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Abstract. We use Lazard Elimination to devise some new bases of the free Lie algebra which (like classical Hall bases) consist of Lie products of left normed basic Lie monomials. Our bases yield direct decompositions of the homogeneous components of the free Lie algebra with direct summands that are particularly easy to describe: they are tensor products of metabelian Lie powers. They also give rise to new filtrations and decompositions of free Lie algebras as modules for groups of graded algebra automorphisms. In particular, we obtain some new decompositions for free Lie algebras and free restricted free Lie algebras over fields of positive characteristic.

1. Introduction

Let $L = L(X)$ be the free Lie algebra with free generating set X over a commutative ring K with 1. Thus

$$L = \bigoplus_{n\geq 1} L_n$$

where $L_n = L_n(X)$ is the homogeneous component of degree n in L (we also say: the n-th Lie power). Assume that the set X is ordered. A left normed basic Lie monomial of degree n over X is a Lie product of the form

$$[x_1, x_2, x_3, \ldots, x_n] \quad \text{with} \quad x_1, x_2, x_3, \ldots, x_n \in X \ \text{ and } \ x_1 > x_2 \leq x_3 \leq \ldots \leq x_n.$$

We write $H_n = H_n(X)$ for the set all left normed basic Lie monomials of degree n, and H for the set of all left normed basic Lie monomials of degree ≥ 2. Notice that the left normed basic Lie monomials are contained in any classical Hall basis of L, and that any such basis consists indeed of Lie products of left normed basic Lie monomials. Moreover, the left normed basic Lie monomials of degree ≥ 2 form a free generating set of the derived algebra $L' = \bigoplus_{n\geq 2} L_n$. It follows that for each $n \geq 1$ the set H_n (more precisely, the set $\{v + L'' ;\ v \in H_n\}$) is a basis of the n-th homogeneous component $M_n = M_n(X)$ of the free metabelian Lie algebra $M = L/L''$ on X. We call M_n the n-th metabelian Lie power.

In this paper we obtain some new bases for the free Lie algebra L, which (like classical Hall bases) consist of Lie products of left normed basic Lie monomials. The advantage of our bases is that they provide direct decompositions of the Lie powers L_n (as K-modules) with direct summands which are particularly easy to describe: they are metabelian Lie powers and tensor products of metabelian Lie

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powers. For example, if $n > m \geq 2$, our bases may be chosen in such a way that they contain all Lie products

$$[u, w_1, w_2, \dots, w_k] \qquad (u \in H_n,\ w_1, \dots, w_k \in H_m). \tag{1.1}$$

The span of these basis elements is (as a free K-module) isomorphic to the tensor product $M_n \otimes M_m \otimes \cdots \otimes M_m$ (with k tensor factors M_m). On contrast, a Hall basis contains only the Lie products (1.1) with $w_1 \leq w_2 \leq \cdots \leq w_k$, and the span of those is isomorphic to $M_n \otimes S_k(M_m)$ where $S_k(M_m)$ is the k-th symmetric power of M_m.

Our bases give rise to filtrations of the L_n, not just as K-modules but as modules for a group G of graded algebra automorphisms. The quotients of theses filtrations are direct sums of tensor products of metabelian Lie powers. The filtrations compare favorably with the (more complicated) filtrations obtained using Hall bases in [18, Section 3.1]. In the case where K is a field of characteristic zero, we can do better and instead of filtrations we actually get decompositions of the L_n as KG-modules. It is interesting to compare these decompositions with somewhat similar decompositions obtained by G.E. Wall [21], where direct summands are tensor products of symmetric powers of metabelian Lie powers (similar to the quotients of the filtration in [18]). We also obtain a number of applications for Lie powers over fields of positive characteristic. There our methods apply not only to free Lie algebras, but also to free restricted Lie algebras (in the case where K is a field of positive characteristic).

When working with the free restricted Lie algebra $R = R(X)$, it is convenient to think of R as the closure of L in its universal envelope under the unary operation $u \mapsto u^p$. The universal envelope of L will be identified with the tensor algebra $T = \bigoplus_{n \geq 0} T_n$ where $T_n = \langle X \rangle^{\otimes n}$, the n-th tensor power of the free K-module on X.

Given a free K-module A, we write $L(A)$ for the free Lie algebra on A, that is the free Lie algebra on X where X is an arbitrary K-basis of A, and $L_n(A)$ for the n-th Lie power. Similar notation will be used for free restricted Lie algebras and free metabelian Lie algebras. If G is a group acting on A by K-linear automorphisms, so that A becomes a KG-module, then the action of G extends uniquely to the whole of $L(A)$ with G acting by graded algebra automorphisms. In particular, the Lie powers $L_n(A)$ become KG-modules. Similarly, $M_n(A)$, $R_n(A)$ and $T_n(A)$ will be regarded as KG-modules. In the most general case G is the full group of graded algebra automorphisms of the free Lie algebra $L = L(X)$, that is $G = GL(V)$ where $V = L_1 = \langle X \rangle$. The actual aim of this paper is to obtain information about the structure of the free Lie algebra as a KG-module, and the greater part of it (Sections 5-10) deals with module structure. Our new bases have been devised to serve this purpose, but we hope they will be of independent interest. The final four sections are concerned with modular Lie powers. In recent years these have been studied intensively by a number of authors, and considerable progress has been made. Some comments on that and related references can be found at the end of Section 7.

The key tool in this paper is Lazard elimination for free Lie algebras (see Section 2), and another important devise is a variation thereof, called restricted elimination, that is peculiar to free restricted Lie algebras (see Section 8). We use the left normed convention for Lie brackets (that is $[u, v, w] = [[u, v], w]$), and we write $\langle U\rangle$ for the span of set U in a K-module.

2. Decomposition by Lazard Elimination

Let X be a countable set of cardinality at least 2, and let $L = L(X)$ be the free Lie algebra on X over a commutative ring K with 1. The Lazard Elimination Theorem (see [2, Chapter 2, Section 2.9, Proposition 10]) reads as follows.

Lazard Elimination Theorem. *Let $X = Y \cup Z$ be the disjoint union of its proper subsets Y and Z, then*

$$L(X) = L(Y \cup Z) = L(Y) \oplus L(Z \wr Y)$$

where

$$Z \wr Y = \{[z, y_1, \dots, y_k]\,;\; z \in Z,\, y_i \in Y,\, k \geq 0\}.$$

We call $Z \wr Y$ the *wreath set* of Y and Z. Our aim is to apply this theorem repeatedly, namely, first to $L(X)$, then to $L(Z \wr Y)$, then to the free Lie algebra on the wreath set resulting from the previous elimination, and so on, to obtain decompositions of $L(X)$ over K.

Definition 2.1. Let $\{E_i\}_{i\geq 1}$ and $\{\hat{E}_i\}_{i\geq 0}$ be sequences of non-empty subsets of the free Lie algebra $L(X)$ satisfying the following recursive conditions.

(i) $\hat{E}_0 = X$.
(ii) For all $i > 0$, E_i is a proper subset of $\hat{E}_{i-1}$ and $\hat{E}_i = (\hat{E}_{i-1} \setminus E_i) \wr E_i$.

Then we call $\{E_i\}_{i\geq 1}$ an *elimination sequence* for $L(X)$, and the sequence $\{\hat{E}_i\}_{i\geq 0}$ its associated *wreath sequence*.

Note that the associated wreath sequence $\{\hat{E}_i\}_{i\geq 0}$ is uniquely determined by its initial term $\hat{E}_0$ and the elimination sequence $\{E_i\}_{i\geq 1}$. Observe also that, by the definition of a wreath set, all elements of E_i and $\hat{E}_i$ are Lie monomials in X, and therefore have a well-defined degree with respect to X. An immediate consequence of the Lazard Elimination Theorem is the following

Lemma 2.1. *Let $\{E_i\}_{i\geq 1}$ be an elimination sequence for $L(X)$ with associated wreath sequence $\{\hat{E}_i\}_{i\geq 0}$. Then for each $k \geq 1$ there is a direct decomposition*

$$L(X) = L(E_1) \oplus L(E_2) \oplus \cdots \oplus L(E_k) \oplus L(\hat{E}_k). \tag{2.1}$$

of $L(X)$ as a free K-module. □

Definition 2.2. An elimination sequence $\{E_i\}_{i\geq 1}$ for $L(X)$ is called *convergent* if

$$L(X) = \bigoplus_{i\geq 1} L(E_i). \tag{2.2}$$

Next we derive some conditions for the convergence of an elimination sequence. These conditions refer to notion of degree in $L(X)$. It will be convenient to state them for the case when the elements of X may have been assigned degrees other than one. Namely, we say that X is a *graded set* if X has a distinguished decomposition as a disjoint union $X = \bigcup_{\alpha\in I} X_\alpha$ of its subsets X_α where α runs over some index set I, and the elements of each subset X_α are assigned the degree $n(\alpha)$ where $n(\alpha)$ is a natural number. If for each $n \geq 1$ there are at most finitely many $\alpha \in I$ with $n(\alpha) = n$, we say that X is *finitely graded.* As in the common case where all elements of X have degree 1, this more general notion of degree gives rise to a decomposition of $L(X)$ into homogeneous components: The n-th homogeneous component L_n of $L(X)$ is spanned by all Lie products $[w_1, w_2, \dots, w_k]$ with $w_i \in X$ such that $\deg w_1 + \cdots + \deg w_k = n$, and we have $L(X) = \bigoplus_{n\geq 1} L_n$. For a set B of homogeneous elements in $L(X)$ we let $\delta(B)$ denote the smallest natural number such that B contains an element of degree $\delta(B)$, and if all elements of B have the same degree we write $\deg B$ for the common degree of all these elements. The definition of the sets $\hat{E}_i$ as $(\hat{E}_{i-1} \setminus E_i) \wr E_i$ implies that for any elimination sequence $\{E_i\}_{i\geq 1}$ the sequence $\{\delta(\hat{E}_i)\}_{i\geq 0}$ is non-decreasing.

Lemma 2.2. *Let $X = \bigcup_{\alpha\in I} X_\alpha$ be a graded set, and let $\{E_i\}_{i\geq 1}$ be an elimination sequence for $L(X)$ with associated wreath sequence $\{\hat{E}_i\}_{i\geq 0}$. Then $\{E_i\}_{i\geq 1}$ is convergent if*

$$\lim_{i\to\infty} \delta(\hat{E}_i) = \infty.$$

Proof. Suppose that $\lim_{i\to\infty} \delta(\hat{E}_i) = \infty$. Since $L(X)$ is the direct sum of its homogeneous components, it is sufficient to show that for each $n \geq 1$ the homogeneous component L_n is contained in $\bigoplus_{i\geq 1} L(E_i)$. Our limit condition implies that there exists a $k \geq 1$ such that $\hat{E}_k$ consists entirely of elements of degree $> n$, and consequently $L(\hat{E}_k) \subseteq \bigoplus_{i>n} L_i$. But then (2.1) implies that L_n is contained in $\bigoplus_{i=1}^{k} L(E_i)$, and hence $L_n \subseteq \bigoplus_{i\geq 1} L(E_i)$ as required. □

We exploit the lemma to derive another sufficient condition for convergence. Suppose that $X = \bigcup_{\alpha\in I} X_\alpha$ is a graded set and let $\beta \in I$. Then the wreath set $(X \setminus X_\beta) \wr X_\beta$ has a distinguished decomposition as the disjoint union of the homogeneous sets

$$X_{\alpha,k} = [X_\alpha, \underbrace{X_\beta, \dots, X_\beta}_{k}] = \{[u, v_1, \dots, v_k]\,;\ u \in X_\alpha, v_i \in X_\beta\}$$

where $\alpha \in I \setminus \{\beta\}, k \geq 0$. We call this decomposition the *natural grading* of the wreath set $(X \setminus X_\beta) \wr X_\beta$, and we refer to the sets $X_{\alpha,k}$ as to the *components* of the natural grading. We call an elimination sequence $\{E_i\}_{i\geq 1}$ for $L(X)$ *natural,* if $E_1 = X_\beta$ for some $\beta \in I$, and each E_i with $i > 1$ is a component of the natural grading of the wreath set $\hat{E}_{i-1} = (\hat{E}_{i-2} \setminus E_{i-1}) \wr E_{i-1}$. A natural elimination sequence is called *regular* if each E_i is a set of smallest possible degree (that is $\deg E_i = \delta(\hat{E}_{i-1})$).

Example. Let $Y = Y_2 \cup Y_3 \cup Y_4 \cup \cdots$ with $\deg Y_i = i$. To produce a regular elimination sequence $\{E_i\}_{i\geq 1}$ for $L(Y)$, we first set $E_1 = Y_2$, the set of smallest degree in the (natural) grading of $\hat{E}_0 = Y$. Then

$$\hat{E}_1 = \{Y_3, Y_4, Y_5, [Y_3, Y_2], Y_6, [Y_4, Y_2], Y_7, [Y_5, Y_2], [Y_3, Y_2, Y_2], \ldots\}.$$

The regularity condition implies that $E_2 = Y_3$, the component of smallest degree in the natural grading of $\hat{E}_1$. Then

$$\begin{aligned}\hat{E}_2 = \{&Y_4, Y_5, [Y_3, Y_2], Y_6, [Y_4, Y_2], Y_7, [Y_5, Y_2], [Y_3, Y_2, Y_2], [Y_4, Y_3], \ldots\\ &Y_{10}, [Y_8, Y_2], [Y_6, Y_2, Y_2], [Y_4, Y_2, Y_2, Y_2], [Y_7, Y_3], [Y_5, Y_2, Y_3],\\ &[Y_3, Y_2, Y_2, Y_3], [Y_4, Y_3, Y_3], \ldots\}.\end{aligned}$$

Next comes $E_3 = Y_4$, but after that there are choices to be made as $\hat{E}_3$ contains two components of minimal degree, namely, Y_5 and $[Y_3, Y_2]$, and there will be even more choice later on. Different choices will result in different elimination sequences. The first terms of the elimination sequence up to degree 8 are as follows:

$$\begin{aligned}&Y_2, \quad Y_3, \quad Y_4, \quad Y_5,\ [Y_3, Y_2], \quad Y_6,\ [Y_4, Y_2],\\ &Y_7,\ [Y_5, Y_2],\ [Y_3, Y_2, Y_2],\ [Y_4, Y_3],\\ &Y_8,\ [Y_6, Y_2],\ [Y_4, Y_2, Y_2],\ [Y_5, Y_3],\ [Y_3, Y_2, Y_3].\end{aligned} \tag{2.3}$$

These sets do not depend on the ordering we are required to choose, but the ordering will have an essential effect in higher degrees. For example, if we choose $Y_5 > [Y_3, Y_2]$, we get the set $[Y_5, [Y_3, Y_2], [Y_3, Y_2], Y_5]$ in degree 20, but this set will not occur if we make the opposite choice.

Lemma 2.3. *Let $X = \bigcup_{\alpha\in I} X_\alpha$ be a finitely graded set. Then every regular elimination sequence for $L(X)$ is convergent.*

Proof. Let $\{E_i\}_{i\geq 1}$ be a regular elimination sequence for $L(X)$, and let $\hat{E}_i = \bigcup_{\alpha\in I_i} U_\alpha$ be the natural grading of $\hat{E}_i$. The assumption that X is finitely graded guarantees that for each $i \geq 1$ and each $n \geq 1$ there are only finitely many $\alpha \in I_i$ such that $\deg U_\alpha = n$. Let J_i denote the subset of I_i such that the U_α with $\alpha \in J_i$ are of minimal degree, that is $\deg U_\alpha = \delta(\hat{E}_i)$. Then the regularity condition on the elimination sequence gives that $E_{i+1} = U_\beta$ for some $\beta \in J_i$. But then $\hat{E}_{i+1}$ consists of the sets U_α with $\alpha \in (J_i \setminus \beta)$ and sets of higher degree. In particular, the number of sets of degree $\delta(\hat{E}_i)$ in the natural grading of $\hat{E}_{i+1}$ is strictly less than the number of sets of degree $\delta(\hat{E}_i)$ in the natural grading of $\hat{E}_i$. This gives that for some $j > i$ there will be no sets of degree $\delta(\hat{E}_i)$ in the natural grading of $\hat{E}_j$, and since the sequence $\{\delta(\hat{E}_i)\}_{i\geq 1}$ is non-decreasing, we have that $\delta(\hat{E}_j) > \delta(\hat{E}_i)$. But this means that our elimination sequence satisfies the condition of Lemma 2.2, and the result follows. □

Convergent elimination sequences can be used to construct homogeneous K-bases of the free Lie algebra $L(X)$. The most straightforward example of such an application comes from convergent elimination sequences $\{E_i\}_{i\geq 1}$ in which each term E_i is a singleton: $E_i = \{w_i\}$ for some $w_i \in L(X)$. In this case each of the direct summands on the right hand side of (2.2) is a free Lie algebra of rank 1,

$L(X_i) = \langle w_i \rangle$, and the right hand side of (2.2) is a decomposition of $L(X)$ into a direct sum of rank 1 submodules $\langle w_i \rangle$ $(i = 1, 2, \ldots)$. In other words, the set $\{w_i\,;\ i \geq 1\}$ is a K-basis of $L(X)$. In particular, if the grading of X in Lemma 2.2 is such that every X_α is a singleton $X_\alpha = \{x_\alpha\}$, then all the sets in the natural gradings of the wreath sets $\hat{E}_i$ are singletons, and, consequently, all elimination sets E_i are singletons. This gives the following

Corollary 2.1. *Let $X = \bigcup_{\alpha \in I} X_\alpha$ be a finitely graded set, and assume in addition that every X_α is a singleton. Then the elimination sets E_i in every regular elimination sequence $\{E_i\}_{i \geq 1}$ are singletons, $E_i = \{w_i\}$ with $w_i \in L(X)$, and the set $\{w_i\,;\ i = 1, 2, 3 \ldots\}$ is a K-basis of $L(X)$.* □

An easy consequence of this corollary is the following result.

Corollary 2.2. *Let X be a graded set such that for each $n \geq 1$ there are at most finitely many elements of degree n in X, and let $\{E_i\}_{i \geq 1}$ be an elimination sequence such that each E_i consists of a single element w_i that is of smallest possible degree in $\hat{E}_{i-1}$. Then $\{E_i\}_{i \geq 1}$ is convergent. In particular, the set $\{w_i\,;\ i = 1, 2, 3 \ldots\}$ is a K-basis of $L(X)$.*

Proof. Write X as a finitely graded set $X = \bigcup_{x \in X} \{x\}$. Then any elimination sequence satisfying the condition in the statement of the corollary will be regular with respect to this grading, and hence it is convergent. □

Corollary 2.2 applies to the standard case where X is a finite, and all elements of X have degree 1. In this case it is not hard to see that if $\{E_i\}_{i \geq 1}$ is an elimination sequence as in Corollary 2.2, then the resulting K-basis $\{w_1, w_2, w_3, \ldots\}$ is precisely a classical Hall basis. Hall basic monomials are defined recursively and depend on an ordering of the basis elements. When we use an elimination sequence as above to get a Hall basis, this order is just the order in which the elements w_i are eliminated: $w_1 < w_2 < w_3 < \cdots$.

Our next aim is to give an alternative description of the sets E_i in a regular elimination sequence. To this end we define an collection of subsets of the free Lie algebra $L(X)$.

Definition 2.3. Let $X = \bigcup_{\alpha \in I} X_\alpha$ be a graded set. A *basis set collection* for $L(X)$ is an ordered set $\mathcal{B} = \mathcal{B}(X)$ of subsets of the free Lie algebra $L(X)$, which we call B-sets, defined inductively as follows. The B-sets of minimal degree are the sets X_α of minimal degree in X, ordered in an arbitrary way. Now suppose that the B-sets of degree $< n$ have been defined and ordered so that the ordering respects the degree. Then the B-sets of degree n are the sets X_α with $\deg X_\alpha = n$ and the sets

$$[U, V] = \{[u, v]; \quad u \in U, v \in V\}$$

such that

(i) U and V are B- sets with $\deg U + \deg V = n$,
(ii) $U > V$,
(iii) if $U = [U_1, U_2]$ for B-sets U_1, U_2 then $V \geq U_2$.

The degree n sets are then ordered arbitrarily, and declared to be greater than the B-sets of degree less than n. We write $\mathcal{B}_n$ for the set of all B-sets of degree n, and $\mathcal{B}(X)$ or simply $\mathcal{B}$ for the set of all B-basis sets.

Note that if X is a finitely graded set, then each $\mathcal{B}_n$ consists of finitely many sets, and hence we may assume that the order of $\mathcal{B}$ is of type $\aleph_0$: $\mathcal{B} = \{E_1, E_2, E_3, \ldots\}$ with $E_1 < E_2 < E_3 < \cdots$.

Theorem 2.1. *Let $X = \bigcup_{\alpha\in I} X_\alpha$ be a finitely graded set, and let $\mathcal{B} = \{E_1, E_2, E_3, \ldots\}$ be a basis set collection for $L(X)$. Then $\{E_i\}_{i\geq 1}$ is a regular elimination sequence for $L(X)$. In particular, there is a direct decomposition*

$$L(X) = \bigoplus_{i\geq 1} L(E_i) = \bigoplus_{n\geq 1} \bigoplus_{U\in\mathcal{B}_n} L(U)$$

of $L(X)$ as a free K-module.

Proof. By construction, E_1, that is one of the X_α with minimal degree, is the initial term of a regular elimination sequence. Hence, to prove the theorem we need to show that each E_i with $i > 1$ is a component of minimal degree in the natural grading of $\hat{E}_{i-1}$. The are two cases to consider. If $E_i = X_\alpha$ for some $\alpha \in I$, then X_α is a component of $\hat{E}_{i-1}$ and it must be of minimal degree since, by induction, $\{E_j\}_{j<i}$ is the initial part of an elimination sequence, and hence the span of the E_j contains all homogeneous components L_m with $m < \deg E_i$. So $\hat{E}_{i-1}$ cannot contain elements of degree less than $\deg E_i$. If, on the other hand, $E_i = [U, V]$ for B-sets U and V, then it follows from Definition 2.3 that $E_i = [E_l, E_k, \ldots, E_k]$ for some $k < l < i$. But then E_i is a component of $\hat{E}_{k-1}$ and hence of all subsequent terms of the wreath sequence until $\hat{E}_{i-1}$. Indeed, since $\{E_j\}_{j<i}$ is the initial part of an elimination E_i cannot be eliminated at an earlier step than the i-th (because otherwise the set E_i would appear more than once in our sequence of B-sets prompting the contradiction $E_i < E_i$), and it must be of minimal degree in $\hat{E}_{i-1}$ by the same argument that was used in the case where $E_i = X_\alpha$. □

An important consequence of the theorem is that the span of each B-set of the form $[U, V]$ is (as a free K-module) isomorphic to the tensor product $\langle U\rangle \otimes \langle V\rangle$. It follows that the span of each basis set in $\mathcal{B}_n$ is isomorphic to a tensor product with tensor factors $\langle X_\alpha\rangle$ ($\alpha \in I$). In view of Corollary 2.1, the number of B-basis sets involving a given set the basis sets $X_{\alpha_1}, X_{\alpha_2}, \ldots, X_{\alpha_q}$ with multiplicities $t_1, t_2, \ldots, t_q$, respectively, is equal to the dimension of the fine homogeneous component of the free Lie algebra $L(X)$ with $X = \{x_\alpha\,;\ \alpha \in I\}$ that is spanned by all Lie products involving $x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_q}$ with multiplicities $t_1, t_2, \ldots, t_q$. This dimension is given by the fine Witt formula (see [16, Theorem 5.11]). We summarize our discussion as follows.

Corollary 2.3. *The number of B-sets U involving the sets $X_{\alpha_1}, X_{\alpha_2}, \ldots, X_{\alpha_q}$ with multiplicities $t_1, t_2, \ldots, t_q$, respectively, is*

$$\frac{1}{t}\sum_{d|(t_1,\ldots,t_q)} \mu(d)\frac{(t/d)!}{(t_1/d)!(t_2/d)!\cdots(t_q/d)!} \tag{2.4}$$

where $t = t_1+t_2+\cdots+t_q$ *and the sum runs over all divisors* d *of the greatest common divisor* $(t_1,\ldots,t_q)$ *of* $t_1,\ldots,t_q$. *For any such basis set* U *there is an isomorphism*

$$\langle U\rangle \cong \langle X_{\alpha_1}\rangle^{\otimes t_1} \otimes \langle X_{\alpha_2}\rangle^{\otimes t_2} \otimes \cdots \otimes \langle X_{\alpha_q}\rangle^{\otimes t_q}$$

as free K*-modules.* □

Example. If $Y = Y_2 \cup Y_3 \cup \cdots$ is as in our previous example, we have two B-basis sets involving Y_2 with multiplicity 3 and Y_3 with multiplicity 2 in degree 12, namely $[Y_3,Y_2,Y_2,Y_2,Y_3]$ and $[[Y_3,Y_2,Y_2],[Y_3,Y_2]]$, and the span of either of them is isomorphic to the tensor product $\langle Y_2\rangle^{\otimes 3} \otimes \langle Y_3\rangle^{\otimes 2}$.

3. The Decomposition Theorem for $L(X)$

From now on X is a finite set , and $L = L(X)$ is the free Lie algebra on X over a commutative ring K with 1. We will assume that each element of X has degree 1 and that X is ordered. Write L as the direct sum

$$L = \langle X\rangle \oplus L'$$

where L' is the derived algebra of L. Then L' is itself a free Lie algebra (of infinite rank). Moreover, L' has a graded free generating set Y of the form

$$Y = Y_2 \cup Y_3 \cup Y_4 \cup \cdots$$

where the elements of Y_n $(n \geq 2)$ have degree n (with respect to X): $\deg Y_n = n$. A graded free generating set of this form will be referred to as a *standard free generating set* for L'. The most prominent example of a standard free generating set is the set of left normed basic Lie monomials of degree ≥ 2 in X, that is $Y = H$ where H is as defined in Section 1 (see, for example, [1, Section 2.4.2], or [6, Section 2.2] where this free generating set is obtained using Lazard elimination). Note that any standard free generating set for L' is finitely graded. The main result of this section involves a basis set collection for the derived algebra L' as a free Lie algebra with a standard free generating set $Y = Y_2 \cup Y_3 \cup \cdots$. The grading of Y is particularly simple, and a basis set collection for $L(Y)$ is relatively straightforward. In fact, such a collection was discussed in the Example in Section 2, and the corresponding B-sets up to degree 8 are listed in (2.3). The main result of this section is an immediate consequence of Theorem 2.1, applied to L'.

Theorem 3.1. *Let* Y *be a standard free generating set for the derived algebra* L', *and let* $\mathcal{B} = \mathcal{B}(Y)$ *be a basis set collection for* $L' = L(Y)$. *Then there is a direct decomposition*

$$L = \langle X\rangle \oplus \bigoplus_{U\in\mathcal{B}} L(U) \tag{3.1}$$

of L *as a free* K*-module.* □

The theorem yields the following result about the homogeneous components of $L(X)$.

Corollary 3.1. *For each $n \geq 2$ there is a direct decomposition*

$$L_n(X) = \bigoplus_{d|n\ d\neq n} \bigoplus_{U\in\mathcal{B}_{n/d}} L_d(U).$$

In particular, for all prime numbers p we have

$$L_p(X) = \bigoplus_{U\in\mathcal{B}_p} L_1(U) = \bigoplus_{U\in\mathcal{B}_p} \langle U\rangle,$$

and hence the union of the B-sets of degree p in $\mathcal{B}(Y)$ is a basis for $L_p(X)$. □

Another consequence of Theorem 3.1 is that the B-sets in $\mathcal{B}(Y)$ are linearly independent in L, and that the K-span of the union of all B-sets in $\mathcal{B}$ is the direct sum (over K) of the spans of the individual B-sets in L. Moreover, the span of Y_n is, as a K-space, isomorphic to the metabelian Lie power $M_n = M_n(X)$ while the span of a basis set of the form $U = [U_1, U_2]$ is isomorphic to the tensor product $\langle U_1\rangle \otimes \langle U_2\rangle$. It follows easily that the K-span of any B-set is isomorphic to a tensor product of metabelian Lie powers. To make this more precise, we introduce the following

Definition 3.1. Let Y and $\mathcal{B}$ as in Theorem 3.1. The *associated tensor product* $t(U)$ of a B-set $U \in \mathcal{B}(X)$ is given by

$$t_B(U) = M_n(X) \quad \text{if } U = Y_n$$

and

$$t_B(U) = t_B(U_1) \otimes t_B(U_2) \quad \text{if } U = [U_1, U_2].$$

It follows from the definition that the associated tensor product $t_B(U)$ of a B-set U of degree n is either a metabelian Lie power M_n or a tensor product of the form $M_{n_1} \otimes M_{n_2} \otimes \cdots \otimes M_{n_k}$ with $n_1 + n_2 + \cdots + n_k = n$. This observation and Corollary 3.1 imply the following

Corollary 3.2. *For each $U \in \mathcal{B}(Y)$ there is an isomorphism (of K-modules)*

$$\langle U\rangle \cong t_B(U),$$

where

$$t_B(U) = M_{n_1} \otimes M_{n_2} \otimes \cdots \otimes M_{n_k}$$

for some k-tuple $(n_1, n_2, \ldots, n_k)$, $k \geq 1$, of natural numbers ≥ 2 with $n_1 + n_2 + \ldots + n_k = n$. The number of B-basis sets of degree n for which $t_B(U)$ is a tensor product in which a given set $M_{n_1}, M_{n_2}, \ldots, M_{n_k}$ occurs as tensor factors with multiplicity $t_1, t_2, \ldots, t_k$, respectively, is given by (2.4). □

4. Bases

Now we exploit Theorem 3.1 to obtain new K-bases for L. In view of Corollary 3.1, the Theorem provides us with K-bases for the homogeneous components of prime degree p : The union of the B-sets U of degree p in $\mathcal{B}(Y)$ is a basis of L_p. To obtain bases for the homogeneous components of arbitrary degree, and hence for the whole of L, we order all the finite basis sets $U \in \mathcal{B}(Y)$ (in an arbitrary way), and apply Theorem 3.1 to all direct summands $L(U)$ in (3.1). This gives a direct decomposition

$$L(X) = \langle X \rangle \oplus \bigoplus_{U \in \mathcal{B}(Y)} \langle U \rangle \oplus \bigoplus_{U \in \mathcal{B}(Y)} \bigoplus_{V \in \mathcal{B}(Y(U))} L(V).$$

where $Y(U)$ is a standard basis of the derived algebra $L'(U)$. After that we can apply Theorem 3.1 to all the direct summands $L(V)$, and by repeating this process indefinitely we obtain in the limit a direct decomposition of the free Lie algebra L whose direct summands are free K-modules with bases obtained from X by repeated application of the B-set construction to standard bases of derived algebras. In order to turn this discussion into a formal theorem, we make the following

Definition 4.1. A *complete basis set collection* $\mathcal{T}$ for $L(X)$ is a union

$$\mathcal{T} = \bigcup_{k \geq 0} \mathcal{T}^{(k)}(X)$$

where each $\mathcal{T}^{(k)}$ is a family of subsets of $L(X)$, which are termed the T-sets of level k ($k = 0, 1, 2, \ldots$), such that $\mathcal{T}^{(0)} = \{X\}$, and for $k > 0$

$$\mathcal{T}^{(k)}(X) = \{U \; ; \; U \in \mathcal{B}(Y(V)) \, , \; V \in \mathcal{T}^{(k-1)}(X)\}$$

where $Y(V)$ is a standard free generating set of the derived algebra $L'(V)$ and $\mathcal{B}(Y(V))$ is a basis set collection for $L(Y(V))$ as defined in Definition 2.3. We write $\mathcal{T}_n$ and $\mathcal{T}_n^{(k)}$ for the set of all T-sets of degree n in $\mathcal{T}$ and $\mathcal{T}^{(k)}$, respectively.

Observe that the T-basis sets of level 1 are precisely the B-sets in $\mathcal{B}(Y)$ where $Y = Y(X)$ is a standard basis of the derived algebra $L' = L'(X)$. With this definition, we can now state the our basis theorem.

Theorem 4.1. *Let $\mathcal{T}$ be a complete basis set collection for $L(X)$. Then the union of all T-sets in $\mathcal{T}$ is a K-basis for L.* □

We use the term *T-basis* for a basis of this sort. In order to domesticate the definition of T-sets, recall that each T-set of level $k \geq 1$ is, by definition, a B-set for a derived algebra $L' = L'(V)$ regarded as a free Lie algebra $L(Y(V))$ with standard free generating set $Y(V) = Y_2(V) \cup Y_3(V) \cup \cdots$, where V is a T-set of level $k-1$. In the case where $V = X$, we write simply Y for $Y(X)$. As mentioned in the previous Section, T-sets of level 1 for $L(X)$ are listed in (2.3). Then the level-2 T-basis sets up to degree 8 are

$$Y_2(Y_2), \quad Y_2(Y_3), Y_3(Y_2), \quad Y_2(Y_4),$$

and the only level-3 T-basis set of degree ≤ 8 is

$$Y_2(Y_2(Y_2)).$$

An important special case arises if all standard free generating sets in the definition of the complete basis set collection $\mathcal{T}$ are canonical free generating sets consisting of left normed basic Lie monomials. In this case the T-basis consists of Lie products of left normed basic Lie monomials in X, as mentioned in the Introduction. For example, if $X = \{x, y, z\}$ with order $x < y < z$, then

$$Y_2 = \{[y, x], [z, x], [z, y]\},$$

and, assuming $[y, x] < [z, x] < [z, y]$, we get

$$Y_2(Y_2) = \{[[z, x], [y, x]], [[z, y], [y, x]], [[z, y], [z, x]]\}$$

and

$$Y_3(Y_2) = \{[[z, x], [y, x], [y, x]], [[z, x], [y, x], [z, x]], \ldots, [[z, y], [z, x], [z, y]]\}.$$

An example of a level-2 T-set in degree 10 is

$$[Y_3, Y_2](Y_2) = \{[[[z, x], [y, x], [y, x]], [[z, x], [y, x]]], \\ [[[z, x], [y, x], [y, x]], [[z, y], [y, x]]], \ldots\}.$$

In the previous section we have defined the associated tensor product of a B-set. If U is a B-set in $\mathcal{B}(Y(V))$ with $\langle U\rangle \cong t(U) = M_{n_1}(V) \otimes \cdots \otimes M_{n_k}(V)$, and V is itself a B-set in $\mathcal{B}(Y(W))$ with $\langle V\rangle \cong t(W) = M_{m_1}(W) \otimes \cdots \otimes M_{m_k}(W)$, then

$$M_{n_i}(V) = M_{n_i}(M_{m_1}(W) \otimes \cdots \otimes M_{m_k}(W)),$$

for each tensor factor in the decomposition of $t(U)$. This motivates the following definition of the associated iterated tensor product of a T-set.

Definition 4.2. Let $\mathcal{T} = \bigcup_{k\geq 0} \mathcal{T}^{(k)}(X)$ be a complete basis set collection for $L(X)$. The *associated iterated tensor product* $t(U)$ for a T-set $U \in \mathcal{T}$ is defined as follows. For X, the only T-set of level 0 we set $t(X) = M_1(X) = \langle X\rangle$, and if U is a T-set of level $k > 0$, that is a B-set in $\mathcal{B}(Y(V))$ where $V \in \mathcal{T}^{(k-1)}(X)$ and $Y(V)$ is a standard free generating set for $L'(V)$, with associated tensor product $t_B(U) = M_{n_1}(V) \otimes \cdots \otimes M_{n_k}(V)$ (as in Definition 3.1) we set

$$t(U) = M_{n_1}(t(V)) \otimes M_{n_2}(t(V)) \otimes \cdots \otimes M_{n_k}(t(V)).$$

Note that $t(U) = t_B(U)$ for any T-set U of level 1. The following is now an immediate consequence of the basis theorem.

Corollary 4.1. *For any $U \in \mathcal{T}$ there is an isomorphism*

$$\langle \mathcal{U}\rangle \cong t(\mathcal{U}).$$

of free K-modules. □

5. Filtrations

In the remaining part of this paper we deal with the structure of the Lie powers L_n with $n \geq 2$ as modules for a group G of graded algebra automorphism of L. The Lie powers L_n, the metabelian Lie powers M_n (and later on the restricted Lie powers R_n) will be regarded as modules for G. *The presence of the group G will be a standing assumption for the rest of the paper.*

Let $Y = \bigcup_{n\geq 2} Y_n$ be a standard free generating set for the derived algebra L'. The lower central series of the derived algebra L' induces on the Lie powers L_n with $n \geq 2$ a filtration

$$L_n = L_{n,1} \geq L_{n,2} \geq L_{n,3} \geq \cdots \geq L_{n,l(n)} \geq L_{n,l(n)+1} = 0 \tag{5.1}$$

where $L_{n,k}$ is spanned by all Lie products in Y of total degree n and degree $\geq k$ with respect to Y. Here $l(n) = n/2$ if n is even, and if n is odd we have $l(n) = (n-1)/2$. Clearly, the $L_{n,k}$ are KG-submodules of L, and we have, in particular, $L_{n,2} = L_n \cap L''$, and hence $L_{n,1}/L_{n,2} \cong M_n$. Let $\mathcal{B} = \mathcal{B}(Y)$ be a basis set collection for $L' = L(Y)$. By construction, all B-sets U in $\mathcal{B}$ are homogeneous with respect to both X and Y, and hence each of them has well defined degrees with respect to both X and Y. We write deg as usual for the degree with respect to X and Deg for the degree with respect to Y. It follows from Corollary 3.1 that the quotient $L_{n,k}/L_{n,k+1}$ is spanned by the homogeneous components $L_d(U)$ where d runs over all common divisors of n and k, and U runs over all B-sets of degree n/d with respect to X and degree k/d with respect to Y. Given n, k with $n \geq 2$ and $1 \leq k \leq l(n)$, and a common divisor d of n and k we set

$$\mathcal{B}_{n,k,d} = \{U \in \mathcal{B}\,;\ \deg U = n/d, \operatorname{Deg} U = k/d\}.$$

Theorem 5.1. *Let Y a standard free generating set of L', and let $\mathcal{B}$ be a basis set collection for $L' = L(Y)$. Then the terms of the filtration* (5.1) *are KG-submodules, and for the quotients there are KG-module isomorphisms*

$$L_{n,k}/L_{n,k+1} \cong \bigoplus_{d|(n,k)} \bigoplus_{U\in\mathcal{B}_{n,k,d}} L_d(t(U)). \tag{5.2}$$

Proof. It follows from Corollaries 3.1 and 3.2 that for the quotients of the filtration (5.1) we have isomorphisms of free K-modules as in (5.2). To see that these are, in fact, isomorphisms of KG-modules, we define for each $U \in \mathcal{B}$ the associated tensor product of Lie powers $\hat{t}(U)$, and a homomorphism $\gamma_U : \hat{t}(U) \to L_n$ where $n = \deg U$ as follows. If $U = Y_n$ we set

$$\hat{t}(Y_n) = L_n \quad \text{and} \quad \gamma_{Y_n} = id : L_n \to L_n,$$

and if $U = [U_1, U_2]$ we set

$$\hat{t}(U) = \hat{t}(U_1) \otimes \hat{t}(U_2)$$

and define γ_U by

$$(u_1 \otimes u_2)\gamma_U = [u_1\gamma_{U_1}, u_2\gamma_{U_2}] \quad (u_i \in \hat{t}(U_i),\, i = 1, 2).$$

Hence each $\hat{t}(U)$ is of the form $\hat{t}(U) = L_{n_1} \otimes \cdots \otimes L_{n_s}$ for some positive integers $n_1, \ldots, n_s$ with $n_i \geq 2$. For each $U \in \mathcal{B}$ there is an obvious surjection $\pi_U : \hat{t}(U) \to t(U)$ stemming from the natural surjections $L_{n_i} \to M_{n_i}$ $(n_i \geq 2)$. It is clear that the kernel of π_U is spanned by all tensors $u = u_1 \otimes \cdots \otimes u_s$ with $u_i \in L_{n_i}$ such that at least one of the factors u_i belongs to $L_{n_i} \cap L''$. This implies, in particular, that $u\gamma_U \in L_{n,s+1}$ where $n = \deg u$ and $s = \operatorname{Deg} u$.

Moreover, for each natural d the homomorphism γ_U induces a homomorphism $\gamma_U^{(d)} : L_d(\hat{t}(U)) \to L_{dn}$ which is, for $v_1, \ldots, v_d \in \hat{t}(U)$ given by

$$[v_1, \ldots, v_d]\gamma_U^{(d)} = [v_1\gamma_U, \ldots, v_1\gamma_U] \in L_{dn}$$

and the homomorphism π_U induces a homomorphism $\pi_U^{(d)} : L_d(\hat{t}(U)) \to L_d(t(U))$ given by

$$[v_1, \ldots, v_d]\pi_U^{(d)} = [v_1\pi_U, \ldots, v_1\pi_U] \in L_{dn}.$$

In particular, $\gamma_U^{(1)} = \gamma_U$ and $\pi_U^{(1)} = \pi_U$. Observe that the kernel of $\pi_U^{(d)}$ is spanned by left normed Lie products $v = [v_1, \ldots, v_d]$ such at least one of the v_i belongs to $\ker \pi_U$, which, in its turn, implies that $v\gamma_U^{(d)} \in L_{dn,s+1}$ where $s = d \operatorname{Deg} U$.

For $n \geq 2$ and $1 \leq k \leq l(n)$, consider the homomorphism

$$\Gamma_{n,k} : \bigoplus_{d|(n,k)} \bigoplus_{U \in \mathcal{B}_{n,k,d}} L_d(\hat{t}(U)) \to L_n$$

which is for each direct summand $U \in \mathcal{B}$ with $\operatorname{Deg} U = n/d$ and $\operatorname{Deg} U = k/d$ of the domain defined as $\gamma_U^{(d)} : L_d(\hat{t}(U)) \to L_n$, and the surjection

$$\Pi_{n,k} : \bigoplus_{d|(n,k)} \bigoplus_{U \in \mathcal{B}_{n,k,d}} L_d(\hat{t}(U)) \to \bigoplus_{d|(n,k)} \bigoplus_{U \in \mathcal{B}_{n,k,d}} L_d(t(U))$$

defined as the direct sum of the appropriate $\pi_U^{(d)}$. In order to establish the isomorphism (5.2) we observe that

(i) the image of the homomorphism $\Gamma_{n,k}$ is contained in $L_{n,k}$,

(ii) the homomorphism $\Gamma_{n,k}$ maps the kernel of the homomorphism $\Pi_{n,k}$ into $L_{n,k+1}$.

By (i), the homomorphism $\Gamma_{n,k}$ induces a homomorphism

$$\bigoplus_{d|(n,k)} \bigoplus_{U \in \mathcal{B}_{n,k,d}} L_d(\hat{t}(U)) \to L_{n,k}/L_{n,k+1}.$$

By (ii), this homomorphisms factors through

$$\bigoplus_{d|(n,k)} \bigoplus_{U \in \mathcal{B}_{n,k,d}} L_d(t(U)).$$

Finally, it remains to observe that the induced homomorphism

$$\bigoplus_{d|(n,k)} \bigoplus_{U \in \mathcal{B}_{n,k,d}} L_d(t(U)) \to L_{n,k}/L_{n,k+1}$$

yields a bijection between K-bases of the domain and the image (which is clear since (5.2) is a K-isomorphism). This completes the proof of the theorem. □

Examples. For $n = 8$ we have $l(8) = 4$ and the filtration (5.1) has the form

$$L_8 = L_{8,1} \geq L_{8,2} \geq L_{8,3} \geq L_{8,4} \geq 0.$$

The B-basis sets of degree up to 8 are listed in (2.3). We have

$$\mathcal{B}_{8,1,1} = \{Y_8\}$$
$$\mathcal{B}_{8,2,1} = \{[Y_6, Y_2], [Y_5, Y_3]\}, \quad \mathcal{B}_{8,2,2} = \{Y_4\}$$
$$\mathcal{B}_{8,3,1} = \{[Y_4, Y_2, Y_2], [Y_3, Y_2, Y_3]\}$$
$$\mathcal{B}_{8,4,1} = \mathcal{B}_{8,4,2} = \varnothing, \quad \mathcal{B}_{8,4,4} = \{Y_2\}.$$

For the quotients of the filtration we have

$$
\begin{aligned}
L_{8,1}/L_{8,2} &\cong M_8 \\
L_{8,2}/L_{8,3} &\cong M_6 \otimes M_2 \quad \oplus \quad M_5 \otimes M_3 \quad \oplus \quad L_2(M_4) \\
L_{8,3}/L_{8,4} &\cong M_4 \otimes M_2 \otimes M_2 \quad \oplus \quad M_3 \otimes M_2 \otimes M_3 \\
L_{8,4} &= L_4(M_2)
\end{aligned}
$$

For $n = 12$ get a more illuminating example. Here $l(12) = 6$, and the filtration (5.1) takes the form

$$L_{12} = L_{12,1} \geq L_{12,2} \geq L_{12,3} \geq L_{12,4} \geq L_{12,5} \geq L_{12,6} = 0$$

with quotients

$$
\begin{aligned}
L_{12,1}/L_{12,2} &\cong t(Y_{12}) \\
L_{12,2}/L_{12,3} &\cong t([Y_{10}, Y_2]) \;\oplus\; t([Y_9, Y_3]) \;\oplus\; t([Y_8, Y_5]) \;\oplus\; t([Y_7, Y_6]) \;\oplus L_2(t(Y_6)) \\
L_{12,3}/L_{12,4} &\cong t([Y_8, Y_2, Y_2]) \;\oplus\; t([Y_7, Y_2, Y_3]) \;\oplus\; t([Y_7, [Y_3, Y_2]]) \oplus \;\cdots \oplus L_3(t(Y_4)) \\
L_{12,4}/L_{12,5} &\cong t([Y_6, Y_2, Y_2, Y_2]) \;\oplus\; \cdots \;\oplus\; L_2(t([Y_4, Y_2])) \;\oplus\; L_4(t(Y_3)) \\
L_{12,5}/L_{12,6} &\cong t([Y_4, Y_2, Y_2, Y_2, Y_2]) \;\oplus\; t([Y_3, Y_2, Y_2, Y_2, Y_3]) \;\oplus\; t([Y_3, Y_2, Y_2, [Y_3, Y_2]]) \\
L_{12,6} &\cong L_6(t(Y_2))
\end{aligned}
$$

There are too many level-1 T-basis sets of degree 12 to list them all, but we have listed the ones with Deg equal to 1,2, and 5 as well as all direct summands stemming from level-1 T-basis sets of smaller degree. Note that in $L_{12,3}/L_{12,4}$ we have the basis sets $[Y_7, Y_2, Y_3]$ and $[Y_7, [Y_3, Y_2]]$ with

$$t([Y_7, Y_2, Y_3]) = M_7 \otimes M_2 \otimes M_3, \quad t([Y_7, [Y_3, Y_2]]) = M_7 \otimes (M_3 \otimes M_2),$$

that is, their associated tensor products are equal up to the order of the tensor factors.

Now we return to the direct decomposition (5.2) in Theorem 5.1. Consider the direct summands $L_d(t(U))$ with $d > 1$ in the quotients of our filtration. Then we can apply the theorem to these homogeneous components to obtain a filtration

$$L_d(t(U)) = L_{d,1}(t(U)) \geq L_{d,2}(t(U)) \geq \cdots \geq L_{d,l(d)}(t(U)) \geq L_{d,l(d)+1}(t(U)) = 0$$

with quotients

$$L_{d,k}(t(U))/L_{d,k+1}(t(U)) \cong \bigoplus_{d_1 \mid (d,k)} \bigoplus_{V \in \mathcal{B}_{d,k,d_1}(Y)} L_{d_1}(t(V))$$

where Y is a standard free generating set for the derived algebra $L'(t(U))$. Now think of the $\mathcal{B}$ in Theorem 5.1 as $T^{(1)}$ in some complete basis set collection $\mathcal{T}$ for $L(X)$. Then every V in under the big direct sum is a level-2 T-set, and $t(V)$ is its associated iterated tensor product as in Definition 4.2. Theorem 5.1 can then be applied to the direct summands $L_{d_1}(t(V(U))$ with $d_1 > 1$, and we can repeat this process until we reach a state where all quotients in the resulting refinement of the filtration (4.1) are of the form $L_1(t(U)) = t(U)$ for some T-set U (of level ≥ 1). It is not hard to see that the direct summands in the ultimate refinement of the original filtration for L_n will be in one-to-one correspondence with the T-basis sets on degree n. We summarize our discussion in the following

Theorem 5.2. *Let $\mathcal{T}$ be a complete T-basis set collection for $L(X)$. Then each Lie power L_n with $n \geq 2$ has a finite filtration whose quotients are isomorphic to direct sums of KG-modules of the form*

$$\bigoplus_U t(U)$$

where U runs over an appropriate subset of the T-basis sets of degree n. Moreover, there is a one-to-one correspondence between the T-sets of degree n and the direct summands that appear in these quotients. □

Examples. Consider the filtration for L_8 in the example after Theorem 5.1. Here all direct summands of the quotients except $L_2(M_4) = L_2(t(Y_2))$ and $L_4(M_2) = L_4(t(Y_2))$ are of the form $t(U)$ for some level-1 T-basis set U. Applying Theorem 5.1 to the exceptional quotients gives

$$L_2(t(Y_2)) \cong t(Y_2(Y_2)) = M_2(M_2)$$

and a filtration of length 2

$$L_4(t(Y_2)) = L_{4,1}(t(Y_2)) \geq L_{4,2}(t(Y_2)) \geq 0$$

with quotients

$$L_{4,1}(t(Y_2))/L_{4,2}(t(Y_2)) \cong t(Y_4(Y_2)) = M_4(M_2)$$

and

$$L_{4,2}(t(Y_2)) = L_2(t(Y_2(Y_2))) = t(Y_2(Y_2(Y_2))) = M_2(M_2(M_2))$$

where the last in this chain of equations is, strictly speaking, obtained by yet another application of Theorem 5.1. Hence the filtration in Theorem 5.2 for L_8 has length 5, and the direct summands of the quotients are precisely the modules $t(U)$ for the eight T-basis sets U of degree 8.

In the filtration for L_{12} the direct summands $L_2(t(Y_6)), L_3(t(Y_4)), L_2(t([Y_4, Y_2]))$, $L_4(t(Y_3))$ and $L_6(t(Y_2))$ require further applications of Theorem 5.1. For $L_4(t(Y_3))$ we get a filtration of length 2 with quotients isomorphic to

$$t(Y_4(Y_3)) = M_4(M_3) \quad \text{and} \quad t(Y_2(Y_2(Y_3))),$$

and for $L_6(t(Y_2))$ we get a filtration of length 3 with top quotient

$$t(Y_6(Y_2)) = M_6(M_2),$$

middle quotient

$$t([Y_4, Y_2](Y_2)) \oplus t(Y_2(Y_3(Y_2))) = M_4(M_2) \otimes M_2(M_2) \oplus M_2(M_3(M_2)),$$

and bottom quotient

$$t(Y_3(Y_2(Y_2))) = M_3(M_2(M_2)).$$

Hence the filtration in Theorem 5.2 for L_{12} has length 9, and its bottom term is the submodule $L_3(L_2(L_2))\rangle \cong M_3(M_2(M_2))$.

6. Module decompositions of $L(X)$ in characteristic zero

Suppose there exists a standard free generating set $Y = Y_2 \cup Y_3 \cup \cdots$ of the derived algebra L' such that the span of each Y_n is invariant under the action of G. In other words, for all $n \geq 2$, $\langle Y_n \rangle$ is a KG-submodule of L'. Then the span of each B-set U in a basis set collection $\mathcal{B}(Y)$ is G-invariant, and the isomorphism $\langle U \rangle \cong t(U)$ in Corollary 3.2 is an isomorphism of KG-modules. Moreover, if $\mathcal{T}$ is a complete basis set collection for $L(X)$ derived by using exclusively G-invariant standard free generating sets, then the span of each T-set $U \in \mathcal{T}$ is G-invariant and the isomorphism $\langle U \rangle \cong t(U)$ in Corollary 4.1 is an isomorphism of KG-modules. Thus any T-basis of L gives rise to a module decomposition in which the direct summands are in one-to-one correspondence with the T-sets in $\mathcal{T}$, provided that there exists a G-invariant standard free generating set for L'. In this section we show that this is always the case when K is a field of characteristic zero.

Lemma 6.1. *Let $L = L(X)$ be a free Lie algebra of finite rank over a commutative ring K with 1, and let $n \geq 2$ be a positive integer such that $(n-2)!$ is invertible in K. Then the derived algebra L' has a standard free generating set $Y = Y_2 \cup Y_3 \cup \cdots$ such that the span of Y_n is G-invariant.*

Proof. Let $\phi_n : L_n \to M_n$ denote the natural surjection from L_n onto M_n. The key ingredient in the proof of this lemma is the fact that there exist a KG-module homomorphism $\psi_n : M_n \to L_n$ such that the composite of ψ_n and ϕ_n amounts to multiplication by $(n-2)!$ in M_n. Such a homomorphisms is exhibited in [5, pp. 349-350]. If $(n-2)!$ is invertible in K, then $\tilde{\psi}_n = 1/(n-2)!\psi_n$ is a splitting map for the natural surjection ϕ_n. Now let $Y = Y_2 \cup Y_3 \cup \cdots$ be a standard free generating set for L'. Then Y_n is a basis of L_n modulo $L'' \cap L_n$, and the set $\tilde{Y}_n = Y_n \phi_n \tilde{\psi}_n$ too is basis of L_n modulo $L'' \cap L_n$. Moreover, $\tilde{Y}_n$ is G-invariant as it is the image in L_n of the KG-module M_n under the G-map $\tilde{\psi}$. Since $L'' \cap L_n = L(Y_2 \cup \cdots \cup Y_{n-1}) \cap L_n$, it follows that for each $y \in Y_n$, the image $\tilde{y} = y\phi_n\tilde{\psi}_n \in \tilde{Y}_n$ is of the form

$$\tilde{y}_n = y\eta + w_y$$

where η is an automorphism of the free K-module $\langle Y_n \rangle$. But then it is easily seen that the set $\tilde{Y} = Y_2 \cup \cdots Y_{n-1} \cup \tilde{Y}_n \cup Y_{n+1} \cup \cdots$ is a free generating set of L' (see [6, Lemma 2.1]), and the lemma follows. □

Now suppose that K is a field of characteristic zero. Then $(n-2)!$ is invertible for all $n \geq 2$, and, by Lemma 6.1, the derived algebra of any free Lie algebra of finite rank over K has a G-invariant standard free generating set. As observed above, this implies that for $L = L(X)$ there exist G-invariant basis set collections $\mathcal{B}(Y)$ for $L' = L'(Y)$ and G-invariant complete basis set collections $\mathcal{T}$. This gives the main result of this section.

Theorem 6.1. *Let $L = L(X)$ be a free Lie algebra of finite rank at least 2 over a field K of characteristic zero. Then there exist complete basis set collections $\mathcal{T}$ for $L(X)$ such that the span of each T-set is a KG-submodule of $L(X)$. For such $\mathcal{T}$, the direct decompositions in Theorem 3.1 and Corollary 3.1 (with $\mathcal{B} = \mathcal{T}^{(1)}$)*

are direct decompositions of KG-modules, and the isomorphisms in Corollary 3.2 *(again with $\mathcal{B} = \mathcal{T}^{(1)}$) and Corollary* 4.1 *are isomorphisms of KG-modules.* □

For example, for the Lie power L_8 we get

$$L_8 \cong M_8 \;\oplus\; M_6 \otimes M_2 \;\oplus\; M_4 \otimes M_2 \otimes M_2 \;\oplus\; M_5 \otimes M_3 \\ \oplus\; M_3 \otimes M_2 \otimes M_3 \;\oplus\; M_2(M_4) \;\oplus\; M_2(M_2(M_2)).$$

Our decompositions are particularly simple in prime degree p as only basis sets of level 1 (that is B-sets) occur (see Corollary 3.1), and the number of B-sets U for which $t(U)$ has a given set of tensor factors (counting multiplicities) is given in Corollary 3.2. Obviously, the relevant tensor products occurring in L_p are in one-to-one correspondence with the partitions of p in which all parts are ≥ 2. Recall that a partition of n is a string $\lambda = (n_1^{t_1}, n_2^{t_2}, \ldots, n_q^{t_q})$ where the n_i and the t_i are positive integers such that $n_1 > n_2 > \cdots > n_q$ and $t_1 n_1 + t_2 n_2 + \cdots + t_q n_q = n$. Define the tensor product M_λ by

$$M_\lambda = (M_{n_1})^{\otimes t_1} \otimes (M_{n_2})^{\otimes t_2} \otimes \cdots \otimes (M_{n_q})^{\otimes t_q}$$

and set

$$m(\lambda) = \frac{1}{t} \sum_{d|(t_1,\ldots,t_q)} \mu(d) \frac{(t/d)!}{(t_1/d)!(t_2/d)! \cdots (t_q/d)!}$$

where $t = t_1 + t_2 + \cdots + t_q$. Also, let $\Lambda(n)$ denote the set of all partitions $\lambda = (n_1^{t_1}, n_2^{t_2}, \ldots, n_q^{t_q})$ of n with $n_q \geq 2$. Then we have for any prime p an isomorphism

$$L_p \cong \bigoplus_{\lambda \in \Lambda(p)} (M_\lambda)^{\oplus m(\lambda)}. \tag{6.1}$$

In the case where G is the group of all graded algebra automorphisms of $L = L(X)$, that is $G = GL(r, K)$ where $r = |X|$, it is well known that the Lie powers L_n are semisimple with the isomorphism types of the simple direct summands indexed by partitions of n with at most r parts (the number of parts for a partition λ as above is $t_1 + \cdots + t_q$). We write $[\lambda]$ for the simple module corresponding to the partition λ (and we adopt the convention that $[\lambda] = 0$ if λ has more than r parts). The n-th metabelian Lie power M_n is known to be simple, and, in fact, $M_n \cong [n-1, 1]$ for $n \geq 3$ and $M_2 \cong [1^2]$. Thus (6.1) expresses L_p as a direct sum of tensor products of simple modules. Consequently, the irreducible constituents of L_p and their multiplicities can be calculated by using the Littlewood-Richardson rule (see [15, p. 68]). For example, for $p = 7$, (6.1) turns into

$$L_7 \cong M_7 \;\oplus\; M_5 \otimes M_2 \;\oplus\; M_3 \otimes M_2 \otimes M_2 \;\oplus\; M_4 \otimes M_3.$$

Here $M_7 \cong [6, 1]$, and one calculates

$$\begin{aligned} M_5 \otimes M_2 &\cong [5, 2] \oplus [5, 1^2] \oplus [4, 2, 1] \oplus [4, 1^3] \\ M_3 \otimes M_2 \otimes M_2 &\cong [4, 3] \oplus [4, 2, 1]^{\oplus 2} \oplus [4, 1^3] \oplus [3^2, 1]^{\oplus 2} \oplus [3, 2^2]^{\oplus 2} \oplus [3, 2, 1^2]^{\oplus 4} \\ &\quad \oplus [3, 1^4]^{\oplus 2} \oplus [2^3, 1]^{\oplus 2} \oplus [2^2, 1^3]^{\oplus 2} \oplus [2, 1^5] \\ M_4 \otimes M_3 &\cong [5, 2] \oplus [5, 1^2] \oplus [4, 3] \oplus [4, 2, 1]^{\oplus 2} \oplus [4, 1^3] \\ &\quad \oplus [3^2, 1] \oplus [3, 2^2] \oplus [3, 2, 1^2] \end{aligned}$$

so that

$$L_7 \cong [6,1] \oplus [5,2]^{\oplus 2} \oplus [5,1^2]^{\oplus 2} \oplus [4,3]^{\oplus 2} \oplus [4,2,1]^{\oplus 5} \oplus [4,1^3]^{\oplus 3} \oplus [3^2,1]^{\oplus 3} \oplus [3,2^2]^{\oplus 3} \oplus [3,2,1^2]^{\oplus 5} \oplus [3,1^4]^{\oplus 2} \oplus [2^3,1]^{\oplus 2} \oplus [2^2,1^3]^{\oplus 2} \oplus [2,1^5],$$

as first published by Thrall [20]. In fact, in this paper of 1942 Thrall published a list of the multiplicities of the irreducibles occurring in the Lie powers L_n for $n \le 10$ (his result for $n = 10$ was later corrected by Brandt [3]). Unfortunately, even in the case where n is a prime it doesn't appear that the method of determining those multiplicities outlined above is more practical than Brandt's (see [3, Corollary I], see also [17, Chapter 8] for an overview of more recent results on multiplicities).

7. Module decompositions in positive characteristic

In this section p is an arbitrary but fixed prime, and K is a field of characteristic p. Then the situation with module decompositions is more complicated than in the characteristic zero case since it is no longer true that the derived algebra L' has a G-invariant standard free generating set. However, by Lemma 6.1, the derived algebra L' of every Lie algebra L over K has a standard free generating set $Y = Y_2 \cup Y_3 \cup \cdots$ such that the spans $\langle Y_n \rangle$ are G-invariant for $2 \le n \le p+1$. Consequently, if $\mathcal{T}$ is a complete basis set collection for $L(X)$ derived by using exclusively standard free generating sets of this kind, then all the T-sets involving exclusively basis sets of degree $2, 3, \dots, p+1$ are G-invariant. It is clear that this is the case for all T-sets of degree $\le p+1$. Hence we have the following

Theorem 7.1. *Let $L = L(X)$ be a free Lie algebra of finite rank at least 2 over a field K of positive characteristic p. Then there exist complete basis set collections $\mathcal{T}$ for $L(X)$ such that the span of each T-set of degree $n \le p+1$ is a KG-submodule of $L(X)$. Moreover,*

(i) *for all n with $2 \le n \le p+1$ there are direct decompositions*

$$L_n(X) = \bigoplus_{d|n\ d\neq n} \bigoplus_{U\in\mathcal{T}^{(1)}_{n/d}} L_d(U) = \bigoplus_{U\in\mathcal{T}_n} \langle U\rangle$$

as KG-modules,

(ii) *for each $U \in \mathcal{T}_n$ with $2 \le n \le p+1$, there is an isomorphism*

$$\langle U\rangle \cong t(U)$$

as KG-modules. □

In particular, if $n = q$ where q is a prime, decompositions of the form (6.1) (with q in place of p) are valid over fields of characteristic p whenever $q \le p$.

As an application of Theorem 7.1 we recover the key result of [7]. Consider the Lie power $L_p(X)$ where p is the characteristic of K. Then, by the theorem, we have

$$L_p = \bigoplus_{U\in\mathcal{T}^{(1)}_p} \langle U\rangle.$$

It is clear that the intersection $L_p \cap L''$ coincides with the span of the T-set $U \in \mathcal{T}_p^{(1)}$ such that $\operatorname{Deg} U \geq 2$, in other words, all T-set in $\mathcal{T}_p^{(1)}$ except Y_p. For any such U, the associated tensor power $t(U)$ is of the form

$$t(U) \cong M_{n_1} \otimes M_{n_2} \otimes \cdots \otimes M_{n_k} \quad (k \geq 2,\ n_1 + \cdots + n_k = p) \tag{7.1}$$

for some suitable $n_1, \ldots, n_k$. For the metabelian Lie power $M_n(X)$ (over an arbitrary commutative ring K with 1), there is a KG-homomorphism $\mu_n : M_n \to T_n$ from M_n into the tensor power $T_n = \langle X \rangle^{\otimes n}$ such that the composite of μ_n with the canonical projection $\rho_n : T_n \to M_n$ given by $x_1 \otimes \ldots \otimes x_n \mapsto [x_1, \ldots, x_n]$ $(x_i \in X)$ amounts to multiplication by $n(n-2)!$ on M_n (see [18, Section 3.2] or [12, Theorem 3.3]). It follows that M_n is a direct summand of T_n (as KG-module) whenever $n(n-2)!$ is invertible in K. In particular, in the case under consideration, where K is a field of characteristic p, M_n is a direct summand of T_n for all $n < p$. This holds for the tensor factors on the right hand side of (7.1), and hence the entire tensor product is a direct summand on the tensor power $T_p = T_{n_1} \otimes \cdots \otimes T_{n_k}$. Since $L_p \cap L''$ is a direct sum of KG-modules $\langle U \rangle$ with $t(U)$ as in (7.1), we have proved the following result.

Theorem 7.2 ([7], Theorem 3.1)**.** *Let $L = L(X)$ be a free Lie algebra of finite rank at least 2 over a field K of positive characteristic p. Then the submodule $L_p \cap L''$ is a direct summand of the tensor power $T_p = \langle X \rangle^{\otimes p}$.* □

An alternative proof of Theorem 7.2 was given by Erdmann and Schocker in [11, Section 6]. In [7] this result was used to determine the indecomposable direct summands of L_p and their Krull-Schmidt multiplicities as modules for $G = GL(r, K)$ in the case where K is an infinite field of characteristic p. For Lie powers L_n with $(n, p) = 1$ this had been accomplished earlier by Donkin and Erdmann [10], while [11] deals with the case where $n = mp$ with $(m, p) = 1$. Further progress has recently been made by Bryant and Schocker [8], [9].

Over the past twelve years there has been remarkable progress in studying modular Lie powers for finite groups (see [4] and the references therein). However, most of the results in this area refer to the case where the p-Sylow subgroup of G is cyclic, and very little is known about Lie powers in characteristic p as modules for finite groups with non-cyclic p-Sylow subgroup. In the smallest possible instants, where K is a field of characteristic 2 and G is the Klein four group, some initial results have been obtained in [14], but these rather confirm how limited our knowledge is. We hope, however, that the results of the present paper will help to remedy the situation, particularly for modular Lie algebras of rank 2. For these our method gives much more detailed information as we will see in the final two sections. There we will work with the free restricted Lie algebras, which will be discussed in the short section that follows.

8. Free restricted Lie algebras and restricted elimination

Let K be a field of positive characteristic p, and let $R = R(X)$ be the free restricted Lie algebra (or p-Lie algebra) on X. As explained in the Introduction,

R can be identified with the closure $L = L(X)$ in the tensor algebra $T = \bigoplus_{n\geq 0} T_n$ with respect to the unary operation $u \to u^p$, and we will take this point of view in what follows. Then $L \leq R \leq T$, and for the degree n homogeneous components we have $L_n \leq R_n \leq T_n$. Here $R_n = R \cap T_n$ and $L_n = L \cap T_n$. In fact, $R_n = L_n$ for all n which are not divisible by p, and if p divides n, then $R_n = L_n + \langle \{u^p\,;\ u \in R_{n/p}\}\rangle$.

As for ordinary free Lie algebras, there is Lazard elimination for free restricted Lie algebras. The corresponding Elimination Theorem is exactly the same as the Lazard Elimination Theorem in Section 2 with R in place of L. We also need the following variation of Lazard elimination that is specific to free restricted Lie algebras.

Restricted Elimination Theorem. *Let $X = \{x\} \cup Z$. Then there is a direct decomposition (over K)*

$$R(X) = \langle x\rangle \oplus R(Z \wr_p \{x\})$$

where

$$Z \wr_p \{x\} = \{x^p\ ,\ [z, \underbrace{x, \ldots, x}_{s}]\ ;\ z \in Z,\, 0 \leq s \leq p-1\}.$$

For a proof of this theorem we refer to the proof of Theorem 2.7.4. in [1].

We conclude this section with an examination of the p-th restricted Lie power R_p. Here we have a direct decomposition (over K)

$$R_p = \langle \{x^p\,;\ x \in X\} \cup H_p\rangle\rangle \oplus (L_p \cap L'').$$

where H_p is the set of all left normed basic Lie monomials of degree p in x and y. However, as we have just seen in the previous section, $L_p \cap L''$ is a direct summand of T_p as a KG-module. But then it is also direct summand of R_p. We let P_p denote the quotient $P_p = R_p/(L_p \cap L'')$. Then

$$R_p \cong P_p \oplus (L_p \cap L'').$$

Clearly P_p has a submodule that is isomorphic to the metabelian Lie power M_p, and for the quotient, which is generated by X^p, there is an isomorphism

$$P_p/M_p \cong \langle X\rangle^F$$

where $\langle X\rangle^F$ is the Frobenius twist of $\langle X\rangle$. However, in general, M_p is not a direct summand of P_p.

9. Free Lie algebras of rank 2 in characteristic 2

Now let K be a field of characteristic 2, let $X = \{x, y\}$ and consider the free restricted Lie algebra $R = R(x, y)$ of rank 2 over K. By applying the Restricted Elimination Theorem twice (that is by eliminating first x and then y) we obtain the direct decomposition

$$R(x, y) = \langle x, y\rangle \oplus R(x^2, y^2, [y, x], [y, x, y], [x^2, y]).$$

On noting that x^2, y^2 and $[y, x]$ span R_2, and that $[y, x, x]$ and $[x^2, y] = [y, x, x]$ span R_3 (and recalling that both R_2 and R_3 are invariant under the action of G) we obtain the following result which is implicitly contained in [19, Section 2].

Theorem 9.1. *For the free restricted Lie algebra $R = R(x,y)$ of rank 2 over a field K of characteristic 2 there is a direct decomposition*

$$R(x,y) = \langle x,y\rangle \oplus R(R_2 \oplus R_3).$$

as a KG-module. □

Now we can apply the methods of Section 2 to obtain a decomposition theorem for $R(x,y)$. For that we use Lazard elimination for free restricted Lie algebras. By applying the restricted analogue of Theorem 2.1 (and Corollary 2.3) to $R(R_2 \oplus R_3)$, we obtain the following result.

Theorem 9.2. *For the free restricted Lie algebra $R = R(x,y)$ of rank 2 over a field K of characteristic 2 there is a direct decomposition as a KG-module*

$$R(x,y) \cong \langle x,y\rangle \oplus R(R_2) \oplus R(R_3) \oplus \bigoplus_{s,t\geq 1} (R(\underbrace{R_2 \otimes \cdots \otimes R_2}_{s} \otimes \underbrace{R_3 \otimes \cdots \otimes R_3}_{t}))^{\oplus m(s,t)}.$$

where the direct sum runs over all ordered pairs s,t of positive integers and

$$m(s,t) = \frac{1}{s+t} \sum_{d|(s,t)} \mu(d) \frac{((s+t)/d)!}{(s/d)!(t/d)!}.$$

□

Corollary 9.1. *For all $n \geq 2$ there is a direct decomposition of $R_n = R_n(x,y)$ as KG-module*

$$R_n \cong R_{n/2}(R_2) \oplus R_{n/3}(R_3) \oplus \bigoplus_{s,t\geq 1} (R_{n/(2s+3t)}(\underbrace{R_2 \otimes \cdots \otimes R_2}_{s} \otimes \underbrace{R_3 \otimes \cdots \otimes R_3}_{t}))^{\oplus m(s,t)}$$

where we adopt the convention that $R_{n/k} = 0$ if n/k is not an integer. □

For Lie powers this implies the following result.

Corollary 9.2. *For all $n \geq 4$ there is a direct decomposition of $L_n = L_n(x,y)$ as KG-module*

$$L_n \cong L_{n/2}(R_2) \oplus L_{n/3}(R_3) \oplus \bigoplus_{s,t\geq 1} (L_{n/(2s+3t)}(\underbrace{R_2 \otimes \cdots \otimes R_2}_{s} \otimes \underbrace{R_3 \otimes \cdots \otimes R_3}_{t}))^{\oplus m(s,t)}$$

where we adopt the convention that $L_{n/k} = 0$ if n/k is not an integer. □

Examples. For small n the decompositions in Corollary 9.2 are as follows:

$L_4 \cong L_2(R_2)$
$L_5 \cong R_3 \otimes R_2$
$L_6 \cong L_2(R_3) \oplus L_3(R_2)$
$L_7 \cong R_3 \otimes R_2 \otimes R_2$
$L_8 \cong R_3 \otimes R_3 \otimes R_2 \oplus L_4(R_2)$
$L_9 \cong R_3 \otimes R_2 \otimes R_2 \otimes R_2 \oplus L_3(R_3)$
$L_{10} \cong R_3 \otimes R_3 \otimes R_2 \otimes R_2 \oplus L_2(R_3 \otimes R_2) \oplus L_5(R_2)$
$L_{11} \cong R_3 \otimes R_2 \otimes R_2 \otimes R_2 \otimes R_2 \oplus R_3 \otimes R_3 \otimes R_3 \otimes R_2$

Remark. Since the dimension of $R_3 = L_3$ is 2, Theorem 9.2 can be applied to the direct summand $R(R_3)$, and then consecutively to all direct summands which are free restricted Lie algebras of rank 2. Further applications of elimination are possible if the 3-dimensional module R_2 has a non-trivial direct decomposition. This happens, for example, if K is the field of order 2 and $G = GL(2, 2)$ acting naturally on $\langle x, y\rangle$. Then R_2 is the direct sum of two simple $GL(2, 2)$-modules (a trivial and a natural). We mention that in this case the module structure of $L(x, y)$ has been completely determined in [13].

10. Free Lie algebras of rank 2 in characteristic $p \geq 3$

Now let K be a field of characteristic $p \geq 3$ and let $L = L(x, y)$ and $R = R(x, y)$ be the free Lie algebra and the free restricted Lie algebra on two free generators x, y. Restricted elimination of x and y gives a direct decomposition

$$R = \langle x, y\rangle \oplus R(N)$$

where

$$\begin{aligned} N =&\{[y, \underbrace{x, \dots, x}_{s}, \underbrace{y, \dots, y}_{t}] \ ; \ 1 \leq s < p, \, 0 \leq t < p\} \\ &\cup \{x^p, y^p\} \cup \{[x^p, \underbrace{y, \dots, y}_{t}] \ ; \ 1 \leq t < p\}. \end{aligned}$$

Note that the degrees of the free generators in N range from 2 to $2p - 1$. For n in this range we write N_n for the subset of elements with degree n in N. Note that

$$N_p = H_p \cup \{x^p, y^p\}$$

where H_p denotes the set of all left normed basic Lie monomials of degree p in x and y. As we have seen in Section 8, the set N_p spans the homogeneous component R_p modulo $L_p \cap L''$. Recall the direct decomposition (8.1), and let $\tilde{N}_p$ be a basis of the direct summand P_p. Of course $\tilde{N}_p$ too spans R_p modulo $L_p \cap L''$, and then it follows (see [6, Section 2.3]) that the subset N_p of the free generating set N can be replaced by $\tilde{N}_p$ so that the resulting set $\tilde{N}$ is again a free generating set:

$$\begin{aligned} R &= \langle x, y\rangle \oplus R(\tilde{N}) \\ &= \langle x, y\rangle \oplus R(N_2 \cup \cdots \cup N_{p-1} \cup \tilde{N}_p \cup N_{p+1} \cup \cdots \cup N_{2p-1}). \end{aligned}$$

Next we use Lazard elimination to eliminate the free restricted Lie algebra $R(\tilde{N}_p) = R(P_p)$. This gives a direct decomposition

$$R = \langle x, y\rangle \oplus R(\tilde{N}_p) \oplus R(W). \tag{10.1}$$

Here W consists of all elements of the form

$$[u, w_1, \dots, w_k]$$

where $u \in N_i$ with $2 \leq i \leq 2p - 1$ but $i \neq p$ while $w_1, \dots, w_k \in \tilde{N}_p$ and $k \geq 0$. Since $P_p = \langle \tilde{N}\rangle$ is a KG-submodule of R_p, the free restricted Lie algebra $R(\tilde{N}_p)$ is a submodule of R. We claim that $R(W)$ too is a KG-submodule of R. To verify the claim we need to show that for any $v \in W$ and any $g \in G$ the element vg is a linear combination of Lie products of elements of W. This will be an obvious consequence of the following three assertions.

(i) For all $u \in N_i$ with $2 \leq i \leq 2p-1$ but $i \neq p$ and all $g \in G$, ug is a linear combination of Lie products of elements of W.
(ii) For all $w \in \tilde{N}_p$ and all $g \in G$, wg is a linear combination elements of $\tilde{N}_p$.
(iii) For all $v_1, \ldots, v_m \in W$ and all $w \in \tilde{N}_p$, $[v_1, \ldots, v_m, w]$ is a linear combination of Lie products of elements of W.

Now, (i) holds because for any $u \in N_i$ with i in the relevant range we have $ug \in R_i$ and in view of (10.1) we have $R_i \subseteq R(W)$ for all i with $2 \leq i \leq 2p-1$ but $i \neq p$ (since $R(\tilde{N}_p)$ consists entirely of elements whose homogeneous components have degrees divisible by p). The assertion (ii) holds since $\tilde{N}_p$ spans a KG-submodule of R_p, namely P_p. Finally, (iii) is an immediate consequence of the identity

$$[v_1, \ldots, v_m, w] = \sum_{j=1}^{m} [v_1, \ldots, [v_j, w], \ldots, v_m],$$

which follows easily from the Jacobi identity, and the fact that, by definition, for any $v_j \in W$ and any $w \in \tilde{N}_p$ the Lie product $[v_j, w]$ is again an element of W.

Now set $B_n = R(W) \cap R_n$. Then we have the following

Theorem 10.1. *Let $R = R(x, y)$ be the free restricted Lie algebra of rank 2 over a field K of positive characteristic $p \geq 3$. For each $n \geq 2$ there exists a KG-submodule B_n of R_n such that $R_n = B_n$ for all $n \geq 2$ which are not divisible by p, and if n is divisible by p then*

$$R_n = R_{n/p}(P_p) \oplus B_n.$$

In particular, for all n which are divisible by p, $R_{n/p}(P_p)$ is a direct summand of R_n as a KG-module. □

For the free Lie algebra L, set $C_n = R(W) \cap L_n = L(W) \cap L_n$. Then we obtain the following

Corollary 10.1. *Let $L = L(x, y)$ be the free Lie algebra of rank 2 over a field K of positive characteristic $p \geq 3$. For each $n \geq 2$ there exists a KG-submodule C_n of L_n such that $L_n = C_n$ for all $n \geq 2$ which are not divisible by p, and if n is divisible by p then*

$$L_n = L_{n/p}(P_p) \oplus C_n.$$

for $n > p$ while

$$L_p = M_p \oplus C_p$$

In particular, for all $n > p$ which are divisible by p, $L_{n/p}(P_p)$ is a direct summand of L_n as a KG-module. □

In the case where $p = 3$, the method used to prove Theorem 10.1 yields particularly simple decompositions of the Lie powers up to degree 9. The decomposition of L_9 is of special interest in view of recent work by Bryant and Schocker [8]. They have shown that the general decomposition problem for Lie powers over fields of characteristic p reduces to the decomposition problem for Lie powers of prime power degree p^k. For $k=1$ this problem has been solved in [7], and so the case of Lie powers of degree p^2 is the smallest case that is open. We conclude this section by spelling out the details. Let K be a field of characteristic 3 and let $L = L(x,y)$ and $R = R(x,y)$ be the free Lie algebra and the free restricted Lie algebra on two free generators x, y. Restricted elimination of x and y gives a direct decomposition

$$R = \langle x, y\rangle \oplus R(N_2 \cup N_3 \cup N_4 \cup N_5).$$

It is easily seen that

$$\langle N_2\rangle = R_2, \ \ \langle N_3\rangle = R_3, \ \ \langle N_4\rangle = R_4,$$

all of which are KG-submodules of R, while

$$N_5 = \{[y,x,x,y,y],[x^3,y,y]\}.$$

We rewrite our decomposition as

$$R = \langle x, y\rangle \oplus R(R_2 \oplus R_3 \oplus R_4 \oplus \langle N_5\rangle)$$

Now elimination of $R(R_3)$ gives a decomposition

$$\begin{aligned} R =& \langle x, y\rangle \oplus R(R_3) \\ & \oplus R(R_2 \oplus R_4 \oplus (\langle N_5\rangle \oplus [R_2,R_3]) \oplus [R_4,R_3] \oplus ([\langle N_5\rangle, R_3] \oplus [R_2,R_3,R_3]) \oplus \cdots). \end{aligned}$$

We will not list free generators of degree greater than 9. Now observe that

$$\langle N_5\rangle \oplus [R_2,R_3]) = R_5 \quad \text{and} \quad [\langle N_5\rangle, R_3] \oplus [R_2,R_3,R_3] = [R_5,R_3].$$

With this the decomposition can be rewritten as

$$R = \langle x, y\rangle \oplus R(R_3) \oplus R(R_2 \oplus R_4 \oplus R_5 \oplus [R_4,R_3] \oplus [R_5,R_3] \oplus \cdots).$$

Now apply elimination of $R(R_2)$. Since R_2 is one-dimensional, this is a free restricted Lie algebra of rank 1.

$$\begin{aligned} R =& \langle x, y\rangle \oplus R(R_3) \oplus R(R_2) \\ & \oplus R(R_4 \oplus R_5 \oplus [R_4,R_2] \oplus ([R_4,R_3] \oplus [R_5,R_2]) \oplus ([R_5,R_3] \oplus [R_4,R_2,R_2]) \\ & \oplus ([R_4,R_3,R_2] \oplus [R_5,R_2,R_2]) \oplus \cdots). \end{aligned}$$

Next observe that

$$[R_4,R_3] \oplus [R_5,R_2] = R_7 \quad \text{and} \quad [R_4,R_3,R_2] \oplus [R_5,R_2,R_2] = [R_7,R_2],$$

so the decomposition turns into

$$\begin{aligned} R =& \langle x, y\rangle \oplus R(R_3) \oplus R(R_2) \\ & \oplus R(R_4 \oplus R_5 \oplus [R_4,R_2] \oplus R_7 \oplus ([R_5,R_3] \oplus [R_4,R_2,R_2]) \oplus [R_7,R_2] \oplus \cdots). \end{aligned}$$

Finally, elimination of $R(R_4)$ gives

$$\begin{aligned} R =& \langle x, y\rangle \oplus R(R_3) \oplus R(R_2) \oplus R(R_4) \\ & \oplus R(R_5 \oplus [R_4,R_2] \oplus R_7 \oplus ([R_5,R_3] \oplus [R_4,R_2,R_2]) \oplus ([R_7,R_2] \oplus [R_5,R_4]) \oplus \cdots). \end{aligned}$$

Here

$$[R_7,R_2] \cong R_7 \otimes R_2 \quad \text{and} \quad [R_5,R_4] \cong R_5 \otimes R_4.$$

Then the decomposition yields the following result for R_9.

Theorem 10.2. *Let $R = R(x, y)$ be the free restricted Lie algebra of rank 2 over a field K of positive characteristic 3. Then there is a direct decomposition of KG-modules*

$$R_9 = R_3(R_3) \oplus [R_7, R_2] \oplus [R_5, R_4]$$

where $[R_7, R_2] \cong R_7 \otimes R_2$ and $[R_5, R_4] \cong R_5 \otimes R_3$ are direct summands of the tensor power T_9. □

For the free Lie algebra $L(x, y)$ this yields the following

Corollary 10.2. *Let $L = L(x, y)$ be the free restricted Lie algebra of rank 2 over a field K of positive characteristic 3. Then there is a direct decomposition of KG-modules*

$$L_9 = L_3(R_3) \oplus [L_7, L_2] \oplus [L_5, L_4]$$

where $[L_7, L_2] \cong L_7 \otimes L_2$ and $[L_5, L_4] \cong L_5 \otimes L_4$ are direct summands of the tensor power T_9. □

These decompositions are peculiar to rank 2. They are not valid for ranks greater than two. A a byproduct of our calculations for R_9 we have the following decompositions for degree 6 in rank 2.

$$R_6 \cong R_3(R_2) \oplus R_2(R_3) \oplus [R_4, R_2], \quad L_6 \cong L_2(R_3) \oplus [L_4, L_2]$$

where $[R_4, R_2] = [L_4, L_2] \cong L_4 \otimes L_2$.

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