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Blocks with trivial intersection defect groups*

This paper is dedicated to Jon Alperin on the occasion of his 65th birthday

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Abstract. We show that each block whose defect groups intersect pairwise trivially either has cyclic or generalised quaternion defect groups, or is Morita equivalent to one of a given list of blocks of central extensions of automorphism groups of non-abelian simple groups. In particular we classify all blocks of automorphism groups of non-abelian simple groups whose defect groups are non-cyclic and intersect pairwise trivially. A consequence is that Donovan's conjecture holds for blocks whose defect groups intersect pairwise trivially.

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1 Introduction and notation

The classification given by Gorenstein and Lyons in [17] of finite simple groups possessing a strongly p-embedded subgroup, together with the classification given by Suzuki in [30], yields a classification of finite simple groups whose Sylow p-subgroups are TI (trivial intersection, meaning that each pair of distinct conjugates intersect trivially). Clifford theory then allows certain questions about groups with TI Sylow p-subgroups to be reduced to questions about finite simple groups (and their covering and automorphism groups). This approach has lead to the verification of various conjectures in representation theory for this class of groups (see [5] and [8]).

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By a well-known theorem of Green, when a finite group has TI Sylow p-subgroups each block must either have defect zero or maximal defect, and hence such blocks must have TI defect groups. A natural generalisation is to consider all blocks with TI defect groups (call these *TI defect blocks*). Here we classify all TI defect blocks of finite groups of the form $M \le G \le \mathrm{Aut}(M)$, where M is quasisimple. Clifford-theoretic arguments then allow us to establish Morita equivalences between any given TI defect block and a member of one of these Morita equivalence classes or a block with cyclic or generalised quaternion defect groups. The Clifford-theoretic arguments are essentially those of Fong, observing that similar arguments apply when the covered block is a block of defect zero, rather than just a block of a p'-group. As such the arguments may be considered elementary.

Behind the reduction lies the fact that TI defect blocks are precisely those with p-local rank one as defined in [3], and that the p-local rank is compatible with any reductions we make. Note that the class of TI defect blocks contains the class of TI blocks (see [4]).

Let $\mathcal{O}$ be a local complete discrete valuation ring containing a primitive $|G|^3$ root of unity, whose residue field $k = \mathcal{O}/J(\mathcal{O})$ is algebraically closed of characteristic p and whose field of fractions K has characteristic zero. Denote by $\mathrm{Blk}(G)$ the set of blocks of G with respect to $\mathcal{O}$ and let $B \in \mathrm{Blk}(G)$. If $H \le G$, then denote by $\mathrm{Blk}(H, B)$ the set of blocks of H with Brauer correspondent B. If $H \lhd G$, write $\mathrm{Blk}_b(G)$ for the set of blocks of G covering $b \in \mathrm{Blk}(H)$.

Let G be a finite group and p a prime. Then $H \le G$ is a TI subgroup if $H^g \cap H = 1$ for each $g \in G - N_G(H)$. Here we also stipulate that $G \ne N_G(H)$.

Our main result is the following:

Theorem 1.1 *Let G be a finite group and $B \in \mathrm{Blk}(G)$ have non-normal TI defect groups. Then B is Morita equivalent to one of the following:*

(a) a block with cyclic or generalised quaternion defect groups;
(b) a 2-block of A_n with Klein-four defect groups, where $n = m^2/2 + m + 4$ or $n = m^2/2 + m + 6$ for some integer m;
(c) the unique block of J_2 or Ru with Klein-four defect groups;
(d) the unique block of $O'N$, $\mathrm{Aut}(O'N)$, $2.Suz$, $\mathrm{Aut}(Suz)$ with defect groups of the form $C_3 \times C_3$;
(e) the principal 3-block of M_{11};
(f) a 5-block of maximal defect of $3.McL$ or $\mathrm{Aut}(McL)$;
(g) the principal 11-block of J_4;
(h) a block of $Sp_{2m}(3)$ with defect groups of the form Q_8, where $m \ge 4$;
(i) a p-block of maximal defect of a p'-central extension of a group X with $Y \le X \le \mathrm{Aut}(Y)$, where $(p, [X : Y]) = 1$ and Y is $PSL_2(p^m)$ or $PSU_3(p^m)$, where $m > 1$. Further the corresponding central extension of Y is perfect;
(j) a 2-block of maximal defect of a group X with $Y \le X \le \mathrm{Aut}(Y)$, where $(2, [X : Y]) = 1$ and Y is ${}^2B_2(2^{2m+1})$, where $m \ge 1$;
(k) a 3-block of maximal defect of a group X with $Y \le X \le \mathrm{Aut}(Y)$, where $(3, [X : Y]) = 1$ and Y is ${}^2G_2(3^{2m+1})$, where $m \ge 1$;
(l) the principal 3-block of $\mathrm{Aut}({}^2G_2(3)')$;
(m) the principal 5-block of ${}^2F_4(2)'$, ${}^2F_4(2)$ or $\mathrm{Aut}({}^2B_2(2^5))$.

(n) a 3-block of maximal defect of a $3'$-central extension of a group X with $Y \leq X \leq \mathrm{Aut}(Y)$*, where* $(3, [X : Y]) = 1$ *and Y is* $PSL_3(4)$*. Further the corresponding central extension of Y is perfect.*

Further (b)–(n) comprise a complete list of non-cyclic TI defect blocks occuring in automorphism groups of quasisimple groups.

Note that we do not assert that the above are representatives of distinct Morita equivalence classes – this is certainly not the case. The main objective is to give a framework within which results about TI defect blocks may be proved.

Recall that Donovan's conjecture states that for a given p-group D and algebraically closed field k of characteristic p, there are only finitely many Morita equivalence classes of blocks (with respect to k) with defect groups isomorphic to D. The following is a consequence of the proof of Theorem 1.1 as well as the result itself, so we delay its proof until section 8 (the conclusion of the theorem is not sufficient since at the time of writing Donovan's conjecture is not yet known for generalised quaternion groups). For this result only we are concerned only with blocks with respect to an algebraically closed field of characteristic p (this is since we make use of results in [10], [11] and [21], for which this is a hypothesis).

Corollary 1.2 *Given a p-group D, there are only finitely many Morita equivalence classes of blocks (with respect to k) containing a block with TI or normal defect group isomorphic to D.*

The paper is structured as follows. In what remains of this section we outline some general notation and definitions. Section 2 contains a collection of properties of the TI defect blocks and the p-local rank. In section 3 we give an elementary account of the correspondences used in the reduction step, and in section 4 we give the reduction itself. Sections 5 to 7 contain the analysis of the alternating groups, sporadic simple groups and groups of Lie type comprising the classification. Section 8 contains the proof of Theorem 1.1 itself.

A radical p-subgroup of G is one with $Q = O_p(N_G(Q))$, where $O_p(H)$ is the unique maximal normal p-subgroup of H. A radical p-chain σ of G is a chain $Q_0 < \cdots < Q_n$ of p-subgroups of G, with strict inclusions, such that Q_0 is radical in G and Q_{i+1} is a radical p-subgroup of $N_G(Q_0) \cap \cdots \cap N_G(Q_i)$ for each i with $0 \leq i \leq n-1$. Let $|\sigma| = n$ be the length of σ and $G_\sigma = N_G(Q_0) \cap \cdots \cap N_G(Q_n)$. Denote by $\mathcal{R}(G)$ the set of radical p-chains of G. Following [3] and [29], define the p-local rank to be

$$plr(B) = max\{|\sigma| : \sigma \in \mathcal{R}(G, B)\},$$

where $\mathcal{R}(G, B) \subseteq \mathcal{R}(G)$ consists of those chains σ for which $\mathrm{Blk}(G_\sigma, B) \neq \emptyset$. By [3, 5.1] $plr(B) = 1$ if and only if B has a defect group D for which $D/O_p(G) \neq 1$ and is a TI subgroup of $G/O_p(G)$.

Write $\mathrm{Irr}(G)$ for the set of irreducible characters of G. If $N \lhd G$ and $\mu \in \mathrm{Irr}(N)$, then write $\mathrm{Irr}(G, \mu)$ for the set of irreducible characters of G covering μ. Denote by $\mathrm{Irr}(G, B)$ the set of irreducible characters belonging to B, and more generally, for $H \leq G$ write $\mathrm{Irr}(H, B)$ for the set of irreducible characters of H belonging to Brauer correspondents of B.

2 General properties of TI defect blocks and the p-local rank

Lemma 2.1 *Let N be a normal subgroup of a finite group G and D a TI radical subgroup of G. If $Q = D \cap N \neq 1$, then $N_G(Q) = N_G(D)$ and Q is a TI radical subgroup of N. In particular, if $b \in \mathrm{Blk}(N)$ is covered by a TI defect block $B \in \mathrm{Blk}(G)$, then b has a TI defect group. If further $O^{p'}(G) \leq N$, then b and B have a defect group in common.*

Proof. Suppose W is a defect group of b. Then $W = N \cap R$ for some defect group R of B (see for example [1, 15.1]). Thus either $W = 1$ or $W \neq 1$. In both cases W is a TI subgroup. The last statement is trivial. □

We observe that the p-local rank of a block is compatible with the reductions of later sections:

Proposition 2.2 *Let $B \in \mathrm{Blk}(G)$ have defect group D. Let $H \leq G$ and $N \lhd G$. Then:*

(i) $plr(B) = 0$ *if and only if* $D \lhd G$; $plr(B) \leq 1$ *if and only if* $D/O_p(G)$ *is TI;*
(ii) *If* $b \in \mathrm{Blk}(H, B)$, *then* $plr(b) \leq plr(G)$;
(iii) *Suppose* $\mu \in \mathrm{Irr}(N)$ *extends to* $\theta \in \mathrm{Irr}(G)$. *If* $\psi \in \mathrm{Irr}(G/N)$ *lies in the block* b_N *of* G/N *and* $\theta\psi \in \mathrm{Irr}(G, B)$, *then* $plr(b_N) \leq plr(B)$. *If further* $plr(B) = 1$ *and* $O_p(G) = 1$, *then* b_N *either has a normal defect group or TI defect groups;*
(iv) *If* $G = G_1 \times G_2$ *and* $B = B_1 \times B_2$, *where* $B_i \in \mathrm{Blk}(G_i)$, *then* $plr(B) = plr(B_1) + plr(B_2)$;
(v) *If* $b \in \mathrm{Blk}(N)$ *is covered by* B, *with* $plr(B) = 1$ *and* $O_p(G) = 1$, *then* $plr(b) \leq 1$ *and* $O_p(N) = 1$;
(vi) *If* $N \leq Z(G)$, $plr(B) = 1$ *and* $\overline{B}$ *is the unique block of* G/N *corresponding to* B *under the natural epimorphism, then* $plr(\overline{B}) = 1$.

Proof. (i) is [3, 5.1]; (ii) is [3, 3.2]; the first part of (iii) is [3, 4.1], and the second part follows from its proof; (iv) is [3, 3.5]; (v) follows immediately from (i) and Lemma 2.1; (vi) follows from (i), noting that $O_p(Z(G)) \leq O_p(G)$ is contained in every defect group. □

Lemma 2.3 *Suppose that $B \in \mathrm{Blk}(G)$ has a TI defect group D. Let $1 \neq x \in Z(D)$. Then $D = O_p(L)$ whenever $C_G(x) \leq L \leq N_G(\langle x \rangle)$, and L possesses a block with Brauer correspondent B and defect group D.*

Proof. Write $Q = \langle x \rangle$ and suppose $C_G(Q) \leq L \leq N_G(Q)$. Then $D \leq L \leq N_G(Q) \leq N_G(D)$, so that $D \lhd L$. But there is a block b_Q of L with $b_Q^G = B$, which must have a defect group $D(b_Q)$ satisfying $D \leq O_p(L) \leq D(b_Q) \leq D$. □

Lemma 2.4 *Let D be a p-subgroup of G, and $N \leq O_{p'}(Z(G))$. If $D \notin \mathrm{Syl}_p(G)$ and $C_{G/N}(DN/N) \leq DN/N$, then there is no block with defect group D.*

Proof. It's well known that if D is a defect group, then $D = P^g \cap P$ for some $P \in \mathrm{Syl}_p(G)$ and some $g \in C_G(D)$ (see, for example [1, 13.6]). If $C_{G/N}(DN/N) \leq DN/N$, then $P^g = P$ for every $P \in \mathrm{Syl}_p(G)$ containing D and every $g \in C_G(D)$, so $D \in \mathrm{Syl}_p(G)$, a contradiction. □

Lemma 2.5 *Let G be a finite group with $O_p(G) = 1$ and P a p-subgroup of G. Then P is a non-trivial TI radical subgroup of G if and only if $M = N_G(P)$ is a maximal p-local subgroup of G with $P = O_p(M)$ a non-trivial TI subgroup of G.*

Proof. Let P be a non-trivial TI radical subgroup of G. Then $N_G(P)$ is p-local and so $N_G(P) \leq M$ for some maximal p-local subgroup M of G. In particular, $O_p(M) \leq O_p(N_G(P)) = P$. If $x \in M$, then $O_p(M) = O_p(M)^x \leq P \cap P^x$, so that $x \in N_G(P)$ and $M = N_G(P)$. Thus $O_p(M) = P$ is TI.

Conversely, suppose M is maximal p-local and $1 \neq P = O_p(M)$ is a TI subgroup. If P is non-radical, then $M < N_G(P)$, so that $N_G(P) = G$ and $P \leq O_p(G)$, which is impossible. □

Note that the following lemma applies in particular to defect groups.

Lemma 2.6 *Let P be a radical p-subgroup of G, where $G = G_1 \times G_2$. Then $P = P_1 \times P_2$ for uniquely defined radical p-subgroups P_1, P_2 of G_1 and G_2 respectively. Conversely, $P_1 \times P_2$ is a radical p-subgroup of G whenever P_1, P_2 are radical subgroups of G_1, G_2.*

Proof. See [27, 2.2]. □

In performing the reductions it may happen that we reduce to the case where the block has a normal defect group. We will see that, once we have reduced to the case where every normal p'-subgroup is central, this happens only when the defect groups intersect trivially with the generalized Fitting subgroup. The following two results show that this implies that the defect group cannot contain an elementary abelian subgroup of order p^2, i.e., that the defect groups are either cyclic or generalised quaternion (cf. [16, Theorem 5.4.10]).

Recall that a *quasisimple* group H is a perfect group where $H/Z(H)$ is non-abelian simple. A *component* of G is a subnormal quasisimple subgroup. $E(G)$ is the normal subgroup of G generated by the components and $F^*(G) = E(G)F(G)$ is the generalised Fitting subgroup.

Lemma 2.7 *Let $N \lhd G$ and $D \leq G$ be a TI defect group for some block $B \in$ Blk(G) such that $D \cap N = 1$. Then $C_N(x) = C_N(D)$ for each $x \in D - \{1\}$. In particular for each $n \in N$, either $C_D(n) = D$ or $C_D(n) = 1$.*

Proof. Let $x \in D - \{1\}$. Then $C_N(D) \leq C_N(x)$. Suppose that $n \in C_N(x) - C_N(D)$. Then there is $y \in D$ such that $n \notin C_N(y)$. Now $y^n = y[y, n]$ and $[y, n] \in N$, so $y^n \notin D$, for otherwise $1 \neq [y, n] \in D \cap N = 1$. Hence $D^n \neq D$ whilst $1 \neq x \in D^n \cap D$, a contradiction. So $C_N(x) = C_N(D)$. The last part follows immediately. □

Proposition 2.8 *Suppose that D is a (non-normal) TI defect group for $B \in$ Blk(G), where $O_{p'}(G) \leq Z(G)$. If $D \cap F^*(G) = 1$, then D is either cyclic or generalised quaternion.*

Proof. Suppose that D possesses a subgroup $Q = \langle x, y : x^p = y^p = [x, y] = 1\rangle \cong C_p \times C_p$ (otherwise D is cyclic or generalised quaternion).

Note that $C_G(F^*(G)) \leq F^*(G)$, that $E(G) \neq 1$ and that every component has order divisible by p. We claim first that Q fixes every component of G.

Write $M_1, \ldots, M_n$ for the components of G, and $H = \cap_{i=1}^n N_G(M_i)$. Suppose that $Q \not\leq H$. Consider the permutation action of Q on $\{1, \ldots, n\}$ induced by permutation of the components (by conjugation). Suppose that $x \in H$ but $y \notin H$. Let M_i lie in an orbit of length p of y. Since $p \mid |M_i|$, x must fix some non-trivial $m \in M_i$. Clearly y does not fix m, a contradiction by Lemma 2.7. Hence $Q \cap H = 1$, so that Q acts faithfully. Without loss of generality we may assume that x moves the points $\{1, \ldots, p\}$ transitively but y doesn't. Now y fixes one of $\{1, \ldots, p\}$, say 1, or moves one of these points to the outside of the x-orbit. In the first case we may choose non-trivial $m \in M_1$ fixed by y, since $p \mid |M_1|$, obtaining a contradiction by Lemma 2.7. In the latter case, x fixes some non-trivial $m \in M_1 * \cdots * M_p$. But $y \notin C_G(m)$, again a contradiction to Lemma 2.7. Hence $Q \leq H$ as claimed.

Since $C_G(F^*(G)) \leq F^*(G)$, there must be a component, say M, not centralized by Q. We examine the various possibilities for $MQ = M \rtimes Q$.

We may assume that M is simple, since otherwise we may just quotient by the centre of M (which is contained in the centre of G). Since $C_{MQ}(M) = 1$, no element of Q (which we may regard as a subgroup of $\mathrm{Aut}(M)$) may act as an inner automorphism.

Since $Q \cong MQ/M$ is a subgroup of $\mathrm{Out}(M)$, it follows that $\mathrm{Out}(M)$ is non-cyclic, so M is a group of Lie type (we are making use of the isomorphism $A_6 \cong PSL_2(9)$). We may write $x = x_i x_o$ and $y = y_i y_o$, where x_i, y_i are inner automorphisms, and $x_o = x_d x_f x_g$, $y_o = y_d y_f y_g$, where x_d, y_d are diagonal automorphisms, x_f, y_f are field automorphisms and x_g, y_g are graph automorphisms. Let r be the characteristic of the field of definition of M. Note that Lemma 2.7 tells us that $\langle C_M(F) : 1 \neq F \leq Q \rangle = C_M(Q) \neq M$.

Suppose first that $p \neq r$. Then by [18, 7.3.4], the above discussion and the structure of $\mathrm{Out}(M)$ we must have $\langle C_M(F) : 1 \neq F \leq Q \rangle = M$ (see the proof Case (1) of Proposition 7.3), a contradiction.

Suppose that $p = r$. Then M possesses no diagonal automorphisms and we may choose $x = x_i x_f$, $y = y_i y_g$. By [18, 4.9.1.d] we may replace D by an M-conjugate if necessary to further assume that $x = x_f$. But then by [18, 7.3.8] we must have $\langle C_M(F) : 1 \neq F \leq Q \rangle = M$ (see the proof Case (2) of Proposition 7.3), again a contradiction, so we are done. □

3 Some correspondences like Fong's

In [13] Fong studies modular representation theory with respect to a normal p'-subgroup, establishing the well-known Fong correspondences. These give a Morita equivalence between a block of the original group and a block of a p'-central extension of a certain section of that group. Now the same general methods may be applied for blocks of defect zero of any normal subgroup (thus generalising the results of [13]). This is nothing new, since this situation has been studied, among others, by Dade, and Külshammer and Puig (see [22], where strong results are proved concerning blocks covering nilpotent blocks). The results of [22] may be applied directly to establish the equivalences we require, but the authors feel that since the full strength of [22] is not being used, readers may benefit from a more elementary treatment. Our starting point is the treatment given in [9], where correspondences like Fong's are given along with an elementary proof that the correspondences respect blocks. Our proof makes use of techniques from [13].

Let $B \in \mathrm{Blk}(G)$, where $O_p(G) = 1$. We suppose that B has positive defect. Let $N \lhd G$. Suppose that B covers $b \in \mathrm{Blk}(N)$, a block of defect zero. Let ζ be the unique irreducible character lying in b. It is well-known (see, for example [12, V.2.5]) that there is a unique block B_I of $I_G(\zeta)$, the inertia subgroup of ζ in G, such that induction gives a Morita equivalence between B and B_I. Hence it suffices to consider the case $I_G(\zeta) = G$.

In the following paragraph we summarise section 1 of [9]:

We may choose a central extension $\hat{G}$ of G (with $\hat{W} \leq Z(\hat{G})$ where $\hat{G}/\hat{W} \cong G$ and $\hat{W} \leq [\hat{G}, \hat{G}]$) such that there is an irreducible character $\hat{\theta} \in \mathrm{Irr}(\hat{G})$ extending ζ. All extensions $\hat{H}$ of H by A derived in this way that we consider in this paper satisfy $A \leq [\hat{H}, \hat{H}]$. We therefore do not make further explicit reference to the fact. Let $\hat{N}$ be the subgroup of $\hat{G}$ identified with N, so that $\hat{N} \cap \hat{W} = 1$. Set $\tilde{G} = \hat{G}/\hat{N}$, a central extension of G/N by $\tilde{W}$ (where $\tilde{W}$ is the image of $\hat{W}$ under the natural epimorphism). Then $\hat{\theta}$ lies over a unique linear character $\hat{\mu}$ of $\hat{W}$. Let $\tilde{\mu}$ be the complex conjugate of $\hat{\mu}$, regarded as a character of $\tilde{W}$. There is a 1-1 correspondence between $\mathrm{Irr}(G, \zeta)$ and $\mathrm{Irr}(\tilde{G}, \tilde{\mu})$, given by $\chi \leftrightarrow \hat{\theta}\tilde{\chi}$, where of course we are identifying χ with its inflation to $\hat{G}$. Now there is a collection of blocks $\tilde{B}_1, \ldots, \tilde{B}_r$ of $\tilde{G}$ (the Dade correspondents of B) so that, writing $\tilde{B} = \tilde{B}_1 + \ldots + \tilde{B}_r$, there is a correspondence between $\mathrm{Irr}(G, B, \zeta)$ and $\mathrm{Irr}(\tilde{G}, \tilde{B}, \tilde{\mu})$, where implicitly we are using the 1-1 correspondence between blocks of G and blocks of $\hat{G}$ covering the principal block of $O_{p'}(\hat{W})$. Note that the Dade correspondence respects the Brauer correspondence as described in [9]. Let S be an $\mathcal{O}N$-module affording ζ and let T be an $\mathcal{O}\hat{G}$-module extending S. We observe that the functor $M \to (T^* \otimes_{\mathcal{O}} M)^{\hat{N}}$ gives rise to a Morita equivalence between B and $\tilde{B}$ (for details see [23]).

We claim that we may choose $\tilde{W}$ to be a p'-group. For consider $\hat{G}/O_p(\hat{W})$. Let $l \neq p$ be a prime. Let $P_l \leq \hat{G}$ contain $\hat{N} \times \hat{W}$ so that $P_l/(\hat{N} \times \hat{W})$ is a Sylow l-subgroup of $\hat{G}/(\hat{N} \times \hat{W})$. Now ζ extends to $\hat{G}$ so extends to $\phi \in \mathrm{Irr}(P_l)$. Let $\alpha \in \mathrm{Irr}(\hat{N} \times \hat{W})$ be the canonical extension of the *complex conjugate* of the unique irreducible constituent of ϕ on restriction to $O_p(\hat{W})$ (i.e., α is trivial on $O_{p'}(\hat{W}) \times \hat{N}$). By [19, 8.16] α extends to, say $\psi \in \mathrm{Irr}(P_l)$. Then $\phi\psi$ is also an extension of ζ to P_l by standard Clifford theory. Note that $O_p(\hat{W}) \leq \mathrm{Ker}(\phi\psi)$, so ζ (after appropriate identifications are made) extends to $P_l/O_p(\hat{W})$. Now let $P_p \leq \hat{G}$ contain $\hat{N} \times O_{p'}(\hat{W})$ so that $P_p/(\hat{N} \times \hat{W})$ is a Sylow p-subgroup of $\hat{G}/(\hat{N} \times \hat{W})$. Identify b with the block of defect zero containing the canonical extension of ζ to $\hat{N} \times O_{p'}(\hat{W})$. By [12, V.3.5] there is a unique block of $P_p/O_p(\hat{W})$ covering b, and so by [1, 15.1] this has a defect group, Q say, such that $Q \cap (\hat{N} \times O_{p'}(\hat{W})) = 1$ and $P_p/O_p(\hat{W}) = Q(\hat{N} \times O_{p'}(\hat{W}))$, i.e., $P_p/O_p(\hat{W}) = (\hat{N} \times O_{p'}(\hat{W})) \rtimes Q$, and ζ extends to $P_p/O_p(\hat{W})$. Hence ζ extends $P_q/O_p(\hat{W})$ for every prime q, and so extends to $\hat{G}/O_p(\hat{W})$ by [19, 11.31] as claimed. In other words we may, and do, assume that $O_p(\hat{W}) = 1$, so that $\hat{W}$ is a p'-group. A consequence of this is that $\mathrm{Irr}(\tilde{G}, \tilde{B}, \tilde{\mu}) = \mathrm{Irr}(\tilde{G}, \tilde{B})$.

We now observe that in fact $r = 1$, so that B is Morita equivalent to a block $\tilde{B}$ of $\tilde{G}$. Further, as a consequence of the construction B and $\tilde{B}$ have isomorphic defect groups. As with all results in this section, we make no claims to originality.

Proof. Let Λ be the subset of $\mathrm{Irr}(G, B, \zeta) = \mathrm{Irr}(G, B)$ of characters corresponding to characters of $\tilde{B}_1$. We show that $\Lambda = \mathrm{Irr}(G, B)$, which gives $r = 1$. For convenience, identify Λ with the set of inflations of its elements to $\hat{G}$.

By block orthogonality (see for example [25, 3.7]), whenever $\tilde{x} \in \tilde{G}_{p'}$ (the set of p-regular elements of $\tilde{G}$) and $\tilde{y} \in \tilde{G} - \tilde{G}_{p'}$ (i.e., $\tilde{y}$ is p-singular), we have

$$\sum_{\tilde{\chi} \in \mathrm{Irr}(\tilde{G}, \tilde{B}_1)} \tilde{\chi}(\tilde{x})\tilde{\chi}(\tilde{y}) = 0.$$

We use the converse (due to Osima) of the block orthogonality result to show that Λ is the set of irreducible characters belonging to a collection of blocks of $\hat{G}$ and hence is $\mathrm{Irr}(\hat{G}, \hat{B})$, which would give the required result.

Let $\hat{x} \in \hat{G}_{p'}$ and $\hat{y} \in \hat{G} - \hat{G}_{p'}$. Recall that $\tilde{G} = \hat{G}/\hat{N}$. Now $\hat{x}\hat{N}$ is p-regular, but we can't guarantee that $\hat{y}\hat{N}$ is p-singular. We have

$$\begin{aligned} \sum_{\hat{\chi} \in \Lambda} \hat{\chi}(\hat{x})\hat{\chi}(\hat{y}) &= \sum_{\tilde{\chi} \in \mathrm{Irr}(\tilde{G}, \tilde{B}_1)} \tilde{\chi}(\hat{x}\hat{N})\hat{\theta}(\hat{x})\tilde{\chi}(\hat{y}\hat{N})\hat{\theta}(\hat{y}) \\ &= \hat{\theta}(\hat{x})\hat{\theta}(\hat{y}) \sum_{\tilde{\chi} \in \mathrm{Irr}(\tilde{G}, \tilde{B}_1)} \tilde{\chi}(\hat{x}\hat{N})\tilde{\chi}(\hat{y}\hat{N}). \end{aligned}$$

If $\hat{y}\hat{N}$ is p-singular, then this is zero by block orthogonality. If $\hat{y}\hat{N}$ is p-regular, and $\hat{y}$ has order c, then $\hat{y}^{c_{p'}} \in \hat{N}$, where $c_{p'}$ is the p'-part of c. But $\hat{y}^{c_{p'}}$ is the unique p-part of $\hat{y}$, and so is non-trivial. But then $\hat{\theta}(\hat{y}^{c_{p'}}) = \zeta(\hat{y}^{c_{p'}}) = 0$ since ζ is in a block of defect zero. Hence $\sum_{\hat{\chi} \in \Lambda} \hat{\chi}(\hat{x})\hat{\chi}(\hat{y}) = 0$, and by [28, Theorem 3] Λ is the set of irreducible characters belonging to a collection of blocks as required. □

Write $\tilde{B} = \tilde{B}_1$ for the unique Dade correspondent. Note that at each stage, B, B_I and $\tilde{B}$ have isomorphic defect groups.

In summary, we have

Theorem 3.1 *Let $B \in \mathrm{Blk}(G)$ be a block of positive defect and $N \lhd G$, with $O_p(G) = 1$. Suppose that B covers a p-block b of defect zero of N. Write $I = I_G(b)$. Then there is a central extension $\tilde{I}$ of I/N by a cyclic p'-group $\tilde{W}$ and a block $\tilde{b}$ (of defect zero) of $\tilde{W}$ such that B is Morita equivalent to a block $\tilde{B} \in \mathrm{Blk}(\tilde{I})$ covering $\tilde{b}$ and with defect groups isomorphic to those of B. Further, if $plr(B) = 1$, then either $plr(\tilde{B}) = 1$ and $O_p(\tilde{I}) = 1$, or $plr(\tilde{B}) = 0$.*

Proof. It remains to observe that each step we have made is compatible with the p-local rank of a block. Since $B_I \in \mathrm{Blk}(I_G(b))$ has Brauer correspondent B, by part (ii) of Proposition 2.2 we have $plr(B_I) \leq plr(B) = 1$. We must rule out the case that $plr(B_I) = 0$, i.e., that B_I has a normal defect group. Note that

$[O_p(I), N] \leq O_p(I) \cap N \leq O_p(N) = 1$. Hence $O_p(I) \leq C_G(N) \leq I$. But then $O_p(I) \lhd C_G(N)$, and $O_p(I) \leq O_p(C_G(N)) \leq O_p(G) = 1$, since $C_G(N) \lhd G$. Hence B_I cannot have a normal defect group, since we are assuming that B (and so B_I) has positive defect.

By part (vi) of Proposition 2.2 $plr(\hat{B}) = plr(B_I) = 1$, where $\hat{B}$ is the unique block of $\hat{G}$ naturally corresponding to B_I. The rest follows from part (iii) of Proposition 2.2. □

4 Reduction to covers of automorphism groups of simple groups

Let G be a finite group and $B \in \mathrm{Blk}(G)$ with TI defect group D. We assume that B does not have a normal defect group, so that $O_p(G) = 1$.

Let $E(G)$ be a central product $M_1 * \cdots * M_s$ of normal subgroups of G, where each M_i is a central product $M_{i1} * \cdots * M_{it}$ of quasisimple groups. For each i, G acts transitively on $M_{i1}, \ldots, M_{it}$.

Lemma 4.1 *Consider B as above, and suppose that D possesses a subgroup of the form $C_p \times C_p$. Let $H \lhd G$ be a p'-group. Then DH/H is a non-normal TI subgroup of G/H.*

Proof. Write $\overline{G}$ for the quotient by H. Since

$$H = \langle C_H(x) : 1 \neq x \in D \rangle \leq N_G(D)$$

(see [16, 6.2.4]), it follows that $H \leq N_G(D)$, so $D \leq C_G(H)$. Then $N_{\overline{G}}(\overline{D}) = \overline{N_G(D)}$ and the result follows. □

Theorem 4.2 *B as above is Morita equivalent to a block with cyclic or generalised quaternion defect groups or a block of a finite group X where $Z(X)$ is a p'-group and $M \leq X/Z(X) \leq \mathrm{Aut}(M)$, where M is non-abelian simple.*

Proof. First observe that by Theorem 3.1 and its proof (with $N = O_{p'}(G)$) B is Morita equivalent to the block B_I, in the notation of Theorem 3.1, and B_I has non-normal TI defect groups which are also defect groups for B. Hence we may assume that $G = I$.

By Theorem 3.1, B is then Morita equivalent to a block $\tilde{B}$ of a group $\tilde{G}$, where $O_{p'}(\tilde{G}) \leq Z(\tilde{G})$ and either $plr(\tilde{B}) = 0$ or $plr(\tilde{B}) = 1$ and $O_p(\tilde{G}) = 1$. Now $\tilde{G} = \hat{G}/\hat{N}$ in the notation of the previous section. If B (and hence $\tilde{B}$) does not have cyclic or generalised quaternion defect groups, then by Lemma 4.1 applied to $\hat{G}$, the image of D in $\tilde{G}$ (a defect group of $\tilde{B}$) is non-normal and trivial intersection. So we assume that $O_{p'}(G) \leq Z(G)$. Let D be a defect group for B. If $E(G) = 1$, then $C_G(F^*(G)) \leq F^*(G) = O_{p'}(G) \leq Z(G)$, and G is abelian, contradicting out assumption that $N_G(D) \neq G$. Hence we may assume that $E(G) \neq 1$, and further that every component has order divisible by p.

If $D \cap E(G) = 1$, then by Proposition 2.8 D is again cyclic or generalised quaternion. Hence we assume that $D \cap E(G) \neq 1$.

Consider $M_i \lhd G$, and let $b_i \in \mathrm{Blk}(M_i)$ be covered by B. By Proposition 2.2 (v) either b_i has positive defect and (non-normal) TI defect groups, or has defect zero. If b_i has defect zero, then by Theorem 3.1 B is Morita equivalent to a block $\tilde{B}$ of a central extension of $I_G(b_i)/M_i$ (where $\tilde{B}$ has TI defect groups or cyclic or generalised quaternion defect groups), and so we may assume that b_i has positive defect, and p-local rank one. We may also assume that b_i is G-stable.

Now $M_i = M_{i1} * \cdots * M_{it}$, and $b_i = b_{i1} \times \cdots \times b_{it}$. But $b_{ij} = b_{ik}$ for all j, k since b_i is G-stable and G acts transitively on the components of M_i. By Proposition 2.2 (iv) we have $1 = plr(b_i) = t(plr(b_{i1}))$, so $t = 1$. Similarly the block $b = b_1 \times \cdots \times b_s$ of $E(G)$ has p-local rank one, and $s = plr(b_1) + \cdots + plr(b_s) = plr(b) = 1$. Hence G has a unique component, say M. Note that M is the unique minimal non-central normal subgroup of G. Hence $C_G(M) = Z(G)$, else G would have more than one component. Hence $M \leq G/Z(G) \leq \mathrm{Aut}(M)$ as required. □

For any $\chi \in \mathrm{Irr}(G)$, denote by $d(\chi)$ and $\kappa(\chi)$ the nonnegative integers such that $1 \leq \kappa(\chi) \leq (p-1)$,

$$p^{d(\chi)} = \frac{|G|_p}{\chi(1)_p} \quad \text{and} \quad \kappa(\chi) \equiv \frac{|G|_{p'}}{\chi(1)_{p'}} \pmod{p}.$$

If $H \leq G$ and ξ is a character of H, then

$$d(\mathrm{Ind}_H^G(\xi)) = d(\xi) \quad \text{and} \quad \kappa(\mathrm{Ind}_H^G(\xi)) = \kappa(\xi).$$

Remark 4.3 Let B and $\tilde{B}$ be the blocks given by Theorem 4.2 such that B is Morita equivalent to $\tilde{B}$. Suppose $\chi \in \mathrm{Irr}(B)$ corresponds to $\tilde{\chi} \in \mathrm{Irr}(\tilde{B})$. Then

$$d(\chi) = d(\tilde{\chi}) \quad \text{and} \quad \kappa(\chi) \equiv \kappa_0 \kappa(\tilde{\chi}) \pmod{p}$$

for some integer $\kappa_0 \not\equiv 0 \pmod{p}$ dependent only on the block B.

Indeed, in the notation of Section 3, $\chi \leftrightarrow \hat{\theta}\tilde{\chi}$ with $\hat{\theta}(1) = \zeta(1)$, so

$$d(\chi) = d(\zeta) + d(\tilde{\chi}) = d(\tilde{\chi}) \quad \text{and} \quad \kappa(\chi) \equiv \kappa_0 \kappa(\tilde{\chi}) \pmod{p},$$

where $\kappa_0 \equiv \kappa(\zeta) \pmod{p}$. Thus the remark follows by the note above.

In the following sections we classify all TI defect blocks of p'-central extensions of groups $M \leq H \leq \mathrm{Aut}(M)$, where M is a non-abelian simple group. For the sake of space, and since blocks with cyclic defect groups are well understood, we only give a classification of TI defect blocks with non-cyclic defect groups.

5 Alternating groups

Let $Z(G) = O_{p'}(G) \leq M \leq G$ where $M/Z(G) \leq G/Z(G) \leq \mathrm{Aut}(M/Z(G))$ and $M/Z(G) \cong A_n$, an alternating group, $n \geq 5$. Write $\overline{H}$ for $Z(G)H/Z(G)$ whenever $H \leq G$, and A^π for the preimage of $A \leq \overline{G}$ in G. We demonstrate that, but for a small list of exceptions, the only blocks $B \in \mathrm{Blk}(G)$ of p-local rank one are those which have defect groups of order p.

For the sake of exposition it is convenient to work first of all with the symmetric groups. For odd primes this is sufficient.

Proposition 5.1 *Suppose that* $\overline{G} = S_n$*, where* $n \geq 5$*, and suppose* $B \in \mathrm{Blk}(G)$ *has a TI defect group* $D \neq 1$*. Then* $|D| = p$ *unless* $(n, p) = (6, 3)$*, in which case* $D \cong C_3 \times C_3$*.*

Proof. Let $x \in Z(D)$ have order p, and write $\overline{x} = xZ(G) \neq Z(G)$. Write $Q = \langle x \rangle$. Note that $C_{\overline{G}}(\overline{x})^\pi \leq N_G(Q)$. Suppose that $\overline{x}$ consists of exactly k cycles of length p and $n - kp$ of length one. Then $C_{\overline{G}}(\overline{x}) \cong (C_p \wr S_k) \times S_{n-pk}$.

Suppose first that $p > 3$, so that in particular $O_p(S_m) = 1$ for every m. By Lemma 2.3 and Lemma 2.6 $D = O_p(C_{\overline{G}}(\overline{x})^\pi) \cong (C_p)^k$. [We will be implicitly using Lemma 2.6 in this way repeatedly throughout this proof, so we will no longer make explicit reference to it]. Suppose that $k > 1$. We may choose an element $\overline{y} \in \overline{D} = Z(\overline{D})$ consisting of precisely one cycle of length p. Then $C_{\overline{G}}(\overline{y}) \cong C_p \times S_{n-p}$, so $D = O_p(C_{\overline{G}}(\overline{y})^\pi) \cong C_p$, a contradiction. Hence $k = 1$ and $D \cong C_p$ as required.

Now suppose that $p = 3$. If $k > 3$, then $D = O_p(C_{\overline{G}}(\overline{x})^\pi) \cong (C_p)^k \times O_3(S_{n-3k})$. As above, there is $\overline{y} \in Z(\overline{D})$ consisting of just one cycle length 3. We have $C_{\overline{G}}(\overline{y}) \cong C_p \times S_{n-3}$, and so $D = O_p(C_{\overline{G}}(\overline{y})) \cong C_p \times O_3(S_{n-3})$. Since $k > 3$ we have $n > 9$, so that $O_3(S_{n-3}) = 1$, giving a contradiction.

If $k = 3$, then $D = O_3(C_{\overline{G}}(\overline{x})^\pi) \cong (C_3 \wr C_3) \times O_3(S_{n-9})$. Write $D = D_1 \times D_2$, where $D_1 \cong C_3 \wr C_3$ and $D_2 \cong O_3(S_{n-9})$. Note that D_1 is isomorphic to a Sylow 3-subgroup of S_9. Now S_9 does not have TI Sylow 3-subgroups (see [5] for example), so in particular D cannot be TI.

If $k = 2$, then $D = O_3(C_{\overline{G}}(\overline{x})^\pi) \cong (C_3)^2 \times O_3(S_{n-6})$. If $n \neq 9$, then $D \cong (C_3)^2$. Let $\overline{y} \in Z(\overline{D})$ consist of just one cycle of length 3. Then $C_{\overline{G}}(\overline{y}) \cong C_3 \times S_{n-3}$, so $D = O_p(C_{\overline{G}}(\overline{y})^\pi) \cong C_3 \times O_3(S_{n-3})$. If $n \neq 6$, then $|D| = 3$ and we are done in this case. If $n = 9$, then again we may choose $y \in Z(D)$ consisting of just one cycle of length 3. Then $C_{\overline{G}}(\overline{y}) \cong C_3 \times S_6$, so $D \cong C_3$, a contradiction. If $n = 6$, then G has self-centralizing TI Sylow 3-subgroups, and so G has (two) blocks with TI defect groups isomorphic to $C_3 \times C_3$.

Finally suppose that $p = 2$. In this case we may take $Z(G) = 1$ unless $n = 6$ or $n = 7$ (since S_n has Schur multiplier of order two when $n = 5$ or $n \geq 8$). We have $D \cong (C_2 \wr O_2(S_k)) \times O_2(S_{n-2k})$. Suppose that $O_2(S_k) = 1$ (i.e., $k = 3$ or $k \geq 5$). As usual we may choose a transposition $\overline{y} \in Z(\overline{D})$. Then $C_{\overline{G}}(\overline{y}) \cong C_2 \times S_{n-2}$, and $D \cong C_2 \times O_2(S_{n-2})$. If $n > 6$, then $O_2(S_{n-2}) = 1$ so that $|D| = 2$. If $n \leq 6$, then it is easy to find examples to show that D is not TI. If $k = 2$, then $C_{\overline{G}}(\overline{x}) \cong (C_2 \wr C_2) \times S_{n-4}$ and $D \cong (C_2 \wr C_2) \times O_2(S_{n-4})$. Hence $C_{\overline{G}}(\overline{D}) \leq \overline{D}$, and if $n \geq 7$, then $D \notin \mathrm{Syl}_2(G)$, so that by Lemma 2.4 D cannot be TI.

If $k = 4$, then $n > 7$ and so $Z(G) = 1$. We have $D \cong (C_2 \wr (C_2 \times C_2)) \times O_2(S_{n-8})$. We have $C_G(x) \cong (C_2 \wr S_4) \times S_{n-8}$, and this must possess a block with defect group D by Lemma 2.3. Write $C_G(x) = H \times L$, where $H \cong C_2 \wr S_4$ and $L \cong S_{n-8}$. Then H must possess a block with defect group $P \cong C_2 \wr (C_2 \times C_2)$. Notice that $P \notin \mathrm{Syl}_2(H)$. Hence since $C_H(P) = Z(P) \leq O_p(H)$, by Lemma 2.4 P cannot be a defect group in H, so D cannot be a defect group of $C_G(x)$. □

Corollary 5.2 *Suppose that* $A_n \leq G/Z(G) \leq \mathrm{Aut}(A_n)$*, where* $n \geq 5$*, and suppose* $B \in \mathrm{Blk}(G)$ *has* p*-local rank one and is of positive defect. If* D *is a defect group of* B*, then* $|D| = p$ *unless* $(n, p) = (6, 3)$ *or* $p = 2$*. If* $(n, p) = (6, 3)$*, then*

$D \cong C_3 \times C_3$. If $p = 2$, then D is of order two or is elementary abelian of order 4. In particular A_n possesses a 2-block with Klein-four TI defect group precisely when $n = m^2/2 + m + 4$ or $n = m^2/2 + m + 6$ for some integer $m \geq 1$.

Proof. Suppose that H is a covering group of A_n, where $n \geq 5$, and G is a covering group of S_n containing H. Let b be a TI defect block of H with defect groups of order greater than p. Let D be a defect group of b.

First suppose $p \neq 2$. If $C_G(D) \leq H$, then the unique block B of G is a Brauer correspondent of b and so $plr(B) = plr(b) = 1$ by Proposition 2.2 (ii). Hence $(n, p) = (6, 3)$ as required. If $C_G(D) \not\leq H$, then it follows that $[G : N_G(D)] = [H : N_H(D)]$, so that B and b have precisely the same defect groups. Hence again $plr(B) = 1$, so that $(n, p) = (6, 3)$.

Now suppose that $p = 2$. We treat the cases $n = 6, 7$ separately, so that we may assume as before that $Z(G) = 1$. Let $x \in Z(D)$ have order two. Suppose that x consists of precisely k cycles of length two, so that k is even. Then $C_{A_n}(x) \cong ((C_2 \wr A_k) \times A_{n-2k})\langle t\rangle$, where t is an involution, so that $H = (C_2 \wr A_k) \times A_{n-2k} \lhd C_{A_n}(x) \lhd (C_2 \wr S_k) \times S_{n-2k} = C_{S_n}(x)$. Hence $D = O_2(C_{A_n}(x)) \leq O_2(C_{S_n}(x)) = (C_2 \wr O_2(S_k)) \times O_2(S_{n-2k})$. We show that we may assume $k = 2$.

Suppose that $k \neq 2$. If further $n - 2k \neq 2$, then $O_2(A_k) = O_2(S_k)$ and $O_2(A_{n-2k}) = O_2(S_{n-2k})$, so that $D \cong (C_2 \wr O_2(S_k)) \times O_2(S_{n-2k})$. Now $C_H(Q)$ must possess a block, b_Q say, with defect group D, where $Q = \langle x\rangle$. Let $\tilde{b}$ be a block of H covered by b_Q. This must also have defect group D. Write $H = H_1 \times H_2$, where $H_1 = C_2 \wr A_k$ and $H_2 = A_{n-2k}$. Write $\tilde{b} = \tilde{b}_1 \times \tilde{b}_2$, where $\tilde{b}_i$ is a uniquely determined block of H_i. Then $D = D_1 \times D_2$, where D_i is a (the) defect group for $\tilde{b}_i$. We have $C_{H_1}(D_1) \leq D_1$, so such a block can only exist when $D_1 \in \mathrm{Syl}_2(H_1)$, i.e., when $k = 4$. Notice that D_1 is isomorphic to a Sylow 2-subgroup of A_8 when $k = 4$, but A_8 does not have TI Sylow 2-subgroups, so in fact D cannot be TI in this case.

Suppose that $n - 2k = 2$. Then $C_2 \wr A_k \leq C_{A_n}(x) \leq (C_2 \wr S_k) \times C_2$. As above, we may assume $n \neq 8$, so that $k \geq 6$, and D is elementary abelian of order 2^k or 2^{k+1}. Hence we may choose $z \in Z(D) = D$ consisting of precisely two cycles of length two, i.e., we may as well have assumed that $k = 2$.

Now suppose $k = 2$. We need to show that, where it exists, B must have defect groups of the form $C_2 \times C_2$. We have $(C_2)^2 \times A_{n-4} \lhd C_{A_n}(x) \lhd (C_2 \wr C_2) \times S_{n-4}$, and $(C_2)^2 \times O_2(A_{n-4}) \leq D \leq (C_2 \wr C_2) \times O_2(S_{n-4})$. There is no block with non-cyclic TI defect groups for A_8. If $n \neq 8$, then $(C_2)^2 \leq D \leq C_2 \wr C_2$. Note that in this instance, $C_2 \wr C_2$ is isomorphic to a subgroup of S_4, but there is no such subgroup of A_4, so $D \cong (C_2)^2$. Now $C_{A_n}(D) \cong D \times A_{n-4}$, and so B does indeed exist whenever A_{n-4} possesses a p-block of defect zero. The precise conditions for the existence of such a block are well-known and easy to derive.

In the cases $n = 6$ and $n = 7$, each linear character of $Z(G)$ extends to every normaliser of a Klein-four subgroup of H, so there is a defect-preserving 1-1 correspondence between blocks of $H/Z(G)$ and blocks of H covering a given block of defect zero of $Z(G)$. Hence it suffices to note that there are no TI defect blocks of A_6 or A_7.

The remaining automorphism group we must consider is $G/Z(G) \cong A_6.2^2$. It suffices to consider the case $p = 2$. By Lemma 2.1, a block B of G with TI defect groups must cover blocks of defect zero of A_n, since the only 2-defect groups for

A_6 are trivial and the Sylow 2-subgroups, which are not trivial intersection. Hence $|D| = 2$ or 4. But by [6] no involution has centralizer H with $|O_p(H)| = 4$, so $|D| \neq 4$ by Lemma 2.3. □

6 Sporadic simple groups

Proposition 6.1 *Suppose that $N/Z(G) \leq G/Z(G) \leq Aut(N/Z(G))$, where $N/Z(G)$ is one of the 26 sporadic simple groups. Then $B \in \mathrm{Blk}(G)$ has non-cyclic TI defect group D if and only if one of the following occurs:*

(i) $D \cong C_2 \times C_2$ and $G/Z(G)$ is isomorphic to one of J_2 or Ru;
(ii) $D \cong C_3 \times C_3$ and $N/Z(G) \cong O'N$ or Suz;
(iii) $p = 3$ and $N/Z(G) \cong M_{11}$;
(iv) $p = 5$ and $N/Z(G) \cong McL$;
(v) $p = 11$ and $N/Z(G) \cong J_4$.

Note that in cases (i) and (ii) D is not a Sylow p-subgroup and in cases (iii)-(v) D is a Sylow p-subgroup.

Throughout we use the notation of [18], including for the labelling of conjugacy classes of sporadic groups. So for instance $xZ(G) \in 2A$ means that $xZ(G)$ is a member of the conjugacy class $2A$ (of whichever group we are considering at the time) as labelled in [18].

As in the previous section write $\overline{H}$ for the quotient by $Z(G)$, and A^π for pre-images. Of course we may take $Z(G) = O_{p'}(G)$.

In this section let $\overline{N}$ be a sporadic simple group and $\overline{N} \leq \overline{G} \leq \mathrm{Aut}(\overline{N})$. Let $B \in \mathrm{Blk}(G)$ have non-cyclic TI defect group D. We use Lemma 2.3 and Lemma 2.4 to eliminate most of the possibilities for conjugacy classes containing non-trivial elements of $Z(D)$. We first suppose that $G = N$.

Suppose that $1 \neq x \in Z(D)$. First, by Lemma 2.3 $D = O_p(N_G(Q))$, where $Q = \langle x \rangle$. Hence we may eliminate the cases where $O_p(N_G(Q))$ is cyclic. We identify these from [18, Table 5.3], using the fact that $N_{\overline{G}}(\overline{Q})^\pi = N_G(Q)$, and list them in Table 1. Of course we consider only cases where $p^2 || G|$.

Of the remaining possibilities we may apply Lemma 2.4 to eliminate all but a small number of cases.

Lemma 6.2 *Let $\overline{G}$ be a sporadic simple group and $xZ(G)$ in one of the conjugacy classes listed in Table 2. Then G cannot have a TI defect defect group D with $x \in Z(D)$.*

Proof. By Lemma 2.4 it suffices in each case to show that $O_p(N_G(\langle x \rangle))$ is not a Sylow p-subgroup of G and that its quotient is self-centralizing. In all but one case this follows from consideration of the orders of centralizers of elements contained in $N_G(\langle x \rangle)$. The only different case is $G = BM$ and $x \in 3B$. In this case $N_G(\langle x \rangle) \cong 3_+^{1+8} : 2_-^{1+6} \cdot U_4(2).2$, and $D \cong 3_+^{1+8} \notin \mathrm{Syl}_3(G)$. The only involutions in G which can centralize a subgroup of G of order 3^9 are those in $2A$. Suppose z is an involution in $C_G(D)$. Then by [6] D is a Sylow 3-subgroup of $C_G(z)$, but Sylow 3-subgroups of $C_G(z)$ have exponent 9 whilst D has exponent 3. Hence $C_G(D) \leq D$. □

Table 1. Conjugacy classes containing $xZ(G)$ with $O_p(N_G(\langle x\rangle))$ cyclic, $p^2||G|$

$\overline{G}$	Conjugacy classes	$\overline{G}$	Conjugacy classes
M_{11}	none	M_{12}	$2A, 3B$
M_{22}	$3A$	M_{23}	$3A$
M_{24}	$3A, 3B$	J_1	$2A$
J_2	$3A, 3B, 5A, 5B$	J_3	$3A$
J_4	$3A$	Co_3	$2A, 2B, 3C, 5B$
Co_2	$3B, 5B$	Co_1	$3A, 3D, 5A, 5B, 7A, 7B$
HS	$2B, 3A, 5B$	McL	$2A$
Suz	$3A, 5A, 5B$	He	$3A, 3B, 5A, 7A, 7B$
Ly	$2A, 3A$	Ru	$3A, 5B$
$O'N$	$2A$	Fi_{22}	$2A, 3A, 5A$
Fi_{23}	$2A, 3A, 5A$	Fi'_{24}	$2A, 3A, 5A, 7A$
HN	$2A, 3A, 5A$	Th	$3A, 7A$
$F_2 = BM$	$2A, 3A, 5A, 7A$	$F_1 = M$	$2A, 3A, 3C, 5A, 7A, 11A, 13A$

Table 2. Conjugacy classes containing x with $O_p(N_G(\langle x\rangle)) \notin \mathrm{Syl}_p(G)$ and self-centralizing

G	Conjugacy classes	G	Conjugacy classes
M_{12}	$2B$	M_{11}, M_{22}	$2A$
M_{23}	$2A$	M_{24}	$2A, 2B$
J_2	$2A$	J_3	$2A$
J_4	$2A, 2B, 11B$	Co_3	$3A, 3B$
Co_2	$2A, 2B, 2C, 3A$	Co_1	$2A, 2C, 3C, 5C$
HS	$2A, 5C$	McL	$3A, 3B, 5B$
Suz	$2A, 3B$	He	$2B, 7D, 7E$
Ly	$3B, 5A, 5B$	Ru	$2A$
$O'N$	$7B$	Fi_{22}	$2B, 2C, 3B, 3C, 3D$
Fi_{23}	$2C, 3B, 3C, 3D$	Fi'_{24}	$2B, 3B, 3C, 3D$
HN	$2B, 3B, 5B, 5C, 5D, 5E$	Th	$2A, 3B, 3C$
$F_2 = BM$	$2B, 2D, 3B, 5B$	$F_1 = M$	$2B, 3B, 5B, 7B$

We already know the classification of sporadic groups with TI Sylow p-subgroups (see [5]). This says that the only such cases are M_{11} with $p = 3$, McL with $p = 5$ and J_4 with $p = 11$ (and their p'-covering groups). Note that in each of these cases we have one block of maximal defect for each block of defect zero of the centre. Hence we may now assume that $O_p(N_G(\langle x\rangle)) \notin \mathrm{Syl}_p(G)$.

We are thus left with, for $p = 2$: $2B$ in J_2; $2B$ in Co_1; $2B$ in Suz; $2A$ in He; $2B$ in Ru; $2B$ in Fi_{23}; and $2C$ in BM; for $p = 3$: $3B$ in Co_1; $3C$ in Suz; $3A$ in $O'N$; and $3E$ in Fi'_{24}.

In the remaining cases $D \cong C_p \times C_p$. Landrock in [24] gives a complete list of non-principal 2-blocks (but not for the covering groups). From this, we are able to eliminate Co_1, He, Fi_{23} and BM for $p = 2$. We must eliminate Suz. If $xZ(G) \in 2B$, then $\overline{N_G(D)} \cong (A_4 \times L_3(4) : 2) : 2$. In this case $N_G(D)$ does not possess a block with defect group D, and so neither does G.

J_2 and Ru each possess a single 2-block with defect group $C_2 \times C_2$. Further neither J_2 nor Ru possesses a radical 2-subgroup of order two, so these blocks are indeed TI defect.

Suppose that $\overline{G} \cong Co_1$ and $xZ(G) \in 3B$ lies in the centre of a non-cyclic TI defect group D. Then $D \cong C_3 \times C_3$. But by [18, Table 5.3l] $O_3(\overline{N_G(\langle x \rangle)})$ contains an element of $3A$, so that $D = \langle x \rangle$, a contradiction.

Suppose that $\overline{G} \cong Fi'_{24}$ and x with $xZ(G) \in 3E$ lies in the centre of a non-cyclic TI defect group D. Then $D \cong C_3 \times C_3$. But by [18, Table 5.3v] $O_3(\overline{N_G(\langle x \rangle)})$ contains an element of $3A$, so that $D = \langle x \rangle$, a contradiction.

If $\overline{G} \cong O'N$ and $xZ(G) \in 3A$, or if $\overline{G} \cong Suz$ and $xZ(G) \in 3C$, then by [18, Table 5.3s] and [18, Table 5.3o], $\overline{N_G(\langle x \rangle)}$ has the form $((C_3 \times C_3) \cdot 2) \times A_6$, and the maximal normal 3-subgroup D of this is TI. By [6] $N_G(\langle x \rangle)$ is maximal, so is equal to $N_G(D)$. This possesses a 3-block with defect group D and so by Brauer's first main theorem a suitable B exists.

It remains to consider the automorphism groups of the sporadic groups. Since $[\mathrm{Aut}(N) : N] | 2$ for each sporadic simple group N, for odd primes, $\mathrm{Aut}(N)$ has non-cyclic TI defect groups if and only if N does (by the same argument as in Corollary 5.2). It thus suffices to consider $p = 2$. Note that by Proposition 2.2 (v) and since blocks of $\mathrm{Aut}(N)$ covering blocks of defect zero of N have defect groups of order dividing two, if $\mathrm{Aut}(N)$ possesses a non-cyclic TI defect group, then N possesses a (possibly cyclic of order two) TI defect group. By [24] this leaves the cases $\mathrm{Aut}(J_2)$, $\mathrm{Aut}(McL)$ and $\mathrm{Aut}(Fi_{22})$.

We may eliminate $\mathrm{Aut}(J_2)$ (noting that here $Z(G) = 1$), since if $x \in Z(D)$, where D is a non-cyclic TI defect group, is not contained in N, then $D = O_p(N_G(\langle x \rangle))$ has order two. But by [6] the non-cyclic TI defect block of J_2 is stabilised by the outer automorphisms, so by [1, 15.1] D must contain an element outside of N.

We may eliminate $\mathrm{Aut}(McL)$ since if D is a TI defect group, then $Z(D)$ must contain an involution outside of N. But then by [18, Table 5.3n] $|D| = 2$.

We may eliminate $\mathrm{Aut}(Fi_{22})$ since if D is a non-cyclic TI defect group, then $|D| = 4$, and D must contain an element x with $xZ(G) \in 2A$. But then $D = \langle x \rangle$. This completes the proof of Proposition 6.1.

7 Finite groups of Lie type

We will follow the notation of [18]. In particular, if K is a finite group of Lie type, then K_u and K_a are the corresponding universal and adjoint finite groups.

Lemma 7.1 *Let* $K \in \{SL^{\epsilon}_m(q), E_7(q)_u, Sp_{2m}(q), \Omega_{2m+1}(q), \Omega^{\epsilon}_{2m}(q)\}$ *with odd* q *and* $m \geq 1$*, and let* B *be a* 2*-block of* K *with a defect group* $O_2(Z(K))$*, where* $\epsilon = \pm = \pm 1$*. Then one of the following holds.*

(i) $K = SL^{\epsilon}_m(q)$ *and there exists a* 2*-element* $\tau \in GL^{\epsilon}_m(q)\backslash Z(GL^{\epsilon}_m(q))K$ *stabilizing* B*.*
(ii) $K = \Omega^{\epsilon}_{2m}(q)$ *and there exists a* 2*-element* $\tau \in SO^{\epsilon}_{2m}(q)\backslash\Omega^{\epsilon}_{2m}(q)$ *stabilizing* B*.*
(iii) There is a 2*-element* $\tau \in Outdiag(K)$ *stabilizing* B*.*
(iv) $K = Sp_{2m}(q)$*,* $B = \mathcal{E}_2(K, (s))$ *for some semisimple* $2'$*-element* s *in the dual group* K^* *such that* $C_{K^*}(s)$ *is a Coxeter torus of* K^* *and* $|C_{K^*}(s)|_2 = 2$*. Moreover,* $C_{K^*}(s) \cong GL_1(q^m)$ *or* $GU_1(q^m)$ *according as* $4|q+1$ *or* $4|q-1$*, and in the former case,* m *is odd. In addition, there exists a* 2*-element* $\tau \in GSp_{2m}(q)\backslash Z(GSp_{2m}(q))K$ *stabilizing* B*.*

Proof. Set $\nu(n) = \log_2(n_2)$ for any integer $n \geq 1$, where n_2 is the 2-part of n.

Suppose $K = E_7(q)_u$, so that $D(B) = Z(K) = 2$. If χ is the canonical character of B, then χ can be viewed as an irreducible character of $K_a = E_7(q)_a$. Let $H = Inndiag(K)$, χ' an irreducible character H covering χ and B_H the 2-block of H containing χ'. Then the dual group H^* is K. Since χ has defect 0 as a character of K_a, it follows that $|D(B_H)| = 1$ or 2, and in the former case $\chi'(1) = 2\chi(1)$ and in the later case $\chi'(1) = \chi(1)$.

If $\chi'(1) = \chi(1)$, then the inertia group $I(B)$ is H and there exists a 2-element $\tau \in H\backslash K_a$ stabilizing B. Suppose $\chi'(1) = 2\chi(1)$, so that $D(B_H) = 1$. If (s, μ) is the pair of semisimple and unipotent labels of χ', then

$$\nu(\chi') = \nu(K\colon C_K(s)) + \nu(\mu).$$

In particular, $\nu(\mu) = \nu(C_K(s)\colon Z(C_K(s)))$ and $|Z(C_K(s))|_2 = 1$, since $\mu(1)$ divides $|C_K(s)\colon Z(C_K(s))|$. This is impossible as $C_2 = Z(K) \leq C_K(s)$.

Suppose K is classical. Let V be the underlying space of K, $I(V) = Isom(V)$ the set of all isometries on V defined in [18] and let $I_0(V)$ be the subset of $I(V)$ consisting of elements of determinant 1.

Suppose $K \neq \Omega_{2m+1}(q)$ and $\Omega^{\epsilon}_{2m}(q)$. Then $K = I_0(V)$ and $B = \mathcal{E}_2(K, (s))$ for some semisimple odd element s of the dual group K^*. If $K = SL^{\epsilon}_m(q)$, then we may suppose $s \in K$ and set $s^* = s$. If $K = Sp_{2m}(q)$, then let s^* be the dual element of s defined in [2], so that $s^* \in K$. Thus $D(B)$ is a Sylow 2-subgroup of $C_K(s^*)$. Since $D(B)$ is cyclic, it follows that $C_K(s^*)$ is a maximal torus T of K.

If $K = Sp_{2m}(q)$, then $T \cong GL^{\eta}_1(q^m)$ is a Coxeter torus and $|Z(K)| = 2$ is the exact power of 2 dividing $q^m - \eta$ (written $2\|q^m - \eta$). If $4|q-1$, then $4|q^m-1$, so that $\eta = -1$ and m is arbitrary. If $4|q+1$, then $2\|q^m+1$ if and only if m is even. But if $2\|q^m+1$, then m must be odd since m is the degree of a monic irreducible polynomial Δ such that ω is a root of Δ if and only if ω^{-q} is its root (cf. [14, p.111]). In both cases, there is a 2-element $\tau \in GSp_{2m}(q)\backslash Z(GSp_{2m}(q))K$ such that τ centralizes s^*. It follows that τ stabilizes B.

Suppose $K = SL^{\epsilon}_m(q)$, so that $I(V) = GL^{\epsilon}_m(q)$. If T is non-cyclic, then $C_{I(V)}(s) = GL^{\epsilon_1}_1(q^{m_1}) \times GL^{\epsilon_2}_1(q^{m_2})$. Thus $O_2(C_{I(V)}(s)) > O_2(Z(I(V)))D(B)$ and there exists some $\tau \in O_2(C_{I(V)}(s))\backslash O_2(Z(I(V)))D(B)$ stabilizing B. If T is cyclic, then $C_{I(V)}(s) = GL^{\eta}_1(q^{m'})$ is a Coxeter torus of $I(V)$, where $(\eta, m') = (1, m)$ when $\epsilon = 1$, and $(\eta, m') = (1, \frac{m}{2})$ or $(-1, m)$ when $\epsilon = -1$.

If m is odd, then $D(B) = 1$ and there exists some $\tau \in O_2(C_{I(V)}(s))\backslash D(B)$ stabilizing B. If $\epsilon = 1$ and $\alpha = \nu(m)$, then $\alpha \geq 1$ and $2^{b+\alpha} \| q^m - 1$, where $b \geq 2$ is an integer such that $2^{b+1} \| q^2 - 1$. In particular, $2^{b+\alpha} > 2^b \geq |D(B)|$. Suppose $\epsilon = -1$. Since m is even and since $q+1$ is not a factor of $q^m + 1$, it follows that $\eta = 1$ and so $m' = m/2$. But $Z(I(V)) = q + 1 \leq C_{I(V)}(s)$, so $q+1 | q^{m'} - 1$ and m' is even. Similarly, if $\alpha' = \nu(m') \geq 1$, then $2^{b+\alpha'} \| q^{m'} - 1$ and $2^{b+\alpha'} > 2^b \geq |D(B)|$. Thus in both cases, there exists some $\tau \in O_2(C_{I(V)}(s))\backslash O_2(Z(I(V)))D(B)$ stabilizing B.

Suppose $K = \Omega_{2m+1}(q)$ or $\Omega^{\epsilon}_{2m}(q)$, so that $D(B) = 1$ or 2. If K is abelian, then $K = \Omega^{\epsilon}_2(q) = C_{\frac{q-\epsilon}{2}}$ and the proof is trivial since $SO^{\epsilon}_2(q) = C_{q-\epsilon}$.

Let $\tilde{B}$ be the unique block of $I_0(V)$ covering B, so that $\tilde{B} = \mathcal{E}_2(I_0(V), (s))$ for some semisimple $2'$-element s of $I_0(V)^*$. Thus $D(\tilde{B}) \cap K = D(B)$ and $|D(\tilde{B})\colon D(B)| \leq 2$.

If $|D(\tilde{B})\colon D(B)| = 2$, then $I(B) = I_0(V)$ and there exists a 2-element $\tau \in I_0(V)\backslash K$ stabilizing B.

Suppose $D(\tilde{B}) = D(B)$, and let $s^* \in I_0(V)$ be a dual element of s in $I_0(V)$. Then $D(\tilde{B})$ is a Sylow 2-subgroup of $C_{I_0(V)}(s^*)$. Let $C_{I_0(V)}(s^*) = \prod_{\Gamma} L_{\Gamma}$, where $L_{\Gamma} = I_0(V_0)$ with V_0 the fixed-point set of s^* on V or $L_{\Gamma} \cong GL^{\eta_{\Gamma}}_{n_{\Gamma}}(q^{m_{\Gamma}})$ for some integers n_{Γ}, m_{Γ} and sign η_{Γ}. Since $D(\tilde{B})$ is cyclic and K is non-abelian, it follows that $C_{I_0(V)}(s^*) \cong GL^{\epsilon}_1(q^m)$ is a Coxeter torus and $D(\tilde{B}) = O_2(C_{I_0(V)}(s^*)) \neq 1$. Thus $D(B) = D(\tilde{B}) = 2$ and $K = \Omega^{\epsilon}_{2m}(q)$.

Since $-1_V \in K$, it follows that $\frac{m(q-1)}{2}$ is even or odd according as $\epsilon = 1$ or -1. If $\epsilon = -1$, then m is odd and $2 \| q-1$, so that $2^b \| q^m + 1$ and $|O_2(C_{I_0(V)}(s^*))| = 2^b \geq 4$. If $\epsilon = 1$ and m is even, then $2^{b+\nu(m)} \| q^m - 1$ and $2^{b+\nu(m)} \geq 8$. If $\epsilon = 1$ and m is odd, then $4 | q-1$ and $2^b \| q^m - 1$. In all cases, $|O_2(C_{I_0(V)}(s^*)| \geq 2|D(B)|$, which is impossible. □

Remark 7.2 (i) In the notation of Lemma 7.1 (i) and (ii), let $K = SL^{\epsilon}_m(q)$ or $\Omega^{\epsilon}_{2m}(q)$. Then either $\tau = -1_V \notin K$ or τ induces a non-trivial element of $Outdiag(K)$.

(ii) If $\tilde{B}$ is a 2-block of $I_0(V) = SO(V)$ such that $D(\tilde{B}) = Z(SO(V))$, then there exists a 2-element $\tau \in GSO(V)\backslash Z(GSO(V))SO(V)$ stabilizing $\tilde{B}$.

Proof. Indeed, in the notation above, $C_{GSO(V)}(s^*) = \langle \sigma, C_{GSO(V)}(s^*) \rangle$, where $\sigma^{q-1} \in Z(C_{GSO(V)}(s^*))$ and σ centralizes $C_{SO(V)}(s^*)$ (cf [15, (1A)]). Thus $C_{GSO(V)}(s^*)$ is abelian and $|C_{GSO(V)}(s^*)|_2 > |Z(GSO(V))C_{SO(V)}(s^*)|_2$, so that there exists such a 2-element τ. □

Let $Z(G) = O_{p'}(G) \leq M \leq G$ where $M/Z(G) \leq G/Z(G) \leq \mathrm{Aut}(M/Z(G))$ and $M/Z(G)$ is a finite non-abelian simple group of Lie type. If D is a TI defect group of a block of G, then

$$N_G(D)/Z(G) = N_{G/Z(G)}(D)$$

and D is a TI radical subgroup of $G/Z(G)$. Thus we first classify TI radical subgroups of $G/Z(G)$.

Proposition 7.3 *Let $K = K_a$ be a finite non-abelian simple group of Lie type and $K \leq G \leq Aut(K)$. Suppose that D is a TI radical subgroup of G. Then one of the following holds.*

(i) D is cyclic
(ii) $C_G(D) \leq D$.
(iii) $K = PSL_3(4)$ and D is a Sylow subgroup of K.
(iv) D is a generalized quaternion with $D \cap K = Q_8$ a quaternion, and

$$K \in \{{}^3D_4(3), F_4(3), E_6^{\epsilon}(3), E_7(3), E_8(3), PSL_n^{\epsilon}(3), PSp_{2n}(3)\},$$

where $n \geq 3$ and $n \neq 4$ when $K = PSL_n^{\epsilon}(3)$.

Proof. Let $H = Inndiag(K)$, $R = H \cap D$ and $Q = K \cap D$.

Case (1) Suppose p and the characteristic r of the underlying field of K are distinct and D contains a non-cyclic abelian subgroup W and let $E = \Omega_1(W)$. Then E is a non-cyclic elementary abelian group acting on K and $N_G(E) \leq N_G(D)$. Let

$$\Gamma = \Gamma_{E,1}^{r'} = \langle O^{r'}(C_K(F)) \mid 1 \neq F \leq E \rangle$$

and $\Gamma' = \Gamma'_{E,1} = \langle C_K(F) \mid 1 \neq F \leq E \rangle$. Then $\Gamma' \leq \Gamma \leq N_G(E) \cap K$.

Suppose $\Gamma = K$ or $\Gamma' = K$. If $Q \neq 1$, then by Lemma 2.1, Q is a radical TI subgroup of K and $N_G(Q) = N_G(D)$. But $K \leq N_G(D)$, so $N_K(Q) = N_G(Q) \cap K = K$ and $Q \leq O_p(K)$, which is impossible. Thus $Q = 1$ and $KD = K \times D$ as $D = O_p(N_G(D))$, which is also impossible. So $\Gamma' \leq \Gamma < K$.

Since $\Gamma < K$, it follows by [18, Theorem 7.3.3] that $C_E(K) = 1$. Applying [18, Theorem 7.3.4] we get a finite list of pairs (E, K). By Lemma 2.5, $N_G(D)$ is a maximal p-local subgroup of G with $E \leq D \lhd N_G(D)$ and $|N_G(D)|_p > |E|$ whenever E is not a Sylow subgroup of G. Note that if $N_G(D)$ is contained in a maximal (non p-local) subgroup N of G, then $N_G(D)$ is a maximal p-local subgroup of both N and G. Using the maximal subgroups of $Aut(K)$ and K given in [6], we can get the possible $N_{Aut(K)}(D)$. The possible triples $(E, K, N_{Aut(K)}(D))$ are given in Table 3.

If $Aut(K) = G = K = C_4(2)$ and $N_G(D) = 3^2\colon D_8 \times S_6 < (S_6 \times S_6).2$, then $D = 3^2$ contains an element y of type $3A$, so that $C_G(y) = 3 \times Sp_6(2) \leq N_G(D)$, which is impossible. If $(E, K) = (3^2, A_2(4))$, then $E \in \mathrm{Syl}_3(K)$ and $Aut(K) = A_2(4).D_{12}$. Thus $D \in \mathrm{Syl}_3(K)$ or $D \in \mathrm{Syl}_3(Aut(K))$. In the latter case, $N_G(E) = N_G(D)$ and so $D = O_3(N_G(E))$, which is impossible. If $(E, K, N_{Aut(K)}(D)) \neq (3^2, C_4(2), 3^2\colon D_8 \times S_6)$ and $(3^2, A_2(4), 3^2\colon 2S_4 \times 2)$, then $C_G(D) \leq C_{N_{Aut(K)}(D)}(D) \leq D$.

Case (2) Suppose $p = r$ and $Q = K \cap D \neq 1$. Then $N_G(Q) = N_G(D)$ and by [18, Corollary 3.1.4],

$$F^*(N_G(Q)) = O_r(N_G(Q)) = D.$$

Thus $C_G(D) = C_{N_G(D)}(D) \leq D$.

Suppose $Q = D \cap K = 1$. If $x \in N_G(D) \cap K$ and $y \in D$, then $x^{-1}y^{-1}xy \in D \cap K = 1$, so that $C_K(D) = N_K(D) \lhd N_G(D)$.

Table 3. The possible triples $(E, K, N_{Aut(K)}(D))$

$\|E\|$	K	$N_{Aut(K)}(D)$
2^2	$A_1(5)$	S_4
2^2	$A_1(7)$	D_{16}, S_4
2^2	$A_1(9)$	$S_4.2 \times 2$
2^2	$B_2(3)$	$2^4\colon S_5, 2.(A_4 \times A_4).2.2$
$2^2, 2^3$	${}^2G_2(3)'$	$2^3\colon 7\colon 3$
3^2	$A_2(4)$	$3^2\colon 2S_4 \times 2$
3^2	$U_4(2)$	$3_+^{1+2}\colon 2S_4, 3^3\colon (S_4 \times 2)$
3^2	$B_2(2)'$	$3^2\colon D_8$
3^2	$C_3(2)$	$3_+^{1+2}\colon 2S_4, 3^3\colon (S_4 \times 2)$
3^2	$C_4(2)$	$3^2\colon D_8 \times S_6, S_3 \wr S_4$
3^2	$G_2(2)'$	$3_+^{1+2}\colon 8\colon 2$
3^2	${}^2F_4(2)'$	$3^2\colon 2S_4, 3_+^{1+2}\colon SD_{16}$
3^2	$L_2(8)$	$9\colon 6$
5^2	${}^2F_4(2)'$	$5^2\colon 4S_4$
5^2	${}^2B_2(2^5)$	$25\colon 20$

In the notation of [18, Theorem 2.5.12], $Out(K) = Outdiag(K)\colon \Phi_K\Gamma_K$ and so $D \leq \Phi_K\Gamma_K$, since $Outdiag(K)$ is a p'-group.

Suppose D is non-cyclic. Then $|\Gamma_K|_p \neq 1$ and $p = 2$ or 3. If $|\Gamma_K|_2 \neq 1$, then $|\Gamma_K|_2 = 2$, and $K = A_m(2^a)$, $B_2(2^a)$, $D_m(2^a)$, $F_4(2^a)$ or $E_6(2^a)$, and moreover, $\Phi_K\Gamma_K$ is cyclic whenever $K = B_2(2^a)$ and $F_4(2^a)$. Thus $K = A_m(2^a)$ with $m \geq 2$, $D_m(2^a)$ or $E_6(2^a)$ with $2|a$, and $\Phi_K\Gamma_K = \Phi_K \times \Gamma_K$. If $|\Gamma_K|_3 \neq 1$, then $|\Gamma_K|_3 = 3$ and $K = D_4(3^a)$ with $3|a$. So D is abelian of 3-rank 2.

Let ϕ be a field of graph-field automorphism of order p. Then $K_0 = C_K(\phi)$ is simple and by [18, Theorem 7.1.4], $C_G(\phi) = \langle\phi\rangle$. It follows that $\phi \notin D$ and D is cyclic, which is impossible.

Case (3) Suppose $p \neq r$ and every abelian subgroup of D is cyclic. Thus D is cyclic or a generalized quaternion group Q_{2^α} of order $2^\alpha \geq 8$ (cf. [16, Theorem 5.4.10]). In the latter case, each subgroup of D is either cyclic or generalized quaternion, and D has a unique element z of order 2 with $\langle z\rangle = \Omega_1(Z(D))$.

Suppose D is non-cyclic, so that $p = 2$, $D = Q_{2^\alpha}$ and r is odd with $q = r^a$.

Case (3.1) Suppose, moreover that $D \cap K = 1$, so that $N_K(D) = C_K(D)$ and D is a quaternion subgroup of $Out(K)$. In particular, a Sylow 2-subgroup of $Out(K)$ is non-abelian, so that K is classical, $K \neq PSp_{2m}(q)$, and $m \geq 3$ when $K = PSL_m^\epsilon(q)$.

Since $Out(P\Omega_m(q))$ is either abelian or $Out(P\Omega_m(q)) \in \{D_8 \times C_a, S_3 \times C_a, S_4 \times C_a\}$, it follows that $Out(P\Omega_m(q))$ contains no quaternion subgroup, so that $K = PSL_m^\epsilon(q)$.

Suppose D contains a field or graph-field automorphism ϕ of order 2. Then $K_0 = C_K(\phi) \leq N_G(D)$ and $C_K(\phi)D = C_K(D) \times D$. In particular, $D \leq C_G(K_0)$. Since r is odd, it follows that K_0 is simple and by [18, Theorem 7.1.4], $C_G(K_0) = \langle \phi \rangle$, so that D is cyclic, which is impossible.

Since $Outdiag(K)$ is cyclic and since D has exactly one involution z, it follows that $R = D \cap H$ is cyclic. If moreover, $R = 1$, then $D \leq \Phi_K \times \Gamma_K$, which is impossible. Thus $R \neq 1$, $z \in R$, $N_G(D) = C_G(z)$.

Let $M = GL^{\epsilon}_m(q)$, so that $H = Inndiag(K) = M/Z(M)$. Then $z \in H \backslash K$, $R = O_2(C_H(z))$ and $C_H(z)$ is given by [18, Table 4.5.2]. If $L = O^{r'}(C_H(z))$, then

$$L \in \{SL^{\epsilon}_{m-1}(q), SL^{\epsilon}_i(q) * SL^{\epsilon}_{m-i}(q), SL^{\epsilon}_{\frac{m}{2}}(q^2)\},$$

where $2 \leq i \leq \frac{m}{2}$ and $SL^{\epsilon}_i(q) * SL^{\epsilon}_{m-i}(q)$ is a central product of $SL^{\epsilon}_i(q)$ and $SL^{\epsilon}_{m-i}(q)$. Let $[D, L] = \langle y^{-1}x^{-1}yx \mid y \in D, x \in L\}$, so that $[D, L] \leq D \cap L = O_2(L)$. Since $O_2(L) \leq D \cap H$ is cyclic, it follows that $O_2(L) \leq Z(L)$. But $y^{-1}x^{-1}yx = 1$ for any $y \in D$ and $2'$-element $x \in L$, so $[D, L] = 1$ and $D \leq C_{C_G(z)}(L)$.

Let ι be the inverse-transpose map or 1 according as $M = GL_m(q)$ or $M = U_m(q)$. We may suppose that

$$C = C_{Aut(K)}(z) = \langle C_H(z), \gamma, \iota \rangle,$$

where γ is a field automorphism of M. Note that if $x\beta$ centralizes L for some $x \in C_H(z)$ and $\beta \in \langle \gamma, \iota \rangle$, then $x \in C_L(\beta)$. It follows that $C_C(L) = C_{C_H(z)}(L)$ and $O_2(C_{C_H(z)}(L)) \leq D \cap H$ is cyclic. This is impossible as $D \leq C_C(z)$.

Case (3.2) Suppose $Q = D \cap K \neq 1$. Then $\Omega_1(Z(Q))$ has order 2 and $z \in Z(Q)$. So $N_G(D) = C_G(z)$, $N_K(Q) = C_K(z)$, $D = O_2(C_G(z))$ and $Q = O_2(C_K(z))$. The proof is similar to above.

The centralizer $C_H(z)$ is given by [18, Table 4.5.1]. If $L = O^{r'}(C_H(z))$, then either $L = L_1$ or $L_1 * L_2$ for some finite groups L_i of Lie type. Since $L \leq C_G(z) = N_G(D)$, it follows that $D \cap L \lhd L$ and $D \cap L = O_2(L)$, so that $[D, L] \leq O_2(L)$.

Note that each $O_2(L_i)$ is cyclic except when $L_i \in \{\Omega_3(3) = A_4, SL_2(3) = SU_2(3) = Sp_2(3) = Q_8 : 3, \Omega_4(3) = SL_2(3) * SL_2(3)\}$.

Since $O_2(L) \leq D$ is cyclic or generalized quaternion, it follows that either $O_2(L_i)$ is cyclic for all i or $L_i = SL_2(3)$ for a unique i. In the former case each L_i is quasisimple or cyclic, $O_2(L_i) \leq Z(L_i)$, and $[D, L] \leq Z(L)$, since L is generated by $2'$-elements. In the latter case $q = 3$ and $O_2(L) = Q_8$.

Suppose $O_2(L) = Q_8$, so that $q = 3$. If K is exceptional, then

$$K \in \{{}^3D_4(3), G_2(3), F_4(3), E^{\epsilon}_6(3), E_7(3), E_8(3)\}.$$

If K is classical, then $K \in \{PSL^{\epsilon}_n(3), PSp_{2n}(3)\}$. By [18, Table 4.5.1], each K has exactly one possible conjugacy class of involutions z, and $L = L_1 * L_2$ with $L_1 = SL_2(3)$ and

$$L_2 = \begin{cases} SL_2(27) & \text{if } K = {}^3D_4(3), \\ SL_2(3) & \text{if } K = G_2(3), \\ Sp_6(3) & \text{if } K = F_4(3), \\ SL_6^{\epsilon}(3) & \text{if } K = E_6^{\epsilon}(3), \\ Spin_{12}^{+}(3)/\langle z_s\rangle & \text{if } K = E_7(3), \\ E_7(3)_u & \text{if } K = E_8(3), \\ SL_{n-2}^{\epsilon}(3) & \text{if } K = PSL_n^{\epsilon}(3), \\ Sp_{2n-2}(3) & \text{if } K = PSp_{2n}(3). \end{cases} \tag{7.1}$$

If $K = G_2(3)$, then $Q \geq O_2(L) = 2_+^{1+4}$, which is impossible. Similarly, we may suppose $K \neq PSL_4^{\epsilon}(3)$ and $PSp_4(3)$. Thus $Q = O_2(L) = Q_8$ and $C_K(Q) = C_H(Q) = L_2$.

Suppose $O_2(L)$ is cyclic and L is non-abelian, so that L is quasisimple, and $D \leq C_{C_G(z)}(L) \leq C_{C_{Aut(K)}(z)}(L)$. In particular, $Q \leq R \leq C_{C_H(z)}(L)$ and by [18, Theorem 4.5.1], R is cyclic.

In the notation of [18], $Aut(K) = H\Phi_K\Gamma_K$. It follows by [18, Theorem 4.5.1] that we may suppose

$$C_{Aut(K)}(z) = C_H(z)\Phi_K\Gamma_K.$$

If $x\beta \in C_{C_{Aut(K)}(z)}(L)$ for some $x \in C_H(z)$ and $\beta \in \Phi_K\Gamma_K$, then $x \in C_L(\beta)$. It follows that $C_{C_{Aut(K)}(z)}(L) \leq C_H(z)$ and so $D \leq C_{C_H(z)}(L)$ is cyclic, which is impossible.

Suppose L is cyclic, so that $K = PSL_2(q)$. Let $\tilde{G}$ be the general semilinear group on the underlying space V of K_u and $A = \tilde{G}\langle\iota\rangle$, where ι is the inverse-transpose map. In addition, let $Z = \{\alpha 1_V : 0 \neq \alpha \in \mathbb{F}_q\}$ be a subgroup of A, so that $Aut(K) = A/Z$ and we may suppose $z = tZ$ with $O_2(Z) \leq \langle t\rangle$, since $Q \neq 1$.

If $xZ \in C_G(z)$, then $x^{-1}tx \in tZ$, so that $x^{-1}tx \in \langle t\rangle$ and $x \in N_A(\langle t\rangle)$. Thus $C_{Aut(K)}(z) = N_A(\langle t\rangle)/Z$ and $C_H(z) = N_{GL_2(q)}(t)/Z$.

Let D_A be a 2-subgroup of $N_A(\langle t\rangle)$ such that $D_A/O_2(Z) = D$. Now $N_A(\langle t\rangle) = \langle GL_1(q^2), \rho, \gamma, \iota\rangle$ and $N_{GL_2(q)}(\langle t\rangle) = \langle GL_1(q^2), \rho\rangle$, where ρ and γ induce field automorphisms of order 2 and a, respectively and ι inverts each element of $GL_1(q^2)$.

We may choose ρ such that $\rho^4 = 1$, ρZ has order 2 in $GL_2(q)/Z$ and $[\rho, \gamma] = [\rho, \iota] = 1$. Suppose $y \in D_A \cap \langle\rho, \gamma, \iota\rangle$ such that $2^w = |y| \geq 4$. Then $y^{2^{w-1}} = \phi \in D_A$, which is impossible. Here ϕ is the field automorphism of order 2. Thus $D_A \cap \langle\rho, \gamma, \iota\rangle \leq \langle\rho, \phi, \iota\rangle$ and

$$O_2(GL_1(q^2)) < D_A \leq \langle O_2(GL_1(q^2)), \rho, \phi, \iota\rangle.$$

Since $\langle\rho, \phi, \iota\rangle Z/Z$ is elementary abelian of order 2^3, it follows that $m_2(D_A/O_2(Z)) \geq 2$, which is impossible. □

Proposition 7.4 *Let $Z(G) = O_{p'}(G) \leq M \leq G$ such that $M/Z(G)$ is a finite non-abelian simple group of Lie type and $M/Z(G) \leq G/Z(G) \leq Aut(M/Z(G))$. Suppose $B \in$ Blk(G) has a trivial intersection defect group $D \neq 1$. Then one of the following holds:*

(i) D is cyclic.

(ii) D is a non-cyclic Sylow p-subgroup of G. If $D \cap M$ is non-cyclic, then the possible pairs $(M/Z(G), p)$ are listed in [5, Proposition 1.3] and moreover, $[G\colon M]_p = 1$, so that $D \leq M$. If $D \cap M$ is cyclic, then $Z(G) = 1$, $(M, p) = ({}^2B_2(2^5), 5)$ or $(L_2(8), 3)$, in which case $G = Aut(M) = M.p$.

(iii) $p = 2$, $Z(G) = 1$, $G = PSp_{2m}(3)$ with even $m \geq 4$ and $D = Q_8$ is a quaternion group. In this case, let B_u be a block of $K_u = Sp_{2m}(3)$ containing B. Then $B_u = \mathcal{E}_2(K_u, (s))$ for some semisimple $2'$-element s of the dual group $K_u^* = SO_{2m+1}(3)$, $D(B_u) \cong Q_8 \times C_2$ and $C_{K_u^*}(s) \cong SO_3(3) \times GL_1(3^{m-1})$.

In particular, if D is non-cyclic, then $G/Z(G)$ has a block with a defect group $D = DZ(G)/Z(G)$.

Proof. Let $K = M/Z(G)$. Since $D = DZ(G)/Z(G)$ is a radical subgroup of $G/Z(G)$, it follows by Proposition 7.3 that D is cyclic, $C_{G/Z(G)}(D) \leq D$, $K = A_2(4)$ with $D = 3^2$ or D is generalized quaternion with $D \cap K = Q_8$ and K given by Proposition 7.3 (iv).

Suppose $C_{G/Z(G)}(D) \leq D$. By Lemma 2.4 D is Sylow of both G and $G/Z(G)$, so that by Lemma 2.1, $Q = D \cap K$ is a TI Sylow subgroup of K.

Suppose $p \neq r$. If Q is non-cyclic, then by [5, Proposition 1.3],

$$(K, p) = \{(PSL_3(4), 3), ({}^2F_2(2)', 5)\}$$

and if Q is cyclic, then by Table 3,

$$(K, p) \in \{(L_2(8), 3), ({}^2B_2(2^5), 5)\}.$$

If $[G\colon M]_p \neq 1$, then $(K, p) \neq ({}^2F_2(2)', 5)$, so that $[G\colon M]_p = p$.

If $(K, p) = (L_3(4), 3)$, then $Aut(K)/K = 2 \times S_3$, $N_K(Q) = 3^2\colon Q_8$ and $N_{Aut(K)}(Q) = 3^2\colon 2S_4 \times 2$. It follows that Q is a radical subgroup of G and D is not TI, which is impossible.

If $(K, p) = (L_2(8), 3)$ or $({}^2B_2(2^5), 5)$, then Q is cyclic and $G/Z(G) = Aut(K) = K.p$. Since $Aut(K)/K$ is cyclic and $M(K) = 1$, it follows by [8, Lemma 3.4] that $M(G) = 1$, so that $Z(G) = 1$ and $G = Aut(K)$. If $(K, p) = (L_2(8), 3)$, then $K = {}^2G_2(3)'$, $G = {}^2G_2(3)$ and $N_G(Q) = 9\colon 6$. If $(K, p) = ({}^2B_2(2^5), 5)$, then $N_G(Q) = 25\colon 20$. Thus Q is non-radical in G and so the Sylow subgroup D is a TI subgroup of G.

Suppose $p = r$. By [5, Proposition 1.3], $K = A_1(p^a), {}^2A_2(p^a), {}^2B_2(2^a)$ or ${}^2G_2(3^a)$. Since $p = r$, it follows that a Sylow p-subgroup of $Out(K)$ is cyclic generated by a field automorphism.

If $|Out(K)|_p = 1$, then $D = Q$ is Sylow in G. In particular, we may suppose $K \neq {}^2B_2(2^a)$ as in this case a is odd.

Suppose $[G\colon M]_p \neq 1$, so that $|Out(K)|_p \neq 1$. Since $C_K(Q) \leq Q$ and since $Out(K)$ is cyclic, it follows that $Q < D$. Thus D contains a field automorphism ϕ of order p. By [18, Proposition 4.9.1], $C_K(\phi) = A_1(p^{a/p})$, ${}^2A_2(p^{a/p})$ and ${}^2G_2(3^{a/3})$, respectively. But $C_K(\phi) \leq N_{Aut(K)}(D)$, so $C_K(\phi)$ is a subgroup of the Borel subgroup $N_K(Q)$. In particular, $C_K(\phi)$ is soluble. Thus $p = 2$, $C_K(\phi) = A_1(2)$ or ${}^2A_2(2)$. In both cases, $|Out(K)|_p \leq p$ and D is Sylow in G.

If $(K, p) = (A_1(4), 2)$, then $Q = 2^2$, $N_K(Q) = A_4$ and $N_{Aut(K)}(Q) = S_4$. If $(K, p) = ({}^2A_2(4), 2)$, then $Q = 2^{2+4}$, $N_K(Q) = 2^{2+4}: 15$ and $N_{Aut(K)}(Q) = 2^{2+4}: (3 \times D_{10}).2$. In both cases, Q is radical in $Aut(K)$ and a Sylow 2-subgroup D of $Aut(K)$ is not a TI defect group.

Suppose D is generalized quaternion with $D \cap K = Q_8$ and

$$K \in \{{}^3D_4(3), F_4(3), E_6^\epsilon(3), E_7(3), E_8(3), PSL_n^\epsilon(3), PSp_{2n}(3)\},$$

where $n \geq 3$ and $n \neq 4$ when $K = PSL_n^\epsilon(3)$. Thus

$$Aut(K) = \begin{cases} {}^3D_4(3).3 & \text{if } K = {}^3D_4(3), \\ F_4(3) & \text{if } K = F_4(3), \\ E_6^\epsilon(3).2 & \text{if } K = E_6^\epsilon(3), \\ E_7(3).2 & \text{if } K = E_7(3), \\ E_8(3) & \text{if } K = E_8(3), \\ PSL_n^\epsilon(3).(n, 3-\epsilon).2 & \text{if } K = PSL_n^\epsilon(3), \\ PSp_{2n}(3).2 & \text{if } K = PSp_{2n}(3). \end{cases} \tag{7.2}$$

In particular, $Aut(K)/K$ is either cyclic or a 2-group. Since $K \neq PSU_4(3)$, it follows that $M(K)$ is a 2-group, and by [8, Lemma 3.4], $M(X)$ is a 2-group for any $K \leq X \leq Aut(K)$. Since $Z(G)$ is a $2'$-group, it follows that we may suppose $Z(G) = 1$.

Let B_K be the block of K covered by B. Then we may suppose $Q = K \cap D = D(B_K)$ and in addition, suppose (Q, b_Q) is a Sylow B_K-subgroup.

If $L = O^{3'}(C_H(Z(Q)))$, then $L = L_1 * L_2$, $Q = O_2(L)$ and $C_K(Q) = L_2$, where $L_1 = SL_2(3)$ and L_2 given by (7.1). Since (Q, b_Q) is a Sylow B_K-subgroup, it follows that $b_Q \in \mathrm{Blk}(L_2)$ with defect group $Z(Q) = Z(L_2)$. Moreover, suppose $K \neq PSp_{2n}(3)$. By [18, Table 4.5.1], $N_K(Q) \geq \langle Q * L_2, \tau\rangle$ for some 2-element τ, and in addition, τ induces a non-trivial element of $Outdiag(L_2)$, except when $L_2 = Spin_{12}^+(3)/\langle z_s\rangle$, in which case we can suppose $\tau \in SO_{12}^+(3)\setminus \Omega_{12}^+(3)$. By Lemma 7.1, τ stabilizes b_Q, so that $\tau \in N_K(Q, b_Q)$, which is impossible as (Q, b_Q) is maximal.

Suppose $K = PSp_{2n}(3)$ and $G \neq K$. Then $G = Aut(K) = K.2 = GSp_{2n}(3)$ and so $C_G(Q) = L_2$, $N_G(Q) = \langle Q * L_2, \tau\rangle$ for some 2-element $\tau \in G$. By Lemma 7.1 again, τ stabilizes b_Q and so $\tau \in N_G(Q, b_Q)$, which is impossible. □

8 Proof of the main theorem

By Theorem 4.2 it suffices to find all TI defect blocks of groups X where $M \leq X/Z(X) \leq \mathrm{Aut}(M)$ for a non-abelian simple group M. Thus we have reduced to p'-central extensions of groups of automorphisms of simple groups. Since it is not always the case that the Schur multiplier of a simple group and that of its automorphism group are the same, we show here that it suffices to consider central extensions which are perfect when considered as an extension of just the simple group. In our case, all the non-abelian simple groups we consider have outer automorphism group which is a product of at most two cyclic groups. We deal with each cyclic group in turn.

Proposition 8.1 *Suppose that M is a non-abelian simple group and $M \leq X \leq \mathrm{Aut}(M)$. Suppose further that $\mathrm{Out}(M)$ is a product of at most two cyclic p'-groups. Let G be a central extension of X by a p'-group, and $B \in \mathrm{Blk}(G)$ a TI defect block. Then there is a central extension L of M by a p'-group and a finite group H with $L \leq H \leq \mathrm{Aut}(L)$ such that H possesses a TI defect block B_H Morita equivalent to B.*

Proof. Let $Z(G) \leq N \lhd T \lhd G$ such that $N/Z(G) = M$, and T/N and G/T are cyclic p'-groups. Let W be the central extension of $T/Z(G)$ by $T' \cap Z(G)$ with $W \leq T$. We repeat first the construction from [5, 3.6].

Choose $u \in G$ such that $uWZ(G) = uT$ generates the cyclic p'-group $G/WZ(G) = G/T$ of order m. Hence $u^m \in WZ(G)$, say $u^m = wz$ where $w \in W$, $z \in Z(G)$. Let V be a cyclic p'-group such that $Z(G) \leq V$, and containing an element $v \in V$ such that $v^m = z$. Write $A = G * V$, where we identify $Z(G)$ with $G \cap V$.

Setting $h = uv^{-1} \in A$, define $H = \langle W, h \rangle \lhd A$. Define $L = N'$. Following the argument in [8, 3.7], and using the fact that $T/Z(G)$ has Schur multiplier contained in that of M (see [8, 3.5]) we see that $Z(L) \leq H'$ and $C_H(L) = Z(L) = Z(H) = O_{p'}(H)$, so that H is indeed an automorphism group of the perfect p'-central extension L of M.

Now there are $Z_G, Z_H \leq Z(A)$ such that $A/Z_G \cong G$ and $A/Z_H \cong H$. By the constructions of section 3 there is a block B_A of A Morita equivalent to B, and a block B_H of H Morita equivalent to B_A. By Proposition 2.2 (vi) B_A and B_H have TI defect groups, and we are done. □

If X/M is cyclic, then by [8, 3.5] X has smaller Schur multiplier than M. Thus using this fact and Proposition 8.1, the result holds when M is a sporadic or alternating group by Corollary 5.2 and Proposition 6.1. Otherwise, if the defect group is not a Sylow p-subgroup, then the result holds by Proposition 7.4. If the defect group is a Sylow p-subgroup (and M is a group of Lie type), then the result holds by Proposition 7.4 and Proposition 8.1, but here we use the fact that G has TI Sylow p-subgroups if and only if $G/Z(G)$ does (given that $Z(G)$ is a p'-group), and that in each case $\mathrm{Out}(M)$ is either cyclic or a direct product of cyclic p'-groups.

Theorem 1.1 follows, noting that $A_6 \cong PSL_2(9)$ and $C_3 \times C_3$ is its Sylow subgroup.

Proof of Corollary 1.2. Let B be a block (with respect to k) with TI or normal defect group D of a finite group G. By Theorem 4.2 and its proof B is Morita equivalent to a block $\tilde{B}$ of a group $\tilde{G}$, where the following hold: (i) $\tilde{B}$ has a defect group $\tilde{D}$ which is isomorphic to D and is TI or normal; (ii) either $\tilde{D}$ is normal in $\tilde{G}$ or $Z(G)$ is a p'-group and $M \leq G/Z(G) \leq \mathrm{Aut}(M)$, where M is non-abelian simple (note that the Morita equivalences of Theorem 4.2 are established with respect to $\mathcal{O}$, but this implies that the corresponding k-blocks are Morita equivalent).

Suppose that $\tilde{D}$ is not normal in $\tilde{G}$. Then by Theorem 1.1 $\tilde{D}$ is either cyclic, Klein-four, quaternion of order eight, or $\tilde{B}$ is a block belonging to one (or more) of (d)–(g), (i)–(n) of Theorem 1.1. Donovan's conjecture is known to hold for cyclic and Klein-four defect groups (see [7], [20] and [11]). Now consider the case $\tilde{D} \cong Q_8$, which occurs in $Sp_{2m}(3)$. Note that here $N_{\tilde{G}}(\tilde{D}) \cong SL_2(3)$.

If $\tilde{D} \cong Q_8$, then by [26] $\tilde{B}$ possesses either one or three Brauer characters, and this number is determined within $N_{\tilde{G}}(\tilde{D})$, so that $l(\tilde{G}, \tilde{B}) = l(N_{\tilde{G}}(\tilde{D}), \tilde{B}) = l(SL_2(3)) = 3$. But Donovan's conjecture is known to hold for blocks with quaternion defect groups and three simple modules (see [10], in which a classification of algebras of quaternion type is given and it is shown that none of the infinite families in the list contains a block of a finite group) and we are done in this case.

Cases (d)-(g), (l) and (m) of Theorem 1.1 each contain only a finite number of (isomorphism classes of) groups. We must show that in each of the cases (i)-(k) and (n) there are only a finite number of Morita equivalence classes of blocks for each possible defect group. In each case, $\tilde{G}$ has a quasisimple normal subgroup H of p' index, and further the distinct H in each case have non-isomorphic TI Sylow p-subgroups $\tilde{D}$. Note that $\langle \tilde{D}^h : h \in H \rangle = H$. By Section 5 of [21], $\tilde{B}$ is Morita equivalent to a crossed product $Y = \bigoplus_{x \in X} Y_x$, where Y_1 is a basic subalgebra of a block b of H with defect group $\tilde{D}$ and X is a finite p'-group whose order divides $|\operatorname{Out}(\tilde{D})|^2$ (a full definition of a crossed product may be found in [21]). Up to isomorphism, there are only a finite number of possibilities for X and Y_1 and so only finitely many such crossed products (see the discussion in Section 5 of [21]), so the result follows in this case.

Suppose that $\tilde{D} \lhd \tilde{G}$. Then by [21] $\tilde{B}$ is Morita equivalent to a crossed product $Y = \bigoplus_{x \in X} Y_x$, where Y_1 is isomorphic to a basic subalgebra of $k\tilde{D}$ and X is a p'-group with order dividing $|\operatorname{Out}(\tilde{D})|^2$. Hence as above there are only finitely many such crossed products, and the result follows. □

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