathbf{j} = \mathbf{0}$ and $i_t = 0$ for $t \geqslant r+1$. The monomials $\Theta_1^{k_1} \cdots \Theta_r^{k_r}$ and $(\mathrm{gr}\,\Theta_1)^{k_1} \cdots (\mathrm{gr}\,\Theta_r)^{k_r}$ with $(k_1,\ldots,k_r) \in \mathbb{Z}_+^r$ form $\mathbb{K}$-bases of H_χ and $\mathrm{gr}\, H_\chi$, respectively. Furthermore, $[\Theta_i,\Theta_j] = \Theta_j \circ \Theta_i - \Theta_i \circ \Theta_j \in H_\chi^{n_i+n_j+2}$ for all $1 \leqslant i,j \leqslant r$.

As explained in [20, Sect. 2], there exists a linear map $\Theta : \mathfrak{g}_e \to H_\chi,\ x \mapsto \Theta_x$ such that $\Theta_{x_i} = \Theta_i$ for all i and

$$[\Theta_{x_i},\Theta_{x_j}] \equiv \Theta_{[x_i,x_j]} + q_{ij}(\Theta_1,\ldots,\Theta_r) \pmod{H_\chi^{n_i+n_j}} \qquad (1 \leqslant i,j \leqslant r), \tag{2}$$

where q_{ij} is a polynomial in r variables such that $\deg_e\big(q_{ij}(\Theta_1,\ldots,\Theta_r)\big) = n_i+n_j+2$ and $\mathrm{in}(q_{ij}) \geqslant 2$ whenever $q_{ij} \neq 0$. Moreover, the map Θ has the property that $\Theta_{[x,y]} = [\Theta_x,\Theta_y]$ for all $x \in \mathfrak{g}_e(0)$ and $y \in \mathfrak{g}_e$. In particular, $\Theta(\mathfrak{g}_e(0))$ is a Lie subalgebra of H_χ with respect to the commutator product.

2.3. Let $m_1,\ldots,m_l$ be the exponents of the Weyl group of $\mathfrak{g}$. By the Chevalley Restriction Theorem, there exist algebraically independent elements $F_1,\ldots,F_l \in \mathcal{S}(\mathfrak{g})^G$ such that $F_i \in S^{m_i+1}(\mathfrak{g})$ for all i and $\mathcal{S}(\mathfrak{g})^G = \mathbb{K}[F_1,\ldots,F_l]$. Let

$$\varphi : \mathfrak{g} \longrightarrow \mathbb{A}^n, \qquad x \mapsto \big(\kappa(F_1)(x),\ldots,\kappa(F_l)(x)\big),$$

be the adjoint quotient map of $\mathfrak{g}$, and let φ_e denote its restriction to the Slodowy slice $\mathcal{S}_e = e + \mathfrak{g}_f$. Composing φ_e with the translation $\tau : \mathfrak{g}_f \xrightarrow{\sim} \mathcal{S}_e,\ x \mapsto e + x$, one obtains a morphism

$$\psi := \varphi_e \circ \tau : \mathfrak{g}_f \longrightarrow \mathbb{A}^n, \qquad x \mapsto (\psi_1(x), \ldots, \psi_l(x)).$$

According to [25, 5.2 & 7.4], the morphism ψ is faithfully flat with normal fibres, while in [19, Sect. 5] it is proved that all fibres of ψ are irreducible complete intersections of dimension $r - l$. It should be mentioned here that each ψ_i is homogeneous of degree $2m_i + 2$ with respect to the Slodowy grading of $\mathbb{K}[\mathfrak{g}_f] \cong \mathbb{K}[\mathcal{S}_e]$.

Let U^k be the kth component of the standard filtration of $U(\mathfrak{g})$. In view of the PBW theorem, the corresponding graded algebra $\operatorname{gr} U(\mathfrak{g})$ identifies with the symmetric algebra $\mathcal{S}(\mathfrak{g})$. We let $Z(\mathfrak{g})$ denote the centre of of $U(\mathfrak{g})$. It well-known that there exist algebraically independent elements $\tilde{F}_1, \ldots, \tilde{F}_l$ in $Z(\mathfrak{g})$ such that $\tilde{F}_i \in U^{m_i+1}$ and $\operatorname{gr} \tilde{F}_i = F_i$ for all i; see [6, 7.4] for example. Moreover, the map taking each F_i to $\tilde{F}_i$ extends uniquely to an algebra isomorphism between $\mathcal{S}(\mathfrak{g})^G$ and $Z(\mathfrak{g})$. Given $F \in \mathcal{S}(\mathfrak{g})^G$ we shall denote by $\tilde{F}$ the image of F under this isomorphism. Note that when $F \in S^k(\mathfrak{g})^G \setminus \{0\}$, we have $\tilde{F} \in U^k \setminus U^{k-1}$.

Each $\tilde{F} \in Z(\mathfrak{g})$ maps into the centre of H_χ via $\tilde{F} \mapsto \tilde{F}(1_\chi)$. By [19, 6.2], this map is injective. To keep the notation simple we shall identify the elements of $Z(\mathfrak{g})$ with their images in $Z(H_\chi)$. Note that $\tilde{F}_i \in H_\chi^{2m_i+2} \setminus H_\chi^{2m_i+1}$; see [19, 6.2]. For $1 \leqslant i \leqslant r$, we denote by ξ_i the restriction of $\kappa(x_i)$ to $\mathfrak{g}_f$, which we regard as a homogeneous polynomial function of degree $n_1 + 2$ on $\mathfrak{g}_f$. We denote by $\tilde{\psi}_i$ the image of $\tilde{F}_i$ in the Poisson algebra $\operatorname{gr} H_\chi$. Clearly, each $\tilde{\psi}_i$ lies in the Poisson centre of $\operatorname{gr} H_\chi$.

2.4. Let M denote the subspace of $\mathfrak{g}$ spanned by $z_1, \ldots, z_s$ and $x_1, \ldots, x_m$. We say that the monomial $x^{\mathbf{a}} z^{\mathbf{b}} \in \mathcal{S}(M)$ has Kazhdan degree $\sum_{i=1}^m a_i(n_i + 2) + \sum_{i=1}^s b_i$. By [19, 6.3], the map δ' taking $\operatorname{gr} \Theta_k$ to $x_k + \sum_{|(\mathbf{i},\mathbf{j})|_e = n_k+2} \lambda_{\mathbf{i},\mathbf{j}}^k\, x^{\mathbf{i}} z^{\mathbf{j}}$ for all $k \leqslant r$ extends to a graded algebra embedding $\operatorname{gr} H_\chi \hookrightarrow \mathcal{S}(M)$. Let $\nu : \mathcal{S}(M) \twoheadrightarrow \mathcal{S}(\mathfrak{g}_e)$ be the graded algebra epimorphism with the property that $z_i, x_j \in \operatorname{Ker} \nu$ for $1 \leqslant i \leqslant s,\ r+1 \leqslant j \leqslant m$ and $\nu(x_k) = x_k$ for $1 \leqslant k \leqslant r$. As in [19, 6.3] we denote by δ'' the restriction of $\nu \circ \delta'$ to $\operatorname{gr} H_\chi$, and set $\delta := \kappa \circ \delta''$.

By [19, Prop. 6.3], the map $\delta\colon \operatorname{gr} H_\chi \to \mathbb{K}[\mathfrak{g}_f]$ is a graded algebra isomorphism satisfying $\delta(\operatorname{gr} \Theta_k) = \xi_k$ for all $k \leqslant r$ and $\delta(\tilde{\psi}_i) = \psi_i$ for all $i \leqslant l$. This implies that $\delta''\colon \operatorname{gr} H_\chi \xrightarrow{\sim} \mathcal{S}(\mathfrak{g}_e)$ is a graded algebra isomorphism with the following properties:

$$\delta''(\tilde{\psi}_i) = \kappa_e^{-1}(\psi_i) \qquad (1 \leqslant i \leqslant l); \qquad \delta''(\operatorname{gr} \Theta_i) = \kappa_e^{-1}(\xi_i) = x_i \qquad (1 \leqslant i \leqslant r). \tag{3}$$

We use δ'' to transport the Poisson algebra structure of $\operatorname{gr} H_\chi$ to the symmetric algebra $\mathcal{S}(\mathfrak{g}_e)$. Combining (1) and (2) one observes easily that the new Poisson bracket of $\mathcal{S}(\mathfrak{g}_e)$ satisfies the following condition:

$$\{x_i, x_j\} \;=\; [x_i, x_j] + q_{ij}(x_1, \ldots, x_r) \qquad (1 \leqslant i, j \leqslant r). \tag{4}$$

Furthermore, each $\kappa_e^{-1}(\psi_i)$ is in the Poisson centre of $\mathcal{S}(\mathfrak{g}_e)$.

2.5. With these preliminaries at hand we are in a position to prove Proposition 0.1.

Proof of Proposition 0.1. Let $F = g(F_1, \dots, F_l)$ be a homogeneous element of $\mathcal{S}(\mathfrak{g})^G$ and let $\tilde{F} = g(\tilde{F}_1, \dots, \tilde{F}_l)$ be the corresponding element of $Z(\mathfrak{g}) \hookrightarrow H_\chi$; see our discussion in (2.3). Since each $\tilde{F}_i$ commutes with h, the definition of the Kazhdan filtration and (2) yield

$$\delta''(\operatorname{gr}\tilde{F}) = \delta''\big(g(\tilde{\psi}_1, \dots, \tilde{\psi}_l)\big) = g\big(\kappa_e^{-1}(\psi_1), \dots, \kappa_e^{-1}(\psi_l)\big) = \kappa_e^{-1}\big(g(\psi_1, \dots, \psi_l)\big),$$

see [19, 6.2] for more detail. Note that $\delta''(\operatorname{gr}\tilde{F}) = \kappa_e^{-1}\big(g(\psi_1, \dots, \psi_l)\big)$ belongs to the Poisson centre of $\mathcal{S}(\mathfrak{g}_e)$, that is $\{x, \delta''(\operatorname{gr}\tilde{F})\} = 0$ for all $x \in \mathfrak{g}_e$. Abusing notation we denote by $\operatorname{ad} x$ the derivation of the algebra $\mathcal{S}(\mathfrak{g}_e)$ induced by the inner derivation of $x \in \mathfrak{g}_e$. Then

$$0 = \{x, \delta''(\operatorname{gr}\tilde{F})\} = (\operatorname{ad} x)\big(\operatorname{in}(\delta''(\operatorname{gr}\tilde{F}))\big) + \text{ terms of higher standard degree,}$$

in view on (3). (One should also keep in mind that $q_{ij} \neq 0$ implies $\operatorname{in}(q_{ij}) \geqslant 2$). Since this holds for all $x \in \mathfrak{g}_e$, we deduce that $\operatorname{in}(\delta''\big(\operatorname{gr}\tilde{F})) \in \mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e}$. But then

$${}^eF := \kappa_e^{-1}\big(\operatorname{in}(\tau^*(\kappa(F)_{|\,\mathcal{S}_e}))\big) = \kappa_e^{-1}\big(\operatorname{in}(g(\psi_1, \dots, \psi_l))\big) = \operatorname{in}(\delta''\big(\operatorname{gr}\tilde{F})) \in \mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e}.$$

We thus obtain ${}^eF \in \mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e} = \mathcal{S}(\mathfrak{g}_e)^{G_e^\circ}$.

Now let $C_e = G_e \cap G_f$. It is well-known that C_e is a reductive subgroup of G_e, and G_e is generated by C_e and the unipotent radical $R_u G_e$; see [10, 3.7] for example. Clearly, both $\mathfrak{g}_f$ and $\mathcal{S}_e = e + \mathfrak{g}_f$ are C_e-stable, and the mappings κ and κ_e are C_e-equivariant. Since $F \in \mathcal{S}(\mathfrak{g})^G$, this entails ${}^eF \in \mathcal{S}(\mathfrak{g}_e)^{C_e}$. But then ${}^eF \in \mathcal{S}(\mathfrak{g}_e)^{C_e \cdot G_e^\circ} = \mathcal{S}(\mathfrak{g}_e)^{G_e}$, completing the proof. □

2.6. Theorem 1.2 will enable us to obtain a differential criterion for regularity of linear functions applicable to a large class of centralisers in $\mathfrak{g}$. Recall that a linear function $\gamma \in \mathfrak{g}_e^*$ is called *regular* if $\dim \mathfrak{g}_e^\gamma = \operatorname{ind}\mathfrak{g}_e$, where $\mathfrak{g}_e^\gamma = \{x \in \mathfrak{g}_e \mid \gamma([x, \mathfrak{g}_e]) = 0\}$ is the stabiliser of γ in $\mathfrak{g}_e$.

Theorem 2.1. *Suppose* $\operatorname{ind}\mathfrak{g}_e = l$. *Then the following are true for any homogeneous generating system* $F_1, \dots, F_l$ *of the invariant algebra* $\mathcal{S}(\mathfrak{g})^G$*:*

(i) $\sum_{i=1}^l \deg {}^eF_i \leqslant (r+l)/2$ *where* $r = \dim \mathfrak{g}_e$.

(ii) *The elements* ${}^eF_1, \dots, {}^eF_l$ *are algebraically independent if and only if* $\sum_{i=1}^l \deg {}^eF_i = (r+l)/2$.

(iii) *Suppose* $\sum_{i=1}^l \deg {}^eF_i = (r+l)/2$. *Then the differentials* $d_\gamma({}^eF_1), \dots, d_\gamma({}^eF_l)$ *are linearly independent at* $\gamma \in \mathfrak{g}_e^*$ *if and only if* γ *is regular in* $\mathfrak{g}_e^*$.

Proof. We are going to apply Theorem 1.2 to the Poisson algebra $\operatorname{gr} H_\chi$. Let π_e denote the Poisson bivector of $\operatorname{gr} H_\chi$ and let π_e^{PL} be the Poisson bivector of the polynomial algebra $\mathcal{A} := \mathcal{S}(\mathfrak{g}_e)$ regarded with its standard Poisson structure. We identify $\operatorname{gr} H_\chi$ with $\mathcal{A}$ by using the recipe described in (2.4) and set $f_i := \kappa_e^{-1}(\tau^*(\kappa(F_i)_{|\,\mathcal{S}_e}))$, $1 \leqslant i \leqslant l$. It follows from [19, Theorem 5.4] that the ideal $(f_1, \dots, f_l) \subset \mathcal{A}$ is radical and its zero locus in $\mathfrak{g}_e^*$ is normal. This implies that $\mathcal{J}(f_1, \dots, f_l)$ has codimension $\geqslant 2$ in $\mathfrak{g}_e^*$.

From the alternative description of the Poisson structure on $\operatorname{gr} H_\chi$ given in [11, Sect. 3] it follows that

$$\operatorname{rk}\pi_e(\gamma) = \dim(\operatorname{Ad} G)\big(e + (\kappa_e^*)^{-1}(\gamma)\big) - \dim(\operatorname{Ad} G)\, e \qquad \big(\forall\, \gamma \in \mathfrak{g}_e^*\big).$$

Consequently, $\gamma \in \operatorname{Sing}\pi_e$ if and only if the adjoint orbit $(\operatorname{Ad}G)\big(e+(\kappa_e^*)^{-1}(\gamma)\big)$ is not of maximal dimension. By Kostant's criterion for regularity, this happens if and only if $e+(\kappa_e^*)^{-1}(\gamma) \in \mathcal{J}\big(\kappa(F_1),\ldots,\kappa(F_l)\big)$. Chasing through the definitions it is easy to see that the latter happens if and only of $\gamma \in \mathcal{J}(f_1,\ldots,f_l)$. Thus, $\operatorname{Sing}\pi_e = \mathcal{J}(f_1,\ldots,f_l)$. Our earlier remarks now show that $\operatorname{Sing}\pi_e$ has codimension $\geqslant 2$ in $\mathfrak{g}_e^*$. As $\operatorname{ind}(\operatorname{gr}H_\chi) = \operatorname{ind}\mathfrak{g}$, we conclude that the subset $\{f_1,\ldots,f_l\}$ is admissible and the Poisson algebra $(\operatorname{gr}H_\chi,\pi_e)$ is quasi-regular.

The standard grading of $\mathcal{A}$ (by total degree) induces gradings of the $\mathbb{K}$-algebras Ω and $\bigwedge_{\mathcal{A}} W$ where we impose that $\deg dx_i = 0$ and $\deg \partial_i = 0$ for all i. Our assumption that $\operatorname{ind}\mathfrak{g}_e = l$ yields $(\pi_e^{PL})^{(r-l)/2} \neq 0$ whereas (4) entails that $\operatorname{in}(\pi_e) = \pi_e^{PL}$. Consequently,

$$\operatorname{in}(\pi_e^{(r-l)/2}) = (\pi_e^{PL})^{(r-l)/2} \neq 0. \tag{5}$$

As $\operatorname{in}(f_i) = {}^eF_i$ for all i, we also have that

$$\deg\big(\operatorname{in}(df_1\wedge\ldots\wedge df_r)\big) \geqslant \deg\, d({}^eF_1)\wedge\ldots\wedge d({}^eF_l). \tag{6}$$

Combining (5) and (6) with Theorem 1.2 we now conclude that

$$\frac{r-l}{2} = \deg\big((\pi_e^{PL})^{(r-l)/2}\big) = \deg\big(\operatorname{in}(j_{df_1\wedge\ldots\wedge df_l}(\partial_1\wedge\cdots\wedge\partial_r)\big) \geqslant -l + \textstyle\sum_{i=1}^l \deg\, {}^eF_i.$$

Statement (i) follows. Now ${}^eF_1,\ldots,{}^eF_l$ are algebraically independent in $\mathcal{S}(\mathfrak{g}_e)$ if and only if $d({}^eF_1)\wedge\ldots\wedge d({}^eF_l) \neq 0$. Since the latter happens if and only if $\deg\big(\operatorname{in}(df_1\wedge\ldots\wedge df_r)\big) = -l + \sum_{i=1}^l \deg\, {}^eF_i$, the above argument also yields (ii).

Finally, suppose $\sum_{i=1}^l \deg\, {}^eF_i = (r+l)/2$. Then $\operatorname{in}(df_1\wedge\ldots\wedge df_r) = d({}^eF_1)\wedge\ldots\wedge d({}^eF_l)$, and Theorem 1.2 forces

$$(\pi_e^{PL})^{(r-l)/2} = \lambda j_{d({}^eF_1)\wedge\ldots\wedge d({}^eF_l)}(\partial_1\wedge\cdots\wedge\partial_r), \qquad \lambda\in\mathbb{K}^\times. \tag{7}$$

Since $\operatorname{ind}\mathfrak{g}_e = l$, the specialisation of $(\pi_e^{PL})^{(r-l)/2}$ at γ is nonzero if and only if γ a regular linear function of $\mathfrak{g}_e$. On the other hand, the RHS of (7) is nonzero at γ if and only if the differentials $d_\gamma({}^eF_1),\ldots,d_\gamma({}^eF_l)$ are linearly independent. This completes the proof. $\square$

2.7. Suppose Elashvili's conjecture holds for $\mathfrak{g}_e$. Simple examples show that the sum of the degrees of ${}^eF_1,\ldots,{}^eF_l$ depends on the choice of homogeneous generators $F_1,\ldots,F_l$ of $\mathcal{S}(\mathfrak{g})^G$. We say that a homogeneous generating system $\{F_1,\ldots,F_l\}\subset\mathcal{S}(\mathfrak{g})^G$ is *good* for e if

$$\textstyle\sum_{i=1}^l \deg {}^eF_i = (\dim\mathfrak{g}_e + \operatorname{rk}\mathfrak{g})/2.$$

For any generating system $\{F_1,\ldots,F_l\}\subset\mathcal{S}(\mathfrak{g})^G$ which is good for e the Jacobian locus $\mathcal{J}({}^eF_1,\ldots,{}^eF_l)$ is a *proper* Zariski closed subset of $\mathfrak{g}_e^*$; see Theorem 2.1. We say that a homogeneous generating system $\{F_1,\ldots,F_l\}\subset\mathcal{S}(\mathfrak{g})^G$ is *very good* for e if the Jacobian locus $\mathcal{J}({}^eF_1,\ldots,{}^eF_l)$ has codimension $\geqslant 2$ in $\mathfrak{g}_e^*$. It follows from Theorem 2.1(ii) that for any very good generating system $\{F_1,\ldots,F_l\}\subset\mathcal{S}(\mathfrak{g})^G$ we have the equality $\sum_{i=1}^l \deg {}^eF_i = (\dim\mathfrak{g}_e + \operatorname{rk}\mathfrak{g})/2$. This shows that very good systems are good.

We are now in a position to prove the main result of this section:

Theorem 2.2. *Suppose e admits a very good generating system $\{F_1, \dots, F_l\} \subset \mathcal{S}(\mathfrak{g})^G$ and assume further that Elashvili's conjecture holds for $\mathfrak{g}_e$, that is $\operatorname{ind}\mathfrak{g}_e = l$. Then*

$$\mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e} = \mathcal{S}(\mathfrak{g}_e)^{G_e} = \mathbb{K}[{}^eF_1, \dots, {}^eF_l].$$

In particular, $\mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e}$ is a graded polynomial algebra in $l = \operatorname{rk}\mathfrak{g}$ variables.

Proof. By Theorem 0.1, the elements ${}^eF_1, \dots, {}^eF_l$ are in $\mathcal{S}(\mathfrak{g}_e)^{G_e}$. Since $\operatorname{ind}\mathfrak{g}_e = l$ and $\mathcal{J}({}^eF_1, \dots, {}^eF_l)$ has codimension $\geqslant 2$ in $\mathfrak{g}_e^*$ by our assumption, the set $\{{}^eF_1, \dots, {}^eF_l\}$ is an admissible for the Poisson algebra $\mathcal{S}(\mathfrak{g}_e)$. Moreover, Theorem 2.1(iii) shows that the Poisson algebra $\mathcal{S}(\mathfrak{g}_e)$ is quasi-regular. Applying Corollary 1.3 to the Poisson algebra $\mathcal{S}(\mathfrak{g}_e)$ regarded with its standard grading we now obtain that $S(\mathfrak{g}_e)^{\mathfrak{g}_e}$ coincides with $\mathbb{K}[{}^eF_1, \dots, {}^eF_l]$. Since $\mathbb{K}[{}^eF_1, \dots, {}^eF_l] \subseteq \mathcal{S}(\mathfrak{g}_e)^{G_e} \subseteq \mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e}$, the result follows. □

Remark 2.1. As explained in the proof of Theorem 2.1, the Poisson algebra $(\operatorname{gr} H_\chi, \pi_e)$ is quasi-regular and $\{f_1, \dots, f_l\}$ is an admissible set for $\operatorname{gr} H_\chi$. Applying Corollary 1.3 to the Poisson algebra $\operatorname{gr} H_\chi$ (regarded with its Slodowy grading) we are able to deduce that the Poisson centre $Z(\operatorname{gr} H_\chi)$ of $\operatorname{gr} H_\chi$ is generated by $f_1, \dots, f_l$. In particular, $Z(\operatorname{gr} H_\chi)$ is a polynomial algebra in l variables. This, in turn, implies that $Z(H_\chi) = Z(\mathfrak{g})$. We thus recover a result of Victor Ginzburg; see the footnote in [20].

2.8. Let ${}^eZ$ denote the $\mathbb{K}$-span of all ${}^eF$ with $F \in \mathcal{S}(\mathfrak{g})^G$, a subalgebra of $\mathcal{S}(\mathfrak{g}_e)^{G_e}$. For later applications we put on record the following consequence of Proposition 1.4:

Corollary 2.3. *For any nilpotent element $e \in \mathfrak{g}$ we have the equality* $\operatorname{tr.deg}_{\mathbb{K}}({}^eZ) = \operatorname{rk}\mathfrak{g}$.

Proof. Recall that ${}^eZ$ coincides with the algebra of initial components of the subalgebra $\kappa_e^{-1}(\tau^*(\kappa(\mathcal{S}(\mathfrak{g})^G)_{|\mathcal{S}_e}))$ of $\mathcal{S}(\mathfrak{g}_e)$, where the latter is regarded with its standard grading. Since $\mathcal{S}(\mathfrak{g})^G$ is spanned by homogeneous elements, Proposition 1.4 implies that

$$\operatorname{tr.deg}_{\mathbb{K}}({}^eZ) = \operatorname{tr.deg}_{\mathbb{K}}\big(\kappa(\mathcal{S}(\mathfrak{g})^G)_{|\mathcal{S}_e}\big) = \operatorname{tr.deg}_{\mathbb{K}}\mathcal{S}(\mathfrak{g})^G = \operatorname{rk}\mathfrak{g},$$

as stated. □

Question 2.1. Is it true that ${}^eZ$ is always finitely generated over $\mathbb{K}$?

3. Regular linear functions on centralisers

3.1. Given a finite dimensional Lie algebra $\mathfrak{q}$ and a linear function γ on $\mathfrak{q}$ we let $\mathfrak{q}^\gamma$ denote the stabiliser of γ in $\mathfrak{q}$. Recall that $\operatorname{ind}\mathfrak{q} = \min_{\gamma\in\mathfrak{q}^*}\dim\mathfrak{q}^\gamma$. We set

$$\mathfrak{q}^*_{\mathrm{sing}} := \{\gamma \in \mathfrak{q}^* \mid \dim\mathfrak{q}^\gamma > \operatorname{ind}\mathfrak{q}\}.$$

The set $\mathfrak{q}^*_{\mathrm{reg}} := \mathfrak{q}^* \setminus \mathfrak{q}^*_{\mathrm{sing}}$ consists of all regular linear fuctions of $\mathfrak{q}$. The main goal of this section is to prove that $(\mathfrak{g}_e^*)_{\mathrm{sing}}$ has codimension $\geqslant 2$ in $\mathfrak{g}_e^*$ for any nilpotent element e in $\mathfrak{g} = \mathfrak{gl}_n$ and $\mathfrak{g} = \mathfrak{sp}_{2n}$, where $n \geqslant 2$. When dealing with $\mathfrak{g} = \mathfrak{gl}_n$ we do not impose any restrictions on the characteristic of $\mathbb{K}$, whilst for $\mathfrak{g} = \mathfrak{sp}_{2n}$ we require that $\operatorname{char}\mathbb{K} \neq 2$.

3.2. Let $\mathbb{V}$ be an n-dimensional vector space over $\mathbb{K}$ and let e be a nilpotent element in $\mathfrak{g} = \mathfrak{gl}(\mathbb{V})$. Let k be the number of Jordan blocks of e and $W \subseteq \mathbb{V}$ a (k-dimensional)

complement of $\operatorname{Im} e$ in $\mathbb{V}$. Let $d_i + 1$ denote the size of the i-th Jordan block of e. We always assume that the Jordan blocks are ordered such that $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_k$. Choose a basis $w_1, w_2, \ldots, w_k$ in W such that the vectors $e^j{\cdot}w_i$ with $1 \leqslant i \leqslant k$, $0 \leqslant j \leqslant d_i$ form a basis for $\mathbb{V}$, and put $\mathbb{V}[i] := \operatorname{span}\{e^j{\cdot}w_i \mid j \geqslant 0\}$. Note that $e^{d_i+1}{\cdot}w_i = 0$ for all $i \leqslant k$. When $k = 1$, the element e is regular in $\mathfrak{g}$, so that $\mathfrak{g}_e$ is abelian of dimension n and $(\mathfrak{g}_e^*)_{\mathrm{sing}} = \varnothing$. So we assume from now on that $k \geqslant 2$.

If $\xi \in \mathfrak{g}_e$, then $\xi(e^j{\cdot}w_i) = e^j{\cdot}\xi(w_i)$, hence ξ is completely determined by its values on W. Each vector $\xi(w_i)$ can be written as

$$\xi(w_i) \;=\; \sum_{j,s} c_i^{j,s} e^s{\cdot}w_j, \qquad c_i^{j,s} \in \mathbb{K}. \tag{8}$$

Thus, ξ is completely determined by the coefficients $c_i^{j,s} = c_i^{j,s}(\xi)$. This shows that $\mathfrak{g}_e$ has a basis $\{\xi_i^{j,s}\}$ such that

$$\begin{cases} \xi_i^{j,s}(w_i) = e^s{\cdot}w_j, \\ \xi_i^{j,s}(w_t) = 0 \;\text{ for } t \neq i, \end{cases} \qquad 1 \leqslant i, j \leqslant k, \;\text{ and }\; \max\{d_j - d_i, 0\} \leqslant s \leqslant d_j\,.$$

Note that $\xi \in \mathfrak{g}_e$ preserves each $\mathbb{V}[i]$ if and only if $c_i^{j,s}(\xi) = 0$ for $i \neq j$.

3.3 Given a collection $a_1, \ldots, a_k$ of scalars in $\mathbb{K}$ we consider the linear function α on $\mathfrak{g}_e$ defined by the formula

$$\alpha(\xi) = \sum_{i=1}^{k} a_i c_i^{i,d_i} \qquad (\forall\, \xi \in \mathfrak{g}_e), \tag{9}$$

where $c_i^{j,s}$ are the coefficients of $\xi \in \mathfrak{g}_e$. Let $\mathfrak{g}_e^\alpha$ denote the stabiliser of α in $\mathfrak{g}_e$. By aesthetic reasons we prefer it to $(\mathfrak{g}_e)^\alpha$.

Proposition 3.1 ([29]). *If the scalars $a_1, \ldots, a_k$ are nonzero and pairwise distinct, then the stabiliser of $\alpha = \alpha(a_1, \ldots, a_k)$ in $\mathfrak{g}_e$ consists of all elements in $\mathfrak{g}_e$ preserving the subspaces $\mathbb{V}[i]$, where $1 \leqslant i \leqslant k$. In other words, $\mathfrak{g}_e^\alpha$ is the linear span of the basis elements $\xi_i^{i,s}$, and $\dim \mathfrak{g}_e^\alpha = n$. In particular, $\alpha \in (\mathfrak{g}_e^*)_{\mathrm{reg}}$.*

A direct computation shows that the following commutator relation holds in $\mathfrak{g}_e$:

$$[\xi, \xi_i^{j,s}] = \sum_{t,\ell} c_t^{i,\ell}(\xi)\xi_t^{j,\ell+s} - \sum_{t,\ell} c_j^{t,\ell}(\xi)\xi_i^{t,\ell+s} \qquad (\forall\, \xi \in \mathfrak{g}_e); \tag{10}$$

see [29] for more detail. To show that $(\mathfrak{g}_e^*)_{\mathrm{sing}}$ has codimension $\geqslant 2$ in $\mathfrak{g}_e^*$, for $\mathfrak{g}_e \subset \mathfrak{gl}(\mathbb{V})$, we have to produce more regular elements in $\mathfrak{g}_e^*$.

Proposition 3.2. *Define $\beta \in \mathfrak{g}_e^*$ by setting $\beta(\xi) = \sum\limits_{i=1}^{k-1} c_{i+1}^{i,d_i}(\xi)$ for all $\xi \in \mathfrak{g}_e$. Then $\dim \mathfrak{g}_e^\beta = n$, so that $\beta \in (\mathfrak{g}_e^*)_{\mathrm{reg}}$.*

Proof. From (10) and the definition of β it follows that $\beta([\xi, \xi_i^{j,s}]) = c_{j+1}^{i,d_j-s}(\xi) - c_j^{i-1,d_{i-1}-s}(\xi)$ for all $\xi \in \mathfrak{g}_e$. Suppose $(\mathrm{ad}^*\xi)\beta = 0$. Then $\beta([\xi, \mathfrak{g}_e]) = 0$ forcing $c_{j+1}^{i,d_j-s}(\xi) = c_j^{i-1,d_{i-1}-s}(\xi)$ for all $i, j \in \{1, \ldots, k\}$ and all s such that $\max(0, d_j - d_i) \leqslant s \leqslant d_j$.

We claim that $c_j^{i,s}(\xi) = 0$ for $i < j$. Suppose for a contradiction that this is not the case and take the maximal j for which there are $i < j$ and $d_i - d_j \leqslant t \leqslant d_i$ such that $c_j^{i,t}(\xi) \neq 0$. Recall that, according to our convention, $d_i \geqslant d_j$. Moreover, $d_{i+1} \geqslant d_j$, because $i+1 \leqslant j$. Set $s := d_i - t$. Then $0 \leqslant s \leqslant d_j$ and $c_{j+1}^{i+1,d_j-s}(\xi) = c_j^{i,d_i-s}(\xi)$. As $j+1 > j$ and $i+1 < j+1$, the coefficients $c_{j+1}^{i+1,d_j-s}(\xi)$ and $c_j^{i,t}(\xi)$ are both zero, hence the claim.

Now take $\xi_{i+1}^{i,s} \in \mathfrak{g}_e$ with $d_i - d_{i+1} \leqslant s \leqslant d_i$. Since $\beta([\xi, \xi_{i+1}^{i,s}]) = 0$, we have $c_{i+1}^{i+1,d_i-s}(\xi) = c_i^{i,d_i-s}(\xi)$. Therefore, $c_i^{i+1,t}(\xi) = c_i^{i,t}(\xi) = c_1^{1,t}(\xi)$ for $0 \leqslant t \leqslant d_{i+1}$. In the same way one can show that $c_i^{i+\ell,t}(\xi) = c_{i-1}^{i+\ell-1,t}(\xi) = c_1^{1+\ell,t}(\xi)$ for $0 \leqslant t \leqslant d_{i+\ell}$. It follows that ξ is completely determined by its effect on w_1. So $\dim \mathfrak{g}_e^\beta \leqslant n$ simply because $\xi(w_1) \in \mathbb{V}$. On the other hand, $\dim \mathfrak{g}_e^\beta \geqslant \operatorname{ind} \mathfrak{g}_e \geqslant \operatorname{ind} \mathfrak{g} = n$ by Vinberg's inequality. The result follows. □

3.4. Let $a\colon \mathbb{K}^\times \to \mathrm{GL}(\mathbb{V})_e$ be the cocharacter such that $a(t)w_i = t^i w_i$ for all $i \leqslant k$ and $t \in \mathbb{K}^\times$, and define a rational linear action $\rho\colon \mathbb{K}^\times \to \mathrm{GL}(\mathfrak{g}_e^*)$ by the formula

$$\rho(t)\gamma = t(\mathrm{Ad}^* a(t))^{-1}\gamma \qquad \big(\forall\, \gamma \in \mathfrak{g}_e^*,\ \forall\, t \in \mathbb{K}^\times\big). \tag{11}$$

Proposition 3.3. $(\mathbb{K}\alpha \oplus \mathbb{K}\beta) \cap (\mathfrak{g}_e^*)_{\mathrm{sing}} = 0$.

Proof. Since $(\mathrm{Ad}\, a(t))(\xi_i^{j,s}) = t^{j-i}\xi_i^{j,s}$, we have $(\mathrm{Ad}^* a(t))(\alpha) = \alpha$ and $(\mathrm{Ad}^* a(t))(\beta) = t\beta$. Hence $\rho(t)\alpha = t\alpha$ and $\rho(t)\beta = \beta$. So $\mathbb{K}\alpha \oplus \mathbb{K}\beta$ is $\rho(\mathbb{K}^\times)$-stable and the induced action of $\rho(\mathbb{K}^\times)$ on this plane is a contraction to $\mathbb{K}\beta$. Since $\dim(\mathfrak{g}_e)^{\rho(t)\gamma} = \dim \mathfrak{g}_e^\gamma$ and $\beta \in (\mathfrak{g}_e^*)_{\mathrm{reg}}$, all linear functions $x\alpha + y\beta$ with $y \neq 0$ are regular. The linear functions $x\alpha$ with $x \neq 0$ are regular by Proposition 3.1. □

Theorem 3.4. *Suppose* $\dim \mathbb{V} \geqslant 2$. *Then for any nilpotent element* $e \in \mathfrak{g}$ *the locus* $(\mathfrak{g}_e^*)_{\mathrm{sing}}$ *has codimension* $\geqslant 2$ *in* $\mathfrak{g}_e$.

Proof. Since $(\mathfrak{g}_e^*)_{\mathrm{sing}}$ is conical and Zariski closed, the assertion follows immediately from Proposition 3.3. □

3.5. Using similar ideas we prove below a symplectic analogue of Theorem 3.4. Our argument in the symplectic case is more involved. We also provide an example showing that Theorem 3.4 does not extend to all nilpotent elements in orthogonal Lie algebras. We begin with some useful facts on $\mathbb{Z}_2$-graded Lie algebras.

Let $\mathfrak{q} = \mathfrak{q}_0 \oplus \mathfrak{q}_1$ be a symmetric decomposition (i.e., a $\mathbb{Z}_2$-grading) of a Lie algebra $\mathfrak{q}$. Then $\mathfrak{q}^* = \mathfrak{q}_0^* \oplus \mathfrak{q}_1^*$. If $\alpha \in \mathfrak{q}^*$, then $\tilde{\alpha}$ denotes its restriction to $\mathfrak{q}_0$.

Proposition 3.5. *Suppose* $\alpha \in \mathfrak{q}^*$ *and* $\alpha(\mathfrak{q}_1) = 0$. *Then* $(\mathfrak{q}_0)^{\tilde{\alpha}} = \mathfrak{q}^\alpha \cap \mathfrak{q}_0$.

Proof. Take $\xi \in \mathfrak{q}_0$. Since $[\xi, \mathfrak{q}_1] \subset \mathfrak{q}_1$, we have that $\tilde{\alpha}([\xi, \mathfrak{q}_0]) = 0$ if and only if $\alpha([\xi, \mathfrak{q}]) = 0$. Hence $(\mathfrak{q}_0)^{\tilde{\alpha}} = (\mathfrak{q}_0)_\alpha$, where $(\mathfrak{q}_0)_\alpha$ is the stabiliser of α in $\mathfrak{q}_0$. Clearly $(\mathfrak{q}_0)_\alpha = \mathfrak{q}^\alpha \cap \mathfrak{q}_0$. □

Each $\gamma \in \mathfrak{q}_0^*$ gives rise to a skew-symmetric bilinear form $\hat{\gamma}$ on $\mathfrak{q}_1$ by $\hat{\gamma}(x, y) = \gamma([x, y])$ for all $x, y \in \mathfrak{q}_1$. The following assertion is taken from [29].

Proposition 3.6. *In the above notation we have* $\operatorname{ind} \mathfrak{q} \leqslant \operatorname{ind} \mathfrak{q}_0 + \min\limits_{\gamma \in \mathfrak{q}_0^*} \dim(\operatorname{Ker} \hat{\gamma})$.

Proof. Take any $\gamma \in \mathfrak{q}_0^*$ and extend it to a linear function on $\mathfrak{q}$ by setting $\gamma(\mathfrak{q}_1) = 0$. Then $\mathfrak{q}^\gamma = (\mathfrak{q}_0)^\gamma \oplus (\mathfrak{q}^\gamma \cap \mathfrak{q}_1) = (\mathfrak{q}_0)^\gamma \oplus \operatorname{Ker} \hat{\gamma}$. There exists a nonempty Zariski open subset $U_1 \subset \mathfrak{q}_0^*$ such that $\dim(\mathfrak{q}_0)^\gamma = \operatorname{ind} \mathfrak{q}_0$ for all $\gamma \in U_1 \subset \mathfrak{q}_0^*$. The linear functions γ on $\mathfrak{q}_0$ for which $\operatorname{Ker} \hat{\gamma}$ has the minimal possible dimension form another nonempty Zariski open subset in $\mathfrak{q}_0^*$, call it U_2. For each $\gamma \in U_1 \cap U_2 \neq \varnothing$, the dimension of $\mathfrak{q}^\gamma$ equals the required sum, hence the result. □

Lemma 3.7. *Suppose $\alpha \in \mathfrak{q}^*$ is such that $\alpha(\mathfrak{q}_1) = 0$ and $\dim \mathfrak{q}^\alpha = \operatorname{ind} \mathfrak{q}$. Then $\dim(\mathfrak{q}_0)^{\tilde{\alpha}} = \operatorname{ind} \mathfrak{q}_0$.*

Proof. By Proposition 3.6 we have:

$$\operatorname{ind} \mathfrak{q}_0 \geqslant \operatorname{ind} \mathfrak{q} - \min_{\gamma \in \mathfrak{q}_0^*} \dim(\operatorname{Ker} \hat{\gamma}) \geqslant \dim \mathfrak{q}^\alpha - \dim(\operatorname{Ker} \hat{\alpha}) = \dim(\mathfrak{q}^\alpha \cap \mathfrak{q}_0).$$

Applying Proposition 3.5 yields the assertion. □

3.6. Let $(\ ,\ )$ be a nondegenerate symmetric or skew-symmetric form on $\mathbb{V}$ and let J be the matrix of $(\ ,\ )$ with respect to a basis B of $\mathbb{V}$. Let X denote the matrix of $x \in \mathfrak{gl}(\mathbb{V})$ relative to B. The linear mapping $x \mapsto \sigma(x)$ sending each $x \in \mathfrak{gl}(\mathbb{V})$ to the linear transformation $\sigma(x)$ whose matrix relative to B equals $-JX^tJ^{-1}$ is an involutory automorphism of $\mathfrak{gl}(\mathbb{V})$ independent of the choice of B. The elements of $\mathfrak{gl}(\mathbb{V})$ preserving $(\ ,\ )$ are exactly the fixed points of σ. We now set $\widetilde{\mathfrak{g}} := \mathfrak{gl}(\mathbb{V})$ and let $\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}_0 \oplus \widetilde{\mathfrak{g}}_1$ be the symmetric decomposition of $\widetilde{\mathfrak{g}}$ with respect to σ. The elements $x \in \widetilde{\mathfrak{g}}_1$ have the property that $(x{\cdot}v, w) = (v, x{\cdot}w)$ for all $v, w \in \mathbb{V}$.

Set $\mathfrak{g} := \widetilde{\mathfrak{g}}_0$ and let e be a nilpotent element of $\mathfrak{g}$. Since $\sigma(e) = e$, the centraliser $\widetilde{\mathfrak{g}}_e$ of e in $\widetilde{\mathfrak{g}}$ is σ-stable and $(\widetilde{\mathfrak{g}}_e)_0 = \widetilde{\mathfrak{g}}_e^\sigma = \mathfrak{g}_e$. This yields the $\mathfrak{g}_e$-invariant symmetric decomposition $\widetilde{\mathfrak{g}}_e = (\widetilde{\mathfrak{g}}_e)_0 \oplus (\widetilde{\mathfrak{g}}_e)_1$.

Suppose that $\dim \mathbb{V} = 2n \geqslant 4$ and our form is skew-symmetric. Then $\widetilde{\mathfrak{g}}_0 \cong \mathfrak{sp}_{2n}$. Since e is a nilpotent transformation of $\mathbb{V}$, we recycle the notation introduced in (3.2). Note that in the present case if d_i is even, that is if the dimension of $\mathbb{V}[i] = \operatorname{span}\{e^j{\cdot}w_i \mid j \geqslant 0\}$ is odd, then the restriction of $(\ ,\ )$ to $\mathbb{V}[i]$ is identically zero. By the same reason as in (3.2) it can be assumed that $k \geqslant 2$.

Lemma 3.8. [13, Sect. 1] *The vectors $\{w_i\}_{i=1}^k$ can be chosen such that the following conditions are satisfied:*

(i) *if d_i is odd, then the restriction of $(\ ,\ )$ to $\mathbb{V}[i]$ is nondegenerate and $(\mathbb{V}[i], \mathbb{V}[j]) = 0$ for any $j \neq i$.*
(ii) *if d_i is even, then there is a unique $i' \neq i$ such that $(\mathbb{V}[i'], \mathbb{V}[i]) \neq 0$.*

We thus obtain a decomposition of the set of Jordan blocks of odd size (i.e., those with d_i even) into pairs $\{i, i'\}$. Note that $d_{i'} = d_i$ necessarily holds and the restriction of $(\ ,\ )$ to $\mathbb{V}[i] \oplus \mathbb{V}[i']$ is nondegenerate. For $i \leqslant k$ such that d_i is odd we put $i' = i$.

Choose vectors $\{w_i\}_{i=1}^k$ according to Lemma 3.8. Since the form $(\ ,\ )$ is $\mathfrak{g}$-invariant, $(e^{d_i}{\cdot}w_i, v) = (-1)^{d_i}(w_i, e^{d_i}{\cdot}v)$ and $e^{d_i}{\cdot}w_i$ is orthogonal to all $e^s{\cdot}w_j$ with either $j \neq i'$ or $s > 0$. Since $(\ ,\ )$ is nondegenerate, we also have that $(e^{d_i}{\cdot}w_i, w_{i'}) \neq 0$ for all i.

3.7. Let $\alpha = \alpha(a_1, \dots, a_k) \in \widetilde{\mathfrak{g}}_e^*$ be as in (9) and assume that $\{a_i\} \subset \mathbb{K}$ are nonzero and pairwise distinct. Assume further that $a_{i'} = -a_i$ whenever $i \neq i'$. Then α vanishes on $(\widetilde{\mathfrak{g}}_e)_1$; see [29, Lemma 2]. By Lemma 3.7 and Proposition 3.1, $\tilde{\alpha} \in (\mathfrak{g}_e^*)_{\mathrm{reg}}$. Unfortunately, the linear function β defined in Proposition 3.2 does not always vanish on $(\widetilde{\mathfrak{g}}_e)_1$. For this reason, we need a more sophisticated construction.

Renumbering the $\mathbb{V}[i]$'s if necessary we may assume without loss of generality that $i' = i \pm 1$ for each pair $\{i, i'\}$. As $d_i = d_{i'}$, our assumption that $d_1 \geqslant d_2 \geqslant \ldots \geqslant d_k$ will not be violated. Note that if $i' \neq i+1$, then $i' \leqslant i$ and $(i+1)' \geqslant i+1$.

For each $i \leqslant k-1$ with $i' \neq i+1$ we now define a linear function γ_i on $\widetilde{\mathfrak{g}}_e$ by setting

$$\gamma_i(\xi) := -\frac{(w_{i+1}, e^{d_{i+1}} \cdot w_{(i+1)'})}{(e^{d_i} \cdot w_i, w_{i'})} c_{i'}^{(i+1)', d_{i+1}}(\xi) \qquad (\forall\, \xi \in \widetilde{\mathfrak{g}}_e),$$

and put $\beta' := \sum_{i \leqslant k-1,\, i' \neq i+1} \gamma_i$. Recall from (3.4) that the map ρ gives us a rational action of $\mathbb{K}^\times$ on $\widetilde{\mathfrak{g}}_e^*$. From Lemma 3.8 and the definition of β it is immediate that $\beta + \beta' \neq 0$.

Lemma 3.9. *For all $i \leqslant k-1$ with $i' \neq i+1$ we have $\rho(t)\gamma_i = t^{s_i}\gamma_i$ where $s_i \geqslant 2$. Moreover, $\beta + \beta'$ vanishes on $(\widetilde{\mathfrak{g}}_e)_1$.*

Proof. Recall that $(w_i, e^{d_i} \cdot w_{i'}) \neq 0$ for all i. Take any $\xi \in (\widetilde{\mathfrak{g}}_e)_1$. Then

$$c_{i+1}^{i,d_i}(\xi)(e^{d_i} \cdot w_i, w_{i'}) = (\xi(w_{i+1}), w_{i'}) = (w_{i+1}, \xi(w_{i'}) = c_{i'}^{(i+1)', d_{i+1}}(\xi)(w_{i+1}, e^{d_{i+1}} \cdot w_{(i+1)'}).$$

For $i' \neq i+1$ this yields $c_{i+1}^{i,d_i}(\xi) = -\gamma_i(\xi)$. Suppose $i' = i+1$. Then also $(i+1)' = i$ and $d_i = d_{i+1}$ is even, hence

$$c_{i+1}^{i,d_i}(\xi)(e^{d_i} \cdot w_i, w_{i+1}) = c_{i+1}^{i,d_i}(\xi)(w_{i+1}, e^{d_i} \cdot w_i) = -c_{i+1}^{i,d_i}(\xi)(e^{d_i} \cdot w_i, w_{i+1}).$$

So $c_{i+1}^{i,d_i}(\xi) = 0$ (recall that $\operatorname{char} \mathbb{K} \neq 2$). But then

$$(\beta + \beta')(\xi) = \sum_{i \leqslant k-1,\, i' \neq i+1} \big(c_{i+1}^{i,d_i}(\xi) + \gamma_i(\xi)\big) = 0.$$

It follows from (11) that $\rho(t)\gamma_i = t^{s_i}\gamma_i$, where $s_i = (i+1)' - i' + 1$. Since $i' \neq i+1$, we have $i' \leqslant i$ and $(i+1)' \geqslant i+1$. Then $s_i \geqslant i+1-i+1 = 2$. □

Combining Lemma 3.9 with [29, Lemma 2], we observe that any $\gamma \in \mathbb{K}\alpha \oplus \mathbb{K}(\beta + \beta')$ vanishes on $(\widetilde{\mathfrak{g}}_e)_1$. Let E denote the $\mathbb{K}$-span of $\tilde{\alpha}$ and $\widetilde{\beta + \beta'}$ in $\mathfrak{g}_e^*$.

Proposition 3.10. *Under the above assumptions,* $\dim E = 2$ *and* $E \cap (\mathfrak{g}_e^*)_{\mathrm{sing}} = 0$.

Proof. Let $\gamma = x\alpha + y(\beta + \beta')$ with $x, y \in \mathbb{K}$. By Lemma 3.9, $\rho(t)\gamma_i = t^{s_i}\gamma_i$, where $s_i \geqslant 2$, while in (3.4) it is shown that $\rho(t)\alpha = t\alpha$ and $\rho(t)\beta = \beta$. Since α and $\beta + \beta'$ are nonzero and $\rho(\mathbb{K}^\times)$ is diagonalisable, it follows that α and $\beta + \beta'$ are linearly independent. As both α and $\beta + \beta'$ vanish on $(\widetilde{\mathfrak{g}}_e)_1$, this yields that $\dim E = 2$.

The above discussion also shows that $\lim_{t \to 0} \rho(t)\gamma = y\beta$. If $y \neq 0$, then Proposition 3.2 gives $\gamma \in (\widetilde{\mathfrak{g}}_e^*)_{\mathrm{reg}}$. By Proposition 3.1, $\alpha \in (\widetilde{\mathfrak{g}}_e^*)_{\mathrm{reg}}$ as well. Then $\dim(\widetilde{\mathfrak{g}}_e)^\gamma = 2n = \operatorname{ind} \widetilde{\mathfrak{g}}_e$ for any nonzero $\gamma \in \mathbb{K}\alpha \oplus \mathbb{K}(\beta + \beta')$. As any such γ vanishes on $(\widetilde{\mathfrak{g}}_e)_1$, applying Lemma 3.7 we now conclude that $E \setminus \{0\} \subset (\mathfrak{g}_e^*)_{\mathrm{reg}}$. Equivalently, $E \cap (\mathfrak{g}_e^*)_{\mathrm{sing}} = 0$. □

The following is an immediate consequence of Proposition 3.10.

Theorem 3.11. *Let e be a nilpotent element of $\mathfrak{g} = \mathfrak{sp}_{2n}$, $n \geqslant 2$. Then $(\mathfrak{g}_e^*)_{\mathrm{sing}}$ has codimension $\geqslant 2$ in $\mathfrak{g}_e^*$.*

Proof. Straightforward (see the proof of Proposition 3.4). □

3.8. We shall see in a moment that there are nilpotent elements e in the orthogonal Lie algebra $\mathfrak{g} = \mathfrak{so}(\mathbb{V})$ for which $(\mathfrak{g}_e^*)_{\mathrm{sing}}$ has codimension 1 in $\mathfrak{g}_e^*$. But first we would like to give two positive examples.

Suppose $\dim \mathbb{V}$ is odd and let e be a nilpotent element in $\mathfrak{so}(\mathbb{V})$ with $2m+1$ Jordan blocks indexed by the integers ranging from $-m$ to m, where $m \geqslant 1$. Similar to the symplectic case we may assume that there is an involution $i \to i'$ on the set of indices such that $i' = i$ if and only if d_i is even and $(\mathbb{V}[i], \mathbb{V}[j]) = 0$ whenever $j \neq i'$. Recall that $d_{i'} = d_i$ necessarily holds.

Suppose that $i' = -i$ and $d_i \leqslant d_j$ for $i > j \geqslant 0$. Then d_0 is even and the other d_i are odd. Choose $\mathbb{K}[e]$-generators $w_i \in \mathbb{V}[i]$ such that $(w_i, e^{d_i} w_{-i}) = 1$ for $i \geqslant 0$, and let $\tilde{\alpha}$ denote the restriction to $\mathfrak{g}_e$ of the linear function α on $\mathfrak{gl}(V)_e$ given by

$$\alpha(\xi) = \sum_{i=-m+1}^{m} c_{i-1}^{i,d_i}(\xi) \qquad \big(\forall\, \xi \in \mathfrak{gl}(V)_e\big).$$

By [29, Section 4], this linear function is regular. Let $g \in \mathrm{GL}(\mathbb{V})$ be such that

$$g(w_i) = w_{-i} \quad \text{for } i \geqslant 0, \quad g(w_i) = -w_{-i} \text{ for } i < 0, \quad g(e^s \cdot w_i) = (-1)^s e^s \cdot w_i \quad \text{for } s \geqslant 1.$$

Then $g \in \mathrm{O}(\mathbb{V})$ and $(\mathrm{Ad}\, g)e = -e$, i.e., g normalises $\mathbb{K}e$. Hence $\mathrm{Ad}\, g$ acts on $\mathfrak{g}_e$ as a Lie algebra automorphism. Set $\tilde{\beta} := (\mathrm{Ad}^*\, g)\tilde{\alpha}$. In coordinates,

$$\tilde{\beta}(\xi) = \sum_{i=-m}^{m-1} c_{i+1}^{i,d_i}(\xi) - 2c_1^{0,d_0}(\xi) \qquad \big(\forall\, \xi \in \mathfrak{g}_e\big).$$

Set $E' := \mathbb{K}\tilde{\alpha} + \mathbb{K}\tilde{\beta}$, a subspace of $\mathfrak{g}_e^*$.

Lemma 3.12. *The subspace E' is 2-dimensional and $E' \cap (\mathfrak{g}_e^*)_{\mathrm{sing}} = 0$. The singular locus $(\mathfrak{g}_e^*)_{\mathrm{sing}}$ has codimension $\geqslant 2$ in $\mathfrak{g}_e^*$.*

Proof. By [29, Section 4], the function $\tilde{\alpha}$ is regular in $\mathfrak{g}_e^*$. Hence so is $\tilde{\beta} = (\mathrm{Ad}^* g)\tilde{\alpha}$. In particular, both $\tilde{\alpha}$ and $\tilde{\beta}$ are nonzero. There exists a cocharacter $a\colon \mathbb{K}^\times \to \mathrm{SO}(\mathbb{V})_e$ such that $a(t)w_i = t^i w_i$ for all i. It has the property that $(\mathrm{Ad}^* a(t))\tilde{\alpha} = t^{-1}\tilde{\alpha}$ and $(\mathrm{Ad}^* a(t))\tilde{\beta} = t\tilde{\beta}$. This implies that $\dim E' = 2$. Since the Zariski closed set $E' \cap (\mathfrak{g}_e^*)_{\mathrm{sing}} = 0$ is conical and $(\mathrm{Ad}^* a(\mathbb{K}^\times))$-stable and both $\tilde{\alpha}$ and $\tilde{\beta}$ are regular, it also follows that $E' \cap (\mathfrak{g}_e^*)_{\mathrm{sing}} = 0$. □

Suppose now that $\tilde{\mathbb{V}} = \mathbb{V} \oplus \mathbb{K}v$ is an even dimensional vector space such that $(\mathbb{V}, v) = 0$ and $(v, v) = 1$. Let $e \in \mathfrak{so}(\mathbb{V})$ be the same nilpotent element as above (one with $2m+1$ Jordan blocks and with $i' = -i$ for all i). We regard e as a nilpotent element of $\mathfrak{so}(\tilde{\mathbb{V}})$ by setting $e(v) = 0$. Then $e \in \mathfrak{so}(\tilde{\mathbb{V}})$ has $2m+2$ Jordan blocks. Assume that the new Jordan block of size 1 is indexed by M with $M > m$ and that v is its generator.

Lemma 3.13. *For e as above, the singular locus $(\mathfrak{g}_e^*)_{\mathrm{sing}}$ has codimension $\geqslant 2$ in $\mathfrak{g}_e^* = \mathfrak{so}(\tilde{\mathbb{V}})_e^*$.*

Proof. Note that $\mathfrak{so}(\mathbb{V})$ is a symmetric subalgebra of $\mathfrak{g} = \mathfrak{so}(\tilde{\mathbb{V}})$. Let $\mathfrak{so}(\tilde{\mathbb{V}}) = \mathfrak{so}(\mathbb{V}) \oplus \mathfrak{p}$ be the corresponding symmetric decomposition. Then we can identify the dual space of the centraliser $\mathfrak{so}(\mathbb{V})_e$ with the annihilator of $\mathfrak{p}_e := \mathfrak{p} \cap \mathfrak{g}_e$ in $\mathfrak{g}_e^*$. Let $\tilde{\alpha}$ and $\tilde{\beta}$ be the same linear functions as in Lemma 3.12. We view them as linear functions of $\mathfrak{g}_e$ vanishing on $\mathfrak{p}_e$. As $\mathrm{O}(\mathbb{V}) \hookrightarrow \mathrm{O}(\tilde{\mathbb{V}})$ and $\mathrm{SO}(\mathbb{V})_e \hookrightarrow G_e$, we still have that $\tilde{\beta} = (\mathrm{Ad}^* g)\tilde{\alpha}$ and $(\mathrm{Ad}^* a(t))\tilde{\alpha} = t^{-1}\tilde{\alpha}$, $(\mathrm{Ad}^* a(t))\tilde{\beta} = t\tilde{\beta}$ for the same cocharacter $a\colon \mathbb{K}^\times \to G_e$ as in Lemma 3.12. Therefore, in order to prove the statement it suffices to show that $\tilde{\alpha} \in (\mathfrak{g}_e^*)_{\mathrm{reg}}$. By construction,

$$\dim \mathfrak{g}_e^{\tilde{\alpha}} = (\dim \tilde{\mathbb{V}})/2 - 1 + \dim \{\xi \in \mathfrak{p}_e \mid \tilde{\alpha}([\xi, \mathfrak{p}_e]) = 0\}.$$

The linear space $\mathfrak{p}_e$ has a basis $\{\xi_i^{M,0} + \epsilon(i)\xi_M^{-i,d_i} \mid -m \leqslant i \leqslant m\}$ where $\epsilon(i) = -1$ for $i \geqslant 0$ and $\epsilon(i) = 1$ for $i < 0$. Using (10) we get

$$\begin{aligned} \tilde{\alpha}\big([\xi_i^{M,0} + \epsilon(i)\xi_M^{-i,d_i},\, \xi_j^{M,0} + \epsilon(j)\xi_M^{-j,d_j}]\big) &= 0 \text{ for } j \neq -i-1; \\ \tilde{\alpha}\big([\xi_i^{M,0} + \epsilon(i)\xi_M^{-i,d_i},\, \xi_{-i-1}^{M,0} + \epsilon(-i-1)\xi_M^{i+1,d_{i+1}}]\big) &= 2\epsilon(i), \quad -m \leqslant i \leqslant m. \end{aligned}$$

It follows that $\tilde{\alpha}$ induces on $\mathfrak{p}_e$ a skew-symmetric bilinear form of rank $2m$. But then

$$\dim \{\xi \in \mathfrak{p}_e \mid \tilde{\alpha}([\xi, \mathfrak{p}_e]) = 0\} = 1$$

and the statement follows. □

3.9. For any simple Lie algebra $\mathfrak{g}$ of type different from **A** and **C** we provide in this subsection a uniform construction of $e \in \mathcal{N}(\mathfrak{g})$ for which $(\mathfrak{g}_e^*)_{\mathrm{sing}}$ has codimension 1 in $\mathfrak{g}_e^*$. We assume for simplicity that $\mathrm{char}\,\mathbb{K} = 0$. The Lie algebras $\mathfrak{sl}_n$ and $\mathfrak{sp}_{2n}$ are distinguished by the property that their highest root is not a fundamental dominant weight. This seemingly insignificant fact is the source of many structural differences between $\mathfrak{sl}_n$ and $\mathfrak{sp}_{2n}$, and the other finite dimensional simple Lie algebras. In our situation, it manifests itself as follows.

Let $G{\cdot}\tilde{e} = \mathcal{O}_{\min}$ be the minimal nilpotent orbit in $\mathfrak{g}$ and $\{\tilde{e}, \tilde{h}, \tilde{f}\}$ an $\mathfrak{sl}_2$-triple. Consider the $\mathbb{Z}$-grading determined by $\tilde{h}$

$$\mathfrak{g} = \bigoplus_{i=-2}^{2} \mathfrak{g}(i).$$

Here $\mathfrak{g}(2) = \mathbb{K}\tilde{e}$ and $\mathfrak{g}(-2) = \mathbb{K}\tilde{f}$. Let $G(0)$ denote the stabiliser of $\tilde{h}$ in G. This is a Levi subgroup of G which acts on $\mathfrak{g}(1)$ with finitely many orbits. If $\mathfrak{g} \neq \mathfrak{sl}_n$, then the centre of $\mathfrak{g}(0)$ is one-dimensional and $\mathfrak{g}(1)$ is a simple $\mathfrak{g}(0)$-module. Furthermore, if $\mathfrak{g} \neq \mathfrak{sp}_{2n}$, then the open $G(0)$-orbit in $\mathfrak{g}(1)$ is *affine*. Let $e \in \mathfrak{g}(1)$ be a point in this orbit.

From now on we assume in this subsection that $\mathfrak{g}$ is not isomorphic to $\mathfrak{sl}_n$ or $\mathfrak{sp}_{2n}$. Our goal is to prove that $(\mathfrak{g}_e^*)_{\mathrm{sing}}$ has codimension 1 in $\mathfrak{g}_e^*$. Set $\mathfrak{l} = [\mathfrak{g}(0), \mathfrak{g}(0)]$ and let K denote the stationary subgroup of e in $G(0)$. Then $\mathfrak{k} := \mathrm{Lie}\, K$ is a Lie subalgebra of $\mathfrak{l}$ acting trivially on $\mathfrak{g}(2)$. The centraliser $\mathfrak{g}_e$ is graded and has the following structure. Its component of degree 0 is $\mathfrak{k}$ and its component of degree 1 is isomorphic as a $\mathfrak{k}$-module to $\mathbb{K}e \oplus W \oplus W^*$, where W is a $\mathfrak{k}$-module of dimension $\frac{\dim \mathfrak{g}(1)}{2} - 1$. The component of degree 2 is still $\mathbb{K}\tilde{e}$.

Consider the hyperplane $\mathcal{H} = \{\gamma \in \mathfrak{g}_e^* \mid (\gamma(\tilde{e}) = 0\}$. We wish to prove that $\mathcal{H} \subset (\mathfrak{g}_e^*)_{\rm sing}$. Because $\tilde{e}$ acts trivially on $\mathcal{H}$, the representation of $\mathfrak{g}_e/\mathbb{K}\tilde{e}$ in $\mathcal{H}$ is equivalent to the coadjoint representation of $\mathfrak{g}_e/\mathbb{K}\tilde{e}$. That is, we have to compute the index of this Lie algebra. Modulo the trivial direct summand $\mathbb{K}e$, this algebra is the semi-direct product of $\mathfrak{k}$ and $W \oplus W^*$, denoted $\mathfrak{k} \ltimes (W \oplus W^*)$. For such semi-direct products, one can use Raïs' formula for the index [21]. As the generic stabiliser for the representation of $\mathfrak{k}$ on $W \oplus W^*$, say $\mathfrak{s}$, is reductive, Raïs' formula yields

$$\operatorname{ind}(\mathfrak{k} \ltimes (W \oplus W^*)) = \operatorname{rk}\mathfrak{s} + \dim(W \oplus W^*)/\!\!/K.$$

It turns out that in all cases of interest for us this number equals $\operatorname{rk}\mathfrak{g}$. Taking into account the direct summand $\mathbb{K}e$ and the passage to $\mathcal{H}$, we see that generic G_e-orbits in $\mathcal{H}$ are of codimension $\operatorname{rk}\mathfrak{g} + 2$ in $\mathfrak{g}_e^*$. On the other hand, it is straightforward to see that for any linear function $\gamma \in \mathfrak{g}_e^* \setminus \mathcal{H}$ satisfying $\gamma_{|\mathfrak{g}_e(1)} = 0$ one has $\dim(\mathfrak{g}_e)_\gamma = \dim \mathfrak{k}_\gamma + 2$. As $\mathfrak{k}$ is reductive with $\operatorname{rk}\mathfrak{k} = \operatorname{rk}\mathfrak{g} - 2$, this implies that $\operatorname{ind}\mathfrak{g}_e = \operatorname{rk}\mathfrak{g}$. Then $\mathcal{H} \subset (\mathfrak{g}_e^*)_{\rm sing}$, as wanted.

In Tables 2 and 3, we provide the necessary information related to these computations. In Table 2, W is always a simple $\mathfrak{k}$-module which is represented by its highest weight.

TABLE 2. DATA FOR THE EXCEPTIONAL LIE ALGEBRAS

$\mathfrak{g}$	$\mathfrak{l}$	$\mathfrak{k}$	W	$\dim W$	$\mathfrak{s}$	$\dim(W \oplus W^*)/\!\!/K$	$\operatorname{ind}(\mathfrak{k} \ltimes (W \oplus W^*))$
$\mathbf{E}_6$	$\mathbf{A}_5$	$2\mathbf{A}_2$	$\varpi_1 + \varpi_1'$	9	T_2	4	6
$\mathbf{E}_7$	$\mathbf{D}_6$	$\mathbf{A}_5$	ϖ_2	15	$(\mathbf{A}_1)^3$	4	7
$\mathbf{E}_8$	$\mathbf{E}_7$	$\mathbf{E}_6$	ϖ_1	27	$\mathbf{D}_4$	4	8
$\mathbf{F}_4$	$\mathbf{C}_3$	$\mathbf{A}_2$	$2\varpi_1$	6	0	4	4
$\mathbf{G}_2$	$\mathbf{A}_1$	0	0	1	0	2	2

TABLE 3. DATA FOR $\mathfrak{so}_n$, $n \geqslant 7$

$\mathfrak{g}$	$\mathfrak{l}$	$\mathfrak{k}$	$\dim W$	$\mathfrak{s}$	$\dim(W\oplus W^*)/\!\!/K$	$\operatorname{ind}(\mathfrak{k}\ltimes(W\oplus W^*))$
$\mathfrak{so}_7$	$\mathfrak{so}_3\times\mathfrak{sl}_2$	$\mathfrak{t}_1$	2	0	3	3
$\mathfrak{so}_n$ $(n\geqslant 8)$	$\mathfrak{so}_{n-4}\times\mathfrak{sl}_2$	$\mathfrak{so}_{n-6}\times\mathfrak{t}_1$	$n-5$	$\mathfrak{so}_{n-8}$	4	$[n/2]$

3.10. Adopt the notations and conventions of (3.9) and let $\tilde{e}$ be an element in the minimal nilpotent orbit $\mathcal{O}_{\min}$. Then $\operatorname{ind}\mathfrak{g}_{\tilde{e}} = \operatorname{rk}\mathfrak{g}$ by [17]. We now wish to investigate the singular locus of $\mathfrak{g}_{\tilde{e}}^*$.

Theorem 3.14. *If* $\operatorname{rk}\mathfrak{g} \geqslant 2$ *and* $\tilde{e} \in \mathcal{O}_{\min}$, *then* $(\mathfrak{g}_{\tilde{e}}^*)_{\rm sing}$ *has codimension* $\geqslant 2$ *in* $\mathfrak{g}_{\tilde{e}}^*$.

Proof. In view of Theorems 3.4 and 3.11 the statement holds when $\mathfrak{g}$ is of type **A** or **C**. So we may assume in this proof that $\mathfrak{g}$ is not isomorphic to $\mathfrak{sl}_n$ or $\mathfrak{sp}_{2n}$. Then

$$\mathfrak{g}_{\tilde{e}} = \mathfrak{l} \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2).$$

Since $\dim \mathfrak{g}(2) = 1$, we have a skew-symmetric bilinear form $\langle\,\cdot\,,\cdot\,\rangle$ on $\mathfrak{g}(1)$ such that $[x,y] = \langle x,y\rangle\tilde{e}$ for all $x,y \in \mathfrak{g}(1)$. This form is nondegenerate.

Given a subset $X \subset \mathfrak{g}_{\tilde{e}}$ we denote by $\mathrm{Ann}(X)$ the annihilator of X in $\mathfrak{g}^*_{\tilde{e}}$, that is

$$\mathrm{Ann}(X) := \{\gamma \in \mathfrak{g}^*_{\tilde{e}} \mid \gamma(X) = 0\}.$$

Then $\mathrm{Ann}(\tilde{e}) := \mathrm{Ann}(\{\tilde{e}\})$ is a hyperplane in $\mathfrak{g}^*_{\tilde{e}}$. We claim that $\mathrm{Ann}(\tilde{e}) \not\subset (\mathfrak{g}^*_{\tilde{e}})_{\mathrm{sing}}$. To prove the claim we are going to argue in the spirit of (3.9).

Let L denote the derived subgroup of $G(0)$. Since $\tilde{e}$ acts trivially on $\mathrm{Ann}(\tilde{e})$, the representation of $\mathfrak{g}_{\tilde{e}}/\mathbb{K}\tilde{e}$ in $\mathrm{Ann}(\tilde{e})$ is equivalent to the coadjoint representation of $\mathfrak{g}_{\tilde{e}}/\mathbb{K}\tilde{e}$. This Lie algebra is the semi-direct product of $\mathfrak{l}$ and $\mathfrak{g}(1)$, denoted $\mathfrak{l} \ltimes \mathfrak{g}(1)$. The generic stabiliser for the representation of $\mathfrak{l}$ on $\mathfrak{g}(1)$ is isomorphic to $\mathfrak{k}$. Since $\mathfrak{k}$ is reductive, Raïs' formula [21] yields

$$\mathrm{ind}\,(\mathfrak{l} \ltimes \mathfrak{g}(1)) = \mathrm{rk}\,\mathfrak{k} + \dim \mathfrak{g}(1)/\!\!/L.$$

As the complement $\mathfrak{g}(1) \setminus G(0){\cdot}e$ is a hypersurface in $\mathfrak{g}(1)$ and the semisimple group L has codimension 1 in $G(0)$, the orbit $L{\cdot}e$ has codimension 1 in $\mathfrak{g}(1)$. This implies that $\dim \mathfrak{g}(1)/\!\!/L = 1$, whereas Tables 2 and 3 yield $\mathrm{rk}\,\mathfrak{k} = \mathrm{rk}\,\mathfrak{g} - 2$. Therefore, $\mathrm{ind}\,(\mathfrak{l} \ltimes \mathfrak{g}(1)) = \mathrm{rk}\,\mathfrak{g} - 1$. Each $\gamma \in \mathrm{Ann}(\tilde{e})$ may be regarded as a linear function on $\mathfrak{l} \ltimes \mathfrak{g}(1)$. Moreover, it is easy to see that $\mathfrak{g}^\gamma_{\tilde{e}} \cong \mathbb{K}\tilde{e} \oplus (\mathfrak{l} \ltimes \mathfrak{g}(1))^\gamma$ for every $\gamma \in \mathrm{Ann}(\tilde{e})$. This implies that for a generic $\gamma \in \mathrm{Ann}(\tilde{e})$ we have $\dim \mathfrak{g}^\gamma_{\tilde{e}} = \mathrm{rk}\,\mathfrak{g} = \mathrm{ind}\,\mathfrak{g}_{\tilde{e}}$. The claim follows.

It remains to deal with the affine open set $Y := \mathfrak{g}^*_{\tilde{e}} \setminus \mathrm{Ann}(\tilde{e})$. Set $\mathfrak{n} := \mathfrak{g}(1) \oplus \mathfrak{g}(2)$ and let $N \subset G_{\tilde{e}}$ be the connected subgroup of G with $\mathrm{Lie}\,N = \mathfrak{n}$. The derived subgroup (N,N) is 1-dimensional with $\mathrm{Lie}\,(N,N) = \mathbb{K}\tilde{e}$, and $N/(N,N) \cong \mathfrak{g}(1)$ as varieties. Let $\alpha \in \mathrm{Ann}(\mathfrak{l} \oplus \mathfrak{g}(1))$ be a nonzero function. The set $\mathrm{Ann}(\mathfrak{g}(1)) \cap Y$ is Zariski closed in Y and can be identified with $\mathfrak{l}^* \oplus \mathbb{K}^\times\alpha$. Let $\gamma = \beta + a\alpha$ be an element of $\mathrm{Ann}(\mathfrak{g}(1)) \cap Y$ with $\beta \in \mathfrak{l}^*$ and $a \neq 0$. Then

$$(\mathrm{Ad}^*N)\gamma = \Big\{\beta + \frac{a}{2}(\mathrm{ad}^*v)^2\alpha + a(\mathrm{ad}^*v)\alpha + a\alpha \mid v \in \mathfrak{g}(1)\Big\}.$$

Since the form $\langle\,\cdot\,,\cdot\,\rangle$ is nondegenerate, it follows that the N-saturation of $Y \cap \mathrm{Ann}(\mathfrak{g}(1))$ is equal to Y, that each N-orbit $(\mathrm{Ad}^*N)\gamma$ is isomorphic to $N/(N,N) \cong \mathfrak{g}(1)$, and that $\mathfrak{g}^\gamma_{\tilde{e}} = \mathfrak{l}_\beta \oplus \mathbb{K}\tilde{e}$. In particular, the action morphism

$$\tau\colon\ \big(N/(N,N)\big) \times \big(\mathrm{Ann}(\mathfrak{g}(1)) \cap Y\big) \to Y$$

is an isomorphism. Suppose $g \in N/(N,N)$ and $\gamma = \beta + a\alpha$, where $\beta \in \mathfrak{l}^*$ and $a \neq 0$. Then $\tau((g,\gamma)) \in (\mathfrak{g}^*_{\tilde{e}})_{\mathrm{reg}}$ if and only if $\beta \in (\mathfrak{l}^*)_{\mathrm{reg}}$. Since $(\mathfrak{l}^*)_{\mathrm{sing}}$ has codimension 3 in $\mathfrak{l}^*$, the intersection $(\mathfrak{g}^*_{\tilde{e}})_{\mathrm{sing}} \cap Y$ is of codimension 3 in Y and also in $\mathfrak{g}^*_{\tilde{e}}$. The result follows. $\square$

4. Degrees of basic invariants

4.1. From now on we assume that $\mathrm{char}\,\mathbb{K} = 0$. Let Q be a connected linear algebraic group with Lie algebra $\mathfrak{q}$. Suppose we are given a rational linear action of Q on a vector space V. The differential of this action at the identity element of Q is a representation of the Lie algebra $\mathfrak{q}$ in V.

Definition 4.1. A vector $x \in V$ (a stabiliser $\mathfrak{q}_x$) is called *a generic point* (*a generic stabiliser*), if there exists a Zariski open subset $U \subset V$ such that $x \in U$ and $\mathfrak{q}_x$ is Q-conjugate to any $\mathfrak{q}_y$ with $y \in U$.

Let e be a nilpotent element in $\mathfrak{g} = \mathfrak{gl}(\mathbb{V})$ and set $G := \mathrm{GL}(\mathbb{V})$. Let $\alpha = \alpha(a_1, \dots, a_k) \in \mathfrak{g}_e^*$ be as in (3.3) and put $\mathfrak{h} := \mathfrak{g}_e^\alpha$.

Proposition 4.1 ([29]). *If all $a_1, \dots, a_k$ are nonzero and pairwise distinct, then $\mathfrak{h}$ is a generic stabiliser for the coadjoint representation of G_e.*

For $1 \leqslant i \leqslant n$, let Δ_i denote the sum of the principal minors of order i of the generic matrix $(x_{ij})_{1 \leqslant i,j \leqslant n}$, a regular function on $\mathfrak{g}$, and set $F_i := \kappa^{-1}(\Delta_i)$. It is well-known that $\{F_1, \dots, F_n\}$ is a generating set of the invariant algebra $\mathcal{S}(\mathfrak{g})^G$. Recall from (0.2) the definition of ${}^eF_1, \dots, {}^eF_n$. Let $(d_1 + 1 \geqslant d_2 + 1 \geqslant \dots \geqslant d_k + 1)$ be the partition of n corresponding to e and put $d_0 = 0$.

Theorem 4.2. *Suppose $\dim \mathbb{V} \geqslant 2$ and let $F_1, \dots F_n$ be as above. Then $\{F_1, \dots, F_n\}$ is a very good generating system for e and $\mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e} = \mathcal{S}(\mathfrak{g}_e)^{G_e} = \mathbb{K}[{}^eF_1, \dots, {}^eF_n]$. Moreover,*

$$\deg({}^eF_{d_0+\dots+d_i+i+1}) = \dots = \deg({}^eF_{d_0+\dots+d_i+d_{i+1}+i+1}) = i + 1 \qquad (0 \leqslant i \leqslant k-1).$$

Proof. Let α be as in Proposition 4.1 and and let $\mathfrak{r}$ be the linear span of all $\xi_i^{j,s}$ with $i \neq j$. Let $\mathfrak{t}$ be the span of all $\xi_i^{i,0}$, a maximal toral subalgebra of $\mathfrak{g}_e$. Then the centraliser $\mathfrak{h} = \mathfrak{c}_{\mathfrak{g}_e}(\mathfrak{t})$ is an abelian Cartan subalgebra of $\mathfrak{g}_e$. Moreover, $\mathfrak{g}_e = \mathfrak{h} \oplus \mathfrak{r}$ and $[\mathfrak{h}, \mathfrak{r}] = \mathfrak{r}$ (this follows from the formula displayed in the proof of Proposition 2 in [29]). We identify $\mathfrak{h}^*$ with $\mathrm{Ann}(\mathfrak{r}) \subset \mathfrak{g}_e^*$. The above implies that $\mathfrak{h}^* = \{\gamma \in \mathfrak{g}_e^* \mid (\mathrm{ad}^* \mathfrak{h})\gamma = 0\}$. Since $\mathfrak{h}$ is a generic stabiliser, we have $\overline{G_e \cdot \mathfrak{h}^*} = \mathfrak{g}_e^*$. Therefore, the restriction map $\varphi \mapsto \varphi_{|\mathfrak{h}^*}$ induces an embedding $\mathbb{K}[\mathfrak{g}_e^*]^{G_e} \hookrightarrow \mathbb{K}[\mathfrak{h}^*]$. It follows that each ${}^eF_{i|\mathfrak{h}^*}$ is nonzero and hence has the same degree as ${}^eF_i$.

Let $\mathfrak{r}^\perp \subset \mathfrak{g}$ be the orthogonal complement to $\mathfrak{r}$ with respect to κ and $\mathfrak{s} := \mathcal{S}_e \cap \mathfrak{r}^\perp$. Then $\mathfrak{s} = e + (\kappa_e^*)^{-1}(\mathrm{Ann}\,\mathfrak{r})$, implying that ${}^eF_{i|\mathfrak{h}^*} = {}^eF_{i|\mathrm{Ann}\,\mathfrak{r}}$ is equal to the component of minimal degree of the restriction of Δ_i to $\mathfrak{s}$. Let $\mathfrak{g}[i] \cong \mathfrak{gl}(\mathbb{V}[i])$ be the subalgebra of $\mathfrak{g}$ consisting of all endomorphisms acting trivially on $\mathbb{V}[j]$ for $j \neq i$, and $\hat{\mathfrak{g}} := \mathfrak{g}[1] \oplus \dots \oplus \mathfrak{g}[k]$. Then $\hat{\mathfrak{g}}$ is a Levi subalgebra of $\mathfrak{g}$ and $\mathfrak{s} \subset \hat{\mathfrak{g}}$.

For $1 \leqslant \ell \leqslant d_j + 1$ we denote by $\Delta_\ell[j]$ the sum of all principal minors of order ℓ of the generic matrix $\left(x_{pq}^{(j)}\right)_{1 \leqslant p,q \leqslant d_j+1}$, a homogeneous element of degree ℓ in $\mathbb{K}[\hat{\mathfrak{g}}]$, and put $\Delta_0[j] = 1$. Since the characteristic polynomial of a block-diagonal matrix is the product of the characteristic polynomials of its blocks, it follows that

$$\Delta_{\ell|\hat{\mathfrak{g}}} = \sum_{\ell_1 + \dots \ell_k = \ell} \Delta_{\ell_1}[1] \cdots \Delta_{\ell_k}[k] \qquad (1 \leqslant \ell \leqslant n).$$

As the restriction of e to $\mathbb{V}[i]$ is a regular nilpotent element of $\mathfrak{gl}(\mathbb{V}[i])$, we have inequality $\deg {}^e(\kappa^{-1}(\Delta_{\ell_i}[i])) \geqslant 1$ whenever $\ell_i > 0$. Hence $\deg {}^eF_\ell \geqslant q$, where q is the minimal number

for which there exists a decomposition $\ell = t_1 + \ldots + t_q$ with $0 < t_i \leqslant d_i + 1$. More precisely,

$$\begin{aligned} \deg {}^eF_i &= 1 \quad \text{for } i = 1, \ldots, d_1 + 1, \\ \deg {}^eF_i &\geqslant 2 \quad \text{for } i = d_1 + 2, \ldots, d_1 + d_2 + 2, \\ \deg {}^eF_i &\geqslant 3 \quad \text{for } i = d_1 + d_2 + 3, \ldots, d_1 + d_2 + d_3 + 3, \end{aligned}$$

and so on. Consequently, $\sum_{i=1}^n \deg {}^eF_i \geqslant \sum_{i=1}^k i(d_i+1)$. On the other hand, the well-known formula for $\dim \mathfrak{g}_e$ implies that

$$\dim \mathfrak{g}_e = \sum_{i=1}^k (2i-1)(d_i+1) = 2\sum_{i=1}^k i(d_i+1) - n;$$

see [13] or [3, p. 398] for example. In view of Theorem 2.1(i) we must have the equalities throughout, and $\sum_{i=1}^n \deg {}^eF_i = (\dim \mathfrak{g}_e + n)/2$.

As $\operatorname{ind} \mathfrak{g}_e = n$ by [29], Theorem 2.1(i) yields that the generating set $\{F_i \mid 1 \leqslant i \leqslant n\}$ is good for e. Combining Theorem 2.1(iii) with Theorem 3.4 shows that this set is actually *very* good for e. But then $\mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e} = \mathcal{S}(\mathfrak{g}_e)^{G_e} = \mathbb{K}[{}^eF_1, \ldots, {}^eF_n]$ in view of Theorem 2.2. $\square$

The degrees of ${}^eF_1, \ldots, {}^eF_n$ can be read off the Young diagram of e, as shown in Figure 1.

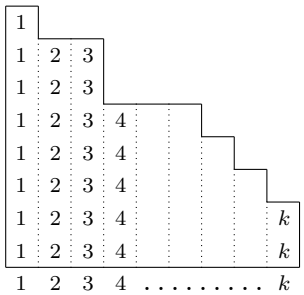


Figure 1

4.2. In this subsection we give a description of ${}^eF_i$ in terms of $\xi_i^{j,s}$. No generality will be lost by assuming that $h \cdot w_i = -d_i w_i$ for $1 \leqslant i \leqslant k$ and $f(e^s \cdot w_i) \in \mathbb{K}(e^{s-1} \cdot w_i)$. Then each $\xi_i^{j,s}$ is an eigenvector for $\operatorname{ad} h$. More precisely, using our discussion in (3.2) it easy to observe that

$$(\operatorname{ad} h)(\xi_i^{j,s}) \;=\; (d_i - d_j + 2s)\xi_i^{j,s}. \tag{12}$$

Given a subset $I \subset \{1, \ldots, k\}$ we denote by $|I|$ the cardinality of I. Given a permutation σ of $I = \{i_1, \ldots, i_m\}$ and a nonnegative function $\bar{s}\colon\ I \to \mathbb{Z}_{\geqslant 0}$ we associate with the triple $(I, \sigma, \bar{s})$ the monomial

$$\Xi(I, \sigma, \bar{s}) \;:=\; \xi_{i_1}^{\sigma(i_1), \bar{s}(i_1)} \xi_{i_2}^{\sigma(i_2), \bar{s}(i_2)} \ldots \xi_{i_m}^{\sigma(i_m), \bar{s}(i_m)} \in \mathcal{S}(\mathfrak{g}_e)$$

of degree $m = |I|$. For every $\Xi = \Xi(I, \sigma, \bar{s})$ we denote by $\lambda(I, \sigma, \bar{s})$ the weight of Ξ with respect to h. Obviously, $\lambda(I, \sigma, \bar{s})$ is the sum of the $\operatorname{ad} h$-eigenvalues of the factors $\xi_{i_j}^{\sigma(i_j), \bar{s}(i_j)}$ of Ξ.

Lemma 4.3. *For each $\ell \leqslant n$, we have*

$$ {}^eF_\ell \;=\; \sum_{|I|=m,\ \lambda(I,\sigma,\bar{s})=2(\ell-m)} a(I,\sigma,\bar{s})\,\Xi(I,\sigma,\bar{s}) $$

for some $a(I,\sigma,\bar{s}) \in \mathbb{K}$.

Proof. 1) Fix a basis $\{y_1,\dots,y_n\} = \{w_1, e{\cdot}w_1,\dots,e^{d_1}{\cdot}w_1, w_2,\dots,w_k,\dots,e^{d_k}{\cdot}w_k\}$ of V and let $E_{ij} \in \mathfrak{gl}(V)$ be such that $E_{ij}(y_k) = \delta_{jk}y_i$ for all $1 \leqslant i,j,k \leqslant n$. View F_ℓ as a polynomial in *variables* E_{ij} and let T be a monomial of F_ℓ for which $\deg {}^eT = \deg {}^eF_\ell$. It can be presented as a product $T = T_1\cdots T_k$, where each T_q involves only only those E_{ij} annihilating $\bigoplus_{t\neq q}\mathbb{V}[t]$. If E_{ij} is such a variable with $j \neq i-1$, then the restriction of E_{ij} to $\kappa(\mathcal{S}_e)$ is either zero or proportional to some $\xi_q^{u,s}$. Note also that if $y_i = e^{d_q}{\cdot}w_q$, i.e., if $y_{i+1} \notin \mathbb{V}[q]$, then the restriction of $E_{i+1,i}$ to $\kappa(\mathcal{S}_e)$ equals $\xi_q^{q+1,d_{q+1}}$. So when we restrict T to $\kappa(\mathcal{S}_e)$, nonzero constants (terms of degree 0) will arise only from those variables $E_{i+1,i}$ with $y_{i+1} \in \mathbb{V}[q]$. But all such variables lie under the main diagonal and the monomial T comes from a principal minor, hence T_q cannot contain only them. Thus, if $\deg T_q > 0$, then either $T_{q|\kappa(\mathcal{S}_e)}$ is zero or $\deg {}^eT_q \geqslant 1$.

On the other hand, $\sum \deg {}^eT_q = \deg {}^eF_\ell$ and each $T_{q|\kappa(\mathcal{S}_e)}$ is nonzero, by our assumption on T. Let $d(T)$ denote the cardinality of $\{q \leqslant k \mid \deg T_q > 0\}$. The above discussion shows that $\deg {}^eT \geqslant d(T)$. Since $\deg T_q \leqslant d_q + 1$ and $\sum \deg T_q = \deg F_\ell$, our discussion in (4.1) yields $\deg {}^eF_\ell \leqslant d(T)$. Hence $\deg {}^eT = d(T)$, forcing $\deg {}^eT_j \leqslant 1$ for all $1 \leqslant j \leqslant k$. This means that each monomial of ${}^eF_\ell$, when expressed via $\{\xi_i^{j,s}\}$, has no factors of the form $\xi_q^{j,s}\xi_q^{i,t}$.

2) Let $\Xi = \xi_{i_1}^{j_1,s_1}\dots\xi_{i_m}^{j_m,s_m}$ be a monomial involved in ${}^eF_\ell$. In part 1) we have proved that all indices $i_1,\dots,i_m$ are distinct. Let $I = \{i_1,\dots,i_m\}$. Suppose there is $j = j_q$ with $j \notin I$. Then Ξ has a positive weight with respect to the semisimple element $\xi_j^{j,0} \in \mathfrak{g}_e$. But ${}^eF_\ell$ is invariant under $\mathfrak{g}_e$, hence Ξ must be of weight zero. This contradiction shows that $j \in I$. Similarly, each i_q must be among $j_1,\dots,j_m$. In other words, $(j_1,\dots,j_m)$ is a permutation of $(i_1,\dots,i_m)$.

3) Since all $\xi_i^{j,s}$ are eigenvectors for $\operatorname{ad} h$, each monomial Ξ involved in ${}^eF_\ell$ has the same weight as ${}^eF_\ell$ itself. Since F_ℓ is h-invariant and f has weight -2, we see that each Ξ has weight $2(\ell-m)$. This completes the proof. □

Conjecture 4.1. *Up to a nonzero constant,*

$$ {}^eF_\ell \;=\; \sum_{|I|=m,\ \lambda(I,\sigma,\bar{s})=2(\ell-m)} (\operatorname{sgn}\sigma)\,\Xi(I,\sigma,\bar{s}), $$

where $m = \deg {}^eF_\ell$.

4.3. In this subsection we use the notation of (3.6) and consider a nilpotent element e of the symplectic Lie algebra $\mathfrak{g} = \widetilde{\mathfrak{g}}_0$ (recall that $\widetilde{\mathfrak{g}}_0 = \widetilde{\mathfrak{g}}^\sigma$ where $\widetilde{\mathfrak{g}} = \mathfrak{gl}(\mathbb{V})$ and $\dim \mathbb{V} = 2n$). It is well-known that $\Delta_{2i|\mathfrak{g}}$ with $1 \leqslant i \leqslant n$ generate the invariant algebra $\mathbb{K}[\mathfrak{g}]^{\mathfrak{g}}$ and the regular functions Δ_{2i+1} vanish on $\mathfrak{g}$. For $1 \leqslant i \leqslant n$ we denote by δ_i the component of

minimal degree of the restriction of Δ_{2i} to $\mathcal{S}_e = e + \mathfrak{g}_f$. Since $e + \mathfrak{g}_f$ is an affine subspace of $e + \widetilde{\mathfrak{g}}_f$ and $\mathfrak{g}_e^*$ identifies with the linear subspace $\kappa_e^*(\mathfrak{g}_f)$ of $\widetilde{\mathfrak{g}}_e^* = \kappa_e^*(\widetilde{\mathfrak{g}}_f)$, one observes easily that either $\deg \delta_i = \deg {}^eF_{2i}$ or the restriction of ${}^eF_{2i}$ to $\mathfrak{g}_e^*$ is zero and $\deg \delta_i > \deg {}^eF_{2i}$.

For $1 \leqslant i \leqslant n$ we denote by $\bar{F}_i \in \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ the preimage of $\Delta_{i|\mathfrak{g}} \in \mathbb{K}[\mathfrak{g}]^{\mathfrak{g}}$ under the Killing isomorphism $\mathcal{S}(\mathfrak{g}) \xrightarrow{\sim} \mathbb{K}[\mathfrak{g}]$. Note that $\deg {}^e\bar{F}_{2i} = \deg \delta_i$ for all $i \leqslant n$.

Theorem 4.4. *Suppose* $\dim \mathbb{V} = 2n \geqslant 4$ *and let* $F_1, \ldots, F_{2n}$ *be as above. Then* $\{\bar{F}_{2i} \mid 1 \leqslant i \leqslant n\}$ *is a very good generating system for any* $e \in \mathfrak{g} \cong \mathfrak{sp}_{2n}$ *and* $\mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e} = \mathcal{S}(\mathfrak{g}_e)^{G_e} = \mathbb{K}[{}^e\bar{F}_2, \ldots, {}^e\bar{F}_{2n}]$. *Furthermore,* $\deg {}^e\bar{F}_{2i} = \deg {}^eF_{2i}$ *for all* $i \leqslant n$.

Proof. From Theorem 4.2 and the formula for $\dim \mathfrak{g}_e$ given in [13, 3.1(3)] we deduce that

$$\dim \mathfrak{g}_e = \tfrac{1}{2}\left(\dim \widetilde{\mathfrak{g}}_e + \sum_{i,\, d_i \text{ even}} 1\right) = \sum_{j=1}^{2n} \deg {}^eF_j - n + \sum_{i,\, i'=i+1} 1\,.$$

On the other hand, applying Theorem 4.2 to $\widetilde{\mathfrak{g}}_e$ yields

$$\sum_{j=1}^{n} \deg {}^eF_{2j} = \sum_{i,\, d_i \text{ odd}} \tfrac{i(d_i+1)}{2} + \sum_{i,\, i'=i+1} \left(i\tfrac{d_i}{2} + (i+1)\tfrac{d_i+2}{2}\right) = \tfrac{1}{2}\left(\sum_{j=1}^{2n} \deg {}^eF_j\right) + \sum_{i,\, i'=i+1} \tfrac{1}{2}\,.$$

To check this equality one takes the Young diagram of shape $(d_1 + 1 \geqslant \cdots \geqslant d_k + 1)$ with all boxes in the j-th column labelled j (as shown in Figure 1) and then sums up all labels assigned to the *even* boxes of the diagram (counted from bottom to top and from left to right). One should also keep in mind that $d_i = d_{i'}$ for all i and $d_i + 1$ is odd whenever $i' \neq i$. Using the above formulae one obtains

$$2\sum \deg {}^eF_{2i} - \dim \mathfrak{g}_e = \sum_{i \leqslant k,\ i'=i} \frac{d_i+1}{2} + \sum_{i \leqslant k,\ i'=i+1} (d_i+1) = n.$$

Since $\deg \delta_i \geqslant \deg {}^eF_{2i}$ for all $i \leqslant n$, by our earlier remarks, we now derive

$$\sum_{i=1}^{n} \deg {}^e\bar{F}_{2i} = \sum_{i=1}^{n} \deg \delta_i \geqslant \sum_{i=1}^{n} \deg {}^eF_{2i} = (\dim \mathfrak{g}_e + n)/2.$$

On the other hand, $\{\bar{F}_{2i} \mid 1 \leqslant i \leqslant n\}$ is a generating system for $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$. As $\operatorname{ind} \mathfrak{g}_e = \operatorname{rk} \mathfrak{g} = n$ by [29], Theorem 2.1(i) shows that $\sum_{i=1}^{n} \deg {}^e\bar{F}_{2i} \leqslant (\dim \mathfrak{g}_e + n)/2$. Hence $\deg {}^e\bar{F}_{2i} = \deg {}^eF_{2i}$ for all i and $\{\bar{F}_{2i} \mid 1 \leqslant i \leqslant n\}$ is a good generating system for e. Combining Theorem 2.1(iii) with Theorem 3.11, we now see that the generating set $\{\bar{F}_{2i} \mid 1 \leqslant i \leqslant n\}$ is very good for e. Then Theorem 2.2 yields $\mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e} = \mathcal{S}(\mathfrak{g}_e)^{G_e} = \mathbb{K}[{}^e\bar{F}_2, \ldots, {}^e\bar{F}_{2n}]$, completing the proof. □

4.4. Now suppose $\mathfrak{g} = \mathfrak{so}(\mathbb{V})$. Recall that $\mathfrak{g}$ is a symmetric subalgebra of $\widetilde{\mathfrak{g}} = \mathfrak{gl}(\mathbb{V})$. Let $F_1, \ldots, F_n$ be as in (4.1) and set $\bar{F}_i := F_{i|\mathfrak{g}^*}$. If $n = \dim \mathbb{V}$ is odd, then the set $\{\bar{F}_{2i} \mid 0 < i < n/2\}$ is a basis of $\mathcal{S}(\mathfrak{g})^G$. If n is even, then $\bar{F}_n = P^2$, where P is the *pfaffian*. Clearly, $({}^eP)^2 = {}^e\bar{F}_n$. Similar to the symplectic case, we have $\deg {}^e\bar{F}_{2i} \geqslant \deg {}^eF_{2i}$. From [13, 3.1(3)] it

follows that

$$\dim \mathfrak{g}_e = \tfrac{1}{2}\left(\dim \widetilde{\mathfrak{g}}_e - \sum_{i,\, d_i \text{ even}} 1\right). \tag{13}$$

Note that $l = \operatorname{rk} \mathfrak{g} = [(\dim \mathbb{V})/2]$. In order to compute $\sum_{i=1}^{l} \deg {}^eF_{2i}$ we again consider our labelled Young diagram (see Figure 1) and sum up the labels assigned to the even boxes. It is important to observe that in the present case neighbouring columns of the same odd size will always have a different number of even boxes. This is illustrated in Figure 2.

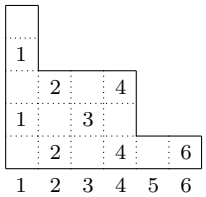


Figure 2

Taking into account (13) and the equality $\sum_{j=1}^{n} \deg {}^eF_j = (\dim \widetilde{\mathfrak{g}}_e + n)/2$ we now arrive at the following:

$$\begin{aligned} \sum_{j=1}^{l} \deg {}^eF_{2j} &= \sum_{i'=i+1} (2i+1)\tfrac{d_i+1}{2} + \sum_{i=i',\, i \text{ odd}} i\tfrac{d_i}{2} + \sum_{i=i',\, i \text{ even}} i\tfrac{d_i+2}{2} \\ &= \tfrac{1}{2}\left(\sum_{j=1}^{n} \deg {}^eF_j - \sum_{i=i',\, i \text{ odd}} i + \sum_{i=i',\, i \text{ even}} i\right) \\ &= \tfrac{1}{2}\left(\dim \mathfrak{g}_e + \tfrac{n}{2} + \sum_{i,\, d_i \text{ even}} \tfrac{1}{2} - \sum_{i=i',\, i \text{ odd}} i + \sum_{i=i',\, i \text{ even}} i\right). \end{aligned} \tag{14}$$

Lemma 4.5. *Let e be a nilpotent element in $\mathfrak{g} = \mathfrak{so}(\mathbb{V})$ such that*

1) *d_1 is even;*
2) *if d_{i-1} is even for i odd, then d_i is even.*

Then either $\bar{F}_2, \bar{F}_4, \ldots, \bar{F}_{n-1}$ or $\bar{F}_2, \bar{F}_4, \ldots, \bar{F}_{n-2}, P$ (depending on the parity of n) is a good generating system for $e \in \mathfrak{g}$.

Proof. Let $t_1, \ldots, t_q$ be the indices of the odd-sized Jordan blocks of e. Note that t_j and j has the same parity.

First suppose n is odd. Then q is also odd, by (13). Recall that $\deg {}^e\bar{F}_{2i} \geqslant \deg {}^eF_{2i}$ for all i. By Theorem 2.1, we have $\sum_{i=1}^{l} \deg {}^e\bar{F}_{2i} \leqslant (\dim \mathfrak{g}_e + \operatorname{rk} \mathfrak{g})/2$. Due to (14) it suffices to prove that

$$\tfrac{n}{2} + \tfrac{q}{2} - t_1 + t_2 - t_3 + \cdots - t_q = \tfrac{n-1}{2}.$$

By the assumptions of the lemma, $t_1 = 1$, $t_3 = t_2 + 1$, $t_5 = t_4 + 1$, and so on. Thus, $\sum_{i=1}^{q} (-1)^i t_i = -1 - (q-1)/2$, which is exactly what we wanted.

Now suppose n is even. In this case q is even, by (13), and $\deg {}^eP \geqslant (\deg {}^eF_n)/2$. Moreover, since d_{t_q+1} cannot be even and odd at the same time, we have $t_q = k$, that is the last Jordan block has odd size. As above, $t_{j+1} = t_j + 1$ for all even $1 < j < q$. Therefore,

$$\begin{aligned}\sum_{j=1}^{n/2-1} \deg {}^e\bar{F}_{2j} + \deg {}^eP &\geqslant \tfrac{1}{2}\left(\dim \mathfrak{g}_e + \tfrac{n+q}{2} + \sum_{i=1}^{q}(-1)^i t_i\right) - \tfrac{k}{2} \\ &= \tfrac{1}{2}\left(\dim \mathfrak{g}_e + \tfrac{n+q}{2} - 1 - \tfrac{q-2}{2} + k - k\right) = \tfrac{1}{2}\left(\dim \mathfrak{g}_e + \tfrac{n}{2}\right),\end{aligned}$$

and we are done. □

Lemma 4.6. *Suppose* $\dim \mathbb{V} = 2l$ *and let* e *be a nilpotent element in* $\mathfrak{g} = \mathfrak{so}(\mathbb{V})$ *such that*

1) d_1 *is odd and* $d_2 = d_1$*;*
2) d_i *is even for* $i \geqslant 3$*.*

Then e *admits a good generating system in* $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$*.*

Proof. Recall that $\deg {}^e\bar{F}_{2i} \geqslant \deg {}^eF_{2i}$ for all i and $\deg {}^eP \geqslant (\deg {}^eF_{2l})/2$. Because k is even, by (13), it follows from (14) that

$$\sum_{i=1}^{l-1} \deg {}^e\bar{F}_{2i} + \deg {}^eP \geqslant \tfrac{1}{2}\left(\dim \mathfrak{g}_e + l + \tfrac{k-2}{2} + \tfrac{k-2}{2}\right) - \tfrac{k}{2} = \tfrac{1}{2}\left(\dim \mathfrak{g}_e + l\right) - 1.$$

Thus, the system $\bar{F}_2, \dots, \bar{F}_{r-2}, P$ is "almost good". Applying Lemma 4.3 we see that in the present case

$${}^eF_{2d_1+2} = a_1\xi_1^{1,d_1}\xi_2^{2,d_2} + a_2\xi_1^{2,d_1}\xi_2^{1,d_1}$$

for some $a_1, a_2 \in \mathbb{K}$. Since ${}^eF_{2d_1+2}$ is irreducible, being a generator of the polynomial algebra $\mathcal{S}(\widetilde{\mathfrak{g}})^{\widetilde{\mathfrak{g}}}$, it must be that $a_1a_2 \neq 0$. Both ξ_1^{2,d_1} and ξ_2^{1,d_2} vanish on $\mathfrak{g}_e$ and so does $\xi_1^{1,d_1} - \xi_2^{2,d_1}$. Therefore, ${}^e\bar{F}_{2d_1+2} = a_1\xi_1^{1,d_1}\xi_2^{2,d_2}$. For $d = (d_1+1)/2$ we have ${}^e\bar{F}_{2d} = \xi_1^{1,d_1}$, up to a nonzero scalar. Consequently, ${}^e\bar{F}_{2d_1+2} = c({}^e\bar{F}_{2d})^2$ for some $c \in \mathbb{K}^\times$.

If $k > 2$, then $d_1 + 1 < l$, and we can replace $\bar{F}_{2d_1+2}$ by $\bar{F}'_{2d_1+2} := \bar{F}_{2d_1+2} - c\bar{F}^2_{d_1+1}$. Since $\deg {}^e\bar{F}'_{2d_1+2} \geqslant \deg {}^e\bar{F}_{2d_1+2} + 1$, Theorem 2.1(i) implies that $\bar{F}_2, \dots, \bar{F}'_{2d_1+2}, \dots, P$ is a good generating system for e.

If $k = 2$, then ${}^eP = c_0{}^e\bar{F}_{2d}$ for some $c_0 \in \mathbb{K}^\times$. In this case we can replace P by $P' := P - c_0\bar{F}_{d_1+1}$. Then $\deg {}^eP' \geqslant \deg {}^eP + 1$, implying that $\bar{F}_2, \dots, \bar{F}_{2r-2}, P'$ is a good generating system for e. □

Combining Lemmas 3.12, 3.13, and 4.5 we obtain the following result:

Theorem 4.7. *Let* $\mathbb{V}$ *be an* n*-dimensional vector space over* $\mathbb{K}$*, where* n *is odd, and let* $\tilde{\mathbb{V}} = \mathbb{V} \oplus \mathbb{K}v$ *be as in (3.8). Let* $(d_1 + 1 \geqslant d_2 + 1 \geqslant \dots \geqslant d_k + 1)$ *be a partition of* n *such that*

1) d_1 *is even and* $k > 1$*;*
2) d_i *is odd for all* $i \geqslant 2$*.*

Let e *and* $\hat{e}$ *be nilpotent elements in* $\mathfrak{g} = \mathfrak{so}(\mathbb{V})$ *and* $\hat{\mathfrak{g}} = \mathfrak{so}(\tilde{\mathbb{V}})$*, respectively, corresponding to the partitions* $(d_1+1, \dots, d_k+1)$ *and* $(d_1+1, \dots, d_k+1, 1)$*. Then* e *and* $\hat{e}$ *admit very good generating systems in* $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ *and* $\mathcal{S}(\hat{\mathfrak{g}})^{\hat{\mathfrak{g}}}$*, respectively, and the invariant algebras* $\mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e}$ *and* $\mathcal{S}(\hat{\mathfrak{g}}_{\hat{e}})^{\hat{\mathfrak{g}}_{\hat{e}}}$ *are free.*

Remark 4.1. Conditions of Lemmas 4.5 and 4.6 are only sufficient for the existence of a good generating system. But we conjecture that the other nilpotent elements in $\mathfrak{g} = \mathfrak{so}(\mathbb{V})$ do not possess good generating systems in $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$.

Example 4.1. Now we wish to exhibit a nilpotent element e in $\mathfrak{g} = \mathfrak{so}(\mathbb{V})$ *without* a good generating system in $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$. Some details will be left to the reader. Let $e \in \mathfrak{so}_{12}$ be a nilpotent element with partition $(5,3,2,2)$. Then $\dim \mathfrak{g}_e = 18$, $\operatorname{ind} \mathfrak{g}_e = 6$, but

$$\sum_{i=1}^{5} \deg {}^eF_{2i} + (\deg {}^eF_{12})/2 = 11 < (18+6)/2 = 12.$$

One can show that $\deg {}^e\bar{F}_{2i} = \deg {}^eF_{2i}$ and $\deg {}^eP = 2$. We have only two ${}^e\bar{F}_{2i}$'s of degree one, but the centre of $\mathfrak{g}_e$ is 3-dimensional and ${}^e\bar{F}_8 = a^2$, where a is a central element of $\mathfrak{g}_e$ linear independent of ${}^e\bar{F}_2$ and ${}^e\bar{F}_4$. Moreover, up to a scalar ${}^e\bar{F}_{10} = a \cdot {}^eP$. We see that ${}^e\bar{F}_{2i}$'s and ${}^eP$ are algebraically dependent. On the other hand, computations show that there is no good way to modify the system of generators $\bar{F}_2, \bar{F}_4, \bar{F}_6, \bar{F}_{10}, P$ of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$.

4.5. Suppose that $\operatorname{rk} \mathfrak{g} \geqslant 2$. Our next goal in this section is to attack Conjecture 0.1 for the elements of the minimal nilpotent orbit $\mathcal{O}_{\min} = G{\cdot}\tilde{e}$ in $\mathfrak{g}$. More precisely, we are going to show that if $\mathfrak{g}$ is not of type $\mathbf{E}_8$, then $\tilde{e}$ admits a good generating system in $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$. Thanks to Theorem 3.14 and Theorem 2.2 this will reduce verifying Conjecture 0.1 for the elements in $\mathcal{O}_{\min}$ to the case where $\mathfrak{g}$ is of type $\mathbf{E}_8$. Some partial results on the $\mathbf{E}_8$ case are obtained in (4.8) where Conjecture 0.1 for $\mathcal{O}_{\min}$ is reduced to a computational problem on polynomial invariants for the Weyl group of type $\mathbf{E}_7$.

We adopt the notation introduced in (3.9) and (3.10), choose a Cartan subalgebra $\tilde{\mathfrak{t}}$ of $\mathfrak{g}$ contained in $\mathfrak{g}(0)$, and denote by Φ the root system of $\mathfrak{g}$ with respect to $\tilde{\mathfrak{t}}$. Choose a positive system Φ_+ in Φ such that for every $\gamma \in \Phi_+$ the root subspace $\mathfrak{g}_\gamma = \mathbb{K}e_\gamma$ is contained in the parabolic subalgebra $\mathfrak{p} := \mathfrak{g}(0) \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2)$. Note that $\Phi = \sqcup_{-2 \leqslant i \leqslant 2} \Phi_i$ where $\Phi_i := \{\gamma \in \Phi \mid \mathfrak{g}_\gamma \subset \mathfrak{g}(i)\}$. Clearly, $\Phi_2 = \{\tilde{\alpha}\}$ where $\tilde{\alpha}$ is the longest root in Φ_+. No generality will be lost by assuming that $\tilde{e} = e_{\tilde{\alpha}}$ and $\tilde{f} = e_{-\tilde{\alpha}}$. Set $\mathfrak{t} := \operatorname{Ker} \tilde{\alpha}$. It is well-known (and easy to see) that $\mathfrak{t}$ is a Cartan subalgebra in $\mathfrak{g}_{\tilde{e}}$ and $\mathfrak{g}_{\tilde{e}} = \mathfrak{t} \oplus \bigoplus_{\gamma \in \Phi_i,\, i \geqslant 0} \mathfrak{g}_\gamma$. For $\beta \in \sqcup_{i \geqslant 0} \Phi_i$ we denote by ξ_β the linear function on $\mathfrak{g}_{\tilde{e}}$ that vanishes on $\mathfrak{t}$ and has the property that $\xi_\beta(e_\gamma) = \delta_{\beta\gamma}$ for all $\gamma \in \sqcup_{i \geqslant 0} \Phi_i$. The dual space $\mathfrak{t}^*$ will be identified with the subspace of $\mathfrak{g}_{\tilde{e}}^*$ consisting of all linear functions ξ such that $\xi(e_\gamma) = 0$ for all $\gamma \in \sqcup_{i \geqslant 0} \Phi_i$. Set $\mathfrak{h} := \mathfrak{t} \oplus \mathbb{K}\tilde{e}$, an abelian subalgebra of $\mathfrak{g}_{\tilde{e}}$. We regard $\mathfrak{h}^* = \mathfrak{t}^* \oplus \mathbb{K}\xi_{\tilde{\alpha}}$ as a subspace of $\mathfrak{g}_{\tilde{e}}^*$.

Choose $\xi_0 \in \mathfrak{t}^*$ such that $\xi_0([\mathfrak{g}_\gamma, \mathfrak{g}_{-\gamma}]) \neq 0$ for all $\gamma \in \Phi_0$ and put $\eta := \xi_0 + \xi_{\tilde{\alpha}}$, an element of $\mathfrak{h}^*$. Since η vanishes on $\mathfrak{g}(1)$, it is immediate from our discussion in (3.10) that $\mathfrak{g}_{\tilde{e}}^\eta = \mathfrak{h}$. In particular, $\eta \in (\mathfrak{g}_{\tilde{e}}^*)_{\mathrm{reg}}$. Moreover, our earlier remarks show that

$$\mathfrak{h}^* = \{\xi \in \mathfrak{g}_{\tilde{e}}^* \mid (\operatorname{ad}^* \mathfrak{h})\xi = 0\} \quad \text{and} \quad \mathfrak{h}^* \cap (\operatorname{ad}^* \mathfrak{g}_{\tilde{e}})\mathfrak{h}^* = 0.$$

It follows that the differential of the coadjoint action morphism $G_{\tilde{e}} \times \mathfrak{h}^* \to \mathfrak{g}_{\tilde{e}}^*$ is surjective at $1 \times \eta$. Then $\overline{G_{\tilde{e}} \cdot \mathfrak{h}^*} = \mathfrak{g}_{\tilde{e}}^*$, implying that the restriction map $\varphi \mapsto \varphi_{|\mathfrak{h}^*}$ induces an embedding $\mathbb{K}[\mathfrak{g}_{\tilde{e}}^*]^{G_e} \hookrightarrow \mathbb{K}[\mathfrak{h}^*]$. Hence, for every nonzero homogeneous $F \in \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ the regular function ${}^{\tilde{e}}F_{|\mathfrak{h}^*}$ is nonzero and thus has the same degree as ${}^{\tilde{e}}F$.

4.6. The Weyl group $W = N_G(\tilde{\mathfrak{t}})/Z_G(\tilde{\mathfrak{t}})$ is generated by the orthogonal reflections s_γ in the hyperplanes $\operatorname{Ker}\gamma$, where $\gamma \in \Phi$. Let $C_W(\tilde{h})$ be the stabiliser of $\tilde{h}$ in W. It is well-known that $C_W(\tilde{h}) = \langle s_\gamma \mid \gamma(\tilde{h}) = 0\rangle$. Obviously, $C_W(\tilde{h})$ preserves $\mathfrak{t}$. We denote by ρ_0 the corresponding representation of $C_W(\tilde{h})$ and put $W_0 := \rho_0(C_W(\tilde{h}))$. Note that W_0 is a finite reflection subgroup of $\mathrm{GL}(\mathfrak{t})$. Since $\mathfrak{t} = \operatorname{Ker}\tilde{\alpha}$ and $s_{\tilde{\alpha}}(\tilde{h}) = -\tilde{h}$, any nonzero $\varphi \in \mathcal{S}(\tilde{\mathfrak{t}})^W$ has the form

$$\varphi = \sum_{i=0}^{\nu} \varphi^{(i)}\tilde{h}^{2i} \qquad \big(\varphi^{(i)} \in \mathcal{S}(\mathfrak{t})^{W_0},\ \ \varphi^{(\nu)} \neq 0,\ \ \nu = \nu(\varphi)\big). \tag{15}$$

For $\psi \in \mathcal{S}(\tilde{\mathfrak{t}})$ we denote by $\tilde{\mathfrak{t}}_\psi$ the set of all $h \in \tilde{\mathfrak{t}}$ such that $\psi(x + \lambda h) = \psi(x)$ for all $x \in \tilde{\mathfrak{t}}$ and all $\lambda \in \mathbb{K}$. If $\psi \in \mathcal{S}(\tilde{\mathfrak{t}})^W$, then $\tilde{\mathfrak{t}}_\psi$ is a W-invariant subspace of $\tilde{\mathfrak{t}}$. As $\tilde{\mathfrak{t}}$ is an irreducible W-module, then for φ as in (15) we must have $\tilde{\mathfrak{t}}_\varphi = 0$. Consequently, $\nu(\varphi) \geqslant 1$.

Proposition 4.8. *If $\{\varphi_1, \varphi_2, \dots, \varphi_l\}$ is a homogeneous generating set in $\mathcal{S}(\tilde{\mathfrak{t}})^W$ with $\deg\varphi_1 = 2$, then $\sum_{i=2}^{l} \nu(\varphi_i) \geqslant \frac{1}{2}\dim\mathfrak{g}(1)$. If $\sum_{i=2}^{l} \nu(\varphi_i) = \frac{1}{2}\dim\mathfrak{g}(1)$, then $\varphi_2^{(\nu)}, \dots, \varphi_l^{(\nu)}$ are algebraically independent and $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ admits a good generating system for $\tilde{e}$.*

Proof. Consider the Levi subalgebra $\tilde{\mathfrak{s}} = \mathbb{K}\tilde{f} \oplus \tilde{\mathfrak{t}} \oplus \mathbb{K}\tilde{e}$ of $\mathfrak{g}$ and put $c := \tilde{h}^2 + 4\tilde{e}\tilde{f}$, an element of $\mathcal{S}(\tilde{\mathfrak{s}})$. Since $\mathfrak{z}(\tilde{\mathfrak{s}}) = \mathfrak{t}$ and $[\tilde{\mathfrak{s}}, \tilde{\mathfrak{s}}] = \mathbb{K}\tilde{f} \oplus \mathbb{K}\tilde{h} \oplus \mathbb{K}\tilde{e}$, we have that $\mathcal{S}(\tilde{\mathfrak{s}})^{\tilde{\mathfrak{s}}} \cong \mathcal{S}(\mathfrak{t}) \otimes_{\mathbb{K}} \mathbb{K}[c]$ as algebras. We identify $\tilde{\mathfrak{s}}^*$ with $\mathbb{K}\xi_{-\tilde{\alpha}} \oplus \mathfrak{t}^* \oplus \mathbb{K}\xi_{\tilde{\alpha}}$. Then $\mathfrak{h}^* = \mathfrak{t}^* \oplus \mathbb{K}\xi_{\tilde{\alpha}} \subset \tilde{\mathfrak{s}}^*$. Since $\tilde{e}$ is regular nilpotent in $\tilde{\mathfrak{s}}$, the restriction map $F \mapsto F_{|\xi_{-\tilde{\alpha}} + \mathfrak{h}^*}$ induces an algebra isomorphism $\iota\colon\ \mathcal{S}(\tilde{\mathfrak{s}})^{\tilde{\mathfrak{s}}} \xrightarrow{\sim} S(\mathfrak{h})$ such that $\iota_{|\mathfrak{t}} = \mathrm{id}$ and $\iota(c) = 4\tilde{e}$.

By the Chevalley Restriction Theorem, there exists a homogeneous generating system $\{F_1, \dots, F_l\} \subset \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ such that $F_{k|\tilde{\mathfrak{t}}^*} = \varphi_k$ for all k. Since $\varphi_k = \sum_{i=0}^{\nu(\varphi_k)} \varphi_k^{(i)}\tilde{h}^{2i}$ by (15), it follows that $F_{k|\tilde{\mathfrak{s}}^*} = \sum_{i=0}^{\nu(\varphi_k)} \varphi_k^{(i)} c^i$. But then $\iota(F_{k|\tilde{\mathfrak{s}}^*}) = \sum_{i=0}^{\nu(\varphi_k)} 4^i \varphi_k^{(i)} e^i$. It is now immediate from the definition of ${}^{\tilde{e}}F$ that

$$({}^{\tilde{e}}F_k)_{|\mathfrak{h}^*} = 4^{\nu(\varphi_k)}\varphi_k^{(\nu)} e^{\nu(\varphi_k)} \qquad (1 \leqslant k \leqslant l). \tag{16}$$

Since $\deg {}^{\tilde{e}}F_k = \deg\,({}^{\tilde{e}}F_k)_{|\mathfrak{h}^*}$ by our concluding remark in (4.5), Theorem 2.1(i) in conjunction with with (16) gives

$$\sum_{i=1}^{l} \deg\varphi_i^{(\nu)} + \sum_{i=1}^{l} \nu(\varphi_i) \leqslant (\dim\mathfrak{g}_{\tilde{e}} + l)/2.$$

On the other hand, (15) shows that

$$\sum_{i=1}^{l} \deg\varphi_i^{(\nu)} + 2\sum_{i=1}^{l} \nu(\varphi_i) = \sum_{i=1}^{l} \deg\varphi_i = \sum_{i=1}^{l} \deg F_i = (\dim\mathfrak{g} + l)/2.$$

As $\dim\mathfrak{g} - \dim\mathfrak{g}_{\tilde{e}} = 2 + \dim\mathfrak{g}(1)$ and $\nu(\varphi_1) = 1$ by our assumption on $\deg\varphi_1$, we are now able to conclude that $\sum_{i=2}^{l} \nu(\varphi_i) \geqslant \frac{1}{2}\dim\mathfrak{g}(1)$.

If $\sum_{i=2}^{l} \nu(\varphi_i) = \frac{1}{2}\dim\mathfrak{g}(1)$, then the above shows that

$$\sum_{i=1}^{l} \deg {}^{\tilde{e}}F_i = \sum_{i=1}^{l} \deg\varphi_i^{(\nu)} + \sum_{i=1}^{l} \nu(\varphi_i) = (\dim\mathfrak{g}_{\tilde{e}} + l)/2.$$

Hence $\{F_1, \dots, F_l\} \subset \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ is a good generating system for $\tilde{e}$, implying that ${}^{\tilde{e}}F_1, {}^{\tilde{e}}F_2, \dots, {}^{\tilde{e}}F_l$ are algebraically independent; see Theorem 2.1(ii). As $\varphi_1^{(\nu)}$ is a nonzero constant, our discussion in (4.5) together with (16) shows that e, $\varphi_2^{(\nu)} e^{\nu(\varphi_2)}$, $\dots$, $\varphi_l^{(\nu)} e^{\nu(\varphi_l)}$ are algebraically independent in $\mathcal{S}(\mathfrak{g}_{\tilde{e}})$. Then $\varphi_2^{(\nu)}, \dots, \varphi_l^{(\nu)}$ must be algebraically independent in $\mathcal{S}(\mathfrak{t})$. This completes the proof. $\square$

4.7. Proposition 4.8 in conjunction with Theorems 2.1(iii), 3.4, 3.11, 3.14 and 2.2 will enable us to show that $\mathcal{S}(\mathfrak{g}_{\tilde{e}})^{\mathfrak{g}_{\tilde{e}}}$ is a graded polynomial algebra in $\operatorname{rk}\mathfrak{g}$ variables in all cases except when $\mathfrak{g}$ is of type $\mathbf{E}_8$. We shall identify $\mathcal{S}(\tilde{\mathfrak{t}})$ with $\mathcal{S}(\tilde{\mathfrak{t}}^*)$ by means of the W-invariant scalar product $(\,\cdot\,|\,\cdot\,)$ used in [1] and [27]. Note that $\tilde{h} = \tilde{\alpha}^\vee$ identifies with a nonzero multiple of $\tilde{\alpha}$. The basis of simple roots contained in Φ_+ will be denoted by Δ.

(1) Suppose $\mathfrak{g}$ is of type $\mathbf{A}_n$, $n \geqslant 2$. Then $\tilde{\mathfrak{t}}^*$ is spanned by $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n+1}$ subject to the relation $\varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_{n+1} = 0$. The Weyl group $W = \mathfrak{S}_{n+1}$ permutes the ε_i's. Put

$$s_k := \sum_{\sigma \in \mathfrak{S}_{n+1}} \varepsilon_{\sigma(1)}\varepsilon_{\sigma(2)} \cdots \varepsilon_{\sigma(k)} \qquad (2 \leqslant k \leqslant n+1).$$

Since $\tilde{\alpha} = \varepsilon_1 - \varepsilon_{n+1}$ and $(\tilde{\alpha}|\varepsilon_i) = 0$ for $2 \leqslant i \leqslant n$, it is routine that $\nu(s_k) = 1$ for $2 \leqslant k \leqslant n+1$. Now set $\varphi_k := s_{k+1}$, $2 \leqslant k \leqslant n+1$. Then $\{\varphi_1, \varphi_2, \dots, \varphi_n\}$ is a homogeneous generating set in $\mathcal{S}(\tilde{\mathfrak{t}}^*)^W$ with $\deg \varphi_1 = 2$. Since $\sum_{i=2}^n \deg \nu(\varphi_i) = n - 1 = \frac{1}{2}\dim \mathfrak{g}(1)$, we derive that $\mathcal{S}(\mathfrak{g}_{\tilde{e}})^{\mathfrak{g}_{\tilde{e}}}$ is a graded polynomial algebra in n variables. The degrees of basic invariants are $1, 2, \dots, n$. Since $\mathfrak{g} = \mathfrak{sl}_{n+1}$ and the partition of $\tilde{e}$ is $(2, 1^{n-1})$, this is consistent with the combinatorial description in Theorem 4.2.

(2) Suppose $\mathfrak{g}$ is of type $\mathbf{C}_n$, $n \geqslant 2$. Then $\tilde{\alpha} = 2\varepsilon_1$, and we can assume that $\varphi_k = \tilde{s}_k$, where

$$\tilde{s}_k := \sum_{\sigma \in \mathfrak{S}_{n+1}} \varepsilon_{\sigma(1)}^2\varepsilon_{\sigma(2)}^2 \cdots \varepsilon_{\sigma(k)}^2 \qquad (1 \leqslant k \leqslant n). \tag{17}$$

As $(\tilde{\alpha}|\varepsilon_i) = 0$ for $2 \leqslant i \leqslant n$, it is clear that $\nu(\varphi_k) = 1$ for all k. Then $\sum_{i=2}^n \deg \nu(\varphi_i) = n-1 = \frac{1}{2}\dim \mathfrak{g}(1)$, which shows that $\mathcal{S}(\mathfrak{g}_{\tilde{e}})^{\mathfrak{g}_{\tilde{e}}}$ is a graded polynomial algebra in n variables. The degrees of basic invariants are $1, 3, \dots, 2n-1$. Since $\mathfrak{g} = \mathfrak{sp}_{2n}$ and the partition of $\tilde{e}$ is $(2, 1^{2n-2})$, this is consistent with our description in Theorem 4.4.

(3) Suppose $\mathfrak{g}$ is of type $\mathbf{B}_n$, $n \geqslant 3$. Then $\tilde{\alpha} = \varepsilon_1 + \varepsilon_2$. For $k \in \{1, 3, \dots, n\}$ put $\varphi_k := \tilde{s}_k$, where $\tilde{s}_k$ is as in (17), and set $\varphi_2 := \tilde{s}_2 - \frac{1}{4}\tilde{s}_1^2$. As $(\tilde{\alpha}|\varepsilon_i) = 0$ for $3 \leqslant i \leqslant n$, it is straightforward to see that $\nu(\varphi_2) = 1$ and $\nu(\varphi_k) = 2$ for $3 \leqslant k \leqslant n$. Then $\sum_{i=2}^n \deg \nu(\varphi_i) = 1 + 2(n-2) = \frac{1}{2}\dim \mathfrak{g}(1)$. Hence $\mathcal{S}(\mathfrak{g}_{\tilde{e}})^{\mathfrak{g}_{\tilde{e}}}$ is a graded polynomial algebra in n variables, and the degrees of basic invariants are $1, 3, 4, \dots, 2n-2$.

(4) Suppose $\mathfrak{g}$ is of type $\mathbf{D}_n$, $n \geqslant 4$. Then again $\tilde{\alpha} = \varepsilon_1 + \varepsilon_2$. For $k \in \{1, 3, \dots, n-1\}$ put $\varphi_k := \tilde{s}_k$ and set $\varphi_2 := \tilde{s}_2 - \frac{1}{4}\tilde{s}_1^2$. Finally, set $\varphi_n := p$ where $p = \prod_{i=1}^n \varepsilon_i$. As in part (3) we obtain $\nu(\varphi_2) = 1$ and $\nu(\varphi_k) = 2$ for $3 \leqslant k \leqslant n-1$. Since $\nu(\varphi_n) = 1$, we have $\sum_{i=2}^n \deg \nu(\varphi_i) = 1 + 2(n-3) + 1 = \frac{1}{2}\dim \mathfrak{g}(1)$. Thus, $\mathcal{S}(\mathfrak{g}_{\tilde{e}})^{\mathfrak{g}_{\tilde{e}}}$ is a graded polynomial algebra in n variables, and the degrees of basic invariants are $1, 3, 4, \dots, 2n-4, n-1$.

(5) Suppose $\mathfrak{g}$ is of type $\mathbf{G}_2$ and assume that $\Delta = \{\alpha, \beta\}$ where β is a short root. Then $\tilde{\alpha} = 2\alpha + 3\beta$ and $(\beta|\tilde{\alpha}) = 0$. The degrees of basic invariants in $\mathcal{S}(\tilde{\mathfrak{t}}^*)^W$ are 2, 6. There exists $\varphi_1 \in \mathcal{S}(\tilde{\mathfrak{t}}^*)^W$ such that $\varphi_1 = \tilde{\alpha}^2 + \lambda_0\beta^2$ for some $\lambda_0 \in \mathbb{K}$. Since $\deg \varphi_1^3 = 6$, we can find a basic W-invariant φ_2 in $\mathcal{S}^6(\tilde{\mathfrak{t}}^*)$ such that $\varphi_2 = \lambda_1\tilde{\alpha}^4\beta^2 + \lambda_2\tilde{\alpha}^2\beta^4 + \lambda_3\beta^6$ for some $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{K}$. Then $\nu(\varphi_2) \leqslant 2 = \frac{1}{2}\dim \mathfrak{g}(1)$. Applying Proposition 4.8 yields $\nu(\varphi_2) = \frac{1}{2}\dim \mathfrak{g}(1)$. Then $\mathcal{S}(\mathfrak{g}_{\tilde{e}})^{\mathfrak{g}_{\tilde{e}}}$ is a graded polynomial algebra in two variables, and the degrees of basic invariants are 1, 4.

(6) Suppose $\mathfrak{g}$ is of type $\mathbf{F}_4$. In this case $\tilde{\alpha} = \varepsilon_1 + \varepsilon_2$ and basic invariants in $\mathcal{S}(\tilde{\mathfrak{t}}^*)^W$ have degrees 2, 6, 8, 12. Let W' denote the subgroup of W generated all reflections s_α corresponding to long roots in Φ. The reflection group W' has type $\mathbf{D}_4$ and acts on the ε-basis of $\tilde{\mathfrak{t}}^*$ in the standard way. Therefore, $\mathcal{S}(\tilde{\mathfrak{t}}^*)^{W'} = \mathbb{K}[\tilde{s}_1, \tilde{s}_2, \tilde{s}_3, p]$ where $\tilde{s}_1, \tilde{s}_2, \tilde{s}_3, p$ are as in part (4). Note that W' is a normal subgroup of W and $W/W' \cong \mathfrak{S}_3$.

Set $\varphi_1 = \tilde{s}_1$. It is easy to see that $\varphi_1 \in \mathcal{S}(\tilde{\mathfrak{t}}^*)^W$. Since $\varphi_1^3 \in \mathcal{S}^6(\tilde{\mathfrak{t}}^*)^W$ and $\nu(\varphi_1^3) = 3$, there exists a basic invariant $\varphi_2 \in \mathcal{S}^6(\tilde{\mathfrak{t}}^*)^W$ for which $\nu(\varphi_2) \leqslant 2$. Next observe that $M := \mathcal{S}^4(\tilde{\mathfrak{t}}^*)^{W'}$ is a W/W'-module with basis $\{\tilde{s}_2, p, \tilde{s}_1^2\}$. We denote by M' the submodule of M spanned by all $(w-1)\cdot m$ with $w \in W$ and $m \in M$. Let $\beta := \varepsilon_1$ and $\gamma := \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)$, short roots in Φ, and put $p' := \prod_{i=1}^4 (\varepsilon_i - \gamma)$. Since $p' = s_\gamma(\prod_{i=1}^4 \varepsilon_i) = s_\gamma(p)$ and $s_\beta(p) = -p$, we have $p, p' \in M'$. Since $s_\beta(\tilde{s}_2) = \tilde{s}_2$ and $\mathcal{S}^4(\tilde{\mathfrak{t}}^*)^W = \mathbb{K}\tilde{s}_1^2$, this shows that M' is isomorphic to the reflection module for $W/W' \cong \mathfrak{S}_3$, and p and p' form a basis for M'.

The above discussion implies that there exist homogeneous polynomials $q_2, q_3 \in \mathbb{K}[X, Y]$ of degree 2 and 3, respectively, such that $q_2(p, p')$ and $q_3(p, p')$ generate the invariant algebra $\mathbb{K}[M']^{\mathfrak{S}_3} \subset \mathcal{S}(\tilde{\mathfrak{t}}^*)^W$. As $(\tilde{\alpha}|\gamma) = (\tilde{\alpha}|\varepsilon_1) = (\tilde{\alpha}|\varepsilon_2) = 1$, one checks easily that $\nu(p) = \nu(p') = 1$. Hence $\nu(q_2(p, p')) \leqslant 2$ and $\nu(q_3(p, p')) \leqslant 3$. Since $\mathcal{S}^6(\tilde{\mathfrak{t}}^*)^{W'}$ is spanned by $\tilde{s}_3$, $\tilde{s}_1\tilde{s}_2$, $\tilde{t}_1 p$, $\tilde{s}_1^3$, there are $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{K}$ such that $\varphi_2 = \lambda_1\tilde{s}_3 + \tilde{s}_1(\lambda_2 p + \lambda_3 p') + \lambda_4\tilde{s}_1^3$. As $(M')^W = 0$, it must be that $\lambda_1 \neq 0$. From this it is immediate that

$$\mathcal{S}(\tilde{\mathfrak{t}}^*)^{W'} \cong \mathbb{K}[\varphi_1, \varphi_2] \otimes_{\mathbb{K}} \mathbb{K}[M']$$

as W-modules. But then we can set $\varphi_3 := q_2(p, p')$ and $\varphi_4 := q_3(p, p')$ to obtain a generating set $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\} \subset \mathcal{S}(\tilde{\mathfrak{t}}^*)^W$ with $\deg \varphi_1 = 2$ and $\sum_{i=2}^4 \nu(\varphi_i) \leqslant 2 + 2 + 3 = 7$. Since in the present case $\dim \mathfrak{g}(1) = 14$, Proposition 4.8 shows that $\nu(\varphi_2) = \nu(\varphi_3) = 2$ and $\nu(\varphi_4) = 3$. Hence $\mathcal{S}(\mathfrak{g}_{\tilde{e}})^{\mathfrak{g}_{\tilde{e}}}$ is a graded polynomial algebra in four variables, and the degrees of basic invariants are 1, 4, 6, 9.

(7) Now suppose $\mathfrak{g}$ is of type $\mathbf{E}_6$ and let σ denote the outer involution in $\mathrm{Aut}(\Phi)$ preserving Δ. In the present case, the degrees of basic invariants in $\mathcal{S}(\tilde{\mathfrak{t}}^*)^W$ are 2, 5, 6, 8, 9, 12. The reflection group W_0 has type $\mathbf{A}_5$ and basic invariants in $\mathcal{S}(\mathfrak{t}^*)^{W_0}$ have degrees 2, 3, 4, 5, 6. We choose a homogeneous generating system $\{\psi_1, \ldots, \psi_5\} \subset \mathcal{S}(\mathfrak{t}^*)^{W_0}$ with $\deg \psi_i = i + 1$ for $1 \leqslant i \leqslant 5$. Since $\sigma(\tilde{\alpha}) = \tilde{\alpha}$, both W_0 and $\mathfrak{t}$ are σ-stable. Set $\tilde{\mathfrak{t}}^\sigma := \{t \in \tilde{\mathfrak{t}} \mid \sigma(t) = t\}$ and $\mathfrak{t}^\sigma := \tilde{\mathfrak{t}}^\sigma \cap \mathfrak{t}$. The groups $W^\sigma = \{w \in W \mid \sigma w = w\sigma\}$ and $W_0^\sigma = W^\sigma \cap W_0$ act on $\tilde{\mathfrak{t}}^\sigma$ and $\mathfrak{t}^\sigma$, respectively, and we shall denote by $\tilde{\rho}$ and ρ the corresponding representations. It is well-known that $\tilde{\rho}(W^\sigma)$ and $\rho(W_0^\sigma)$ are reflection groups of type $\mathbf{F}_4$ and $\mathbf{C}_3$, respectively.

Note that $\tilde{\mathfrak{t}}^\sigma = \mathfrak{t}^\sigma \oplus \mathbb{K}\tilde{h}$. To make use of the results obtained in part (6) we shall restrict functions from $\mathcal{S}(\tilde{\mathfrak{t}}^*)^W$ to $\tilde{\mathfrak{t}}^\sigma$. Let $\bar{\psi}_i$ denote the restriction of ψ_i to $\mathfrak{t}^\sigma$. Since $\rho(W_0^\sigma)$ is a reflection group of type C_3, we have that $\bar{\psi}_2 = \bar{\psi}_4 = 0$ and $\mathbb{K}[\mathfrak{t}^\sigma]^{W_0^\sigma} = \mathbb{K}[\bar{\psi}_1, \bar{\psi}_3, \bar{\psi}_5]$.

Observe that $\dim \mathcal{S}^5(\tilde{\mathfrak{t}}^*)^W = 1$. Let $\tilde{\varphi}_2$ be a nonzero element in $\mathcal{S}^5(\tilde{\mathfrak{t}}^*)^W$. By our remarks in (4.6) we have $\nu(\tilde{\varphi}_2) \geqslant 1$. Thus, it can be assumed that $\tilde{\varphi}_2 = \tilde{\alpha}^2\psi_2 + \tilde{\varphi}_2^{(0)}$ where $\tilde{\varphi}_2^{(0)} \in \mathcal{S}^5(\mathfrak{t}^*)^{W_0}$. Clearly, $\nu(\tilde{\varphi}_2) = 1$. Next note that $\dim \mathcal{S}^9(\tilde{\mathfrak{t}}^*)^W = 2$. As $\nu(\tilde{\varphi}_2\tilde{\varphi}_1^2) = 3$, we can find $\tilde{\varphi}_5 \in \mathcal{S}^9(\tilde{\mathfrak{t}}^*)^W \setminus \mathbb{K}\tilde{\varphi}_2\tilde{\varphi}_1^2$ for which $\nu(\tilde{\varphi}_5) \leqslant 2$. This element is a basic invariant of $\mathcal{S}(\tilde{\mathfrak{t}}^*)^W$.

Let $\{\varphi_1, \varphi_2, \varphi_3, \varphi_4\} \subset \mathbb{K}[\tilde{\mathfrak{t}}^\sigma]^{W^\sigma}$ be the generating set obtained in part (6). Choose $\tilde{\varphi}_1 \in \mathcal{S}^2(\tilde{\mathfrak{t}}^*)^W$ such that $\tilde{\varphi}_1 = \tilde{\alpha}^2 + \tilde{\varphi}_1^{(0)}$ where $\tilde{\varphi}_1^{(0)} \in \mathcal{S}^2(\mathfrak{t}^*)^{W_0}$. As $\tilde{\varphi}_1^3 \in \mathcal{S}^6(\tilde{\mathfrak{t}}^*)^W$ we can find a nonzero $\tilde{\varphi}_3 \in \mathcal{S}^6(\tilde{\mathfrak{t}}^*)^W$ such that $\tilde{\varphi}_3 = \tilde{\alpha}^4 a + \tilde{\alpha}^2 b + c$ for some $a, b, c \in \mathcal{S}(\mathfrak{t}^*)^{W_0}$. Suppose $a = 0$. Since $\nu(\tilde{\varphi}_3) \geqslant 1$, we then have $b \neq 0$. Since b is a W_0-invariant of degree 4, it is a polynomial in ψ_1 and ψ_3. Then $b_{|\mathfrak{t}^\sigma} \neq 0$. Consequently, $\tilde{\varphi}_{3|\tilde{\mathfrak{t}}^\sigma} = \lambda\varphi_2 + \mu\varphi_1^3$ where either $\lambda \neq 0$ or $\mu \neq 0$. Part (6) now yields $\nu(\tilde{\varphi}_3) \geqslant 2$ forcing $a \neq 0$, a contradiction. Thus, $\nu(\tilde{\varphi}_3) = 2$, and it can be assumed without loss that $a = \psi_1$.

Next we observe that $\dim \mathcal{S}^8(\tilde{\mathfrak{t}}^*)^W = 3$. Because $\nu(\tilde{\varphi}_1^4) = 4$ and $\nu(\tilde{\varphi}_3\tilde{\varphi}_1) = 3$ by the above, the set $\mathcal{S}^8(\tilde{\mathfrak{t}}^*)^W \setminus \{\mathbb{K}\tilde{\varphi}_1^4 \oplus \mathbb{K}\tilde{\varphi}_2\tilde{\varphi}_1\}$ contains an element of the form $\tilde{\alpha}^4 a' + \tilde{\alpha}^2 b' + c'$ with $a', b', c' \in \mathcal{S}(\mathfrak{t}^*)^{W_0}$, say $\tilde{\varphi}_4$. The element $\tilde{\varphi}_4$ is a basic invariant of $\mathcal{S}(\tilde{\mathfrak{t}}^*)^W$. As $\nu(\tilde{\varphi}_1^6) = 6$ and $\nu(\tilde{\varphi}_3\tilde{\varphi}_1^3) = 5$, we can find a basic invariant $\tilde{\varphi}_6 \in \mathcal{S}^{12}(\tilde{\mathfrak{t}}^*)^W$ for which $\nu(\tilde{\varphi}_6) \leqslant 4$.

Suppose for a contradiction that $a' = 0$. In view of our remarks in (4.6) we then have $b' \neq 0$ and $\nu(\tilde{\varphi}_4) = 1$. Consequently,

$$\sum_{i=2}^{6} \nu(\tilde{\varphi}_i) \leqslant 1 + 2 + 1 + 2 + 4 = 10.$$

Since in the present case $\dim \mathfrak{g}(1) = 20$, Proposition 4.8 shows that we have equalities everywhere and the elements $\tilde{\varphi}_i^{(\nu)}$ with $2 \leqslant i \leqslant 6$ are algebraically independent in $\mathcal{S}(\mathfrak{t}^*)^{W_0}$. But then $\nu(\tilde{\varphi}_5) = 2$ and $\nu(\tilde{\varphi}_6) = 4$, forcing $\tilde{\varphi}_5^{(\nu)} \in \mathbb{K}\psi_1\psi_2 \oplus \mathbb{K}\psi_4$ and $\tilde{\varphi}_6^{(\nu)} \in \mathbb{K}\psi_1^2 \oplus \mathbb{K}\psi_3$. As $\nu(\tilde{\varphi}_4) = 1$, we have $\tilde{\varphi}_4^{(\nu)} = \mu_1\psi_5 + \mu_2\psi_1\psi_3 + \mu_3\psi_2^2 + \mu_4\psi_1^3$ for some $\mu_i \in \mathbb{K}$. Because $\tilde{\varphi}_2^{(\nu)}, \ldots, \tilde{\varphi}_6^{(\nu)}$ are algebraically independent, the above shows that $\mu_1 \neq 0$. In conjunction with our earlier remarks this yields that for $\tilde{\varphi}_{4|\tilde{\mathfrak{t}}^\sigma} \in \mathbb{K}[\tilde{\mathfrak{t}}^\sigma]^{W^\sigma}$ we have $\nu\big(\tilde{\varphi}_{4|\tilde{\mathfrak{t}}^\sigma}\big) = 1$. On the other hand, $\tilde{\varphi}_{4|\tilde{\mathfrak{t}}^\sigma}$ is a linear combination of φ_3, $\varphi_1\varphi_2$ and φ_1^4. Since $\nu(\varphi_3) = 2$, $\nu(\varphi_1\varphi_2) = 3$ and $\nu(\varphi_1^4) = 4$, this is impossible. Therefore, $a' \neq 0$ and $\nu(\tilde{\varphi}_4) = 2$.

Since a' is a W_0-invariant of degree 4, we have $a' = \lambda'\psi_3 + \mu'\psi_1^2$. Hence $a'_{|\mathfrak{t}^\sigma} = \lambda'\bar{\psi}_3 + \mu'\bar{\psi}_1^2 \neq 0$. It follows that $\nu\big(\tilde{\varphi}_{4|\tilde{\mathfrak{t}}^\sigma}\big) = 2$. Then the above implies that $\tilde{\varphi}_{4|\tilde{\mathfrak{t}}^\sigma} = \eta\varphi_3$ for some $\eta \in \mathbb{K}^\times$. Since $\varphi_3^{(\nu)} = \eta^{-1}a'_{|\mathfrak{t}^\sigma}$ is algebraically independent of $\varphi_2^{(\nu)}$ by part (6), we now derive that $\lambda' \neq 0$. Since $\tilde{\varphi}_6^{(\nu)} \in \mathbb{K}\psi_3 \oplus \mathbb{K}\psi_1^2$, it follows that we can adjust $\tilde{\varphi}_6$ by a suitable linear combination of $\tilde{\varphi}_1^2\tilde{\varphi}_4$ and $\tilde{\varphi}_3^2$ to achieve $\nu(\tilde{\varphi}_6) \leqslant 3$. Then

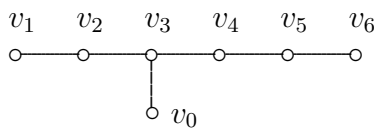
$$\sum_{i=2}^{6} \nu(\tilde{\varphi}_i) \leqslant 1 + 2 + 2 + 2 + 3 = \frac{1}{2}\dim \mathfrak{g}(1).$$

Proposition 4.8 now shows that $\nu(\tilde{\varphi}_5) = 2$, $\nu(\tilde{\varphi}_6) = 3$, and $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ admits a good generating system for $\tilde{e}$. Hence $\mathcal{S}(\mathfrak{g}_{\tilde{e}})^{\mathfrak{g}_{\tilde{e}}}$ is a graded polynomial algebra, and the degrees of basic invariants are $1, 4, 4, 6, 7, 9$.

(8) Finally, suppose $\mathfrak{g}$ is of type $\mathbf{E}_7$. The degrees of basic invariants in $\mathcal{S}(\tilde{\mathfrak{t}})^W$ are $2, 6, 8, 10, 12, 14, 18$, and our arguments in part (7) are not easily adapted to the present situation. Fortunately, this will not be necessary because a suitable for us system of basic invariants in $\mathcal{S}(\tilde{\mathfrak{t}})^W$ is already recorded in the literature. It has been constructed in [14] with the help of computer-aided calculations.

We have to adopt the notation of [14]. So let $\Delta' = \{v_0, v_1, \dots, v_6\}$ be a basis of the root system Φ with the simple roots numbered as follows:

(18)



Since all roots in Φ are conjugate under W, we may (and will) assume that $\tilde{\alpha} = v_1$. Let $\{v_0^*, v_1^*, \dots, v_6^*\}$ be the basis of $\tilde{\mathfrak{t}}$ such that $v_i(v_j^*) = \delta_{ij}$ for all $0 \leqslant i, j \leqslant 6$. As $(v_1|v_1) = 2$, it follows from (18) that $\tilde{h} = 2v_1^* - v_2^*$, whilst our choice of $\tilde{\alpha}$ ensures that $v_i^* \in \operatorname{Ker}\tilde{\alpha}$ for $i \in \{0, 2, \dots, 6\}$. For a root system type $\mathbf{E}_7$, the *distinguished functionals* $t_1, t_2, \dots, t_7$ are defined in [14] by the following formulae:

$$t_1 = -\frac{2}{3}v_0^* + v_1^*, \qquad t_2 = -\frac{2}{3}v_0^* - v_1^* + v_2^*, \quad t_3 = -\frac{2}{3}v_0^* - v_2^* + v_3^*,$$
$$t_4 = \frac{1}{3}v_0^* - v_3^* + v_4^*, \quad t_5 = \frac{1}{3}v_0^* - v_4^* + v_5^*, \qquad t_6 = \frac{1}{3}v_0^* - v_5^* + v_6^*, \qquad t_7 = \frac{1}{3}v_0^* - v_6^*.$$

We are particularly interested in the basic invariants $A_2, A_6, A_8, A_{10}, A_{12}, A_{14}, A_{18}$ of $\mathcal{S}(\tilde{\mathfrak{t}})^W$ displayed in [14, Appendix 2]. These are presented as polynomials in the elementary symmetric functions $s_1, s_2, \dots, s_7$ of the distinguished functionals $t_1, t_2, \dots, t_7$. The coefficients of these polynomials are of no importance to us, but we need to examine the monomials in $s_1, s_2, \dots, s_7$ that occur in the A_k's.

Note that $\tilde{\alpha}(t_1) = v_1(t_1) = 1$, $\tilde{\alpha}(t_2) = -v_1(v_1^*) = -1$, and $\tilde{\alpha}(t_i) = v_1(t_i) = 0$ for $3 \leqslant i \leqslant 7$. It follows that $\nu(s_1) = \nu(s_1(t_1, \dots, t_7)) = 0$ and $\nu(s_i) = \nu(s_i(t_1, \dots, t_7)) = 1$ for $2 \leqslant i \leqslant 7$. Therefore,

$$\nu(s_1^{j_1} s_2^{j_2} \cdots s_7^{j_7}) = j_2 + \cdots + j_7 \qquad (\forall\, j_k \in \mathbb{Z}_+,\ 1 \leqslant k \leqslant 7).$$

Taking this into account and using the explicit formulae for $A_2, A_6, A_8, A_{10}, A_{12}, A_{14}, A_{18}$ in [14, Appendix 2] one finds out that $\nu(A_2) = 1$, $\nu(A_6) \leqslant 2$, $\nu(A_8) \leqslant 2$, $\nu(A_{10}) \leqslant 2$, $\nu(A_{12}) \leqslant 3$, $\nu(A_{14}) \leqslant 3$ and $\nu(A_{18}) \leqslant 4$. It follows that

$$\nu(A_6) + \nu(A_8) + \nu(A_{10}) + \nu(A_{12}) + \nu(A_{14}) + \nu(A_{18}) \leqslant 2 + 2 + 2 + 3 + 3 + 4 = 16.$$

Since in the present case the derived subalgebra of $\mathfrak{g}(0)$ has codimension 1 in $\mathfrak{g}(0)$ and is isomorphic to $\mathfrak{so}_{12}$, we have $\frac{1}{2}\dim\mathfrak{g}(1) = (\dim\mathfrak{g} - \dim\mathfrak{so}_{12} - 3)/4 = (133 - 66 - 3)/4 = 16$. Proposition 4.8 now shows that $\nu(A_6) = \nu(A_8) = \nu(A_{10}) = 2$, $\nu(A_{12}) = \nu(A_{14}) = 3$ and

$\nu(A_{18}) = 4$. This implies that $\mathcal{S}(\mathfrak{g}_{\tilde{e}})^{\mathfrak{g}_{\tilde{e}}}$ is a graded polynomial algebra in seven variables, and the degrees of basic invariants are $1, 4, 6, 8, 9, 11, 14$.

We summarise the results of this subsection:

Corollary 4.9. *If $\mathfrak{g}$ is not of type $\mathbf{E}_8$, then $\mathcal{S}(\tilde{\mathfrak{t}})^W$ contains a homogeneous generating system $\varphi_1, \varphi_2, \ldots, \varphi_l$ such that $\deg \varphi_1 = 2$ and $\mathcal{S}(\mathfrak{t})^{W_0} = \mathbb{K}[\varphi_2^{(\nu)}, \ldots, \varphi_l^{(\nu)}]$.*

Proof. We have shown that under the above assumption on $\mathfrak{g}$ there exists a homogeneous system of basic invariants $\varphi_1, \varphi_2, \ldots, \varphi_l$ in $\mathcal{S}(\tilde{\mathfrak{t}})^W$ such that $\deg \varphi_1 = 2$ and the elements $\varphi_2^{(\nu)}, \ldots, \varphi_l^{(\nu)}$ are algebraically independent in $\mathcal{S}(\mathfrak{t})^{W_0}$. So the result follows by comparing the Hilbert series of the graded polynomial algebra $\mathcal{S}(\mathfrak{t})^{W_0}$ and its graded subalgebra $K[\varphi_2^{(\nu)}, \ldots, \varphi_l^{(\nu)}]$. $\square$

Remark 4.2. If $\mathfrak{g}$ is of type $\mathbf{E}_8$, then one can show by using *ad hoc* arguments that $\mathcal{S}(\mathfrak{g}_{\tilde{e}})^{\mathfrak{g}_{\tilde{e}}}$ contains an element of degree 4 linearly independent of $\tilde{e}^4$. Looking at the degrees of basic invariants in $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ and taking into account (16) one can observe that this element is not of the form ${}^{\tilde{e}}F$ with $F \in \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$. It follows that in type $\mathbf{E}_8$ the elements in $\mathcal{O}_{\min}$ do not admit good generating systems in $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$. Combining this with Proposition 4.8 one obtains that for any homogeneous generating system $\varphi_1, \varphi_2, \ldots, \varphi_8$ in $\mathcal{S}(\tilde{\mathfrak{t}})^W$ with $\deg \varphi_1 = 2$ the elements $\varphi_2^{(\nu)}, \ldots, \varphi_l^{(\nu)}$ are algebraically dependent in $\mathcal{S}(\mathfrak{t})^{W_0}$. This is in sharp contrast with Corollary 4.9.

4.8. In this subsection we assume that $\mathfrak{g}$ is of type $\mathbf{E}_8$, so that $l = \operatorname{rk} \mathfrak{g} = 8$. We adopt the notation introduced in (3.9) and (3.10). In particular, $\mathfrak{n} = \mathfrak{g}(1) \oplus \mathfrak{g}(2)$. As before, we identify $\mathfrak{l}^*$ with $\operatorname{Ann}(\mathfrak{n}) \subset \mathfrak{g}_{\tilde{e}}^*$ and $\mathfrak{g}(1)^*$ with $\operatorname{Ann}(\mathfrak{l} \oplus \mathfrak{g}(2))$.

In the course of proving Theorem 3.14 we established that the principal open subset $Y = \mathfrak{g}_{\tilde{e}}^* \setminus \operatorname{Ann}(\tilde{e})$ of $\mathfrak{g}_{\tilde{e}}^*$ decomposes as $Y \cong \big((N/(N,N)\big) \times \big(\operatorname{Ann}(\mathfrak{g}_1) \cap Y\big)$. It follows that restricting regular functions on Y to $\operatorname{Ann}(\mathfrak{g}(1)) \cap Y$ we get algebra isomorphisms

$$\big(\mathcal{S}(\mathfrak{g}_{\tilde{e}})[1/\tilde{e}]\big)^N \cong \mathcal{S}(\mathfrak{l})[\tilde{e}, 1/\tilde{e}] \quad \text{and} \quad \mathcal{S}(\mathfrak{g}_{\tilde{e}})^{\mathfrak{g}_{\tilde{e}}}[1/\tilde{e}] = \big(\mathcal{S}(\mathfrak{g}_{\tilde{e}})[1/\tilde{e}]\big)^{\mathfrak{g}_{\tilde{e}}} \cong (\mathcal{S}(\mathfrak{l})^L[\tilde{e}, 1/\tilde{e}].$$

The standard Poisson bracket of $\mathcal{S}(\mathfrak{g}_{\tilde{e}})$ (induced by Lie product) gives $\mathcal{S}(\mathfrak{g}_{\tilde{e}})[1/\tilde{e}]$ a Poisson algebra structure. As $\mathfrak{n}$ is a Heisenberg Lie algebra, the subspace $\mathcal{S}^2(\mathfrak{g}(1))/\tilde{e}$ is closed under the Poisson bracket of $\mathcal{S}(\mathfrak{g}_{\tilde{e}})[1/\tilde{e}]$, i.e., $\mathcal{S}^2(\mathfrak{g}(1))/\tilde{e}$ is a Lie subalgebra of $\mathcal{S}(\mathfrak{g}_{\tilde{e}})[1/\tilde{e}]$. This Lie algebra acts faithfully on $\mathfrak{g}(1)$ and is isomorphic to $\mathfrak{sp}(\mathfrak{g}(1))$. Since the bilinear form $\langle \cdot, \cdot \rangle$ is $(\operatorname{ad} \mathfrak{l})$-invariant, $\mathfrak{l}$ acts on $\mathfrak{g}(1)$ as a Lie subalgebra of $\mathfrak{sp}(\mathfrak{g}(1))$. From this it follows that for every $x \in \mathfrak{l}$ there exists a unique $\omega(x) \in \mathcal{S}^2(\mathfrak{g}(1))$ for which $x + \omega(x)/\tilde{e} \in \big(\mathcal{S}(\mathfrak{g}_{\tilde{e}})[1/\tilde{e}]\big)^N$. Since the restriction of $\omega(x)$ to $\operatorname{Ann}(\mathfrak{g}(1))$ is zero, $x + \omega(x)/\tilde{e}$ is the preimage of x in $(\mathcal{S}(\mathfrak{g}_{\tilde{e}})[1/\tilde{e}])^N$. It is straightforward to see that the map $\omega\colon \mathfrak{l} \to \mathcal{S}^2(\mathfrak{g}(1))$ is linear.

Let $x_1, \ldots, x_m$ be a basis of $\mathfrak{l}$. Given an L-invariant $H = Q(x_1, \ldots, x_m)$ in $\mathcal{S}(\mathfrak{l})$ we define

$$\widehat{H} := Q(x_1 + \omega(x_1)/\tilde{e}, \ldots, x_m + \omega(x_m)/\tilde{e}).$$

Clearly, $\widehat{H} \in \mathcal{S}(\mathfrak{g}_{\tilde{e}})^{\mathfrak{g}_{\tilde{e}}}[1/\tilde{e}]$. Let $k = k(H)$ be the smallest integer for which $\tilde{e}^k \widehat{H} \in \mathcal{S}(\mathfrak{g}_{\tilde{e}})^{\mathfrak{g}_{\tilde{e}}}$, and set $\widetilde{H} := \tilde{e}^k \widehat{H}$. Let $\omega(H)$ denote the "constant term" of $\widetilde{H}$ with respect to $\tilde{e}$, so that $\omega(H)$ equals the restriction of $\widetilde{H}$ to $\mathrm{Ann}(\tilde{e})$. Note that $k = \deg \omega(H) - \deg H$.

Let $\{H_1, \dots, H_{l-1}\}$ be a homogeneous generating set for $\mathcal{S}(\mathfrak{l})^L$. Then both $\{\widehat{H_1}, \dots, \widehat{H_{l-1}}\}$ and $\{\widetilde{H_1}, \dots, \widetilde{H_{l-1}}\}$ generate the $\mathbb{K}[\tilde{e}, 1/\tilde{e}]$-algebra $(\mathcal{S}(\mathfrak{g}_{\tilde{e}})^{\mathfrak{g}_{\tilde{e}}}[\tilde{e}, 1/\tilde{e}]$.

Lemma 4.10. *The algebra $\mathcal{S}(\mathfrak{g}_{\tilde{e}})^{\mathfrak{g}_{\tilde{e}}}$ is free if and only if $\mathcal{S}(\mathfrak{l})^L$ contains a homogeneous generating system $H_1, \dots, H_{l-1}$ such that the elements $\omega(H_1), \dots, \omega(H_{l-1})$ are algebraically independent.*

Proof. First suppose that $\mathcal{S}(\mathfrak{l})^L$ contains a required set of generators $H_1, \dots, H_{l-1}$, and let $H \in \mathcal{S}(\mathfrak{g}_{\tilde{e}})^{\mathfrak{g}_{\tilde{e}}}$. Then H is a polynomial in $\widetilde{H_i}$, $\tilde{e}$ and $1/\tilde{e}$, hence can be presented as a finite sum $H = \sum_{p \in \mathbb{Z}} \tilde{e}^p Q_p$, where Q_i are nontrivial polynomials in $\widetilde{H_i}$. Since $\omega(H_i)$ are algebraically independent by our assumption, all Q_i are coprime to $\tilde{e}$. This implies that $H = \sum_{p \geqslant 0} \tilde{e}^p Q_p$, that is H is a polynomial in $\widetilde{H_i}$ and $\tilde{e}$.

Now suppose that $\mathcal{S}(\mathfrak{g}_{\tilde{e}})^{\mathfrak{g}_{\tilde{e}}}$ is a free algebra generated over $\mathbb{K}$ by $T_1, \dots, T_l$. Without loss of generality we may (and will) assume that all T_i are homogeneous and $T_l = \tilde{e}$. As $(\mathcal{S}(\mathfrak{g}_{\tilde{e}})[1/\tilde{e}])^{\mathfrak{g}_{\tilde{e}}} \cong \mathcal{S}(\mathfrak{l})^L[\tilde{e}, 1/\tilde{e}]$, there exist $H_1, \dots, H_{l-1} \in \mathcal{S}(\mathfrak{l})^L$ and $b_1, \dots, b_{l-1} \in \mathbb{Z}$ such that $T_i = e^{b_i} \widehat{H_i}$ for $1 \leqslant i \leqslant l-1$. Moreover, $H_1, \dots, H_{l-1}$ generate $\mathcal{S}(\mathfrak{l})^L$. Because $e^{b_i} \widehat{H_i}$ is both irreducible and regular, it must be that $b_i = k_i$. Hence $T_i = \widetilde{H_i}$ for all $i < l$.

Assume for a contradiction that $P(\omega(H_1), \dots, \omega(H_{l-1})) = 0$ for a nonzero polynomial $P \in \mathbb{K}[X_1, \dots, X_{l-1}]$. Then $H' := P(T_1, \dots, T_{l-1})/\tilde{e}$ is a *regular* $\mathfrak{g}_{\tilde{e}}$-invariant. On the other hand, H' is *uniquely* expressed as a polynomial in $T_1, \dots, T_{l-1}$ with coefficients in $\mathbb{K}[\tilde{e}, 1/\tilde{e}]$, and $\mathcal{S}(\mathfrak{g}_{\tilde{e}})^{\mathfrak{g}_{\tilde{e}}} = \mathbb{K}[T_1, \dots, T_{l-1}, \tilde{e}]$ by our assumption. But then $H' \notin \mathcal{S}(\mathfrak{g}_{\tilde{e}})^{\mathfrak{g}_{\tilde{e}}}$. By contradiction, the result follows. $\square$

It is well-known that in the present case L has type $\mathbf{E}_7$ and the stationary subgroup $K = L \cap G_e$ is a simple algebraic group of type $\mathbf{E}_6$. Recall from (3.9) that e is a generic point of the L-module $\mathfrak{g}(1)$ and K is a genetic stabiliser in L; see Definition 4.1. It is also known that K is the derived subgroup of the intersection of two opposite maximal parabolics of L. More precisely, $K = (L^+ \cap L^-, L^+ \cap L^-)$, where L^+ (resp., L^-) is the the normaliser in L of the line spanned by a highest (resp., lowest) weight vector of the L-module $\mathfrak{g}(1)$. These primitive vectors will be denoted by e^+ and e^-, respectively. Note that $[e^+, e^-]$ is a nonzero multiple of $\tilde{e}$ (equivalently, $\langle e^+, e^- \rangle \neq 0$). Choose a maximal torus $\hat{\mathfrak{t}}$ in the Levi subalgebra $\mathrm{Lie}\,(L^+ \cap L^-)$ of $\mathfrak{l}$ and set $\mathfrak{t} := \hat{\mathfrak{t}} \cap \mathfrak{k}$. It is easy to see that $\mathfrak{t}$ is a maximal torus in $\mathfrak{k} = \mathrm{Lie}\, K$.

It follows from the above description that $\mathfrak{g}(1)^K = \mathbb{K}e^+ \oplus \mathbb{K}e^-$. Hence it can be assumed without loss of generality that $e = e^+ + e^-$. Since the nondegenerate skew-symmetric form $\langle \cdot, \cdot \rangle$ is L-invariant, $\mathfrak{g}(1) \cong \mathfrak{g}(1)^*$ as L-modules. Set $w_+ := \langle e^+, \cdot \rangle$, $w_- := \langle e^-, \cdot \rangle$, and $v := \langle e, \cdot \rangle$. As explained in the proof Theorem 3.14, the orbit $(\mathrm{Ad} L)e$ has codimension 1 in $\mathfrak{g}(1)$ and $(\mathrm{Ad}\, L)(\mathbb{K}^\times e) = (\mathrm{Ad}\, G(0))e$ is Zariski open in $\mathfrak{g}(1)$. Hence the tangent space $\mathfrak{l}{\cdot}v$ (at v) to the orbit $L{\cdot}v$ has codimension 1 in $\mathfrak{g}(1)^* = \mathfrak{g}(0){\cdot}v = \hat{\mathfrak{t}}{\cdot}v + \mathfrak{l}{\cdot}v$. As K is reductive and $(\mathfrak{g}(1)^*)^K = \mathbb{K}w \oplus \mathbb{K}w^*$ is $\hat{\mathfrak{t}}$-stable, we have that $\mathfrak{l}{\cdot}v = \mathbb{K}h_0{\cdot}v \oplus V_0$, where $h_0 \in \hat{\mathfrak{t}}$ is orthogonal to $\mathfrak{k}$ with respect to the Killing form and $V_0 = \big\{\langle x\,, \cdot \rangle \mid \langle x, e^+ \rangle = \langle x, e^- \rangle = 0\big\}$.

As in (4.5), we regard the dual space $\hat{\mathfrak{t}}^*$ as a subspace of $\mathfrak{l}^* \subset \mathfrak{g}_{\tilde{e}}^*$. We identify $\mathfrak{t}^*$ with the subspace $\{\gamma \in \hat{\mathfrak{t}}^* \mid \gamma(h_0) = 0\}$ and view $v \in \mathfrak{g}(1)^*$ as a linear function on $\mathfrak{g}_{\tilde{e}}$ vanishing on $\mathfrak{l} \oplus \mathbb{K}\tilde{e}$. Set $W' := N_L(\hat{\mathfrak{t}})/Z_L(\hat{\mathfrak{t}})$ and $W'_0 := N_K(\mathfrak{t})/Z_K(\mathfrak{t})$ (these are reflection groups of type $\mathbf{E}_7$ and $\mathbf{E}_6$, respectively).

Lemma 4.11. *Let $H_1, \dots, H_{l-1}$ be a homogeneous generating set in $\mathcal{S}(\mathfrak{l})^L$. Then the elements $\omega(H_1), \dots, \omega(H_{l-1})$ are algebraically independent if and only if their restrictions to $\mathfrak{t}^* \oplus \mathbb{K}v$ are.*

Proof. Recall that $\widehat{H_i} \in \mathcal{S}(\mathfrak{g}_{\tilde{e}})^{\mathfrak{g}_{\tilde{e}}}$ and $\omega(H_i) = \widetilde{H_i}_{|\mathrm{Ann}(\tilde{e})}$ for $1 \leqslant i \leqslant l-1$. It follows that all $\omega(H_i)$ are invariant under the coadjoint action of the semidirect product $\mathfrak{l} \ltimes \mathfrak{g}(1)$, where $\mathfrak{g}(1)$ is considered as a commutative Lie algebra.

By our earlier remarks, the L-saturation of $\mathbb{K}v$ is dense in $\mathfrak{g}(1)^*$. Also, for the same v, but regarded as an element of $(\mathfrak{l} \ltimes \mathfrak{g}(1))^*$, we have $\big(\mathrm{ad}^*\mathfrak{g}(1)\big)v \cong (\mathfrak{l}/\mathfrak{k})^*$. Combining this two facts we obtain natural embeddings

$$\mathbb{K}[\omega(H_1), \dots, \omega(H_{\ell-1})] \hookrightarrow \mathbb{K}[\mathfrak{l}^* \oplus \mathbb{K}v]^{\mathfrak{k} \ltimes \mathfrak{g}(1)} \hookrightarrow \mathbb{K}[\mathfrak{k}^* \oplus \mathbb{K}v]^{\mathfrak{k}} \hookrightarrow \mathbb{K}[\mathfrak{t}^* \oplus \mathbb{K}v].$$

As the composition of these embeddings is also an embedding, the result follows. □

Now we wish to express $\omega(H_i)$ in terms of polynomial invariants for W'. Let $\alpha \in \mathfrak{g}_{\tilde{e}}^*$ be such that $\alpha(\tilde{e}) = 1$ and $\alpha(\mathfrak{l} \oplus \mathfrak{g}(1)) = 0$, and set

$$\mathfrak{s} := \mathfrak{t}^* \oplus \mathbb{K}v \oplus \mathbb{K}\alpha.$$

Then the restriction of $\omega(H_i)$ to $\mathfrak{t}^* \oplus \mathbb{K}v$ is equal to the "constant term" (with respect to $\tilde{e}$) of $\widehat{H_i}_{|\mathfrak{s}}$. We thus need to describe the restrictions of $\widehat{H_i}$ to $\mathfrak{s}$. Let $\hat{\mathfrak{t}}^\perp \subset \mathfrak{l}$ be the orthogonal complement to $\hat{\mathfrak{t}} = \mathfrak{t} \oplus \mathbb{K}h_0$ with respect to the Killing form, so that $\mathfrak{l} = \mathfrak{t} \oplus \mathbb{K}h_0 \oplus \hat{\mathfrak{t}}^\perp$. Since $\hat{\mathfrak{t}}^\perp$ is spanned by root vectors of $\mathfrak{l}$ with respect to $\hat{\mathfrak{t}}$ and $e = e^+ + e^-$, it is straightforward to see that $[[\hat{\mathfrak{t}}^\perp, e], e] = 0$.

Lemma 4.12. *The following statements are true:*

(a) $(x + \omega(x)/\tilde{e})_{|\mathfrak{s}} = 0$ *for all* $x \in \hat{\mathfrak{t}}^\perp$;

(b) $(x + \omega(x)/\tilde{e})_{|\mathfrak{s}} = x$ *for all* $x \in \mathfrak{t}$;

(c) $(h_0 + \omega(h_0)/\tilde{e})_{|\mathfrak{s}} = a(e^+ + e^-)^2/\tilde{e}$ *for some* $a \in \mathbb{K}^\times$.

Proof. Let $x \in \mathfrak{l}$ and let $\beta = \gamma + \lambda v + \mu\alpha \in \mathfrak{s}$, where $\gamma \in \mathfrak{t}^*$ and $\lambda, \mu \in \mathbb{K}$. We shall calculate the value of $x + \omega(x)/\tilde{e}$ at β. Without loss of generality we may assume that both λ and μ are nonzero. Recall that $e = e^+ + e^-$. Since $x + \omega(x)/\tilde{e}$ is N-invariant, we can replace β by $\big(\mathrm{Ad}^*(\exp \frac{\lambda}{\mu} \mathrm{ad}\, e)\big)\beta$. Because $v = \langle e, \cdot \rangle = -(\mathrm{ad}^* e)\alpha$ and $[e, [e, \hat{\mathfrak{t}}^\perp]] = 0$, we have that $\big(\mathrm{Ad}^*(\exp \frac{\lambda}{\mu} \mathrm{ad}\, e)\big)\beta = \gamma + \delta + \mu\alpha$, where δ is a nonzero linear function on $\mathfrak{g}_{\tilde{e}}$ which vanishes on $\mathfrak{t} \oplus \hat{\mathfrak{t}}^\perp \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2)$ and has the property that

$$\delta(h_0) = \frac{\lambda^2}{2\mu} v([h_0, e^+ + e^-]).$$

Thus, $x + \omega(x)/\tilde{e}$ is zero on $\mathfrak{s}$ for all $x \in \hat{\mathfrak{t}}^\perp$, proving (a). If $x \in \mathfrak{t}$, then $(x + \omega(x)/\tilde{e})(\beta) = x(\gamma) = x(\beta)$, hence (b). Finally, $(h_0 + \omega(h_0)/\tilde{e})(\beta)$ is a nonzero multiple of λ^2/μ, showing that the restriction of $h_0 + \omega(h_0)/\tilde{e}$ is a nonzero multiple of $(e^+ + e^-)^2/\tilde{e}$. □

For $1 \leqslant i \leqslant l-1$, set $\varphi_i := H_{i|\hat{\mathfrak{t}}^*}$. Then φ_i is homogeneous element in $\mathcal{S}(\hat{\mathfrak{t}})^{W'}$. It can be presented uniquely as

$$\varphi_i = \sum_{j=0}^{\nu} \varphi_i^{(j)} h_0^j \qquad \Big(\varphi_i^{(j)} \in \mathcal{S}(\mathfrak{t})^{W_0'}, \quad \varphi_i^{(\nu)} \neq 0, \quad \nu = \nu(i)\Big).$$

Recall that h_0 spans the orthogonal complement to $\mathfrak{t}$ in $\hat{\mathfrak{t}}$ with respect to the Killing form.

Corollary 4.13. *In the above notation,*

$$\omega(H_i)_{|\mathfrak{t}^* \oplus \mathbb{K}v} = a^{\nu(i)} \varphi_i^{(\nu)} (e^+ + e^-)^{2\nu(i)} \qquad (1 \leqslant i \leqslant l-1).$$

Proof. This follows from Lemmas 4.11 and 4.12. □

Summing up the material of this subsection we obtain the following result:

Theorem 4.14. *The algebra $\mathcal{S}(\mathfrak{g}_{\tilde{e}})^{\mathfrak{g}_{\tilde{e}}}$ is free if and only if there is a homogeneous generating system $\varphi_1, \ldots, \varphi_7$ in $\mathcal{S}(\hat{\mathfrak{t}})^{W'}$ such that the elements $\varphi_1^{(\nu)} h_0^{\nu(1)}, \ldots, \varphi_7^{(\nu)} h_0^{\nu(7)}$ are algebraically independent.*

In type $\mathbf{E}_7$ it is difficult to calculate Weyl invariants by hand, and the system of basic invariants used in the final part of (4.7) is not very helpful in the present situation. Since this paper is already quite long, we leave the $\mathbf{E}_8$ case open for the time being.

4.9. Assume now that $\mathfrak{g}$ is not of type $\mathbf{A}_n$ or $\mathbf{E}_8$. Let $\tilde{e}$ be as before and put $\mathfrak{p} := \mathfrak{n}_{\mathfrak{g}}(\mathbb{K}\tilde{e})$. Recall that $\mathfrak{p} = \mathfrak{g}(0) \oplus \mathfrak{g}(1) \oplus \mathfrak{g}(2)$ is a parabolic subalgebra of $\mathfrak{g}$. We are now going to apply our results on $\mathcal{S}(\mathfrak{g}_{\tilde{e}})^{\mathfrak{g}_{\tilde{e}}}$ to prove that the semi-centre of the universal enveloping algebra $U(\mathfrak{p})$ is a polynomial algebra. This will confirm a conjecture of Joseph for the parabolic subalgebra $\mathfrak{p}$.

Corollary 4.15. *Under the above assumptions, the semi-centre $U(\mathfrak{p})^{[\mathfrak{p},\mathfrak{p}]}$ is a polynomial algebra in $l = \operatorname{rk} \mathfrak{g}$ variables.*

Proof. Since $\mathfrak{g}$ is not of type $\mathbf{A}$, we have $[\mathfrak{p},\mathfrak{p}] = \mathfrak{g}_{\tilde{e}}$ and $\mathfrak{p} = \mathbb{K}\tilde{h} \oplus \mathfrak{g}_{\tilde{e}}$. Let $v \in \mathcal{S}(\mathfrak{p})^{[\mathfrak{p},\mathfrak{p}]}$ and write $v = \tilde{h}^k v_k + \tilde{h}^{k-1} v_{k-1} + \cdots + v_0$ with $v_i \in \mathcal{S}(\mathfrak{g}_{\tilde{e}})$. Since $\tilde{e} \in \mathfrak{z}(\mathfrak{g}_{\tilde{e}})$ and $\tilde{e}{\cdot}\tilde{h}^i = -2i\tilde{h}^{i-1}\tilde{e}$ for all $i > 0$, we get $0 = \tilde{e}{\cdot}v = -\sum_{i=1}^{k} 2i\tilde{h}^{i-1}\tilde{e}v_i$. This yields $\mathcal{S}(\mathfrak{p})^{[\mathfrak{p},\mathfrak{p}]} = \mathcal{S}(\mathfrak{g}_{\tilde{e}})^{[\mathfrak{p},\mathfrak{p}]} = \mathcal{S}(\mathfrak{g}_{\tilde{e}})^{\mathfrak{g}_{\tilde{e}}}$. Arguing in a similar fashion we obtain

$$U(\mathfrak{p})^{[\mathfrak{p},\mathfrak{p}]} = U(\mathfrak{g}_{\tilde{e}})^{\mathfrak{g}_{\tilde{e}}} = Z(\mathfrak{g}_{\tilde{e}}),$$

where $Z(\mathfrak{g}_{\tilde{e}})$ stands for the centre of $U(\mathfrak{g}_{\tilde{e}})$. As $\mathcal{S}(\mathfrak{g}_{\tilde{e}})^{\mathfrak{g}_{\tilde{e}}}$ is a polynomial algebra in l variables, there exist algebraically independent homogeneous elements $v_1, \ldots, v_l \in \mathcal{S}(\mathfrak{p})$ such that $\mathcal{S}(\mathfrak{p})^{[\mathfrak{p},\mathfrak{p}]} = \mathbb{K}[v_1, \ldots, v_l]$.

Let $r_i = \deg v_i$, where $1 \leqslant i \leqslant l$, and let $(U_k)_{k \geqslant 0}$ denote the standard filtration of $U(\mathfrak{p})$. Using the symmetrisation map $\mathcal{S}(\mathfrak{p}) \xrightarrow{\sim} U(\mathfrak{p})$ it is easy to observe that there exist $u_1, \ldots, u_l \in U(\mathfrak{p})^{[\mathfrak{p},\mathfrak{p}]}$ such that $u_i \in U_{r_i}$ and $\operatorname{gr}_{r_i}(u_i) = v_i$ for all i. Since the u_i's are central in $U(\mathfrak{g}_{\tilde{e}})$, the standard filtered-graded techniques now shows that $U(\mathfrak{p})^{[\mathfrak{p},\mathfrak{p}]} = \mathbb{K}[u_1, \ldots, u_l]$ is a polynomial algebra in l variables. □

5. THE NULL-CONES IN TYPE A

5.1. In this section we assume that $\mathfrak{g} = \mathfrak{gl}(\mathbb{V})$ where $\dim \mathbb{V} \geqslant 2$. Our goal is to prove that for every $e \in \mathcal{N}(\mathfrak{g})$ the null-cone $\mathcal{N}(e) \subset \mathfrak{g}_e^*$ has the expected codimension, i.e., $\dim \mathcal{N}(e) = \dim \mathfrak{g}_e - n$. According to Theorem 4.2, the variety $\mathcal{N}(e)$ is the zero locus of ${}^eF_1, \ldots, {}^eF_n$, where $F_i = \kappa^{-1}(\Delta_i)$. Thanks to the Affine Dimension Theorem, in order to compute $\dim \mathcal{N}(e)$ it suffices to find an n-dimensional subspace $W \subset \mathfrak{g}_e^*$ such that $W \cap \mathcal{N}(e) = 0$. This will be achieved in a somewhat roundabout way: first we shall construct a larger subspace $V^* \subset \mathfrak{g}_e^*$ for which the restrictions ${}^eF_{i|V^*}$ can be described more or less explicitly and then show that V^* contains an n-dimensional subspace transversal to $\mathcal{N}(e)$.

For $m \in \{1, \ldots, k\}$, we partition the set $\{1, \ldots, m\}$ into pairs $(j, m-j+1)$. If m is odd, then there will be a "singular pair" in the middle consisting of the singleton $\{(m+1)/2\}$. We denote by V_m the subspace of $\mathfrak{g}_e$ spanned by all $\xi_i^{j,s}$ with $i + j = m + 1$, and set $V := \bigoplus_{m \geqslant 1} V_m$. Using the basis $\{(\xi_i^{j,s})^*\}$ of $\mathfrak{g}_e^*$ dual to the basis $\{\xi_i^{j,s}\}$, we shall regard the dual spaces V_i^* and V^* as subspaces of $\mathfrak{g}_e^*$. Since $\mathbb{K}[V^*] \cong \mathcal{S}(V)$, the restrictions $\hat{\varphi}_i := {}^eF_{i|V^*}$ are elements of $\mathcal{S}(V)$. For $\bar{s} := (s_1, \ldots, s_k)$ with $s_i \in \mathbb{Z}_{\geqslant 0}$ we set $|\bar{s}| := s_1 + s_2 + \ldots + s_k$.

Lemma 5.1. *Suppose $0 \leqslant q \leqslant d_k$. Then $\hat{\varphi}_{n-q} \in \mathcal{S}(V_k)$. More precisely,*

$$\hat{\varphi}_{n-q} = \sum_{|\bar{s}|=q} a(\bar{s})\, \xi_1^{k,d_k-s_k} \xi_2^{k-1,d_{k-1}-s_{k-1}} \cdots \xi_k^{1,d_1-s_1} \quad \textit{for some } \ a(\bar{s}) \in \mathbb{K}^\times .$$

Proof. (a) According to Lemma 4.3, ${}^eF_{n-q}$ is a sum of monomials $\xi_1^{\sigma(1),t_1} \ldots \xi_k^{\sigma(k),t_k}$, where σ is a permutation of $\{1, \ldots, k\}$ and $t_1, \ldots, t_k$ are nonnegative integers. Such a monomial does not vanish on V^* only if $\sigma(k) = 1$, $\sigma(k-1) \in \{1, 2\}$ and $\sigma(j) \leqslant k + 1 - j$ for all $j \leqslant k$. Since σ is a permutation, we then have $\sigma(k-1) = 2$, $\sigma(k-2) = 3$ and, in general, $\sigma(j) = k + 1 - j$.

From (12) we see that $\xi_j^{k-j+1,\, d_{k-j+1}-s_{k-j+1}}\, \xi_{k-j+1}^{j,\, d_j-s_j}$ has weight $2(d_j + d_{k-j+1} - s_j - s_{k-j+1})$ with respect to $\operatorname{ad} h$. As a consequence, the h-weight of

$$\xi_1^{k,d_k-s_k} \xi_2^{k-1,d_{k-1}-s_{k-1}} \cdots \xi_k^{1,d_1-s_1}$$

equals $2(n - k - |\bar{s}|)$. Since $\deg F_{n-q} = n - q$ and $\deg {}^eF_{n-q} = k$, this implies that only monomials with $|\bar{s}| = q$ can occur in ${}^eF_{n-q}$. Because $|\bar{s}| = q \leqslant d_k \leqslant d_i$ and all s_i are nonnegative, we have that $s_i \leqslant d_j$ for all i, j. This means that every ξ_{k-i+1}^{i,d_i-s_i} is a nonzero element of $\mathfrak{g}_e$.

(b) We now prove by induction on k that every $a(\bar{s})$ is nonzero. If $k = 1$, then $V = \mathfrak{g}_e$, $\hat{\varphi}_{n-q} = {}^eF_{n-q} = a(q)\xi_1^{1,d_1-q}$; and clearly $a(q) \neq 0$. If $k = 2$, then $\big(\mathrm{ad}^*\xi_1^{1,1}\big) \cdot V^* \subset V^*$. From this it follows that the Poisson bracket $\{\xi_1^{1,1}, \hat{\varphi}_{n-q}\}$ is zero. On the other hand,

$$\{\xi_1^{1,1},\, \hat{\varphi}_{n-q}\} \;=\; \sum_{i=0}^{q-1} \big(a(q-i, i) - a(q-i-1, i+1)\big) \xi_1^{2,d_2-i} \xi_2^{1,d_1-q+i+1} .$$

As the monomials $\xi_1^{2,d_2-i} \xi_2^{1,d_1-q+i+1}$ with $0 \leqslant i \leqslant q-1$ are nonzero in $\mathcal{S}(\mathfrak{g}_e)$, all coefficients $a(\bar{s})$ with $|\bar{s}| = q$ must be equal. If one of them is zero, then all are zeros. Assume that this

is the case. By Lemma 4.3, we then have

$$ {}^eF_{n-q} \;=\; \sum_{|\bar{s}|=q} b(\bar{s})\, \xi_1^{1,d_1-s_1}\xi_2^{2,d_2-s_2}\,, \quad \text{where } \; b(\bar{s}) \in \mathbb{K}. $$

Let s_2 be the largest integer with $b(s_1,s_2)\neq 0$. As $s_2 \leqslant q \leqslant d_2$, the element $\xi := \xi_2^{1,d_1-d_2+s_2}$ is nonzero in $\mathfrak{g}_e$. As ${}^eF_{n-q}$ belongs to the Poisson centre of $\mathcal{S}(\mathfrak{g}_e)$, we have $\{\xi,\, {}^eF_{n-q}\} = 0$. On the other hand,

$$ \{\xi,\, {}^eF_{n-q}\} \;=\; b(s_1,s_2)\,\xi_1^{1,s_1}\xi_2^{1,d_1} \;+\; (\text{multiples of monomials of the form } \xi_2^{1,*}\xi_2^{2,*}\,). $$

Since the RHS is nonzero, we reach a contradiction, proving the lemma in case $k=2$.

(c) Now suppose $k>2$, and set $\mathfrak{g}' := \mathfrak{gl}\big(\mathbb{V}[1]\oplus\mathbb{V}[k]\big)$ and $\mathfrak{g}'' := \mathfrak{gl}\big(\mathbb{V}[2]\oplus\cdots\oplus\mathbb{V}[k-1]\big)$. These are Lie subalgebras of $\mathfrak{g}$ (embedded diagonally), and $e=e'+e''$ where e' and e'' are the restrictions of e to the e-stable subspaces $\mathbb{V}[1]\oplus\mathbb{V}[k]$ and $\mathbb{V}[2]\oplus\cdots\oplus\mathbb{V}[k-1]$.

We adopt the notation introduced in the course of proving Lemma 4.3 and express F_{n-q} as a polynomial in the variables E_{ij}. Let T be a monomial of F_{n-q} such that $T_{|V^*}$ is a nonzero multiple of a monomial of degree k in ξ_{k-j+1}^{j,d_j-s_j}. Then $T=T'T''$, where T' and T'' are polynomials in the variables coming from $\mathfrak{g}'$ and $\mathfrak{g}''$, respectively. Suppose the restriction of T' to V^* equals $a'\xi_1^{k,d_k-s_k}\xi_k^{1,d_1-s}$, where $a'\in\mathbb{K}^\times$. Then T' is a monomial of $F'_{p'}\in\mathcal{S}(\mathfrak{g}')^{\mathfrak{g}'}$ for $p'=d_1+d_2+2-s_1-s_2$. Likewise, T'' is a monomial of $F_{p''}\in\mathcal{S}(\mathfrak{g}'')^{\mathfrak{g}''}$ for $p''=n-q-p'$. It follows that $a(\bar{s})=a(s_1,s_k)a(s_2,\dots,s_{k-1})$ where the coefficients $a(s_1,s_k)$ and $a(s_2,\dots,a_{k-1})$ are related to the nilpotent elements $e'\in\mathfrak{g}'$ and $e''\in\mathfrak{g}''$, respectively.

Note that $e'\in\mathfrak{g}'$ has two Jordan blocks of sizes d_1+1 and d_2+1, and $a(s_1,s_k)$ is the coefficient of $\xi_1^{k,d_k-s_k}\xi_k^{1,d_1-s_1}$ in the expression for $\hat{\varphi}_{p'}$. This coefficient is nonzero by part (b). The coefficient $a(s_2,\dots,s_{k-1})$ arises in a similar way from the nilpotent element $e''\in\mathfrak{g}''$. Since $\mathfrak{g}''\cong\mathfrak{gl}_{n-d_1-d_k-2}$ we can apply the inductive hypothesis to conclude that $a(s_2,\dots,s_{k-1})\neq 0$. Therefore every $a(\bar{s})$ is nonzero, as wanted. $\square$

5.2. Our next goal is to describe the zero locus $X=X^{(d_k)}$ of $\hat{\varphi}_n,\hat{\varphi}_{n-1},\dots,\hat{\varphi}_{n-d_k}$ in V_k^*. Denote by $X_{\bar{s}}$ the subspace of V_k^* consisting of all $\gamma\in V_k^*$ such that $\xi_{k-i+1}^{i,d_i-t}(\gamma)=0$ for $0\leqslant t<s_i$. Let $\mathbf{e}_i$ be the k-tuple whose i-th component equals 1 and the other components are zero.

Lemma 5.2. *The variety X is a union of subspaces. More precisely, $X=\bigcup_{|\bar{s}|=d_k+1} X_{\bar{s}}$.*

Proof. Let $X^{(q)}\subset V_k^*$ be the zero locus of $\hat{\varphi}_n,\hat{\varphi}_{n-1},\dots,\hat{\varphi}_{n-q}$. We are going to prove by induction on q that $X^{(q)}$ is a union of subspaces in V_k^* and the irreducible components of $X^{(q)}$ correspond bijectively to the k-tuples $\bar{s}$ with $|\bar{s}|=q+1$. When $q=0$, our set of functions is a singleton containing $\hat{\varphi}_n=\xi_k^{1,d_1}\xi_{k-1}^{2,d_2}\cdots\xi_1^{k,d_k}$. Therefore, $X^{(0)}$ is the union of k hyperplanes in V_k^* defined by the equations $\xi_{k-i+1}^{i,d_i}=0$, where $1\leqslant i\leqslant k$.

Assume that $X^{(q-1)}$ is a union of subspaces of V_k^* parametrised by the k-tuples of size q. Let $\bar{s}$ be a k-tuple of size $q-1$ and let $X_{\bar{s}}$ be the irreducible component of $X^{(q-1)}$ corresponding to $\bar{s}$. Now consider an arbitrary monomial $f:=\xi_1^{k,d_k-t_k}\xi_2^{k-1,d_{k-1}-t_{k-1}}\cdots\xi_k^{1,d_1-t_1}$

with $\sum t_i = q$, i.e., a typical summand of $\hat{\varphi}_{n-q}$. If $\bar{t} = (t_1, \dots, t_k) \neq \bar{s}$, then there exists an index i such that $t_i < s_i$. But then ξ_{k-i+1}^{i,d_i-t_i}, and hence f, vanishes on $X_{\bar{s}}$. This shows that the restriction of $\hat{\varphi}_{n-q}$ to $X_{\bar{s}}$ coincides, up to a nonzero multiple, with that of $\xi_1^{k,d_k-s_k}\xi_2^{k-1,d_{k-1}-s_{k-1}}\cdots\xi_k^{1,d_1-s_1}$. As a consequence, the zero locus of $\hat{\varphi}_n, \hat{\varphi}_{n-1}, \dots, \hat{\varphi}_{n-q}$ in $X_{\bar{s}}$ is the union of k linear subspaces $X_{\bar{s}+\mathbf{e}_i}$, where $1 \leqslant i \leqslant k$. Then $X^{(q)} = \bigcup_{|\bar{s}|=q+1} X_{\bar{s}}$, and the statement follows by induction on q. □

5.3. By Lemma 5.2, all irreducible components of the variety $X^{(d_k)} \subset V_k^*$ have dimension equal to $\dim V_k - (d_k + 1)$. Hence there is a linear subspace $W_k \subset V_k^*$ such that $\dim W_k = d_k + 1$ and $W_k \cap X^{(d_k)} = 0$.

Proposition 5.3. *There exists an n-dimensional linear subspace $W = \bigoplus_{m\geqslant 1} W_m$ in V^* such that $W_m \subset V_m^*$ for all m and $W \cap \mathcal{N}(e) = 0$.*

Proof. We argue by induction on k. If $k = 1$, then $\mathcal{N}(e) = 0$ and there is nothing to prove. So assume that $k \geqslant 2$, and set $\mathfrak{g}_k := \mathfrak{gl}\big(\mathbb{V}[k]\big)$ and $\bar{\mathfrak{g}} := \mathfrak{gl}\big(\mathbb{V}[2] \oplus \cdots \oplus \mathbb{V}[k]\big)$. These Lie algebras are embedded diagonally into $\mathfrak{g}$, and we regard the dual spaces $\bar{\mathfrak{g}}^*$ and $\mathfrak{g}_k^*$ as subspaces of $\mathfrak{g}^*$. Note that $e = e_k + \bar{e}$ where e_k and $\bar{e}$ are the restrictions of e to $\mathbb{V}[k]$ and $\mathbb{V}[2] \oplus \cdots \oplus \mathbb{V}[k]$, respectively. Clearly, e_k is a regular nilpotent element in $\mathfrak{g}_k \cong \mathfrak{gl}_{d_k+1}$ and $\bar{e} \in \bar{\mathfrak{g}} \cong \mathfrak{gl}_{n-d_k-1}$ is a nilpotent element with Jordan blocks of sizes $d_1 + 1, \dots, d_{k-1} + 1$. For $1 \leqslant i \leqslant n - d_k - 1$, put $\bar{F}_i := F_{i|\bar{\mathfrak{g}}^*}$. Restricting the principal minors Δ_i from $\mathfrak{g}$ to $\bar{\mathfrak{g}}$ it is easy to see that the homogeneous generating system $\bar{F}_i, \dots, \bar{F}_{n-d_k-1}$ of $\mathcal{S}(\bar{\mathfrak{g}})^{\bar{\mathfrak{g}}}$ is good for $\bar{e} \in \bar{\mathfrak{g}}$.

Next we observe that $\bar{\mathfrak{g}}_{\bar{e}}$ is a Lie subalgebra of $\mathfrak{g}_e$ spanned by all $\xi_i^{j,s}$ with $1 \leqslant i, j < k$. Hence we may identify the dual space $(\bar{\mathfrak{g}}_{\bar{e}})^*$ with the linear span of $\big\{(\xi_i^{j,s})^* \mid 1 \leqslant i, j < k\big\}$ in $\mathfrak{g}_e^*$. For every $i \in \{1, \dots, n - d_k - 1\}$ the restriction of ${}^eF_i$ to $(\bar{\mathfrak{g}}_{\bar{e}})^*$ equals ${}^{\bar{e}}\bar{F}_i$.

Note that $V_m^* \subset (\bar{\mathfrak{g}}_{\bar{e}})^*$ for $m < k$ and $V_k^* \cap (\bar{\mathfrak{g}}_{\bar{e}})^* = 0$. By our inductive hypothesis, there exists a subspace $\overline{W} = \bigoplus_{m=1}^{k-1} W_i$ such that $\dim \overline{W} = n - d_k - 1$ and $\overline{W} \cap \mathcal{N}(\bar{e}) = 0$. Choose a $(d_k + 1)$-dimensional subspace W_k in V_k^* with $W_k \cap X^{(d_k)} = 0$. Such a subspace exists by Lemma 5.2. Now set $W := \overline{W} \oplus W_k$. Then $\dim W = n$.

We claim that $W \cap \mathcal{N}(e) = 0$. By Lemma 5.1, for $n - d_k \leqslant i \leqslant n$ the restriction $\hat{\varphi}_i = {}^eF_{i|V^*}$ belongs to $\mathcal{S}(V_k)$. Therefore, the zero locus of $\hat{\varphi}_n, \dots, \hat{\varphi}_{n-d_k}$ in V^* coincides with $\big(\bigoplus_{m=1}^{k-1} V_m^*\big) \times X^{(d_k)}$. Since $W_k \cap X^{(d_k)} = 0$, we obtain $W \cap \mathcal{N}(e) \subset \bigoplus_{m=1}^{k-1} V_m^* \subset (\bar{\mathfrak{g}}_{\bar{e}})^*$. But then $W \cap \mathcal{N}(e) \subset \overline{W} \cap \mathcal{N}(\bar{e}) = 0$, and we are done. □

The following is the main result of this section:

Theorem 5.4. *Let e be an arbitrary nilpotent element in $\mathfrak{g} = \mathfrak{gl}_n$. Then all irreducible components of the null-cone $\mathcal{N}(e)$ have codimension n in $\mathfrak{g}_e^*$ and hence ${}^eF_1, \dots, {}^eF_n$ is a regular sequence in $\mathcal{S}(\mathfrak{g}_e)$.*

5.4. Let $X \subset \mathbb{A}_{\mathbb{K}}^d$ be a Zariski closed set and let $x = (x_1, \dots, x_d)$ be a point of X. Let I denote the defining ideal of X in the coordinate algebra $\mathcal{A} = \mathbb{K}[X_1, \dots, X_d]$ of $\mathbb{A}_{\mathbb{K}}^d$. Each nonzero $f \in \mathcal{A}$ can be expressed as a polynomial in $X_1 - x_1, \dots, X_d - x_d$, say $f = f_k + f_{k+1} + \cdots$, where f_i is a homogeneous polynomial of degree i in $X_1 - x_1, \dots, X_d - x_d$ and $f_k \neq 0$. We set $\mathrm{in}_x(f) := f_k$ and denote by $\mathrm{in}_x(I)$ the linear span of all $\mathrm{in}_x(f)$ with $f \in I \setminus \{0\}$. This

is an ideal of $\mathcal{A}$, and the affine scheme $TC_x(X) := \operatorname{Spec} \mathcal{A}/\mathrm{in}_x(I)$ is called the *tangent cone* to X at x. Note that $(I \cap \mathfrak{m}_x^k)_{k\geqslant 1}$ is a descending filtration of I, and the scheme $TC_x(X)$ is nothing but the prime spectrum of the graded algebra $\mathrm{gr}_{\mathfrak{m}_x}\mathcal{A}/\mathrm{gr}\, I$. It is well-known that the projectivised tangent cone $\mathbb{P}TC_x(X) \subset \mathbb{P}T_x(X)$ is isomorphic to the special divisor of the blow-up of X at x; see [9, Ex. IV-24] for example. Consequently, for X irreducible, all irreducible components of $TC_x(X)$ have dimension equal to $\dim X$.

Corollary 5.5. *Let $\mathcal{N}$ be the nilpotent cone of $\mathfrak{g} = \mathfrak{gl}_n$ and $F_i = \kappa^{-1}(\Delta_i)$ where $1 \leqslant i \leqslant n$. Let $e \in \mathcal{N}$ and $r = \dim \mathfrak{g}_e$. Then $TC_e(\mathcal{N}) \cong \mathbb{A}_{\mathbb{K}}^{n^2-r} \times \operatorname{Spec} \mathcal{S}(\mathfrak{g}_e)/({}^eF_1, \dots, {}^eF_n)$ as affine schemes.*

Proof. Since the map $x \mapsto (x, \cdot\,)$ takes e to χ and $\mathcal{N}$ isomorphically onto the zero locus of the ideal $J = (F_1, \dots, F_n) \subset \mathcal{S}(\mathfrak{g})$, the scheme $TC_e(\mathcal{N})$ is isomorphic to $\operatorname{Spec} \mathcal{S}(\mathfrak{g})/\mathrm{in}_\chi(J)$. As $\chi(f) = 1$, we have $\mathfrak{g} = \mathbb{K}f \oplus e^\perp$ where $e^\perp$ is the orthogonal complement to $\mathbb{K}e$ in $\mathfrak{g}$. For $1 \leqslant i \leqslant n$ write $F_i = f^{k(i)}p_{0,i} + f^{k(i)-1}p_{1,i} + \dots + p_{k(i),i}$, where $p_{j,i} \in \mathcal{S}(e^\perp)$ and $p_{0,i} \neq 0$. According to Corollary 7.2, we have $p_{0,i} = {}^eF_i$. Since $e^\perp$ and $f - \chi(f)$ lie in the maximal ideal of χ in $K[\mathfrak{g}^*] = \mathcal{S}(\mathfrak{g})$, it follows that $\mathrm{in}_\chi(F_i) = {}^eF_i$ for all $1 \leqslant i \leqslant n$.

By Theorem 5.4, ${}^eF_1, \dots, {}^eF_n$ is a regular sequence in $\mathcal{S}(\mathfrak{g}_e)$. Therefore, it is also a regular sequence in $\mathcal{S}(\mathfrak{g})$. Since $J = (F_1, \dots, F_n)$, it follows that the ideal $\mathrm{in}_\chi(J)$ is generated by ${}^eF_1, \dots, {}^eF_n$; see [26, Prop. 2.1]. As a consequence,

$$TC_e(\mathcal{N}) \cong \operatorname{Spec} \mathcal{S}(\mathfrak{g})/({}^eF_1, \dots, {}^eF_n) \cong \operatorname{Ann}(\mathfrak{g}_e) \times \operatorname{Spec} \mathcal{S}(\mathfrak{g}_e)/({}^eF_1, \dots, {}^eF_n)$$

as affine schemes. Since $\dim \operatorname{Ann}(\mathfrak{g}_e) = n^2 - r$, the result follows. □

Conjecture 5.1. *If $\mathfrak{g} = \mathfrak{gl}_n$, then for any $e \in \mathcal{N}$ the scheme $TC_e(\mathcal{N})$ is reduced.*

Remark 5.1.
1. It can be shown that in the subregular $\mathbf{G}_2$ case the variety $TC_e(\mathcal{N}(\mathfrak{g}))_{\mathrm{red}}$ is isomorphic to an affine space, but the scheme $TC_e(\mathcal{N}(\mathfrak{g}))$ is *not* reduced. Thus, one cannot expect Conjecture 5.1 to be true for any simple Lie algebra.

2. It follows from Corollary 5.5 that for $\mathfrak{g} = \mathfrak{gl}_n$ the affine variety $TC_e(\mathcal{N}(\mathfrak{g}))_{\mathrm{red}}$ is isomorphic to $\mathbb{A}_{\mathbb{K}}^m \times \mathcal{N}(e)$ where $m = \dim \mathfrak{g} - \dim \mathfrak{g}_e$. It is possible that this isomorphism continues to hold for any reductive Lie algebra $\mathfrak{g}$. If this is the case, then the variety $\mathcal{N}(e)$ is always equidimensional.

3. Although the variety $\mathcal{N}(e)$ is irreducible in some cases, in general it has many irreducible components. Due to Theorem 2.1(iii), in order to prove Conjecture 5.1 it would be sufficient to show that every irreducible component of $\mathcal{N}(e)$ intersects with $(\mathfrak{g}_e^*)_{\mathrm{reg}}$. Describing the irreducible components of $\mathcal{N}(e)$ for $\mathfrak{g} = \mathfrak{gl}_n$ appears to be an interesting combinatorial problem.

6. Miscellany

6.1. In this section, Conjecture 0.1 will be verified in some special cases. The idea is that, for some $e \in \mathcal{N}(\mathfrak{g})$, we can prove that the algebra $\mathbb{K}[\mathfrak{g}_e]^{\mathfrak{g}_e}$ is graded polynomial. If, in addition, it is known that $\mathfrak{g}_e \simeq \mathfrak{g}_e^*$ as $\mathfrak{g}_e$-modules, then we conclude that Conjecture 0.1 holds for such e.

We briefly recall the structure of the centralizer $\mathfrak{g}_e$ of a nilpotent element $e \in \mathfrak{g}$ as described by the Dynkin–Kostant theory; see e.g. [5, Ch. 4]. Let $\{e, h, f\}$ be an $\mathfrak{sl}_2$-triple and $\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}(i)$ the corresponding $\mathbb{Z}$-grading. Then $\mathfrak{g}_e = \bigoplus_{i\geqslant 0} \mathfrak{g}_e(i)$ and $\mathfrak{g}_e(0)$ is a maximal reductive subalgebra of $\mathfrak{g}_e$. Moreover, $\mathfrak{g}_e(0) = \mathfrak{z}_{\mathfrak{g}}(e, f) = \mathfrak{z}_{\mathfrak{g}}(e, h, f)$. The element e is called *even* if all the eigenvalues of $\operatorname{ad} h$ are even, i.e., if $\mathfrak{g}(i) = 0$ for i odd. By a classical result of Dynkin, e is even if and only if $\mathfrak{g}(1) = 0$; see [7, Thm. 8.3]. In this case the weighted Dynkin diagram of e contains only labels 0 and 2.

In the following theorem, we use some concepts and results on (1) semi-direct products of Lie algebras and (2) contractions of Lie algebras. All the necessary definitions can be found in [16, Sect. 4] and [28, Ch. 7], respectively.

Theorem 6.1. *Suppose that a principal nilpotent element in $\mathfrak{g}_e(0)$ is also principal in $\mathfrak{g}(0)$ and e is even. Then $\mathbb{K}[\mathfrak{g}_e]^{\mathfrak{g}_e}$ is a polynomial algebra and the degrees of basic invariants (=free homogeneous generators) are the same as those for $\mathbb{K}[\mathfrak{g}(0)]^{\mathfrak{g}(0)}$.*

Proof. Associated to the triple (e, h, f) and the corresponding $\mathbb{Z}$-grading, we have three Lie algebras: $\mathfrak{g}(0)$, $\mathfrak{g}_e$, and $\mathfrak{q} := \mathfrak{g}_e(0) \ltimes \big(\bigoplus_{i\geqslant 2} \mathfrak{g}_e(i)\big)$. Here the sign $\ltimes$ refers to the semi-direct product of Lie algebras and the space $\bigoplus_{i\geqslant 2} \mathfrak{g}_e(i)$ in $\mathfrak{q}$ is regarded as commutative Lie algebra. Clearly, $\dim \mathfrak{q} = \dim \mathfrak{g}_e$. The equality $\dim \mathfrak{g}(0) = \dim \mathfrak{g}_e$ is equivalent to the fact that e is even. Thus, all three Lie algebras have the same dimension. Here we obtain the chain of Lie algebra contractions:

$$\mathfrak{g}(0) \rightsquigarrow \mathfrak{g}_e \rightsquigarrow \mathfrak{q}\ .$$

The first contraction can be described as follows. Consider the curve $e(t) := e + tf \in \mathfrak{g}$, $t \in \mathbb{K}$. For $t \neq 0$, the element $e(t)$ is G-conjugate to h. Therefore, $\mathfrak{g}_{e(t)}$ is isomorphic to $\mathfrak{g}_h = \mathfrak{g}(0)$. Hence $\lim_{t\to 0} \mathfrak{g}_{e(t)} = \mathfrak{g}_e$ yields a contraction of $\mathfrak{g}(0)$ to $\mathfrak{g}_e$. Using the terminology of [16, Sect. 9], one can say that the passage $\mathfrak{g}_e \rightsquigarrow \mathfrak{q}$ is an *isotropy contraction* of $\mathfrak{g}_e$. By [16, Theorem 6.2], the algebra of invariants of the adjoint representation of $\mathfrak{q}$ is polynomial. Moreover, if a regular nilpotent element of $\mathfrak{g}_e(0)$ is also regular in $\mathfrak{g}(0)$, then by [16, Theorem 9.5] the invariant algebras $\mathbb{K}[\mathfrak{g}(0)]^{\mathfrak{g}(0)}$ and $\mathbb{K}[\mathfrak{q}]^{\mathfrak{q}}$ have the same Krull dimension and the same degrees of basic invariants. It is easily seen that the algebra of invariants of the adjoint representation can only become larger under contractions. Since $\mathbb{K}[\mathfrak{g}(0)]^{\mathfrak{g}(0)}$ and $\mathbb{K}[\mathfrak{q}]^{\mathfrak{q}}$ appear to be "the same", the intermediate algebra $\mathbb{K}[\mathfrak{g}_e]^{\mathfrak{g}_e}$ must also be polynomial with the same degrees of basic invariants. $\square$

6.2. By a result of Elashvili–Panyushev (Appendix to [12]), the assumptions on e in Theorem 6.1 precisely mean that e is a member of a *rectangular principal nilpotent pair*. The general theory of principal nilpotent pairs (to be abbreviated as *pn-pairs* from now) was developed by Victor Ginzburg [12]. Because the general notion is not needed here, we only recall the definition of a rectangular pn-pair.

Definition 6.1. A pair of nilpotent elements $\mathbf{e} = (e_1, e_2)$ is called a *rectangular pn-pair* if $\dim(\mathfrak{g}_{e_1} \cap \mathfrak{g}_{e_2}) = \operatorname{rk} \mathfrak{g}$ and there are pairwise commuting $\mathfrak{sl}_2$-triples (e_1, h_1, f_1) and (e_2, h_2, f_2).

We say that a nilpotent orbit $G{\cdot}e$ is *very nice* if e is a member of a rectangular pn-pair and $\mathfrak{g}_e \simeq \mathfrak{g}_e^*$ as $\mathfrak{g}_e$-modules.

Corollary 6.2. *Suppose $G{\cdot}e$ is very nice. Then Conjecture 0.1 holds for $\mathfrak{g}_e$ and $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ admits a good generating system for e.*

A classification of rectangular pn-pairs is obtained by Elashvili and Panyushev in [12, Appendix]. From that classification one derives a description of very nice orbits. It is worth mentioning that for a pn-pair (e_1, e_2) the condition that $G{\cdot}e_1$ is very nice does not in general guarantee that so is $G{\cdot}e_2$.

Although there are not too many very nice nilpotent orbits (especially in the exceptional Lie algebras), this approach does provide new examples supporting Conjecture 0.1. The examples for $\mathfrak{sl}_n$ and $\mathfrak{sp}_{2n}$ are not new; see Section 4.

6.3. Below we list the very nice nilpotent orbits in exceptional Lie algebras. For each such orbit we give the Dynkin-Bala-Carter label, the weighted Dynkin diagram, and the degrees of basic invariants for $\mathcal{S}(\mathfrak{g}_e)^{\mathfrak{g}_e}$.

$\boxed{\mathbf{E}_6}$	$\mathbf{D}_4$	0–0–2–0–0 | 2	$1,1,2,2,3,3$
$\boxed{\mathbf{E}_7}$	$\mathbf{E}_6$	0–2–0–2–2–2 | 0	$1,1,1,1,2,2,2$
	$\mathbf{A}_2{+}\mathbf{A}_4$	0–0–0–2–0–0 | 0	$1,2,2,2,3,3,4$
	$\mathbf{A}_6$	0–2–0–2–0–0 | 0	$1,1,2,2,2,2,3$

Let us give some details on the unique orbit for $\mathbf{E}_6$. Here $\dim \mathfrak{g}_e = 18$ and $\mathfrak{g}_e$ is the direct sum of the 2-dimensional centre and the Takiff Lie algebra $\mathfrak{s}$ modelled on $\mathfrak{sl}_3$. Namely, $\mathfrak{s}$ is just the semi-direct product $\mathfrak{sl}_3 \ltimes \mathfrak{sl}_3$.

6.4. The very nice nilpotent orbits in classical Lie algebras are described below.

1°. $\mathfrak{g} = \mathfrak{sl}_n$. Here e is a member of a rectangular pn-pair if and only if the corresponding partition of n is a rectangle (i.e., all the parts are equal). That is, we may assume that $n = rs$ and the partition of e is $(r, \dots, r)$, with s parts. We also write $e \sim (\underbrace{r, \dots, r}_{s})$ for this. It is harmless but technically easier to work with $\mathfrak{g} = \mathfrak{gl}_n$ in place of $\mathfrak{sl}_n$. Then $\mathfrak{g}_e$ is a generalised Takiff Lie algebra modelled on $\mathfrak{gl}_s$. More precisely, consider the Lie algebra $\mathfrak{gl}_s \otimes \mathbb{K}[t]$ (t is an undeterminate) and take the quotient with respect to the ideal generated by t^r. It is easily seen that $\mathfrak{g}_e \simeq \mathfrak{g}_e^*$. (See [22] and [16, Sect. 11] for more results on generalised Takiff Lie algebras.) The second member of the rectangular pn-pair is given by the conjugate partition $(s, \dots, s)$, with r parts. This situation is symmetric and both nilpotent orbits are very nice.

2°. $\mathfrak{g} = \mathfrak{sp}_{2n}$. Here e is a member of a rectangular pn-pair if and only if the corresponding partition of $2n$ is a rectangle whose sides have different parity. That is, we may assume that $2n = rs$, where r is even and s is odd. The situation here is not symmetric. Only the orbit corresponding to the partition $(s, \dots, s)$ with r parts is very nice.

3°. $\mathfrak{g} = \mathfrak{so}_n$. Here we have to distinguish the series **B** and **D**.

- If n is odd, then the only suitable partitions are the rectangles whose both sides are odd. That is, $n = rs$, where r and s are odd. Then $e \sim (\underbrace{s, \dots, s}_{r})$. Here both members of the rectangular pn-pair give rise to very nice orbits.
- For n even, there are more possibilities for rectangular pn-pairs.

(1) If a partition of n is rectangle with both even sides, then neither of the respective orbits is very nice.

(2) If $n = rs + 1$, where r, s are odd, the there is a rectangular pn-pair (e_1, e_2) with $e_1 \sim (\underbrace{s, \dots, s}_{r}, 1)$ and $e_2 \sim (\underbrace{r, \dots, r}_{s}, 1)$. Here both members of the rectangular pn-pair give rise to very nice orbits.

(3) If $n = r + s$, where r, s are odd, then there is a rectangular pn-pair (e_1, e_2) with $e_1 \sim (s, \underbrace{1, \dots, 1}_{r})$ and $e_2 \sim (r, \underbrace{1, \dots, 1}_{s})$. Here neither of the respective orbits is very nice.

7. Appendix

Here we give an alternative (elementary) proof of Proposition 0.1, which is inspired by an unpublished result of J.-Y. Charbonnel (private communication).

Let $e^\perp \subset \mathfrak{g}$ be the orthogonal complement of $\mathbb{K}e$. Since $(e, f) = 1$, we have $\mathfrak{g} = \mathbb{K}f \oplus e^\perp$. Take a homogeneous $F \in \mathcal{S}(\mathfrak{g})^G$ and express it as

$$F = f^k p_0 + f^{k-1} p_1 + \dots + p_k,$$

where $p_i \in \mathcal{S}(e^\perp)$.

Lemma 7.1. *For any homogeneous $F \in \mathcal{S}(\mathfrak{g})^G$ we have that $p_0 \in \mathcal{S}(\mathfrak{g}_e)^{G_e}$.*

Proof. If $g \in G_e$, then $(\operatorname{Ad} g)e^\perp \subset e^\perp$ and $(\operatorname{Ad} g)f \in f + e^\perp$. Therefore,

$$F = g{\cdot}F = (g{\cdot}p_0) f^k + f^{k-1} p_1' + \dots + p_k'$$

for some $p_i' \in \mathcal{S}(e^\perp)$. Since $g{\cdot}p_0 \in \mathcal{S}(e^\perp)$, this shows that p_0 is G_e-invariant.

Recall that $\mathfrak{g} = \mathfrak{g}_e \oplus \operatorname{Im} \operatorname{ad} f$. Choose a basis $y_1, \dots, y_t$ of $e^\perp \cap \operatorname{Im} \operatorname{ad} f$. If p_0 is not an element of $\mathcal{S}(\mathfrak{g}_e)$, then renumbering the y_i's if necessary we may assume that $p_0 = y_1^s q_0 + y_1^{s-1} q_1 + \dots + q_s$, where $q_i \in \mathbb{K}[y_2, \dots, y_t]$ and $q_0 \neq 0$. Since the Killing form of $\mathfrak{g}$ induces a nondegenerate pairing between $\operatorname{Im} \operatorname{ad} e$ and $\operatorname{Im} \operatorname{ad} f$, there is a $z \in \mathfrak{g}$ such that $([e, z], y_1) \neq 0$, $([e, z], y_i) = 0$ for all $i \neq 1$, and $([e, z], f) = 0$. Note that $([z, y_1], e) = (y_1, [e, z]) \neq 0$, but $([z, y_i], e) = 0$ for all $i \neq 1$, and $([z, x], e) = -(z, [e, x]) = 0$ for all $x \in \mathfrak{g}_e$. Rescaling z if need be, we may assume that $([z, y_1], e) = 1$. Then $[z, y_1] = f + e^\perp$ and

$[z, f] \in e^{\perp}$, implying

$$\{z, F\} = (sy_1^{s-1}q_0 + (s-1)y_1^{s-2}q_1 + \cdots + q_1)f^{k+1} + \text{ (terms with smaller powers of } f).$$

This, however, contradicts the equality $\{z, F\} = 0$. $\square$

Corollary 7.2. *For any homogeneous $F \in \mathcal{S}(\mathfrak{g})^G$ we have that ${}^eF = p_0$ and ${}^eF \in \mathcal{S}(\mathfrak{g}_e)^{G_e}$.*

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