***Geometric methods for anisotopic inverse boundary value problems.***

Lionheart, William R.B.

2004

MIMS EPrint: **2006.411**



ISSN 1749-9097

# Geometric methods for anisotopic inverse boundary value problems.

Paper for the proceedings of:

## Recent Developments in the Wave Field and Diffuse Tomographic Inverse Problems EuroConference. Edinburgh 3-5 August 2000.

W.R.B. Lionheart
Department of Mathematics,
UMIST,
P.O. Box 88,
Manchester M60 1QD,
E-mail: Bill.Lionheart@umist.ac.uk

December 5, 2000

# 1 Introduction

Electromagnetic fields have a natural representation as differential forms. Typically the measurement of a field involves an integral over a submanifold of the domain. Differential forms arise as the natural objects to integrate over submanifolds of each dimension. We will see that the (possibly anisotropic) material response to a field can be naturally associated with a Hodge star operator.

This geometric point of view is now well established in computational electromagnetism, particularly by Kotiuga [13], and by Bossavit and and others (see for example [26],[22]). The essential point is that Maxwell's equations can be formulated in a context independent of the ambient Euclidean metric. This approach has theoretical elegance and leads to simplicity of computation.

In this paper we will review the geometric formulation of the (scalar) anisotropic inverse conductivity problem, amplifying some of the geometric points made in Uhlmann's paper in this volume [28]. We will go on to consider generalizations of this anisotropic inverse boundary value problem to systems of Partial Differential Equation, including the result of Joshi and the author on the inverse boundary value problem for harmonic $k$-forms [8].

# 2 Review of geometric concepts and notation

The context for this paper will be a smooth compact orientable $n$ dimensional manifold with boundary. We will review briefly some concepts and notation from differential geometry essential to the geometric study of inverse problems.

A differential $k$-form is a section of the bundle of skew symmetric $k$-linear maps on the tangent space to $M$. The space of smooth $k$ forms is denoted by $\Omega^k(M)$. $\Omega^0(M)$ consists simply of smooth functions, and $\Omega^1(M)$ co-vector fields. The wedge product $\alpha\wedge\beta$ of a $k$ form $\alpha$ and an $\ell$-form $\beta$ is a $k+\ell$-form equal to the skew symmetric part of $\alpha\otimes\beta$

The derivative $d:\Omega^0(M)\to\Omega^1(M)$ has a natural extension, the exterior derivative $d:\Omega^k(M)\to\Omega^{k+1}(M)$ as a derivation on the complete algebra of differential forms

$$d(\alpha\wedge\beta) = d\alpha\wedge\beta+\alpha\wedge d\beta. \tag{2.1}$$

The exterior derivative satisfies $d^2=0$.

Given a local coordinate chart $x$, a $k$-form $\omega\in\Omega^k(M)$ can be expressed in coordinates as

$$\omega = \sum_{1\le i_1\le\cdots\le i_k\le n} \omega_{i_1\ldots i_k} dx_{i_1}\wedge\cdots\wedge dx_{i_k}$$

The *raison d'être* for studying differential forms is that a $k$-form is the natural object to integrate over $k$-dimensional submanifolds without the need for any metric or measure. We have the generalized Stoke's (or perhaps Newton-Leibnitz-Gauss-Green-Ostrogradski-Stokes-Poincaré) formula

$$\int_N d\omega = \int_{\partial N}\omega \tag{2.2}$$

for a $k$-form $\omega$ and a $k+1$-dimensional submanifold $N$ (or more generally a chain).

The space of smooth vector fields on $M$ will be denoted by $\mathfrak{X}(M)$ and $\mathfrak{X}_0(M)$ will denote vector fields vanishing on $\partial M$. The covariant derivative of a tensor field $T$ will be denoted by $\nabla T$, with components $T^{i_1\ldots i_k}_{j_1\ldots j_\ell;j}$. The space of smooth symmetric tensors of covariant rank two will be denoted by $S^2(M)$. The operator Sym is the symmetric part of a covariant rank two tensor: $\mathrm{Sym}(T)_{ij} = (T_{ij}+T_{ji})/2$. We denote the space of sections of a tensor bundle with ($L^2$ based) Sobolev class $s$ by the prefix $H^s$, otherwise sections will be assumed to be smooth.

The 'musical isomorphisms' $\sharp$ and $\flat$ associated with a metric tensor $g$ raise a covariant and lower a contravariant index respectively, so that $\alpha^\sharp$ is a vector field when $\alpha$ is a one-form. We denote the contravariant metric tensor, with components $g^{ij}$ by $g^\sharp$. We will denote contractions of tensor products over a covariant and contravariant index by a dot so that $\alpha^\sharp = g^\sharp\cdot\alpha$, in components $\alpha^i = \sum_k g^{ik}\alpha_k$. As a special case we will denote the contraction of a $k$-form $\alpha$ with a vector field $X$ by $X \lrcorner\, \alpha$ (often this is denoted by $i_X\alpha$, but we have too many uses for $i$). The operator $X\lrcorner$ is an anti-derivation on the algebra of differential forms:

$$X \lrcorner (\alpha\wedge\beta) = (X\lrcorner\alpha)\wedge\beta + (-1)^k\alpha\wedge(X\lrcorner\beta) \tag{2.3}$$

The metric tensor induces a volume form $\mu \in \Omega^n(M)$ and Hodge star isomorphism $*:\Omega^k(M)\to\Omega^{n-k}(M)$ is defined by the property

$$*\omega\wedge\omega = g^\sharp(\omega,\omega)\mu, \tag{2.4}$$

We can consider the Hodge star on $k$-forms as a contraction of the tensor

$$\overbrace{g^\sharp\otimes\cdots\otimes g^\sharp}^{k}\otimes\mu. \tag{2.5}$$

When there is more than one metric under consideration we use $*_g$ and $\mu_g$ to indicate the dependence on the metric. The Hodge star satisfies $**\alpha = (-1)^{k(n-k)}\alpha$, for any $k$-form $\alpha$.

The formal adjoint with respect to a metric of the exterior derivative on $k$-forms is $\delta = (-1)^{(nk+n+1)} * d *$.

Given a smooth map $f: M\to N$ between manifolds (not necessarily of the same dimension) any covariant rank $k$-tensor field $T$ on $N$ gives rise to a tensor of the same rank $f^*T$ on $M$, and a contravariant tensor field $S$ on $M$ gives rise to the push forward $f_*S$ on $N$. The components of the pull-back and push forward are given by the 'classical' transformation formulas for components of tensor fields. For example let $y$ be a coordinate chart on $N$ and $x = y\circ f$, for a one form $\alpha$

$$(f^*\alpha)_i = \sum_k \frac{\partial y_i}{\partial x_j}\alpha_j. \tag{2.6}$$

Where $f$ is a diffeomorphism, both pull-backs and push-forwards of any type of tensor are defined. Pull-backs and push-forwards have the following 'functorial' behavior under composition $(fg)_* = f_*g_*$ and $(fg)^* = g^*f^*$

The Lie derivative, with respect to a vector field $X$, on a tensor field $T$ is defined by $L_X T = \frac{d}{dt}\Phi_t^* T\big|_{t=0}$. Here $\Phi_t$ the flow of the vector field $\frac{d}{dt}\big|_{t=0}\Phi = X$. The

Lie derivative is a derivation on the tensor algebra, it preserves the rank and commutes with contraction. An important special cases for a Riemannian manifold is $(L_X g)_{ij} = X_{i;j} + X_{j;i} = 2\,\mathrm{Sym}\,\nabla X^\flat$ where indicies following the semi-colon indicate covariant derivatives. Also $L_X \mu_g = (\operatorname{div} X)\mu_g$, where $\operatorname{div} X = \sum_i X^i_{;i} = -\delta X^\flat$.

The operators $\wedge, d, X \lrcorner, L_X$ all commute with pull-backs and push-forwards where these are defined.

A partial differential operators $P = p(x,D)$ defined on vector bundles over Euclidean space $\mathbb{R}^n$, has a full symbol $p(x,\xi)$. Here $D_\alpha = -i\partial_\alpha$ for a multi index $\alpha$ and $\xi$ is in the dual space $(\mathbb{R}^n)^*$. The symbol is a linear map valued multinomial in $\xi$ at each $x$. For an operator of order $m$ the principal symbol $\sigma_P(x,\xi)$ is the $m$-th order part. Partial differential operators on sections of vector bundles over manifolds have an invariantly defined principal symbol. The full symbol can either be defined with reference to a particular coordinate chart, or more elaborate methods on a Riemannian manifold [29]. The principal symbol of $d$ is $\sigma_d(\xi)\omega = i\xi \wedge \omega$. The principal symbol of $\delta$ is $\sigma_\delta(\xi) = -i\xi^\sharp \lrcorner$ .

# 3 Scalar anisotropic inverse conductivity problem

Let us consider an anisotropic electrical conductor with a known, time invariant current density applied to the boundary. The inverse problem of recovering the anisotropic conductivity from all possible measurements of current density and voltage at the boundary is discussed in depth in [28]. Here we will contribute some additional geometric details.

Electric field is naturally formulated as a 1-form $E$ as the work done in moving a charged particle is the integral over its path, and one-forms are the natural objects to integrate over one-dimensional submanifolds. The current density $J$, by contrast is is a rate of of charge flux across a surface. This reveals the true nature of $J$ as a $n-1$-form. In an Ohmic material $E$ and $J$ are linearly related by the conductivity $J = \sigma E$ where $\sigma : \Omega^1 \to \Omega^{n-1}$ is linear on fibres (of course we have $n = 3$ for the physical situation). The use of variable names in electromagnetics is goverened by rigid conventions and we hope the use of $\sigma_P$ for the principal symbol, as well as $\sigma$ for conductivity will not cause confusion. The density of Ohmic power dissipation is $J \wedge E = \sigma E \wedge E$ — an $n$-form which can be integrated over a volume to give the power dissipated over that volume. Physical considerations show that power is non-negative and that for any two one-forms $E_1$ and $E_2$, $\sigma E_1 \wedge E_2 = \sigma E_2 \wedge E_1$. It is important to note that this invariant formulation makes no reference to measurement of length. It is this metric invariance which simplifies the formulation of computational electromagnetism [13]. When one introduces a coordinate chart, in particular the natural chart on a domain in $\mathbb{R}^n$, one can introduce a length scale. If the charts are measured in metres the components $E_i$, where $E = \sum_i E_i dx_i$ will be in volts/metre as expected.

In dimensions $n > 2$ the conductivity gives rise to a Riemannian metric $g$ for which $\sigma$ is the Hodge star. It is an unusual viewpoint for the geometer, who is accustomed to considering the metric structure of a manifold as primary. In electrical geometry the Hodge is the star! Indeed it is hard to attribute physical significance to the geodesics and curvature which play such a fundamental roll in Riemannian geometry. The derivation of the metric in terms of the conductivity for $n > 2$ is given in [28] in terms of coordinates. Let us see how it can be done invariantly.

In our orientable manifold, $\Omega^n(M)$ is isomorphic to functions $\Omega^0(M)$. Ratios of non-vanishing $n$-forms are well defined functions. Let us choose $\mu_0$ as an aribitrary positively oriented $n$-form.

We define $\tilde{g}^\sharp(E_1,E_2) = (\sigma E_1 \wedge E_2)/\mu_0$ and it is simple to verify that this defines a metric tensor. Now for any metric $g$ and positive scalar field $\alpha$ the Hodge star on one-forms satisfies $\alpha *_{\alpha g} = \alpha^{(2-n)/n} *_g$. Then for $n > 2$ metric $g = (\mu_{\tilde{g}}/\mu_0)^{n/(2-n)}\tilde{g}$ has $\sigma = *_g : \Omega^1(M) \to \Omega^{n-1}(M)$.

The anomalous case $n = 2$ has $\sigma : \Omega^1(M) \to \Omega^1(M)$. The Hodge star is an endomorphism of one-forms with $*^2 = -\mathbf{1}$ (here $\mathbf{1}$ is the identity on one-forms). It has determinant one but there is no such restriction on anisotropic conductivities. We have instead $\sigma = \det(\sigma)*_g$.

The absence of interior current sources is expressed by the Kirchoff law $d(\sigma E) = 0$, and for the static case (see Maxwell's equations in Section 8) an irrotational electric field $dE = 0$. Poincaré's Lemma then tells us that for $M$ simply connected $E = du$ for some 0-form $u$ defined up to an additive constant. We then have the familiar conductivity equation $d\sigma du = 0$, which for $n > 2$ is equivalent to the Laplace-Beltrami equation on the Riemannian manifold

$$*d*du = 0 \text{ or } \delta du = 0. \tag{3.1}$$

For the case $n = 2$ we have $d\gamma * du = 0$, for a positive scalar $\gamma$.

We now look at the invariant formulation of boundary conditions. Let $\mathbf{i} : \partial M \hookrightarrow M$ be the inclusion of the boundary. A more careful version of Stoke's formula [24] is

$$\int_N d\omega = \int_{\partial N} \mathbf{i}^*\omega \tag{3.2}$$

where the pull-back $\mathbf{i}^*\omega$ is the restriction of the form $\omega$ to the boundary. The Green's formula, obtained by applying Stoke's Theorem to $d(*du \wedge u)$,

$$\int_M *du \wedge du = \int_{\partial M} \mathbf{i}^*(*du) \wedge \mathbf{i}^* u \tag{3.3}$$

expresses the conservation of power. We see from the weak formulation of Equation 3.1 that natural Dirichlet data is $\mathbf{i}^* u = u|_{\partial M}$ and the natural Neumann data is $\mathbf{i}^* * du$. Our Dirichlet-to-Neumann mapping $\Lambda_\sigma : H^s\Omega^0(\partial M) \to H^{s-1}\Omega^{n-1}(\partial M)$ is $\Lambda_\sigma u|_{\partial M} = \mathbf{i}^* * du$. The constraint of finite power dissipation requires $u \in H^1(M)$ and the trace formula then gives $s = 1/2$. This invariant formulation of the Neumann data $\mathbf{i}^*(*du)$ makes no reference to a normal vector field of an embedding of $\partial M$ in $\mathbb{R}^n$. To make a measurement of current on the boundary one would measure the total current over some part of $\partial M$, this is an integral of the 2-form $\mathbf{i}^*(*du)$ over a two-dimensional submanifold of $\partial M$, which is defined without reference to the embedding. As all the operations we have used commute with pull-backs and push-forwards we see that for a diffeomorphism $\Phi$ on $M$ with $\phi = \Phi|_{\partial M}$

$$\begin{aligned}
\Lambda_{\Phi^*\sigma}\phi^* f &= \mathbf{i}^*(\Phi^*\sigma)d\Phi^* u) \\
&= \mathbf{i}^*\Phi^*(\sigma du) \\
&= (\Phi\mathbf{i})^*(\sigma du)
\end{aligned}$$

As $\mathbf{i}\phi = \Phi\mathbf{i}$ we have $(\Phi\mathbf{i})^* = \phi^*\mathbf{i}^*$ and so

$$\Lambda_{\Phi^*\sigma} = \phi_* \Lambda_\sigma \phi^*.$$

When $\phi$ is the identity $\Lambda_{\Phi^*\sigma} = \Lambda_\sigma$. We now see the non-uniqueness in the anisotopic inverse conductivity problem first pointed out by Tatar [12].

In a neighbourhood of any point on the boundary of the manifold we can define boundary normal coordinates $(x_1, \dots, x_{n-1}, x_n) = (x', x_n)$ where $x_n$ is the geodesic distance to the boundary. For a function $u$ we have at the boundary

$$\partial_n u = \partial_n \lrcorner\, du = *_{\partial M} \mathbf{i}^* * du,$$

where $*_{\partial M}$ is the Hodge star on the boundary, recovering the traditional view of the Neumann data as the normal derivative.

More general classes of self adjoint elliptic equations on manifolds will have Dirichlet-to-Neumann mappings invariant under a smaller group of diffeomorphisms. For example in the case of the stationary Schrödininger equation $\delta du + cu = 0$, the Dirichlet-to-Neumann mapping is invariant under the more restrictive class volume preserving diffeomorphisms $\Phi^* \mu = \mu$ whose restriction to the boundary is the identity.

# 4 Linearization

Practical reconstruction algorithms typically require linearization of the forward mapping. In anisotropic problems where the invariance of boundary data under the action of a group of diffeomorphisms, we will see a non-trivial kernel in the Fréchet derivative of the forward mapping. A simple case of this is explored in [25]. Let us consider the scalar anisotropic inverse conductivity problem where the forward mapping is $F : g \mapsto \Lambda_g$. Let $G_g : H^{-1/2}(\partial M) \mapsto H^1(M)$ be the Greens operator so that $\langle f, \Lambda_g f \rangle = \int_M g(dG_g f, dG_g f)\mu_g$ then the Fréchet derivative applied to a symmetric contravariant rank two tensor field h is

$$\langle f, DF_g h f \rangle = \int_M -(g^\flat \cdot h \cdot g^\flat)(dG_g f, dG_g f)\mu_g.$$

A family $g_t$ of metrics defined by $g_t = \Phi_t^* g_0$, where $\Phi_t$ is a family of diffeomorphisms fixing points on the boundary, will have identical boundary data $\Lambda_{g_t} = \Lambda_{g_0}$. Hence $dg/dt|_{t=0}$ will be in the kernel of $DF_{g_0}$. In practical algorithms for the anisotropic inverse problem it is important to characterise the kernel, in particular one needs to find extra information which gives constraints on the solution transverse to the kernel.

We see that any $h$ which are of the form $h = L_X g$ for some $X \in \mathfrak{X}_0(M)$ is in the kernel of $DF_g$. Similarly for the Hodge star we see that perturbations of the Hodge of the form $L_X *$ are in the kernel of the derivative of the forward mapping. Taking the Lie derivative of Equation 2.5 for 1-forms we see

$$L_X(g^\sharp \otimes \mu) = -g^\sharp \cdot L_X g \cdot g^\sharp \otimes \mu + \mathrm{div}(X) g^\sharp \otimes \mu.$$

Applying to a 1-form $\alpha$ and contracting gives

$$(L_X *)\alpha = 2 * \mathrm{Sym}(\nabla X^\flat) \cdot g^\sharp \cdot \alpha - \mathrm{div}(X) * \alpha$$

in agreement with the Euclidean case given in [25]. More details, and more general results about deformations of the Hodge star are given in [27].

The operator $A : X \mapsto L_X g$ is easily seen to be an elliptic operator, and standard results on elliptic splittings of sections of bundles over manifolds with boundaries can be used to give an $L^2$ orgthogonal direct sum [3]

$$H^s S^2(M) = imA \oplus kerA^*$$

where $A : H^{s+1}\mathfrak{X}_0(M) \to H^s S^2(M)$ and $A^*$ is the formal adjoint $(A^* h)_i = (\mathrm{Div} h)_i := -\sum_{jk} h_{ij;k} g^{jk}$ is its formal adjoint We can regard this as an orthogonal decomposition of a perturbation of the metric into a component in the image of $A$ which is invisible at the boundary, and a component in the kernel of $A^*$ we at least have a hope of identifying. A similar splitting can be derived for perturbations of the Hodge.

One has to be cautious in numerical implementation of anisotropic reconstruction algorithms. Vauhkonen and the author (unpublished) implemented two and three dimensional finite element forward solvers with piecewise constant anisotropic conductivity. We calculated the matrix of the Fréchet derivative, and while we observed a typical decay of the singular values of this matrix, we did not see the rank deficiency we expected from the image of $A$. In the two dimensional case, with sufficiently obtuse triangles, the finite element method gives the same equations as a planar resistor mesh. The work of Colin de Verdière [5] gives necessary and sufficient conditions for uniqueness of solution. Indeed a triangular resistor mesh is an invariant formulation of the finite element model (an embedded piecewise linear manifold). When we treated the embedding, as well as the conductivity as variables, we saw the expected kernel in the Fréchet derivative. This suggests that there may uniqueness results to prove for the anisotropic inverse conductivity problem in the piece-wise linear (or perhaps more general finite element) category.

# 5 Constrained anisotropic problems

In practical situations, such as medical imaging, recovering the anisotropic conductivity 'up to diffeomorphism' may not be enough. For example if one needed to locate a problematic area of tissue for surgical treatment it is of very little use at all. The important point is that the electric fields see only electrical geometry, but the surgical exploration sees the ambient Euclidean metric. We can regard this situation from two viewpoints, one is that the manifold with the electrical metric is embedded in Euclidean space, other other is that we have an abstract manifold with two metrics.

An isotropic conductivity $\sigma$ is one which can be expressed as $\sigma = \gamma *_e$, where $*_e$ is the Hodge star on 1-forms associated with the ambient Euclidean metric. A conductivity might be associated with a conformally flat metric, that is $g = \gamma g_0$ where $g_0$ is flat, but this is not the same as being isotropic. The flat metric $g_0 = \Phi_* e$ for some diffeomorphism $\Phi$ but $\Phi$ will change the shape of the embedded domain. For details, including the surprising possibility of recovering both the boundary shape and the conductivity in the isotropic case, see [17]. The wealth of uniqueness results for the isotropic case ([28] for a summary) suggests that some constrained anisotropic problems where the conductivity is parametrized by one unknown function might also have a unique solution.

An early success in this area was the result by Kohn and Vogelius [12]. If the eigenvectors, and all but one of the eigenvalues, of the conductivity were known, and the conductivity were piece-wise analytic, then that unknown eigenvalue could be recovered from the Dirichlet-to-Neumann mapping. This work was done in a Euclidean context, in coordinates. The eigenvectors and eigenvalues in question are those of 'conductivities in Euclidean coordinates' $a = (-1)^{n-1} *_e \sigma$, which are well defined as endomorphisms of 1-forms.

The work of the present author [16] showed that the piece-wise analytic uniqueness results of Kohn and Vogelius can be generalized to the case where $\sigma = \gamma\sigma_0$ for some known conductivity $\sigma_0$. This is expected from a geometric viewpoint, and can be proved in any category where it is known that smooth diffeomorphisms are the only obstruction to uniqueness. Given $g_0$ we are restricted to conformal diffeomorphims $\Phi_* g_0 = \alpha g_0$. The group of conformal diffeomorphisms of a Riemannian manifold is a finite dimensional Lie group [11] so we have already that the ambiguity in conductivity is finite dimensional. Furthermore, the only conformal diffeomorphism which is the identity on the boundary is the identity, so the conductivity is uniquely determined (see [16] for details).

This suggests the following strategy. First define some constraints on the conductivity then write down the equation for a diffeomorphism which preserves this. The result will typically be a system of partial differential equations. If one is one might find that the solution space is finite dimensional. The in the conformal case discussed above one can use the apparatus of $G$-structures on principal bundles [11], and the argument could be extended to other $G$-structures which are either elliptic (for $M$ compact), or of finite type.

Alessandrini and Gaburro [2] have proved uniqueness results for a family of anisotropic conductivity inverse boundary value problems with one unknown function. When formulated geometrically the idea is as follows. Let $a(x,t)$ be a family of conductivities in Euclidean coordinates such that

$$\frac{\partial a}{\partial t} \geq C\mathbf{1}$$

for some constant $C$. This monotone family of conductivities then gives rise to distinct Dirichlet-to-Neumann maps for distinct piecewise analytic functions $\gamma$ with the conductivity $a(x,\gamma(x))$.

# 6 Laplacians on Forms

We will now consider generalizations of the scalar conductivity equation in the setting of Riemannian geometry. Let $u$ be a 1-form then the 'rough Laplacian' is the operator expressed in coordinates as $-\sum_{ij} g^{ij} u_{k;ij}$. The principal symbol in this case is $g(\xi,\xi)\mathbf{1}$. We shall see that the rough Laplacian has the same principal symbol as the Laplacian. The Laplace-Beltrami operator on $k$-forms is $\Delta = d\delta + \delta d$. Using Equation 2.3 we see

$$\sigma^2_\Delta(\xi)\omega = \xi^\sharp \lrcorner (\xi \wedge \omega) + \xi \wedge (\xi^\sharp \lrcorner \omega) = (\xi^\sharp \lrcorner \xi)\omega = g^\sharp(\xi,\xi)\omega.$$

The connection between the Laplacian and the rough Laplacian, as well as an alternative way to calculate the principal symbol of the former, is given by the coordinate expression for the Laplacian

$$(\Delta u)_{i_1\dots i_k} = \sum_{ij}\left(-g^{ij}u_{i_1\dots i_k;ij} + \sum_{\alpha=1}^{k} R^j_{i_p} u_{i_1\dots i_{\alpha-1} j i_{\alpha+1}\cdots i_k}\right.$$
$$\left. + \frac{1}{2}\sum_{\alpha=1}^{k}\sum_{\beta=1}^{k} R^{ij}{}_{i_\beta i_\alpha} u_{i_1\dots i_{\alpha-1} j i_{\alpha+1}\dots i_{\beta-1} i i_{\beta+1}\dots i_k}\right).$$

In particular for a flat metric the Laplacian and rough Laplacian coincide. A differential form $u$ satisfying Laplace's equation $\Delta u = 0$ is called a harmonic form. On a compact manifold without boundary, this is equivalent to the condition that the form is a harmonic field, that is it is both exact, $du = 0$, and co-exact, $\delta u = 0$ as

$$\langle u, \Delta u\rangle = ||du||^2 + ||\delta u||^2 + \int_{\partial M} \delta u \wedge *u + \int_{\partial M} u \wedge *du \tag{6.1}$$

However on manifolds with boundary there can be harmonic forms which are not harmonic fields. Duff and Spencer [6] show that the Dirichlet data $(\mathbf{i}^* u, \mathbf{i}^* * u)$ together with specification of the integral of $u$ on a basis for the relative homology $H^k(M, \partial M)$, gives a unique solution to $\Delta u = 0$. Similarly for consistent Neumann data $(\mathbf{i}^* d * u, \mathbf{i}^* du)$.

We now look at closely related systems of elliptic partial differential equations occur in electro-magnetics (the vector Helmholtz equation) and in linear elasticity.

# 7 Linear Elasticity

In a linear elastic solid with metric tensor $g$ and with no body forces, the displacement field $u \in \mathfrak{X}(M)$ satisfies the equation $\mathrm{Div}(CL_u g) = 0$ The elastic tensor $C$ is a field of automorphisms of the symmetric tensors on each fibre. The principal symbol of the elastic operator is $C$. For an isotropic solid $C = \lambda g \otimes g^\sharp + \mu I$ where $I$ is the identity operator on symmetric tensor fields. The problem considered by Nakamura and Uhlmann in [18] was the recovery of the Lamé parameters $\lambda$ and $\mu$ for an isotropic solid. We will discuss their work on the more general isotropic case in Section 10.

# 8 Maxwell's Equations

In electro-magnetic theory the electric field $E$ and magnetic filed $H$ are naturally defined as 1-forms, as to take measurements of these fields one must integrate over curves. The resulting electric and magnetic fluxes, $D$ and $B$ are naturally two forms as one must integrate them over surfaces to make a measurement. The material properties (for simplicity we consider a non-chiral, linear, insulating material) are the permittivity $\varepsilon$ and permeability $\mu$, these map one forms to two forms and the Hodge star operators for an associated electric and magnetic Riemannian metric. Assuming all fields to be time harmonic with angular frequency $\omega$ and the electric charge density to be constant we have Maxwell's equations

$$\begin{aligned} dB &= 0, & dD &= 0 \\ dE &= -i\omega\mu H, & D &= \varepsilon E \\ dH &= i\omega\varepsilon E, & B &= \mu H \end{aligned} \tag{8.1}$$

For a conductive body we can replace the permittivity by a complex permittivity $\varepsilon - i\sigma/\omega$.

There are a variety of physical situations where one attempts to recover a selection of the material parameters $\sigma$, $\varepsilon$ and $\mu$ form electromagnetic measurements at the boundary. In medical and industrial applications of electrical imaging one often has a relatively low frequency so that $\omega\mu$ is negligible. As in the static case we have $dE = 0$ and so $E = du$ for a simply connected manifold. The complex conductivity equation

$$d\left((\sigma + i\omega\varepsilon)du\right) = 0$$

is then a good approximation. The anisotropy of $\sigma$ and $\varepsilon$ could be unrelated. For example if one were known to be isotropic a diffeomorphism preserving this property (a conformal mapping of the associated metric) and fixing points on the boundary would be the identity, as shown by Lionheart [16].

Other applications include electromagnic imaging [4], where the full time harmonic Maxwell's equations must be used. And integrated photoelasticity [1] where the permittivity is linearly related to stress in a transparent material.

In many cases the permeability $\mu$ will be isotropic, and even a known constant close to the permeability of a vacuum. Again this means that there is no ambiguity from diffeomorphisms fixing points on the boundary, and we might expect a unique solution to the anisotropic inverse problem.

So far uniqueness results for inverse boundary value problems for time harmonic Maxwell's equations have concentrated on the isotropic case [21] [10].

In the special case where $\mu = \varepsilon = *$ (obviously after units have been scaled) we notice that $E$ and $H$ satisfy the vector Helmholz equations $\Delta E = \omega^2 E$ and $\Delta H = \omega^2 H$.

# 9 Symbols and Pseudo-differential operators

The Dirichlet to Neumann mapping is an example of a classical Pseudo-differential operator, essentially a generalization of differential operators to non-polynomial symbols while retaining some polynomial-like features. A classical pseudo-differential operator of order $m$ has a full symbol which is an asymptotic sum of terms $p_{m-j}(x,\xi)$ which are smooth in $\xi \neq 0$ and for $\lambda > 0$ are homogeneous of degree $m - j$

$$p_{m-j}(x,\lambda\xi) = \lambda^{m-j} p_{m-j}(x,\xi).$$

The principal symbol is $p_m$ also denoted by $\sigma_m(P)$. The class of classical pseudo-differential operators is denoted by $\Psi\mathrm{DO}^m_{\mathrm{cl}}(\mathbb{R}^n)$. There are more general classes of pseudo-differential operators based on more general symbols, but we shall not need them here. These classes form a graded algebra under composition. To obtain the principal symbol of the composite one takes the product: $\sigma_{PQ} = \sigma_P\sigma_Q$ however the full symbol of the product is rather more complicated. Operators in $\Psi\mathrm{DO}^{-\infty} = \bigcap_{m\in\mathbb{R}} \Psi\mathrm{DO}^m$ are called smoothing operators. The full symbol of a pseudo-differential operator determines the operator modulo smoothing operators. For brief introduction to pseudo-differential operators we recommend the notes [9] and for more detail Shubin [23]. We note that the definition of pseudo-differential operators can be extended to smooth manifolds using coordinate charts. As for differential operators, the principal symbol is invariantly defined as a function on the cotangent bundle while the total symbol depends on choice of coordinates.

# 10 Factorization and symbol calculus

The factorisation method of Lee and Uhlmann ([15], see also [28]) extends naturally to the $k$-form Laplace's equation. In the scalar case there is a factorisation modulo smoothing operators

$$\Delta = (D_n + E + iB)(D_n - iB) \tag{10.1}$$

where $D_n = -i\partial_n$ and $B(x,D')$ is a first order pseudo differential operator with principal symbol $\sigma_B(x',\xi') = \sqrt{g^\sharp(\xi',\xi')}$.

At the boundary $*_{\partial M}B(0,D'_x)$ is equal modulo smoothing with the Dirichlet-to-Neumann mapping. This comes from considering the factors in 10.1 as forwards and backwards generalized heat equations so that $\partial_n u = B(0,D'_x)$ modulo smoothing. The theorem of Lee and Uhlmann, that the full symbol of the Dirichlet-to-Neumann mapping determines the Taylor series of the metric at the boundary, follows from this factorization and an inductive argument using the composition formula for full symbols when Equation 10.1 is expanded.

The simplicity of the $k$-form case comes from the principal symbol of the $k$-form Laplacian being diagonal as an endomorphism of $k$-forms $\sigma_\Delta(x,\xi) = g(\xi,xi)\mathbf{1}$. This results in a factorization

$$\Delta = (D_n\mathbf{1} + E + iB)(D_n\mathbf{1} - iB) \tag{10.2}$$

where now $E$ and $B$ operate on $k$-forms. The same heat equation argument leaves us with rather unnatural Neumann data $\partial_n u$, which is taken to mean the normal derivative of each component of $u$.

$$\partial_n u = \sum_I (\partial u_I / dx_n) dx_I$$

Here $I$ are multi-indices $(i_1,\ldots,i_k)$ and $dx_I = dx_{i_1}\wedge\cdots\wedge dx_{i_k}$.

One might expect that the full symbol of $B(0,D')$, for $0 < k < n$ might contain more than enough data to determine the Taylor series of the metric, and this is indeed the case. Joshi and the present author [8] proved

**Theorem 10.1.** *Let $M$ be a smooth compact orientable Riemannian manifold with boundary, with* $\dim(M) > 2$. *Suppose that the full symbol of the Dirichlet-to-Neumann mapping $u \mapsto \partial_n u$ for the $k$-form Laplace's equation $\Delta u = 0$ is given and for $0 < k < n$.*

*Then the Taylor series of the metric at the boundary in boundary normal coordinates is uniquely determined by this data. For $0 < k < n$ only one diagonal component of the full symbol is needed corresponding to but for $k = (n+1)/2$ the multi-index $I = (i_1,\ldots,i_k)$ must exclude $n$ and for $k = (n-1)/2$, $I$ must include $n$.*

The proof given in [8] differs from Lee and Uhlmann's is in its use of families of operators parameterized by the normal distance $x_n$. This technique was also used by Joshi and McDowall [10] in their uniqueness results for isotropic and chiral isotropic time harmonic Maxwell's equations. In general one might consider pseudo-differential operators on a smooth manifold $Y$ depending smoothly on a parameter $t$. For our purposes we will have $Y = \partial M$ and $t$ the normal distance from the boundary. We say that $P \in \Psi\mathrm{DO}^{m,r}(Y, \mathbb{R}^+)$ if it is a family of pseudo-differential operators of order $m$ on $Y$, varying smoothly up to $t = 0$, and such that

$$P = \sum_{j=0}^{r} t^{r-j} P_j$$

with $P_j$ a smooth family of operators on $Y$ of order $m - j$. This definition extends naturally to operators on bundles, in our case the bundle of $k$-forms being the important example.

The proof of Theorem 10.1 starts with two metrics $g_1$ and $g_2$ and it is assumed that they share the same boundary distance coordinate $x_n$ in some neighbourhood. There is no loss of generality in this assumption as we are only interested in metrics up to diffeomorphisms fixing points on the boundary. Assuming that the (non-natural) Dirichlet-to-Neumann data agree we then suppose that the metrics agree up to order $l$ in $x_n$ and using symbol calculations in special coordinates we conclude that they must agree to order $l+1$. The economy of this method lies in working modulo $x_n^l$ which simplifies otherwise daunting calculations. On the other hand the method is less explicit – it does not yeald a formula for the full symbol of the Dirichlet-to-Neumann mapping.

Any $k$-form can be split in to a normal component (the terms containing $dx_n$ and a tangential component (those with no $dx_n$). Contracting with the normal vector field $\partial_n$ annihilates the tangential component. Now the normal component of $\partial_n u$ is $(\partial_n \lrcorner \partial_n u) \wedge dx_n$ and

$$\partial_n \lrcorner du = \partial_n \lrcorner \partial_n u - \partial_n \lrcorner \sum_{I \ni n} \sum_{i \notin I} \partial_i u_I dx_i \wedge dx_I$$

Suppose we are given the Dirichlet data (both normal and tangential components of $u$ at the boundary) then we know all tangential partial derivatives $\partial_i u_I, i < n$ and any $I$, at the boundary. If in addition we are given the normal part of $du$, $\mathbf{i}^* * du$ at the boundary we can recover the normal part of the non-natural Neumann data $\partial_n u$.

Inverse boundary value problems for elliptic systems of equations where the principal symbol is not diagonal present more difficulties. The anisotropic elasticity equation being an example. Nakamura and Uhlmann [20] show that a factorization of the elasticity operator exists

$$\mathrm{Div}(CL_u g) = (D_n \mathbf{1} + E + B) T (D_n \mathbf{1} + B^*) \tag{10.3}$$

where $T$ and $E$ are endomorphisms and $B$ is classical 1st order pseudo differential operator. Nakamura and Uhlmann show that the full symbol of the Dirichlet-to-Neumann map determines the Taylor series of the principal symbol of $B$ at the boundary, however the relationship between this principal symbol and the elasticity tensor $C$ is not so simple. They are able to recover the Taylor series or the 'surface impedance tensor'.

# References


[1] Aben H, Integrated Photoelasticity (New York: McGraw-Hill) 1979.

[2] G Alessandrini and R Gaburro Determining conductivity with special anisotropy by boundary measurements, to appear SIAM J. Math. Analysis.

[3] M Berger, D Ebin, Some decompositions of the space of symmetric tensors on a Riemannian manifold. J. Differential Geometry 3 1969 379–392.

[4] AR Borges et al, Development of Electromagnetic Tomography (EMT) for Industrial Applications, Proc. 1st World Congress on Industrial Process Tomography (ed T York), Buxton, 219-225, 1999.

[5] Y. Colin de Verdière Spectres de Graphes,SMF Cours Spécialisés 4 ,1998

[6] GFD. Duff and DC. Spencer, Harmonic Tensors on Riemannian manifolds with boundary, Ann. of Math, 56, 128-156, 1952

[7] PB Gilkey, Invariance theory, the heat equation and the Atiya-Singer index theorem, Publish or Perish, 1984, electronic reprint `http://www.mi.sanu.ac.yu/EMIS/monographs/gilkey/index.html`.

[8] MS Joshi and WRB. Lionheart, An inverse boundary value problem for harmonic differential forms, submitted to J. Asym Annal. Pre-print `http://www.arxiv.org/abs/math.AP/9911212`.

[9] MS Joshi, Introduction to Pseudo-differential operators, arXiv.org e-print math.AP/9906155,1999, `http://arXiv.org/abs/math.AP/9906155`.

[10] MS Joshi, S McDowall, Total determination of material parameters from electromagnetic boundary information, Pacific J. Math, 193, 107-129,2000

[11] S Kobayashi Transformation groups in differential geometry. Springer-Verlag, 1972.

[12] RV Kohn and M Vogelius, Identification of an unknown conductivity by means of measurement at the boundary , Proc SIAM-AMS Symp. on Appl. Maths,113-123, 1983

[13] PR Kotiuga Metric dependent aspects of inverse problems and functionals based on helicity, J Appl Phys 73: (10) 5437-5

[14] M Lassas, G Uhlmann, On determining a Riemannian manifold from the Dirichlet-to-Neumann map, preprint 1999.

[15] J Lee, G Uhlmann, Determining anisotropic real-analytic conductivities by boundary measurements, Comm. Pure Appl. Math. 42, 1097-1112, 1989

[16] WRB Lionheart, Conformal uniqueness results in anisotropic electrical impedance imaging. Inverse Problems Volume 13, February,125-134, 1997

[17] WRB Lionheart, Boundary Shape and Electrical Impedance Tomography, Inverse Problems,Vol 14, No 1, 139-147,1998

[18] G Nakamura and G Uhlmann, Identification of Lamé parameters by boundary measurement. American J. math. 115, 1161-1187,1993

[19] G Nakamura and G Uhlmann, Inverse problems at the boundary for an elastic medium, SIAM J. Math Anal, 26, 263-279, 1995

[20] Nakamura, Gen; Uhlmann, Gunther A layer stripping algorithm in elastic impedance tomography. Inverse problems in wave propagation (Minneapolis, MN, 1995), 375–384, IMA Vol. Math. Appl., 90, Springer, New York, 1997

[21] P. Ola, L Päivärinta, E Somersalo An inverse boundary value problem in electrodynamics. Duke Math. J. 70, 617-653, 1993.

[22] Z Ren , A Razek Computation of 3-D electromagnetic field using differential forms based elements and dual formulations Int J Numer Model EL 9: (1-2) 81-98 JAN-APR 1996

[23] MA Shubin, Pseudo-differential Operators and Spectral Theory, Springer-Verlag, 1985.

[24] M Spivak, Calculus on manifolds. A modern approach to classical theorems of advanced calculus. W. A. Benjamin, Inc., New York-Amsterdam 1965

[25] J Sylvester, Linearizations of anisotropic inverse problems. Inverse problems in mathematical physics (Saariselkä, 1992), 231–241, Lecture Notes in Phys., 422, Springer, Berlin, 1993.

[26] T Tarhasaari ,L Kettunen ,A Bossavit Some realizations of a discrete Hodoge operator: A reinterpretation of finite element techniques IEEE T MAGN 35: (3) 1494-1497 Part 1 MAY 1999

[27] A. Trautman Deformations of the Hodge map and optical geometry. J. Geom. Phys. 1, , 85–95, 1984.

[28] G Uhlmann, These proceedings.

[29] T. Voronov, Quantization of Forms on the Cotangent Bundle. Commun. Math. Phys., 205 , 315-336, 1999