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Symmetric Hamiltonian Bifurcations

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Based on lectures by James Montaldi

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1 Introduction

The purpose of these notes is to give a brief survey of bifurcation theory of Hamiltonian systems with symmetry; they are a slightly extended version of the five lectures given by JM on Hamiltonian Bifurcations with Symmetry. We focus our attention on bifurcation theory near equilibrium solutions and relative equilibria. The notes are composed of two parts. In the first, we review results on nonlinear normal modes in equivariant Hamiltonian systems, generic movement of eigenvalues in equivariant Hamiltonian matrices, one and two parameter bifurcation of equilibria and the Hamiltonian-Hopf Theorems with symmetry. The second part is about local dynamics near relative equilibria. Particular topics discussed are the existence, stability and persistence

of relative equilibria, bifurcations from zero momentum relative equilibria and examples.

We begin with some basic facts on Lie group actions on symplectic manifolds and Hamiltonian systems with symmetry. The reader should refer to Ratiu's lectures for more details and examples.

Semisymplectic actions A Lie group G acts *semisymplectically* on a symplectic manifold $(\mathcal{P}, \omega)$ if $g^*\omega = \pm\omega$. In this case the choice of sign determines a homomorphism $\chi : G \to \mathbb{Z}_2$ called the *temporal character*, such that $g^*\omega = \chi(g)\omega$. We denote the kernel of χ by G_+; it consists of those elements acting symplectically, and if G does contain antisymplectic elements then G_+ is a subgroup of G of index 2. Some details on semisymplectic actions can be found in [MR00].

Not every semisymplectic action contains an antisymplectic element of order 2, but if it does then we can write $G = G_+ \rtimes \mathbb{Z}_2(\rho)$, where ρ is the element in question.

We write $K < G$ to mean K is a closed subgroup of G. The *fixed point set* of a subgroup $K < G$ is

$$\mathrm{Fix}(K, \mathcal{P}) = \{x \in \mathcal{P} \mid g \cdot x = x,\ \forall g \in K\};$$

it is a closed submanifold of $\mathcal{P}$. If $K < G_+$ is compact then $\mathrm{Fix}(K, \mathcal{P})$ is a symplectic submanifold. That compactness is necessary can be seen from the simple example of $t \in \mathbb{R}$ acting on $\mathbb{R}^2$ by $\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$: the fixed point space is then just the x-axis.

Throughout these lectures, we assume that G acts properly on $\mathcal{P}$. Let $\mathfrak{g}$ be the Lie algebra of G and $G_x = \{g \in G \mid g \cdot x = x\}$ the *isotropy subgroup* of $x \in \mathcal{P}$. The properness assumption implies in particular that G_x is compact.

To each element $\xi \in \mathfrak{g}$ there is an associated vector field on $\mathcal{P}$:

$$\xi_{\mathcal{P}}(x) = \frac{d}{dt} \exp(t\xi) \cdot x|_{t=0}$$

The tangent space at x to the group orbit through x is $\mathfrak{g} \cdot x = \{\xi_{\mathcal{P}}(x) \mid \xi \in \mathfrak{g}\}$.

The *adjoint action* of g on $\mathfrak{g}$ denoted $\xi \mapsto \mathrm{Ad}_g\, \xi$, is the tangent map of $I_g : G \to G$, $h \mapsto ghg^{-1}$ at e, $T_e I_g(\xi)$. In the case of matrix groups, this is just

$$\mathrm{Ad}_g\, \xi = g\xi g^{-1}.$$

Finally, dual to the adjoint action on $\mathfrak{g}$ is the coadjoint action on $\mathfrak{g}^*$:

$$\langle \mathrm{Coad}_g\, \mu, \eta \rangle := \left\langle \mu, \mathrm{Ad}_{g^{-1}}\, \eta \right\rangle.$$

In the case of matrix groups, if we identify $\mathfrak{g}^*$ with matrices via $\langle \mu, \xi \rangle = \operatorname{tr}(\mu^T \xi)$, then the coadjoint action becomes

$$\operatorname{Coad}_g \mu = g^{-T} \mu g^T.$$

where $g^{-T} = (g^{-1})^T = (g^T)^{-1}$. For compact groups and for semisimple groups, the adjoint and coadjoint actions are isomorphic, but in general they can be quite different—this is already the case for the 3-dimensional Euclidean group $\mathbf{SE}(2)$. See Section 5 (in Part II) for how the momentum map relates to a semisymplectic action.

Hamiltonian formalism A Hamiltonian system with symmetry is a quadruple $(\mathcal{P}, \omega, G, H)$ where:

- $(\mathcal{P}, \omega)$ is a symplectic manifold,
- G is Lie group acting smoothly and semisymplectically on $\mathcal{P}$,
- $H : \mathcal{P} \to \mathbb{R}$ is a G-invariant smooth function.

The Hamiltonian vector field X_H is defined implicitly by $\omega(-, X_H) = dH$ and of course defines a dynamical system on $\mathcal{P}$ by

$$\dot{x} = X_H(x). \tag{1.1}$$

When working in the neighbourhood of a point $x \in \mathcal{P}$, the equivariant Darboux theorem states that there exists a coordinate system such that the symplectic form is locally constant. Therefore, without loss of generality we can reduce the Hamiltonian to a vector space by identifying a neighbourhood of x in $\mathcal{P}$ with $V = T_x\mathcal{P}$ and the G action on $\mathcal{P}$ with the G action on $T_x\mathcal{P}$. Note however that for symmetric Hamiltonian systems, Montaldi *et al.* [MRS88] and Dellnitz and Melbourne [DM92] show that symplectic forms are not always locally isomorphic if the isotypic decomposition of the space contains irreducible representations of complex type.

If G is formed of symplectic elements, then X_H is G-equivariant; that is, if $x(t)$ is a solution curve of X_H then so is $g \cdot x(t)$ for all $g \in K$.

On the other hand, suppose that $\rho \in G$ is an *antisymplectic* symmetry, that is $\rho^*\omega = -\omega$ or $\omega(\rho u, \rho v) = -\omega(u, v)$, then it is time-reversing; that is, $x(t)$ is an integral curve of the X_H vector field implies $\rho \cdot x(-t)$ is also an integral curve of the vector field.

If K is compact and formed of symplectic symmetries, then $\operatorname{Fix}(K, \mathcal{P})$ is invariant under the flow of the dynamical system. If in addition K is compact, then $\operatorname{Fix}(K, \mathcal{P})$ is a Hamiltonian subsystem with Hamiltonian given by the restriction of H to $\operatorname{Fix}(K, \mathcal{P})$.

The remainder of these notes is structured as follows. There are two main

parts. The first covers local dynamics near equilibria and the second local dynamics near relative equilibria.

In Section 2, we begin with the local dynamics near equilibria when the Hamiltonian has a nondegenerate quadratic part, and present the equivariant Weinstein-Moser Theorem on the existence of nonlinear normal modes in equivariant Hamiltonian systems. In Section 3 we look at bifurcations near equilibria. We start with a brief review of generic movement of eigenvalues in equivariant Hamiltonian matrices depending on parameters. Then we look at steady-state bifurcations in parameter families of Hamiltonian systems, and in particular the one-parameter case. We conclude this part with the Hamiltonian-Hopf Theorem with symmetry.

The second part deals with bifurcations from relative equilibria. In Section 5 the momentum map is defined and its equivariance is shown. Using this information, it is shown how to define reduced spaces for the dynamics using the momentum map. Then in Section 6, relative equilibria are defined and we explain how to find relative equilibria, and determine their stability and their persistence. Section 7 discusses bifurcations from zero-momentum states and in Section 8 three examples of bifurcations from zero-momentum are presented: relative equilibria of molecules, relative equilibria in point vortex models in the plane, and relative equilibria in point vortex models on the sphere.

PART I: LOCAL DYNAMICS NEAR EQUILIBRIA

2 Nonlinear normal modes

Suppose that the Hamiltonian system (1.1) has a steady-state (equilibrium) solution at some $x_0 \in \mathcal{P}$. Such solutions are critical points of H. The linearized vector field at x_0 is

$$\dot{v} = L_{x_0} v.$$

The matrix L_{x_0} is *Hamiltonian*; a matrix A is Hamiltonian if $\omega(Av, w) + \omega(v, Aw) = 0$. The set of Hamiltonian matrices on $\mathbb{R}^{2n}$ is denoted $\mathfrak{sp}(2n)$. The set $\mathfrak{sp}_G(2n) \subset \mathfrak{sp}(2n)$ is the subspace of matrices that commute with G. The eigenvalues of Hamiltonian matrices arise in quadruplets $\{\lambda, -\lambda, \overline{\lambda} - \overline{\lambda}\}$, see Lemma 4.1 of Meyer's lectures in this volume or Meyer and Hall [MH92].

Suppose that $\mathrm{Re}(\lambda) \neq 0$ for all eigenvalues λ of L_{x_0} then by the Hartman-Grobman theorem the vector field (1.1) is homeomorphic to its linear part $\dot{v} = L_{x_0} v$ in a neighbourhood of x_0; x_0 is a hyperbolic saddle point. In generic (non-hamiltonian) systems, this is usually enough to describe the local dynamics, since the eigenvalues do not (generically) lie on the imaginary

axis. However, for Hamiltonian systems this is no longer true: having pure imaginary eigenvalues is a structurally stable property.

A linear Hamiltonian system with a simple nonresonant imaginary eigenvalue has a family of periodic solutions of constant period in the eigenspace of the imaginary eigenvalue. These families of periodic solutions are called *normal modes*. In nonlinear Hamiltonian systems, the search for families of periodic solutions near a steady-state or *nonlinear normal modes* has attracted a lot of interest since the seminal work of Lyapunov [L]. The Lyapunov Centre Theorem, see Meyer's lectures or [AM78] states that for each simple nonresonant eigenvalue there exists a nonlinear normal modes. A normal mode is a family of periodic orbits in a linear system, of constant period, and sweeping out the eigenspace corresponding to an imaginary eigenvalue; a nonlinear normal mode is a family of periodic orbits parametrized by energy containing a steady-state solution and tangent to the eigenspace of the imaginary eigenvalue, with period close to that of the linear system. There have been many particular extensions of this theorem, but the most general results are due to Weinstein [W73] and Moser [Mos76] who allow for multiple eigenvalues and resonance relations. Montaldi *et al* [MRS88] extend the results of Weinstein and Moser to take account of symmetry.

As we have already noted, if a compact subgroup K of G acts symplectically, then $\mathrm{Fix}(K, \mathcal{P})$ is a sub-hamiltonian system and so Lyapunov's theorem can be applied to this subsystem. The resulting periodic orbits are said to have *spatial* symmetry: the solution $\gamma(t)$ satisfies $g \cdot \gamma(t) = \gamma(t)$ for each t, and for each $g \in K$. However, using *spatio-temporal* symmetries one can go further, and we now describe this idea.

Let $v(t)$ be a 2π-periodic solution of the G-invariant Hamiltonian system (1.1) then, $g.v(t)$ is also a periodic solution of (1.1) for all $g \in G$. By uniqueness of solutions of differential equations, either $\{v(t)\} \cap \{g.v(t)\} = \{v(t)\}$ or $\{v(t)\} \cap \{g.v(t)\} = \emptyset$. In the former case, $g.v(t) = v(t - \theta)$ for some phase shift θ. We identify phase shifts with elements of the circle group $\mathbf{S}^1$ using the identification $\mathbf{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$. The set

$$\Sigma_{v(t)} = \{(g, \theta) \in G \times \mathbf{S}^1 \mid g.v(t) = v(t - \theta)\} < G \times \mathbf{S}^1$$

is the (spatio-temporal) symmetry group of $v(t)$. Therefore, when searching for periodic solutions with spatio-temporal symmetries of equivariant dynamical systems we look for isotropy subgroups $\Sigma \subset G \times \mathbf{S}^1$.

We now describe the equivariant result of Montaldi, Roberts and Stewart [MRS88].

Suppose that the linearized equation $\dot{v} = L_{x_0} v$ has eigenvalue $i\alpha$ and all imaginary eigenvalues $ri\alpha$ ($r \in \mathbb{Q}$) have larger modulus. Let V_α be the set

of all points that lie on $2\pi/\alpha$-periodic trajectories of the linearized equation. Then $V_{ri\alpha} \subset V_\alpha$ and V_α is called the *resonance subspace* of α. Let L_α be the restriction of L_{x_0} to V_α. Elphick *et al.* [ETBCI] show that if L_α is semisimple then $\overline{\{\exp(tL_\alpha^T)|t \in \mathbb{R}\}}$ is isomorphic to $\mathbf{S}^1$, giving rise to an action of $G \times \mathbf{S}^1$ on V_α.

Theorem 2.1 (Equivariant Weinstein-Moser Theorem) *Let $\Sigma \subset G \times \mathbf{S}^1$. Suppose that the restriction $d^2H(0)$ to $\mathrm{Fix}(\Sigma, V_\alpha)$ is definite, then on each energy level near the origin there are at least $\frac{1}{2}\dim \mathrm{Fix}(\Sigma, V_\alpha)$ periodic orbits, with period close to $2\pi/\alpha$ and symmetry at least Σ.*

Theorem 2.1 gives sufficient conditions for the existence of families of periodic solutions with spatio-temporal symmetries for H. Montaldi *et al.* [MRS90] prove a stronger theorem about the existence or nonexistence of all periodic solutions with spatio-temporal symmetries near a steady-state with imaginary eigenvalues. Moreover, their result includes the case when the Hamiltonian system also has time-reversal symmetries.

Let $\rho : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a time-reversing symmetry. Define

$$\tilde{\mathbf{S}}^1 = \begin{cases} \mathbf{S}^1 & \text{if } H \text{ is not time-reversible} \\ \mathbf{S}^1 \rtimes \mathbb{Z}_2^\rho \simeq \mathbf{O}(2) & \text{if } H \text{ is time-reversible} \end{cases}$$

The proof of existence of nonlinear normal modes uses a variational approach. Let $u(s) \in \mathbb{C}^1(\mathbf{S}^1, \mathbb{R}^{2n})$ be a loop in phase space of period 2π. For each real number $\alpha \neq 0$ we define a functional $S_\alpha : \mathcal{C}^1(\mathbf{S}^1, \mathbb{R}^{2n}) \times \mathbb{R} \to \mathbb{R}$ by

$$S_\alpha(u) = S(u, \alpha) = \oint u^*\beta - \alpha \int_0^{2\pi} H(u(s))ds \tag{2.1}$$

where u^* denotes the pull-back of β to $\mathbf{S}^1$ and β is a primitive of the symplectic form; $\omega = d\beta$. The functional S_α can be chosen to be $G \rtimes \tilde{\mathbf{S}}^1$-invariant.

Lemma 2.2 *$u \in \mathcal{C}^1(\mathbf{S}^1, \mathbb{R}^{2n})$ is a critical point of S_α if and only if $z(t) \equiv u(\alpha t)$ is a periodic solution of period $2\pi/\alpha$ of the Hamiltonian system defined by H.*

Theorem 2.3 *For fixed α and for sufficiently small τ the critical points of $S_{\alpha(1+\tau)}$ in a neighbourhood U of $0 \in \mathcal{C}^1(\mathbf{S}^1, \mathbb{R}^{2n})$ are in one-to-one correspondence with the critical points of a smooth finite dimensional $G \rtimes \mathbf{S}^1$-invariant mapping $F_\tau : V_\alpha \to \mathbb{R}$. Moreover this correspondence preserves symmetry groups.*

An interesting observation concerns the class of classical mechanical systems where q is position and p is momentum and the Hamiltonian is $K_q(p) + V(q)$, where $K_q(p)$—the kinetic energy—is quadratic in p. The time-reversal symmetry is $\rho.(p,q) = (-p,q)$. If a periodic solution with spatio-temporal symmetry group Σ contains a conjugate of $\mathbb{Z}_2^\rho$ then it intersects $\mathrm{Fix}(\mathbb{Z}_2^\rho) = \{(0,q)\}$ in two points. Since the velocity vanishes on $\mathrm{Fix}(\mathbb{Z}_2^\rho)$, in time-reversible systems these periodic solutions are called *brake orbits*.

Theorem 2.3 states that the search for nonlinear normal modes reduces to finding critical points of F_τ. To find the critical points it is necessary to have a convenient expression for F_τ. Such an expressions is obtained by putting the Hamiltonian function into $G \rtimes \tilde{\mathbf{S}}$-invariant Birkhoff normal form to sufficiently high order where

$$\tilde{\mathbf{S}} = \begin{cases} \overline{\{\exp(tL^T_{x_0}) : t \in \mathbb{R}\}} & \text{if } H \text{ is not time-reversible} \\ \\ \overline{\{\exp(tL^T_{x_0}) : t \in \mathbb{R}\}} \rtimes \mathbb{Z}_2^\rho & \text{if } H \text{ is time-reversible.} \end{cases}$$

A discussion of Birkhoff normal form can be found in Section 7 of Meyer's lectures in this volume; for a complete description see Elphick *et al.* [ETBCI] or Golubitsky *et al.* [GSS88], or Cushman and Sanders [CS86] for a different approach. The next result gives the expression for F_τ in terms of the Hamiltonian function in Birkhoff normal form.

Theorem 2.4 *If H is in Birkhoff normal form to degree k then*

$$\frac{1}{2\pi} j^k F_\tau(v) = (1+\tau) j^2 H(v) - j^k H(v)$$

for $v \in V_\alpha$ where j^k is the k-jet.

However, we do not know a priori to what order the truncation of the Birkhoff normal form that defines F_τ yields all possible nonlinear normal modes for the full system. Montaldi *et al.* [MRS90] obtain further results using singularity theory to answer the question of generic finite determinacy of the F_τ equation. We do not discuss these results here and refer the reader to the paper.

Here we present briefly one example where the equivariant Weinstein-Moser theorem gives only some nonlinear normal modes while the study of the F_τ equation yields all solutions: the time-reversible 1:1 resonance with $\mathbb{Z}_2$ symmetry.

Example 2.5 Consider Hamiltonian systems H on $\mathbb{C}^2$ where κ is the symmetry and ρ the time-reversible symmetry acting as

$$\kappa.(z_1,z_2) = (z_2,z_1) \qquad \text{and} \qquad \rho.(z_1,z_2) = (-\overline{z_1}, -\overline{z_2}).$$

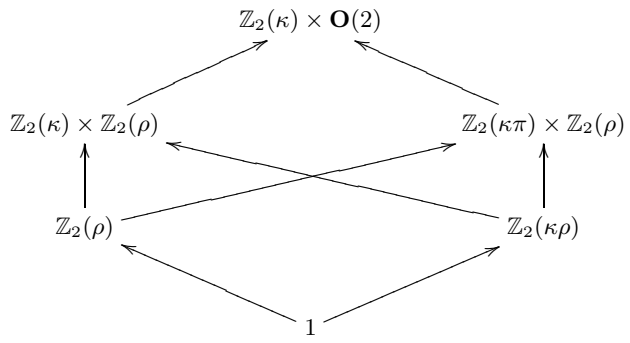

Fig. 2.1. Isotropy lattice of the $\mathbb{Z}_2(\kappa) \times \mathbf{O}(2)$ action on $\mathbb{C}^2$.

At a $1:1$ resonance the linearization $L_0 = (dX_H)_0$ has double eigenvalue $\pm i$ with positive definite quadratic part H_2. We shall see in Section 3 the $1:1$ resonance in more details. Since L_0 commutes with $\mathbb{Z}_2(\kappa)$, L_0 is semisimple and generates the $\mathbf{S}^1$ action

$$\theta.(z_1, z_2) = (e^{i\theta} z_1, e^{i\theta} z_2).$$

These actions combine to give the action of the group $\mathbb{Z}_2(\kappa) \times \mathbf{O}(2)$.

The isotropy subgroups for the $\mathbb{Z}_2(\kappa) \times \mathbf{O}(2)$ action are given in Figure 2.1. It is easy to check that

$$\begin{aligned} \mathrm{Fix}(\mathbb{Z}_2(\kappa) \times \mathbb{Z}_2(\rho)) &= \{z_1 = z_2 \in i\mathbb{R}\}, \text{ and} \\ \mathrm{Fix}(\mathbb{Z}_2(\kappa\pi) \times \mathbb{Z}_2(\rho)) &= \{z_1 = -z_2 \in i\mathbb{R}\} \end{aligned}$$

therefore in the complexification these are two-dimensional. From the equivariant Weinstein-Moser theorem, since H_2 is definite we know immediately that there are nonlinear normal modes with symmetry corresponding to the maximal isotropy subgroups $\mathbb{Z}_2(\kappa) \times \mathbb{Z}_2(\rho)$ and $\mathbb{Z}_2(\kappa\pi) \times \mathbb{Z}_2(\rho)$. Now, $\mathrm{Fix}(\mathbb{Z}_2(\kappa\rho)) = \{z_1, z_2 \in i\mathbb{R}\}$ and $\mathrm{Fix}(\mathbb{Z}_2(\rho)) = \{z_1 = -\overline{z_2}\}$ are four-dimensional, Theorem 2.1 guarantees the existence of two nonlinear normal modes with isotropy containing $\mathbb{Z}_2(\kappa\rho)$ and $\mathbb{Z}_2(\rho)$. However, these may be the solutions with maximal isotropy found above. Therefore, Theorem 2.1 cannot guarantee the existence of solutions with submaximal symmetry in this case.

We look at the F_τ equation where the system is in Birkhoff normal form to degree 4, $H = H_2 + H_4$ where

$$H_2 = N, \qquad H_4 = \alpha_1 N^2 + \alpha_2 P + \beta_1 Q^2 + \beta_2 NQ,$$

$N = |z_1|^2 + |z_2|^2$, $P = |z_1|^2 |z_2|^2$ and $Q = \mathrm{Re}(z_1 \overline{z_2})$. The critical points of

F_τ are given by

$$\begin{aligned}\phi_1(z_1,z_2,\tau) &= i[(-\tau+2\alpha_1 N+\alpha_2|z_2|^2+\beta_2 Q)z_1+\left(\tfrac{1}{2}\beta_2 N+\beta_1 Q\right)z_2\\ \phi_2(z_1,z_2,\tau) &= i[(-\tau+2\alpha_1 N+\alpha_2|z_1|^2+\beta_2 Q)z_2+\left(\tfrac{1}{2}\beta_2 N+\beta_1 Q\right)z_1.\end{aligned} \tag{2.2}$$

We solve for solutions in $\mathrm{Fix}(\kappa\rho)$. Set $z_1 = ix$ and $z_2 = iy$ then (2.2) becomes

$$\begin{aligned}(-\tau+2\alpha_1(x^2+y^2)+\alpha_2 y^2+\beta_2 xy)x+\left(\tfrac{1}{2}\beta_2(x^2+y^2)+\beta_1 xy\right)y &= 0\\ (-\tau+2\alpha_1(x^2+y^2)+\alpha_2 x^2+\beta_2 xy)y+\left(\tfrac{1}{2}\beta_2(x^2+y^2)+\beta_1 xy\right)x &= 0.\end{aligned}$$

After simplification we see that nonlinear normal modes with $x \neq 0$ and $y \neq 0$ are solutions found by setting $\tau = 2\alpha_1(x^2+y^2)+\beta_2 xy$ and solving

$$\frac{1}{2}\beta_2(x^2+y^2)+(\alpha_2+\beta_1)xy = 0. \tag{2.3}$$

We can solve (2.3) if $|\beta_2| < |\alpha_2+\beta_1|$. Thus, for an open set of values of the coefficients $(\alpha_1,\alpha_2,\beta_1,\beta_2)$ nonlinear normal modes with isotropy exactly $\mathbb{Z}_2(\kappa\rho)$ exist in Birkhoff normal form. The same is true for nonlinear normal modes with isotropy exactly $\mathbb{Z}_2(\rho)$. In [MRS90] it is shown that for an open and dense set of values of the coefficients, the truncation of the Birkhoff normal form to degree four $H = H_2 + H_4$, is sufficient to determine all nonlinear normal modes in the full system.

Remark 2.6 Using these methods, one can show that in a 2 degree of freedom system in $1:1$ resonance, with no assumptions on the symmetry, generically there are 2, 4 or 6 pairwise transverse nonlinear normal modes, depending on the coefficients in H_4. Each of the three possibilities is obtained for an open set in the space of coefficients of H_4.

An application of these ideas to a symmetry breaking problem can be found in [Mo99], where a description is given of the nonlinear normal modes obtained after adding a magnetic term to the spherical pendulum (which breaks the reflexional symmetry).

3 Generic bifurcations near equilibria

The quadratic form $d^2H(x_0)$ is degenerate if and only if L_{x_0} has a zero eigenvalue. In this case, there exist arbitrarily small perturbations of H which remove the zero eigenvalues of L_{x_0}. However, in families of Hamiltonian matrices zero eigenvalues are inevitable; that is, they occur stably. In this section, we look at such families of Hamiltonian systems.

3.1 Generic movement of eigenvalues

In parametrized families of Hamiltonian matrices, it is typical for eigenvalues to cross the imaginary axis at the origin or for some eigenvalues to enter in resonance. In this section, we study the generic movement of eigenvalues for one-parameter families of matrices in $\mathfrak{sp}_G(2n)$. We look at the case of zero eigenvalues and the case of $1:1$ resonance. First, we need to introduce some concepts from group representation theory.

Suppose that the compact group G acts linearly on the vector space V. A subspace $W \subset V$ is *G-invariant* if $G(W) = W$. Moreover, if W does not contain any proper G-invariant subspaces, then W is an *irreducible representation* of G.

Let V be an irreducible representation of G. If the only linear mappings on V that commute with G are scalar multiples of the identity, then the representation V is *absolutely irreducible*. More precisely, if V is an irreducible representation and $\mathcal{D}$ is the space of linear mappings from V to itself commuting with G then $\mathcal{D}$ is a division algebra isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, where $\mathbb{H}$ is the group of quaternions. Thus, if $\mathcal{D}$ is isomorphic to $\mathbb{C}$ or $\mathbb{H}$ then V is nonabsolutely irreducible.

If V is a symplectic representation, a subspace $W \subset V$ is *G-symplectic* if it is G-invariant and symplectic, and a symplectic representation is *irreducible* if it contains no proper G-symplectic subspace. Irreducible symplectic representations arise in several types depending on the underlying ordinary representation (i.e. forgetting the symplectic structure): firstly if the underlying representation is not irreducible, then $V = V_0 \oplus V_0$ where V_0 is an absolutely irreducible subspace (which is Lagrangian and one can identify $V = T^*V_0$). Secondly, if the underlying representation is irreducible, it must be either complex or quaternionic, and for a given representation of complex type there are two distinct symplectic representations, which are said to be *dual*. Any two symplectic representations whose underlying representations are equivalent quaternionic are equivalent as symplectic representations. See [MRS88] for more details.

Zero eigenvalues

Proposition 3.1 *Suppose that $L \in \mathfrak{sp}_G(2n)$ has a nonzero kernel. Then E_0, the generalized eigenspace of 0, is a G-symplectic subspace of $\mathbb{R}^{2n}$.*

The structure of the generalized eigenspace and corresponding movement of eigenvalues is described in the next result.

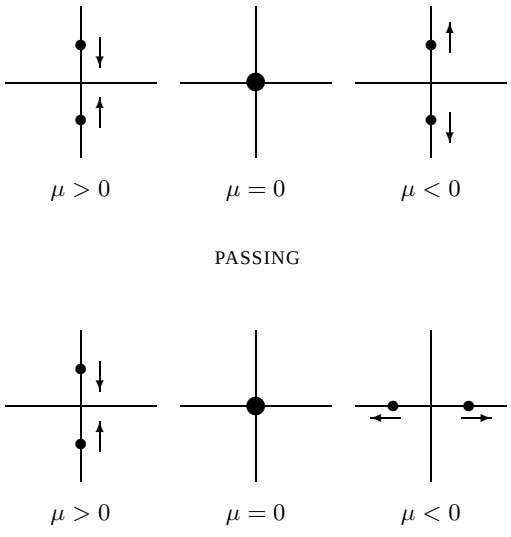

Fig. 3.1. The two scenarios for the generic movement of eigenvalues in steady-state bifurcation: see Theorem 3.2.

Theorem 3.2 (Golubitsky and Stewart [GS86]) *Let L_μ be a generic one-parameter family in $\mathfrak{sp}_G(2n)$ such that 0 is an eigenvalue of L_0. Then either*

(i) *the action of G on E_0 is nonabsolutely irreducible (in which case $L_0|_{E_0} = 0$), and the eigenvalues of L_μ lie on the imaginary axis and cross through 0 with nonzero speed ('passing' in Figure 3.1), or*

(ii) *$E_0 = V \oplus V$, where V is absolutely irreducible (in which case $L_0|_{E_0} \neq 0$ but $L_0^2|_{E_0} = 0$) and the eigenvalues cross through 0 going from purely imaginary to real or vice-versa ('splitting' in Figure 3.1).*

1:1 resonance

Suppose now that L is part of a one-parameter family and has a pair of purely imaginary eigenvalues $\pm i\eta$. By rescaling we can always assume that $\eta = 1$. The generalized eigenspace is denoted $E_{\pm i}$; it is a G-symplectic subspace. We consider two cases, either the eigenvalue i is G-simple ($E_{\pm i}$ is symplectic irreducible) or it is of G-multiplicity 2 also called a $1:1$ *resonance*. It is customary in Hamiltonian systems to distinguish two classes of $1:1$ resonance: the $1:1$ and the $1:-1$ resonances, depending on whether the Hamiltonian is

definite or indefinite, respectively. The structure of the generalized eigenspace in the $1:\pm 1$ resonance is given in the following theorem.

Theorem 3.3 (Dellnitz *et al.* [DMM92], van der Meer [vdM90]) *Let L_μ be a generic one-parameter family in $\mathfrak{sp}_G(2n)$ such that L_0 has eigenvalues $\pm i$ with G-multiplicity 2. Then $E_{\pm i} = U_1 \oplus U_2$ where for $j = 1, 2$, either*

(i) *U_j is nonabsolutely irreducible; or*
(ii) *$U_j = V \oplus V$, with V absolutely irreducible.*

Understanding how eigenvalues may move as a system passes through a $1:\pm 1$ resonance requires a combination of group-theoretic results along with the analysis of the Hamiltonian quadratic form defined on the generalized eigenspace $E_{\pm i}$. Since $E_{\pm i}$ is a symplectic subspace of $\mathbb{R}^{2n}$, the restriction of the symplectic form ω to $E_{\pm i}$ is nondegenerate and thus $\omega_i = \omega|_{E_{\pm i}}$ is a symplectic form on $E_{\pm i}$. The Hamiltonian $Q(z) = \omega(z, Lz)$ is therefore a non-degenerate quadratic form on $E_{\pm i}$.

Recall that there are precisely two isomorphism classes of irreducible symplectic representations for a given complex underlying representation; these representations are dual to each other [MRS88].

Theorem 3.4 (Dellnitz *et al.* [DMM92]) *With the same hypotheses as in the theorem above, and with Q the Hamiltonian quadratic form induced on $E_{\pm i}$, precisely one of the following occurs:*

(i) *U_1 and U_2 are not isomorphic and the eigenvalues pass independently along the imaginary axis; Q may be indefinite or definite.*
(ii) *$U_1 = U_2 = V \oplus V$, V real, or $U_1 = U_2 = W$, W quaternionic, the eigenvalues split, and Q is indefinite.*
(iii) *U_1 and U_2 are complex of the same type, the eigenvalues pass and Q is indefinite.*
(iv) *U_1 and U_2 are complex duals and the eigenvalues pass or split depending on whether Q is definite or indefinite.*

The two splitting cases (ii) and (iv) of the theorem correspond to the Hamiltonian-Hopf bifurcation which is the Hamiltonian version of the Hopf bifurcation theorem for dissipative systems, see Section 4 below.

In the nonsymmetric case $G = 1$, $E_{\pm i} = V \oplus V \oplus V \oplus V$ where V is the one-dimensional trivial representation. The one-parameter unfolding of the

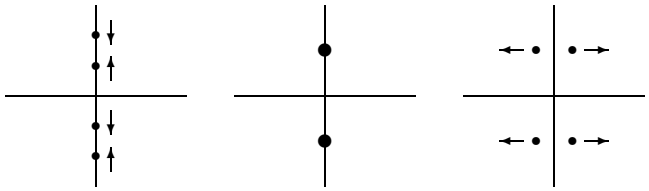

Fig. 3.2. Generic movement of eigenvalues in the $1 : -1$ resonance: the splitting case.

normal form for the $1 : -1$ resonance is given by

$$M(\mu) = \begin{bmatrix} 0 & -1 & \rho & 0 \\ 1 & 0 & 0 & \rho \\ \mu & 0 & 0 & -1 \\ 0 & \mu & 1 & 0 \end{bmatrix}. \tag{3.1}$$

where $\rho = \pm 1$. This normal form is used in the Hamiltonian-Hopf theorem without symmetry, Section 4. Note that without symmetry the $1 : 1$ resonance is of codimension 3 (and codimension 2 in time-reversible systems) and so is not usually considered.

Further information can be found in [MD93] and [Me93], where Melbourne and Dellnitz extend to symmetric systems both Williamson's results on normal forms for linear Hamiltonian systems and Galin's results on their versal deformations. For example in [Me93] one finds that the normal form for case (ii) of the theorem above is also given by (3.1), where each scalar is interpreted as scalar multiplication in V.

3.2 Bifurcation of equilibria

In multiparameter families of Hamiltonian systems, the eigenvalues of the linearization typically cross the imaginary axis leading to bifurcations of equilibria or periodic solutions.

In this section, we look at bifurcations of equilibria. We begin with bifurcations of equilibria in one-parameter families of one degree of freedom Hamiltonian systems. Then, we explain how this information is used in bifurcation diagrams for multiparameter families of one-degree of freedom Hamiltonian systems. We conclude with some comments on bifurcations in many-degrees of freedom Hamiltonians. An important tool for the study of Hamiltonian systems is the Splitting Lemma (or Morse Lemma with parameters) [BG92], which separates out a nondegenerate part of the function from the remainder.

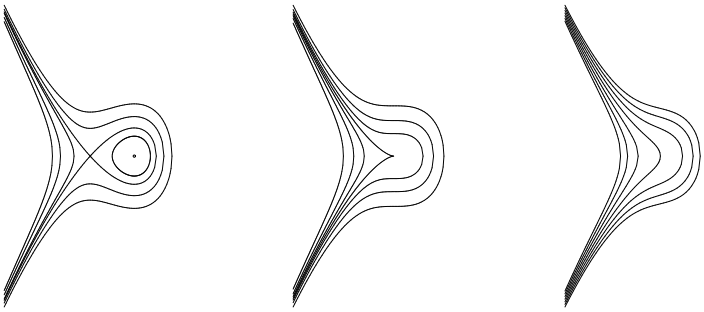

Fig. 3.3. Saddle-centre bifurcation with $G = 1$.

Theorem 3.5 (Splitting Lemma) *Let $F : \mathbb{R}^N \times \mathbb{R}^l \to \mathbb{R}$ be a smooth function. Denote a point in $\mathbb{R}^N \times \mathbb{R}^l$ by $(x, \lambda) = (x_1, \ldots, x_N, \lambda_1, \ldots, \lambda_l)$, and suppose that $d_x F(0,0) = 0$ and that the Hessian matrix $d_x^2 F(0,0)$ is non-degenerate. Then in a neighbourhood of the origin, there is a change of coordinates of the form $\Psi(x, \lambda) = (\psi(x, \lambda), \lambda)$ with $\psi(0,0) = 0$, such that*

$$F \circ \Psi(x, \lambda) = \sum_j \varepsilon_j x_j^2 + h(\lambda),$$

where $\varepsilon_j = \pm 1$ and h is a smooth function with $h(0) = F(0,0)$.

In practice $h(\lambda)$ can be found by solving the equation $d_x F(x, \lambda) = 0$ for $x = c(\lambda)$ (by the implicit function theorem), and then $h(\lambda) = F(c(\lambda), \lambda)$. (Of course, the change of coordinates cannot in general be taken to be symplectic!)

3.2.1 One degree of freedom: one-parameter family

Let $H : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ be a smooth generic one degree of freedom Hamiltonian function depending on a single parameter μ; such a family will have just codimension-1 bifurcations. As always in bifurcation theory, the meaning of "codimension-1" depends on the context, for example on the symmetry. We illustrate this with three simple examples: $G = 1$, $G = \mathbb{Z}_2$ and $G = \mathbf{SO}(2)$. Let L_μ be the linearization of $J\nabla H$ and suppose that L_0 has a zero eigenvalue. This corresponds to Theorem 3.2(ii). We discuss the bifurcation diagram for several group actions.

$G = 1$: Saddle-centre bifurcation. The trivial group has only a one-dimensional irreducible representation which is of course absolutely irreducible.

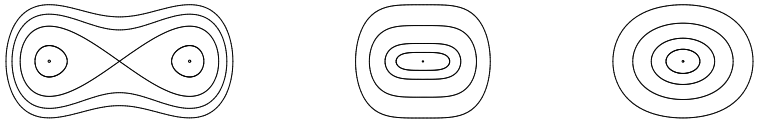

Fig. 3.4. Supercritical $\mathbb{Z}_2$-pitchfork bifurcation.

Therefore, generically $E_0 = \mathbb{R} \times \mathbb{R}$ and L_0 is nilpotent. Suppose that

$$L_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \tag{3.2}$$

then $H(p, q, 0) = q^2 + \tilde{H}$ where $\tilde{H}$ contains terms of degree three and up. By the Splitting Lemma above, $H(p, q, \mu)$ is equivalent to $f(p, \mu)+q^2$. The map f has a fold catastrophe at $\mu = 0$, see Poston and Stewart [PS78]. The universal unfolding is $p^3 + \mu p$. The local dynamics for $p^3 + \mu p + q^2$ near $(0, 0, 0)$ is illustrated by Figure 3.3. This is called the *saddle-centre* bifurcation.

$G = \mathbb{Z}_2$: Pitchfork bifurcation. The group $\mathbb{Z}_2$ has one nontrivial irreducible representation which is of dimension one. Therefore it is absolutely irreducible and $E_0 = \mathbb{R} \oplus \mathbb{R}$ where $\mathbb{Z}_2$ acts by -1 on each copy of $\mathbb{R}$. Thus, H is $\mathbb{Z}_2$-invariant: $H(-p, -q, \mu) = H(p, q, \mu)$. Let L_0 be given by (3.2) then $H(p, q, 0) = q^2+\tilde{H}$ where $\tilde{H}$ contains terms of degree higher than two. Using the Splitting Lemma above, H is equivalent to $f(p, \mu) + q^2$ as in the $G = 1$ case. However, the universal unfolding must be $\mathbb{Z}_2$-invariant, thus the family of maps is $\pm p^4 + \mu p^2 + q^2$. The family $p^4 + \mu p^2 + q^2$ is a supercritical pitchfork bifurcation where a pair of stable centres for $\mu < 0$ coalesce at $\mu = 0$ into one centre at the origin, see Figure 3.2.2. The $-p^4 + \mu p^2 + q^2$ case is a subcritical pitchfork bifurcation, see Figure 3.5, where the bifurcating branch of equilibria are unstable.

Generally, a bifurcation is *subcritical* if at the instant of bifurcation (here $\mu = 0$) the equilibrium is unstable, and *supercritical* if it is stable.

$G = SO(2)$: The only nontrivial irreducible representation of $\mathbf{SO}(2)$ is two-dimensional of complex type. The action on $\mathbb{C}$ is $\theta.z = e^{mi\theta}z$ for $\theta \in \mathbf{SO}(2)$ and some $m \in \mathbb{Z}$. By Theorem 3.2, $E_0 = \mathbb{C} = \mathbb{R}^2$, the eigenvalues are purely imaginary and cross 0 with nonzero speed. The general $\mathbf{SO}(2)$-invariant function of two variables is $H(p, q) = f(p^2+q^2)$ for some smooth function f. Since L_0 is identically zero at $\mu = 0$, then $H(p, q) = \mu(p^2+q^2)+o(p^2+q^2)$.

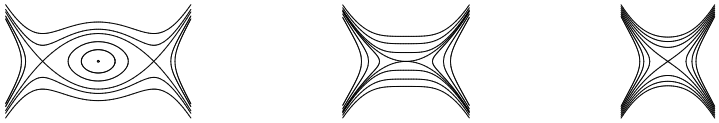

Fig. 3.5. Subcritical $\mathbb{Z}_2$-pitchfork bifurcation.

In polar coordinates, $r^2 = p^2 + q^2$, we obtain $f(r) = \mu r^2 + o(r^2)$ and so the unfolding of the generic singularity is given (up to sign) by $\mu r^2 + r^4$. For $\mu < 0$ the origin is surrounded by periodic solutions. At $\mu = 0$ the bifurcation occurs and an $\mathbf{SO}(2)$-orbit of equilibria of amplitude $\sqrt{\mu}$ for $\mu > 0$ is created.

3.2.2 One degree of freedom: two-parameter family

$G = 1$: the pitchfork revisited. In $\mathbb{Z}_2$-symmetric systems the pitchfork bifurcation is of codimension 1, while if there is no symmetry present then it is of codimension 2, as we see now. Consider a generic two-parameter family $H(p, q, \mu_1, \mu_2)$ of Hamiltonian functions. One can[1] take $H(p, q, 0, 0) = q^2 + \tilde{H}$ where $\tilde{H}$ has terms of degree three and up. By the Splitting Lemma H is equivalent to $f(p, \mu_1, \mu_2) + q^2$. To be of codimension 2, we use $f_\pm(p, 0, 0) = \pm p^4$, and then a two-parameter unfolding is $f_\pm(p, \mu_1, \mu_2) = \pm p^4 + \mu_1 p^2 + \mu_2 p$. Notice that $\mu_2 = 0$ corresponds to the $\mathbb{Z}_2$-pitchforks considered above.

Since $dH^\pm(x) = (\pm 4p^3 + 2\mu_1 p + \mu_2, q)$, equilibria are solutions of $q = 0$ and $f'(p, \mu_1, \mu_2) = \pm 4p^3 + 2\mu_1 p + \mu_2 = 0$. The Hessian is

$$\begin{bmatrix} 2 & 0 \\ 0 & \pm 12p^2 + 2\mu_1 \end{bmatrix}.$$

The Hessian is degenerate at $\mu_1 = \mp 6p^2$. Replacing in f' we obtain $\mu_2 = \pm 8p^3$, thus the cusp $\Delta \equiv 8\mu_1^3 \pm 27\mu_2^2 = 0$ is the bifurcation set. Since f' is a cubic polynomial it has at least one real root. The Hamiltonian system has a saddle-centre on the cusp curve where the number of roots of $f'(p, \mu_1, \mu_2) = 0$ (ie, of equilibria) jumps from 1 to 3 (or vice-versa).

Typical level contours of H^+ are shown in Figure 3.6 as (μ_1, μ_2) crosses the bifurcation set from the region with 3 equilibria to the region with 1. The analogous figure for H^- is left to the reader.

$G = \hat{\mathbb{Z}}_2$: the reversible umbilic. Up to now we have only considered bifurcations with *symplectic* symmetries. Hanßmann [H98] studies two-parameter

[1] If $H(p, q, 0, 0)$ has no quadratic terms then it is of codimension at least 3

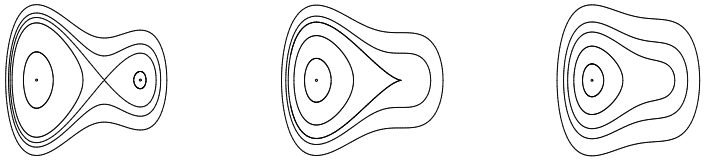

Fig. 3.6. Level contours of $H^+(p, q, \mu_1, \mu_2)$ (supercritical pitchfork) for different values of (μ_1, μ_2) as it crosses the bifurcation curve $\Delta = 0$.

families of one degree of freedom Hamiltonian with the *reversing* symmetry $\kappa.(q, p) = (q, -p)$ (whence the hat in $\hat{\mathbb{Z}}_2$). In two-parameter families the zero linearization occurs generically for reversible Hamiltonian systems. The least degenerate singularities in this case are given by $H^{\pm}(q, p) = p^2 q \mp \frac{1}{3} q^3$. Hanßmann shows that the versal unfolding in this case is a restriction of the umbilic catastrophe [PS78] given by

$$H^{\pm}_{\mu}(q, p) = p^2 q \mp \frac{1}{3} q^3 + \mu_1 (p^2 \pm q^2) + \mu_2 q$$

(where $\mu = (\mu_1, \mu_2)$) and called the *reversible umbilic*. The upper sign corresponds to the elliptic umbilic and the lower one to the hyperbolic umbilic. We only consider the hyperbolic reversible umbilic in these notes.

The critical points of H^-_{μ} are given by solutions of $p^2 + q^2 - 2\mu_1 q + \mu_2 = p(q + \mu_1) = 0$. The Hessian is given by

$$\begin{bmatrix} 2(q - \mu_1) & 2p \\ 2p & 2(q + \mu_1) \end{bmatrix}.$$

Solving for critical points with degenerate quadratic form yields the bifurcation set given by the union of the parabolas $\{\mu_2 = \mu_1^2\}$ and $\{\mu_2 = -3\mu_1^2\}$. On the curve $\{\mu_2 = \mu_1^2\}$ the system has a unique equilibrium at $(q, p) = (\mu_1, 0)$ with degenerate Hessian

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Now $(q, p) = (\mu_1, 0) \in \mathrm{Fix}(\kappa)$ and the kernel of the Hessian lies in $\mathrm{Fix}(\kappa)$, therefore the $\mathbb{Z}_2$-reversing symmetry is not broken and the parabola $\{\mu_2 = \mu_1^2\}$ is a saddle-centre bifurcation curve. For $\mu_1 \neq 0$, the system goes from no equilibria when $\mu_2 > \mu_1^2$ to a saddle and a centre for $\mu_2 < \mu_1^2$.

On the curve $\{\mu_2 = -3\mu_1^2\}$, the system has a unique equilibrium at $(q, p) = (-\mu_1, 0) \in \mathrm{Fix}(\kappa)$ with the kernel of the Hessian transverse to $\mathrm{Fix}(\kappa)$. Hence,

the bifurcation on the parabola $\{\mu_2 = -3\mu_1^2\}$ breaks the $\mathbb{Z}_2(\kappa)$-symmetry and so is a curve of Hamiltonian pitchfork bifurcation.

The reversing symmetry forces some interesting dynamics. Since the bifurcating equilibria are related by the κ symmetry they lie on the same energy level. By reversibility, they must be connected by a heteroclinic connection. See Hanßmann [H98] for details and the full bifurcation picture.

3.2.3 Many degrees of freedom Hamiltonian

For Hamiltonian systems with symmetry having more than one degree of freedom, one can use reduction methods to determine part or all of the dynamics in the neighbourhood of an equilibrium point. A symmetry-based reduction (sometimes called discrete reduction, although the group in question need not be discrete) is given by the following result.

Proposition 3.6 *Let $H : V \to \mathbb{R}$ be a G-invariant Hamiltonian and let X_H be its associated Hamiltonian vector field. Let K be a compact subgroup of G acting symplectically. Then, $\mathrm{Fix}(K)$ is symplectic and X_H leaves it invariant; moreover $X_H\,|_{\mathrm{Fix}(K)}$ is a Hamiltonian vector field with Hamiltonian $H\,|_{\mathrm{Fix}(K)}$.*

In particular, if $\dim\,\mathrm{Fix}(K) = 2$, then the restricted system is of one degree of freedom and the dynamics/bifurcations are readily obtained as above. Note that the compactness of K is essential; for example if $K \simeq \mathbb{R}$ acts on $(\mathbb{R}^2, \omega)$ by $t \cdot (x, y) = (x + ty, y)$ then $\mathrm{Fix}(K) = \mathbb{R}$ (the x-axis).

Other reductions to one degree of freedom can be obtained for example by centre-manifold reduction.

4 Hamiltonian-Hopf bifurcation

A Hamiltonian-Hopf bifurcation occurs when two nonzero imaginary eigenvalues of an elliptic equilibrium collide in a $1 : -1$ resonance and move into the left and right half-planes, see Figure 3.2. It is named in analogy with the Hopf Bifurcation Theorem of dissipative systems where small amplitude periodic solutions bifurcate from an equilibrium that loses stability as a pair of complex eigenvalues cross the imaginary axis.

The existence of periodic solutions in the Hamiltonian-Hopf bifurcation was first established by Meyer and Schmidt [MS71], and then later by van der Meer [vdM85] who was the first to study its equivariant version [vdM90]. Recently, Chossat, Ortega and Ratiu [COR02] extended the Hamiltonian-Hopf Theorem to include relative periodic orbits.

We begin with the nonsymmetric case. For a generic one-parameter family of Hamiltonians, the generalized eigenspace $E_{\pm i}$ at a $1:-1$ resonance is of dimension four. We restrict our study of this bifurcation to the family $(\mathbb{R}^4, \omega, 1, H_\mu)$ with $\mu \in \mathbb{R}$ (this reduction can be obtained by Lyapunov-Schmidt reduction or restriction to the centre manifold). Since the eigenvalues are far from 0, the equilibrium is non-degenerate and there is no bifurcation of equilibria in this family; we can therefore assume the origin is an equilibrium point for all μ. The following points are used implicitly in the statement of the theorem.

• Let $(x, y) \in \mathbb{R}^4$ where $x = (x_1, x_2)$ and $y = (y_1, y_2)$, with symplectic form $\omega = dx_1 \wedge y_1 + dx_2 \wedge dy_2$, and the origin is an equilibrium point of H_μ for all values of μ. Let $H_{2,\mu}$ denote the quadratic part of H_μ, then the linearization at $\mu = 0$, see matrix (3.1), implies that

$$H_{2,\mu}(x, y) = S + N + \mu P.$$

where $S = x_1 y_2 - x_2 y_1$ (the semisimple part of $H_{2,0}$), $N = \frac{1}{2}(x_1^2 + x_2^2)$ (the nilpotent part) and $P = \frac{1}{2}(y_1^2 + y_2^2)$. For $\mu < 0$ the linear system has two distinct pairs of imaginary eigenvalues, so the nonlinear system has 2 nonlinear normal modes, while for $\mu > 0$ the eigenvalues all have non-zero real part, so there are no nonlinear normal modes; indeed no periodic orbits in a neighbourhood of the origin. The problem is to describe this transition.

• The dynamics near equilibrium solutions is understood using Birkhoff normal form [CS86, ETBCI]. We denote by $(\mathbb{R}^4, \omega, \mathbf{S}^1, \tilde{H}_\mu)$ the Hamiltonian system of the symplectic Birkhoff normal form $\tilde{H}$ of $(\mathbb{R}^4, \omega, 1, H_\mu)$ up to some finite order k.

$$\tilde{H}(x, y, \mu) = H_2(x, y, \mu) + \tilde{H}_4(S, P) + \cdots + \tilde{H}_k(S, P), \qquad (4.1)$$

where $S = S(x, y) = x_1 y_2 - x_2 y_1$ and $P = P(x, y) = \frac{1}{2}(y_1^2 + y_2^2)$, and $\tilde{H}_k$ is homogeneous of degree k in x, y. We also write $N = \frac{1}{2}(x_1^2 + y_1^2)$.

Theorem 4.1 (Hamiltonian-Hopf Bifurcation) *Suppose that the family $(\mathbb{R}^4, \omega, 1, H_\mu)$ of Hamiltonian systems has, at $\mu = 0$, imaginary eigenvalues in $1:-1$ resonance. If the coefficient, a of P^2 in $\tilde{H}_4$ is nonzero then for each $k > 0$ there is a neighbourhood of the origin in $\mathbb{R}^4 \times \mathbb{R}$ in which the set of short periodic solutions of the system $(\mathbb{R}^4, \omega, 1, H_\mu)$ is C^k-diffeomorphic to the set of short periodic solutions of the system $(\mathbb{R}^4, \omega, \mathbf{S}^1, S + G_\nu)$ with*

$$G_\nu(x, y) = N + \nu P + aP^2. \qquad (4.2)$$

A point z lies on a short periodic solution of $(\mathbb{R}^4, \omega, \mathbf{S}^1, S + G_\nu)$ if and only if it is a critical point of the "energy-momentum map" $(S, S + G_\nu)$, and so of

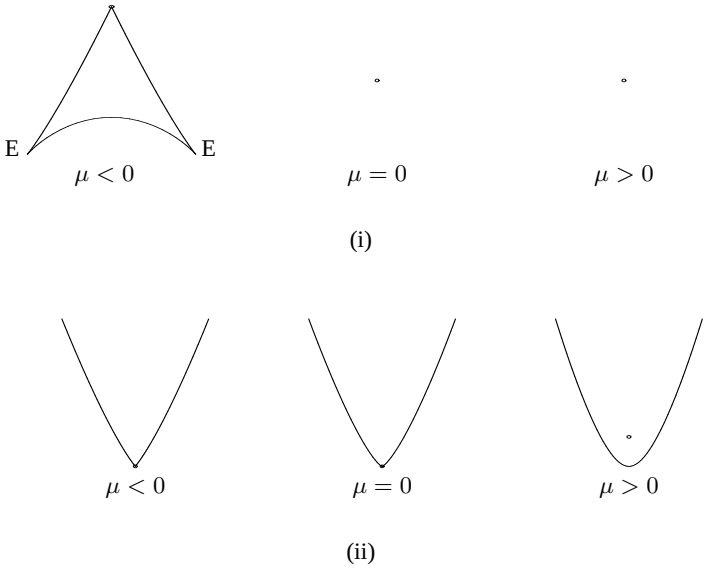

Fig. 4.1. Families of periodic orbits in the two scenarios of the Hamiltonian-Hopf bifurcation. The small dot represents the origin.

the map (S, G_ν). There are two possible scenarios for the structure of the set of periodic orbits, according to the sign of a.

Theorem 4.1 (continued) *Let a be the coefficient of P^2 in the normal form (4.1) for H_μ.*

(i) *If $a > 0$, then for $\mu < 0$ the 2 nonlinear normal modes are globally connected in a single compact family; as $\mu \to 0^-$ this family collapses to the origin and disappears.*

(ii) *If $a < 0$, then for $\mu < 0$ the two nonlinear normal modes are distinct in a neighbourhood of the origin, intersecting only at the origin. As μ passes through 0 they pull away from the origin as a single family.*

The "pulling away from the origin" in case (ii) is similar to that of a 1-sheeted hyperboloid pulling away from the origin as it deforms from a cone, though the analogy cannot be taken very far.

The two cases (i) and (ii) are illustrated in Figures 4.1 (i) and (ii) respectively and are sometimes referred to as the subcritical and supercritical Hamiltonian-Hopf bifurcations. However, the reader should beware that the nomenclature is not universally consistent. Iooss and Pérouème [IP93] refer to (i) as supercritical and (ii) as subcritical, while Hanßmann and van der Meer [HM02] have it

the other way round. The lectures of Cushman in this book use Hanßmann and van der Meer's convention. The illustrations in Figure 4.1 show the images in energy-momentum space of the families of periodic orbits.

Recall that for a non-resonant elliptic equilibrium, the nonlinear normal modes are also elliptic sufficiently close to the equilibrium point. In the bifurcation of type (i) for fixed $\mu < 0$, when the equilibrium is elliptic, if one follows the compact family of periodic orbits emanating from the origin along one nonlinear normal mode and returning along the other, there is a transition from elliptic to hyperbolic periodic orbits, and then back to elliptic again; Iooss and Pérouème [IP93] refer to the transition points as *Eckhaus points*, marked E on Figure 4.1(i). There are therefore hyperbolic periodic orbits in any neighbourhood of the origin $(x, y, \mu) = (0, 0, 0)$ in $\mathbb{R}^4 \times \mathbb{R}$ (and Iooss and Pérouème also show that in the reversible setting there are orbits homoclinic to certain hyperbolic periodic orbits; presumably this would be true also in the Hamiltonian setting, but to our knowledge this has not been checked). An example of both supercritical and subcritical Hamiltonian-Hopf bifurcations are mentioned in Cushman's lectures (Section C.4). Finally, Sokol′skiĭ has shown [So74][1] that at the bifurcation point $\mu = 0$, the Hamiltonian-Hopf case (ii) scenario ($a < 0$) has an unstable equilibrium, while in the other scenario the origin is *formally stable*, meaning that the equilibrium is stable for the dynamics of the normal form approximation at any order.

Symmetric Hamiltonian-Hopf

As in the equivariant Weinstein-Moser theorem (Theorem 2.1) periodic solutions with spatio-temporal symmetries are found by considering the action of $G \times \mathbf{S}^1$, this time on the bifurcation eigenspace $E_{\pm i}$. The equivariant version of Theorem 4.1 is obtained by finding four dimensional fixed point subspaces of the action of $G \times \mathbf{S}^1$ and showing that the hypotheses of Theorem 4.1 are satisfied for the fixed point subspace. The $\mathbf{S}^1$ action is generated by the semisimple part of $H_{2,0}$ on $E_{\pm i}$.

So, we consider a one-parameter family of Hamiltonian systems $(\mathbb{R}^{2n}, \omega, G, H_\mu)$ with nontrivial symmetry G, and we will suppose for simplicity that $E_{\pm i} = \mathbb{R}^{2n}$.

Lemma 4.2 *Let Σ be an isotropy subgroup of $G \times \mathbf{S}^1$, so that $\mathrm{Fix}(\Sigma) \neq \{0\}$. Then $\dim \mathrm{Fix}(\Sigma) \geq 4$ and the restriction of $X_{H_{2,0}}$ to $\mathrm{Fix}(\Sigma)$ gives rise to a $1 : -1$-resonance.*

Lemma 4.2 guarantees that on each nonzero fixed point subspace of the

[1] the lecturer would like to thank Ken Meyer for pointing this out to him

action of $G \times \mathbf{S}^1$ the movement of eigenvalues on the subspace does correspond to a Hamiltonian-Hopf bifurcation. The result is the following.

Theorem 4.3 (Hamiltonian-Hopf Theorem with symmetry [vdM90]) *Let Σ be an isotropy subgroup of $G \times \mathbf{S}^1$ with $\dim \mathrm{Fix}(\Sigma) = 4$. Let a_Σ be the coefficient of P^2 in the normal form of H_0 on $\mathrm{Fix}(\Sigma)$. Then, provided $a_\Sigma \neq 0$, the same two scenarios occur as for the ordinary Hamiltonian-Hopf bifurcation, according to the sign of a_Σ. Moreover the resulting periodic orbits all have spatio-temporal symmetry at least Σ.*

An example of Hamiltonian-Hopf bifurcation with symmetry occurs in models of point vortices on the sphere, see Laurent-Polz [LP00].

Remark 4.4 In the Hamiltonian-Hopf bifurcation, for $\mu > 0$ the eigenvalues have non-zero real parts so there is a stable manifold and an unstable manifold for the equilibrium point, while for $\mu < 0$ the system is elliptic and there are no such manifolds. In an interesting recent paper McSwiggen and Meyer [MM03] have studied this transition and shown that there are again two scenarios that mirror quite remarkably the two scenarios for the nonlinear normal modes. One would expect to have similar behaviour in the symmetric case, but to our knowledge this has not been checked.

Hamiltonian-Hopf and Relative Periodic Orbits

Here we describe very briefly a result of Chossat, Ortega and Ratiu [COR02] which extends the results described above to finding *relative* periodic orbits. A trajectory $\gamma(t)$ of a dynamical system is periodic if $\gamma(T) = \gamma(0)$ for some $T > 0$, and it is a *relative periodic orbit* (or RPO) if there is an element $g \in G$ and $T > 0$ such that $\gamma(T) = g \cdot \gamma(0)$. As usual T is called the period, and g is called the *phase*.

The authors investigate the situation in which a point x_0 is a non-degenerate equilibrium point of a G-invariant Hamiltonian system H_0 with eigenvalues $\pm i\nu$, for which the generalized eigenspace $E_{\pm i\nu}$ is such that it decomposes as the sum of two symplectic irreducibles of complex dual type for the $G \times S^1$-action, where the S^1-action is that derived from the linearization on $E_{\pm i\nu}$; write $E_{\pm i\nu} = U_1 \oplus U_2$. As described in Theorem 3.4(iv) this hypothesis is generic for a 1-parameter family of G-invariant Hamiltonians. Furthermore H_μ is assumed to be a *generic* G-invariant deformation of H_0, which is an assumption on the movement of the eigenvalues as they collide (Figure 3.2 (p. 369)).

Let S be the unit sphere of V; it is $G \times S^1$ invariant and of odd-dimension.

For $\xi \in \mathfrak{g}$ let G_ξ denote the isotropy subgroup of ξ under the adjoint action. The main result of Chossat, Ortega and Ratiu is

Theorem 4.5 *With the setup and genericity assumptions described above then for each $\xi \in \mathfrak{g}$ sufficiently small there is a smooth $G_\xi \times S^1$-equivariant vector field on the sphere S such that on each energy level near x_0 and for each relative equilibrium of the vector field, there is a value of $\mu \approx 0$ for which there is a relative periodic point of H_μ with phase $\exp(T\xi)$ for some $T \approx 2\pi$.*

It is possible to use topological methods to estimate the minimal possible numbers of relative equilibria of $G_\xi \times S^1$ equivariant vector fields. In particular it is always positive, since S^1 acts freely on S, so the $G_\xi \times S^1$ equivariant vector field on S descends to a G_ξ-equivariant vector field on S/S^1. This orbit space S/S^1 has non-zero Euler characteristic so every vector field there has a zero (it is diffeomorphic to a complex projective space).

The proof is based on the reduction method of Vanderbauwhede and van der Meer [VvdM]. The reader should beware that the paper [COR02] mis-states the result by saying that each RPO exists for every value of μ rather than for some value of μ. It would be interesting to understand better the behaviour of the RPOs as μ is varied.

As we have seen, a considerable amount is known about the Hamiltonian-Hopf bifurcation (the "splitting" cases of Theorem 3.4). On the other hand, very little is known about the passing cases, probably because it is of higher codimension if there is no symmetry present: namely codimension 3, and codimension 2 in reversible systems.

PART II: LOCAL DYNAMICS NEAR RELATIVE EQUILIBRIA

5 Momentum map and reduction

5.1 Noether's theorem

Emmy Noether's theorem associates to any 1-parameter group of symmetries a conserved quantity for the dynamics. For a "several-parameter" group, there are correspondingly several conserved quantities, which together are called the momentum map.

Given a symplectic action of a group G, a map $\mathbf{J} : \mathcal{P} \to \mathfrak{g}^*$ is called a *momentum map* if $X_{\mathbf{J}^\xi} = \xi_{\mathcal{P}}$ for each $\xi \in \mathfrak{g}$, where $\mathbf{J}^\xi(x) = \langle \mathbf{J}(x), \xi \rangle$, $x \in \mathcal{P}$. The defining equation for the momentum map is

$$\langle d\mathbf{J}_x(v), \xi \rangle = \omega_x(v, \xi_{\mathcal{P}}(x))$$

for all $x \in \mathcal{P}$, $v \in T_x\mathcal{P}$ and $\xi \in \mathfrak{g}$ (the Lie algebra of G). The momentum map is thus defined up to a constant, and $\mathrm{Im}(\mathrm{d}\mathbf{J}_\mathrm{x}) = \mathfrak{g}_\mathrm{x}^\circ \subset \mathfrak{g}^*$ (where $\mathfrak{g}_x^\circ$ denotes the annihilator of $\mathfrak{g}_x$ in $\mathfrak{g}^*$). It follows that the momentum map is a submersion in a neighbourhood of any point where the action is locally free (i.e., where $\mathfrak{g}_x = 0$).

The momentum map always exists locally, but to ensure the global existence of the momentum map one needs some hypothesis such as semisimplicity of the group, or simple connectedness of the phase space (see [GS84]).

Theorem 5.1 (Noether) *Let H be a G-invariant Hamiltonian on $\mathcal{P}$ with a momentum map $\mathbf{J}$. Then $\mathbf{J}$ is conserved on the trajectories of the Hamiltonian vector field X_H.*

Proof. Differentiating the G-invariance condition, we get $dH \cdot \xi_{\mathcal{P}} = 0$. Since $dH \cdot \xi_{\mathcal{P}} = \{H, \mathbf{J}^\xi\} = -\{\mathbf{J}^\xi, H\} = -d\mathbf{J}^\xi \cdot H$, the functions $\mathbf{J}^\xi$ are conserved on the trajectories of X_H for every ξ in $\mathfrak{g}$. ■

5.2 Equivariance of the momentum map

Given a symplectic action of G on $\mathcal{P}$ and a momentum map $\mathbf{J} : \mathcal{P} \to \mathfrak{g}^*$, one can construct an action of G on $\mathfrak{g}^*$ such that the momentum map is equivariant with respect to these actions. Usually, but not always, this turns out to be the coadjoint action of G on $\mathfrak{g}^*$. The construction was found by Souriau [S70], and proceeds as follows. Let θ be the *cocycle*

$$\begin{array}{rcl} \theta : G & \to & \mathfrak{g}^* \\ g & \mapsto & \mathbf{J}(g \cdot x) - \mathrm{Coad}_g \mathbf{J}(x) \end{array}$$

This map is well-defined if $\theta(g)$ is independent of x, which it is provided $\mathcal{P}$ is connected. We then define the *modified coadjoint action* by

$$\mathrm{Coad}^{\theta}_{g}\,\mu := \mathrm{Coad}_{g}\,\mu + \theta(g)$$

A short calculation shows that it is indeed an action.

Theorem 5.2 (Souriau) *Let the Lie group G act on the connected symplectic manifold $\mathcal{P}$ in such a way that there exists a momentum map $\mathbf{J} : \mathcal{P} \to \mathfrak{g}^*$. Then $\mathbf{J}$ is equivariant with respect to the modified coadjoint action on $\mathfrak{g}^*$:*

$$\mathbf{J}(g \cdot x) = \mathrm{Coad}^{\theta}_{g}\,\mathbf{J}(x)$$

Furthermore, if G is either semisimple or compact then the momentum map can be chosen such that $\theta = 0$.

For proofs see [S70], and [GS84] for semisimple groups, and [Mo97] for compact groups.

Remark 5.3 The above arguments can be extended to semisymplectic actions (see the introduction to these lectures). If $\chi : G \to \mathbb{Z}_2$ is the temporal character, then one defines the χ-twisted coadjoint action by

$$\mathrm{Coad}^{\chi}_{g}\,\mu = \chi(g)\,\mathrm{Coad}_{g}\,\mu, \tag{5.1}$$

and similarly $\mathrm{Coad}^{\chi,\theta}_{g}\,\mu$ and everything then follows as before [MR00].

5.3 Reduction

By Noether's theorem the dynamics preserve the level sets of the momentum map $\mathbf{J}$. It is then natural to study the dynamics on one level set at a time. However, these level sets are not in general symplectic manifolds and the induced dynamics are therefore not Hamiltonian. But, if one passes to the orbit space[1] of one of these level sets, then the resulting space is symplectic (provided the action is free and proper).

Let $\mu \in \mathfrak{g}^*$ and G_μ be the isotropy subgroup of the modified coadjoint action:

$$G_\mu = \{g \in G \mid \mathrm{Coad}^{\theta}_{g}\,\mu = \mu\}.$$

For example, if $G = \mathbf{SO}(3)$ and $\mu \neq 0$, G_μ is the set of rotations with axis $\langle\mu\rangle$ and so is isomorphic to $\mathbf{SO}(2)$, while $G_0 = \mathbf{SO}(3)$.

[1] that is, identify points in the level sets which lie in the same group orbit

By the equivariance of the momentum map, G_μ acts on the level set $\mathbf{J}^{-1}(\mu)$. We can then define the (Meyer-Marsden-Weinstein) *reduced space* $\mathcal{P}_\mu$ to be:

$$\mathcal{P}_\mu = \mathbf{J}^{-1}(\mu)/G_\mu.$$

Refer to Ratiu's lectures for details (and see also [MR] and [OR]). Since G acts freely, $\mathcal{P}_\mu$ is a smooth manifold. Moreover it is symplectic with symplectic form ω_μ given by $\omega_\mu(\pi(u), \pi(v)) = \omega(u, v)$, where $u, v \in T_p\mathcal{P}$ and π is the projection $T_p\mathcal{P} \to T_{\overline{p}}\mathcal{P}_\mu$.

Given an invariant Hamiltonian H, its restriction to $J^{-1}(\mu)$ is invariant under G_μ, and so determines a well-defined function on $\mathcal{P}_\mu$, the *reduced* Hamiltonian denoted H_μ. The dynamics induced on the reduced space is determined by a vector field X_μ which is defined by $dH_\mu = \omega_\mu(-, X_\mu)$.

Recall that $\mathcal{P}_\mu$ is diffeomorphic to $\mathbf{J}^{-1}(\mathcal{O}_\mu)/G$ where $\mathcal{O}_\mu$ is the coadjoint orbit through μ. One defines the *orbit momentum map* $\mathcal{J} : \mathcal{P}/G \to \mathfrak{g}^*/G$ by:

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\mathbf{J}} & \mathfrak{g}^* \\ \downarrow & & \downarrow \\ \mathcal{P}/G & \xrightarrow{\mathcal{J}} & \mathfrak{g}^*/G \end{array}$$

where the vertical arrows are the quotient maps. Then the reduced spaces are the fibers of $\mathcal{J}$.

Now, we introduce the notion of *symplectic slice* to provide a local model for the reduced spaces. Recall that a slice to a group action at a point $p \in \mathcal{P}$ is a submanifold S through p satisfying $T_pS \oplus \mathfrak{g} \cdot p = T_p\mathcal{P}$. If G_p is compact, it can be chosen to be G_p-invariant and then S/G_p provides a local model for the orbit space $\mathcal{P}/G$. If G acts by isometries, one usually chooses $T_pS = N = \mathfrak{g}.p^\perp$ (the normal space to the group orbit).

Definition 5.4 Suppose G_p is compact. Let N be a G_p-invariant subspace satisfying $T_x\mathcal{P} = N \oplus \mathfrak{g} \cdot p$. We then define the *symplectic slice* to be

$$N_1 := N \cap \mathrm{Ker}(d\mathbf{J}(p)).$$

Again, if G acts by isometries, one usually chooses $N = (\mathfrak{g} \cdot p)^\perp$. Note that $(\mathfrak{g} \cdot p)^\perp \cap \mathrm{Ker}(d\mathbf{J}(p)) = (\mathfrak{g}_\mu \cdot p)^\perp \cap \mathrm{Ker}(d\mathbf{J}(p))$, and so one can choose the right hand space to be the symplectic slice.

6 Relative equilibria

6.1 Definition and properties of relative equilibria

A point $x_e \in \mathcal{P}$ is called a *relative equilibrium* if for all t there exists $g_t \in G$ such that $x_e(t) = g_t \cdot x_e$, where $x_e(t)$ is the dynamic orbit of X_H with $x_e(0) = x_e$. In other words, the trajectory is contained in a single group orbit. There are different ways to define relative equilibria as the following proposition shows.

Proposition 6.1 *Let* $\mathbf{J}$ *be a momentum map for the* G*-action on* $\mathcal{P}$ *and let* H *be a* G*-invariant Hamiltonian on* $\mathcal{P}$*. Let* $x_e \in \mathcal{P}$ *and* $\mu = \mathbf{J}(x_e)$*. The following assertions are equivalent:*
i) x_e *is a relative equilibrium*
ii) the group orbit $G \cdot x_e$ *is invariant under the dynamics*
iii) there is a $\xi \in \mathfrak{g}$ *such that* $x_e(t) = \exp(t\xi) \cdot x_e$
iv) there is a $\xi \in \mathfrak{g}$ *such that* x_e *is a critical point of the* ***augmented Hamiltonian****:*

$$H_\xi(x) = H(x) - \langle \mathbf{J}(x),\, \xi \rangle$$

v) x_e *is a critical point of the restriction of* H *to* $\mathbf{J}^{-1}(\mu)$
vi) the image $\overline{x_e} \in \mathcal{P}_\mu$ *of* x_e *is a critical point of the reduced Hamiltonian* H_μ*.*

Remarks 6.2 • The vector ξ appearing in (iii) is called a *velocity* of the relative equilibrium, it is the same as the vector appearing in (iv). Of course, for all $\eta \in \mathfrak{g}_{x_e}$, $\xi + \eta$ is also a velocity of x_e. However, with $N = N_{G_\mu}(G_{x_e})$, the normalizer of G_{x_e} in G_{μ_e}, and given a G_{x_e}-invariant inner product on $\mathfrak{n}_\mu := \mathrm{Lie(N)}$, one can define *the angular velocity* of x_e to be the component of ξ in $\mathfrak{g}_{x_e}^\perp$, the orthogonal complement of $\mathfrak{g}_{x_e}$ in $\mathfrak{n}_\mu$. With this setting, the angular velocity is unique (see [Or98]).
• Note that (iii) implies that relative equilibria cannot meander around a group orbit: they must move in a rigid fashion. For N-body problems in space, the relevant group is $\mathbf{SO}(3)$ and relative equilibria are therefore motions where the shape of the body doesn't change, and these motions are always rigid rotations about some axis.
• If $\mathbf{J}^{-1}(\mu)$ is singular, then it has a natural stratification (see [SL91]) and the condition in assertions (iv) and (v) should be interpreted as being a *stratified* critical point; that is all derivatives of H along the stratum containing x_e vanish at x_e.

Proof. The logic goes as follows:
$i) \Rightarrow ii) \Rightarrow iv) \Rightarrow iii) \Rightarrow i)$ and $iv) \Rightarrow v) \Rightarrow vi) \Rightarrow i)$.

First assume (i), let x_e be a relative equilibrium and let $x = k \cdot x_e$, $k \in G$. By G-equivariance of X_H, $x(t) = k \cdot x_e(t)$ and then $x(t) = kg_tk^{-1}x$, x is a relative equilibrium , which is (ii).

Next, assume (ii). From (ii), we have $x_e(t) \in G \cdot x_e$ for all t. So $X_H(x_e) = T_{x_e}(G \cdot x_e) = \mathfrak{g} \cdot x_e$ and there is a $\xi \in \mathfrak{g}$ such that $X_H(x_e) = \xi_{\mathcal{P}}(x_e)$. By definition of the momentum map, $X_{\mathbf{J}^\xi} = \xi_{\mathcal{P}}$ and then $X_{H-\mathbf{J}^\xi}(x_e) = 0$. Since $w(X_H, \cdot) = dH$, it turns that x_e is a critical point of H_ξ, which is (iv).

Assume (iv). Let φ_t and ψ_t^ξ be the flows of H and $\mathbf{J}^\xi$ respectively, so $\psi_t^\xi(x_e) = \exp(t\xi) \cdot x_e$. Since H is G-invariant, φ_t and ψ_t^ξ commute, it follows that $\varphi_t \circ \psi_{-t}^\xi$ is the flow of $H - \mathbf{J}^\xi$. The critical point x_e of H_ξ is therefore fixed by $\varphi_t \circ \psi_{-t}^\xi$, and so $\varphi_t(x_e) = \psi_t^\xi(x_e) = \exp(t\xi) \cdot x_e$ which is (iii).

Clearly, (iii) implies (i).

Assume (iv). If $\mathbf{J}^{-1}(\mu)$ is a manifold, (v) follows from the Lagrange multipliers theorem. If $\mathbf{J}^{-1}(\mu)$ is singular, then $\mathfrak{g}_{x_e} \neq 0$ since $\mathrm{Im}(d\mathbf{J}(x)) = \mathfrak{g}_x^\circ \subset \mathfrak{g}^*$. By the theorem of Sjamaar and Lerman [SL91], $\mathbf{J}^{-1}(\mu)$ is stratified by the subsets $\mathcal{P}^{(K)} = \{x \in \mathcal{P} \mid G_x \text{ is conjugate to } K\}$ where K is a subgroup of G. Let $\mathcal{P}_{G_x}$ be the set of points with isotropy precisely G_x, this is an open symplectic submanifold of $\mathrm{Fix}(G_x, \mathcal{P})$ containing x. Let $N(G_x)$ be the normalizer of G_x in G and $L = N(G_x)/G_x$, L acts on $\mathcal{P}_{G_x}$. The subsystem $(\mathcal{P}_{G_x}, \tilde{H}, \tilde{\omega}, L)$ is Hamiltonian. Let $\mathbf{J}_L : \mathcal{P}_{G_x} \to \mathrm{Lie(L)}^*$ the corresponding momentum map, $\mathbf{J}_L$ is a submersion since L acts freely on $\mathcal{P}_{G_x}$, thus $\mathbf{J}_L^{-1}(\mu)$ is a manifold, and so we can apply the regular case. The result follows then from the Principle of Symmetric Criticality (see Section 6.2 for a statement).

That (v) implies (vi) follows by passing to the quotient. Finally (vi) implies that the equivalence class $\overline{x_e}$ is a fixed equilibrium of the reduced dynamics. Then $x_e(t)$ lies in $G \cdot x_e$ for all t and this is (i). ■

Proposition 6.3 *Let x_e be a relative equilibrium with angular velocity ξ and $\mu = \mathbf{J}(x_e)$. Then*

$$\mathrm{Coad}^\theta_{\exp(t\xi)}\, \mu = \mu.$$

Proof. This is simply because $\mathbf{J}$ is equivariant, $\exp(t\xi)$ generates the motion and μ is conserved. ■

For $G = \mathbf{SO}(3)$, this implies that ξ and the momentum vector are parallel vectors, but for the Euclidean group the corresponding relationship is more complicated.

6.2 How does one locate relative equilibria?

From Proposition 6.1, relative equilibria are critical points of the Hamiltonian restricted to the level sets of the momentum map, so results on critical point of G-invariant functions are of particular interest. Mainly, these results are due to Palais [P79] and Michel [Mi71].

Let G be a Lie group and $H : \mathcal{P} \to \mathbb{R}$ a G-invariant function. Assume that G is either compact or acts isometrically on $\mathcal{P}$ Riemannian. The *Principle of Symmetric Criticality* [P79] claims that if the directional derivatives $dH_x(u)$ vanish for all directions u at x tangent to $\mathrm{Fix}(K, \mathcal{P})$, then directional derivatives in directions transverse to $\mathrm{Fix}(K, \mathcal{P})$ also vanish. In particular, any isolated point of $\mathrm{Fix}(K, \mathcal{P})$ is a critical point of H. In our context of Hamiltonian system with symmetry, one obtains the following theorem as a corollary of the Principle of Symmetric Criticality. Recall that if G acts semisymplectically, we denote by G_+ the subgroup (of index 2) of elements acting symplectically.

Theorem 6.4 *Let G act semisymplectically on $\mathcal{P}$. Suppose $x \in \mathrm{Fix}(K, \mathcal{P})$ for some subgroup K of G, and $\mu = \mathbf{J}(x)$. If x is an isolated point in $\mathrm{Fix}(K, \mathcal{P}) \cap \mathbf{J}^{-1}(\mu)$, then x is a relative equilibrium. If in addition $K < G_+$, then x is a fixed equilibrium.*

Note that equilibria derived by this theorem do not depend on the form of the Hamiltonian, they depend only on the action of the symmetry group on the phase space. Note also that this result uses the fact that H is G-invariant, and not that G acts symplectically. In the examples of Section 8, we use this theorem with antisymplectic symmetries ($g^*\omega = -\omega$) as well as symplectic ones.

This theorem provides relative equilibria with large isotropy subgroups, and hence is not usually sufficient to determine bifurcating branches of relative equilibria since symmetry-breaking occurs at a bifurcation.

One way to find these relative equilibria with less symmetry is to determine critical points of the restriction of $H_{|\mathrm{Fix}(K,\mathcal{P})}$ to $\mathbf{J}^{-1}(\mu)$ (using Lagrange multipliers for example). Indeed by the Principle of Symmetric Criticality, we have determined critical points of the restriction of H to $\mathbf{J}^{-1}(\mu)$ which are precisely relative equilibria.

As we shall see in Section 7, one can also determine relative equilibria in a neighbourhood of a zero-momentum relative equilibrium by a bifurcation argument.

6.3 Stability

When one has found a relative equilibrium, a natural question arises: is it *stable*? We first review different definitions of stability in Hamiltonian systems of finite dimension.

Let x_0 be a fixed equilibrium of an Hamiltonian dynamical system and L_0 the matrix of the linearized system at x_0. The equilibrium x_0 is said to be *spectrally stable* if the eigenvalues of L_0 all lie on the imaginary axis. If in addition L_0 is semisimple, the equilibrium is said to be *linearly stable*. To end, an equilibrium x_0 is said to be *Lyapunov stable* if for any neighbourhood U of x_0, there is a neighbourhood V of x_0, $V \subset U$ such that any trajectory which intersects V remains in U for all time. Note that Lyapunov stability is interesting for nonlinear dynamics; for a linear system Lyapunov stability is equivalent to linear stability.

These different concepts are related:

• Linear stability implies spectral stability, but the converse is not true as resonance can generate instabilities.

• Lyapunov stability implies spectral stability. Note that spectral instability implies Lyapunov instability, and this is therefore a way to prove Lyapunov instability.

In order to prove Lyapunov stability, one has the *Lagrange-Dirichlet* criterion: *If the Hessian matrix $d^2H(x_0)$ is positive- or negative-definite, then the equilibrium x_0 is Lyapunov stable.* Indeed, a use of the Morse Lemma states that the level sets of H near the equilibrium are topologically spheres, and by conservation of energy, if a trajectory starts on one of these spheres, it remains on it. (See the lectures of Meyer.)

For relative equilibria, the previous definitions of stability are not suitable since relative equilibria which are not fixed equilibria are unstable in some directions tangent to the group orbit. The appropriate concept of stability is *stability modulo a subgroup* as follows. Let K be a subgroup of G; a relative equilibrium x_e is said to be *stable modulo K*, if for all K-invariant open neighbourhoods V of $K \cdot x_e$ there is an open neighbourhood $U \subseteq V$ of x_e such that the trajectory through any point of U is entirely contained in V.

One would like an analogue of the Lagrange-Dirichlet criterion for the stability of relative equilibria. In the case of a free and proper action with μ regular, this was obtained by the energy-momentum method of Arnold and Marsden (see [Ma92] and reference therein). This result was extended by Patrick [Pa92] to allow locally free actions μ not regular and more importantly concluding not only G-stability but G_μ-stability. More recently, the freeness assumption was dropped as the following theorem shows. The lack of freeness

means that the velocity ξ of a relative equilibrium x_e is no longer unique, but only unique modulo $\mathfrak{g}_{x_e}$. Given a G_μ-invariant inner product on $\mathfrak{g}$ the *orthogonal* angular velocity is the unique angular velocity $\xi \in \mathfrak{g}_{x_e}^{\perp}$.

Theorem 6.5 (Lerman, Singer [LS98], Ortega, Ratiu [OR99])
Let G act properly on $\mathcal{P}$, x_e be a relative equilibrium with orthogonal angular velocity ξ, and $\mu = \mathbf{J}(x_e)$. Suppose further that:

i) G_μ acts properly on $\mathcal{P}$ and $\mathfrak{g}$,

ii) the restriction of the Hessian $d^2H_\xi(x_e)$ to the symplectic slice N_1 is definite.

Then x_e is Lyapunov stable modulo G_μ.

6.4 Persistence

Given a relative equilibrium, one asks if the relative equilibrium *persists*; that is, if there are also nearby relative equilibria for nearby values of the momentum map. First, we give two definitions:

Definition 6.6 A relative equilibrium x_e is said to be *non-degenerate* if the restriction of the Hessian $d^2H_\xi(x_e)$ to the symplectic slice N_1 is a non-degenerate quadratic form.

Definition 6.7 A point $\mu \in \mathfrak{g}^*$ is a *regular point* of the (modified) coadjoint action if in a neighbourhood of μ all the isotropy subgroups are conjugate.

For $G = \mathbf{SO}(3)$, all points of $\mathfrak{g}^* \simeq \mathbb{R}^3$ are regular except the origin. The following theorem due to Arnold [A78] was the first result on persistence. The proof is an application of the implicit function theorem.

Theorem 6.8 (Arnold) *Let x_e be a non-degenerate relative equilibrium and suppose that G acts freely in a neighbourhood of x_e, and that $\mu = J(x_e)$ is a regular point of the (modified) coadjoint action. Then in a neighbourhood of x_e there exists a smooth family of relative equilibria parametrized by $\mu \in \mathfrak{g}^*$.*

Again, this result was extended by Patrick [Pa95] who showed one need not assume that μ is regular, but only that $G_\mu \cap G_\xi$ is abelian (it always contains a maximal torus of G) together with non-degeneracy of the reduced hamiltonian. He concludes that the set of relative equilibria forms a submanifold of dimension $\dim G + \operatorname{rank} G$. This has been extended further by Lerman and Singer [LS98].

Theorem 6.9 (Lerman, Singer [LS98]) *Let G act properly on $\mathcal{P}$, x_e be a relative equilibrium with angular velocity ξ, and $\mu = \mathbf{J}(x_e)$. Suppose further that:*

i) G_μ is a compact Abelian group,

ii) the restriction of the Hessian $d^2H_\xi(x_e)$ to $\operatorname{Ker} d\mathbf{J}(\mu)$ is of maximal rank. Then there exists a symplectic manifold M of relative equilibria passing through x_e of dimension $\dim G + \dim G_\mu - 2\dim G_{x_e}$. Furthermore, if the restriction of the Hessian $d^2H_\xi(x_e)$ to the symplectic slice N_1 is definite, then a neighbourhood of x_e in M consists of relative equilibria which are Lyapunov stable modulo subgroups conjugate to G_μ.

Note that if G is compact, then for a generic $\mu \in \mathfrak{g}^*$, G_μ is a torus and so hypothesis (i) of the theorem holds for generic μ.

These theorems do not provide information about the symmetry of the relative equilibria of the manifold. Such a result is given in the thesis of Ortega [Or98]. Another extension of Arnold's and Patrick's theorems has been obtained by Patrick and Roberts [PR00] which, again for free actions, describes a stratification of the set of relative equilibria near a given one, depending on G_μ and G_ξ. A statement of their theorem would take us too far afield.

For the case of extrema of H_μ, one has the following persistence result which does not rely on any regularity hypotheses; however it gives no information about the structure of the set of relative equilibria.

Theorem 6.10 (Montaldi [Mo97], Montaldi, Tokieda [MT03]) *Let G act properly on $\mathcal{P}$ and $x_e \in \mathbf{J}^{-1}(\mu)$ be a relative equilibrium, with G_μ compact. Suppose $[x_e] \in \mathcal{P}_\mu$ is an extremum of the reduced Hamiltonian H_μ. Then there is a G-invariant neighbourhood U of x_e such that, for all $\mu' \in \mathbf{J}(U)$ there is a relative equilibrium in $U \cap \mathbf{J}^{-1}(\mu')$.*

7 Bifurcation from zero-momentum state

In this section we present some methods to analyse bifurcations from a relative equilibrium with momentum zero. For details and complements, we refer to papers of Montaldi [Mo97] and Montaldi-Roberts [MR99].

Let x_e be a non-degenerate relative equilibrium with $\mathbf{J}(x_e) = 0$ and $G_{x_e} = 0$ (i.e. locally a free action). Since the momentum is zero, the symplectic slice can be identified with $\mathcal{P}_0$. From the Marle-Guillemin-Sternberg normal form ([GS84]), we have locally near x_e:

$$\mathcal{P}/G \simeq \mathcal{P}_0 \times \mathfrak{g}^*$$

The orbit momentum map becomes:

$$\begin{array}{rcl} \mathcal{J} : \mathcal{P}/G \simeq \mathcal{P}_0 \times \mathfrak{g}^* & \longrightarrow & \mathfrak{g}^*/G \\ (y, \nu) & \longmapsto & \mathcal{O}_\nu \end{array}$$

and (locally) $\mathcal{P}_\mu \simeq \mathcal{P}_0 \times \mathcal{O}_\mu$ since $\mathcal{P}_\mu = \mathcal{J}^{-1}(\mathcal{O}_\mu)$. In many applications, in this decomposition $\mathcal{P}_0$ corresponds to shape dynamics and $\mathfrak{g}^*$ to rigid body dynamics. Of course, the two are highly coupled.

By hypothesis, x_e is a non-degenerate critical point of the restriction of $H : \mathcal{P}_0 \times \mathfrak{g}^* \to \mathbb{R}$ to $\mathcal{P}_0 \times \{0\}$. Thus by the implicit function theorem, for each ν, the function $H(\cdot, \nu)$ has a critical point $y = y(\nu)$. Then define $h : \mathfrak{g}^* \to \mathbb{R}$ by

$$h(\nu) = H(y(\nu), \nu).$$

An easy exercise shows that the restriction of h to $\mathcal{O}_\mu$ has a critical point at ν if and only if the restriction of H to $\mathcal{P}_\mu$ has a critical point at $(y(\nu), \nu)$. The problem is therefore reduced to one of finding critical points of a function on coadjoint orbits. One can use Morse theory or Lyusternik-Schnirelman techniques to estimate the number of critical points, and so the number of bifurcating relative equilibria.

We assumed that G_{x_e} is trivial. However, if G_{x_e} is finite, the same argument applies, but now the resulting function $h : \mathcal{O}_\mu \to \mathbb{R}$ is G_{x_e} invariant. Here G_{x_e} is acting (semi)symplectically by the modified coadjoint action. Moreover, if ν is a critical point of h with isotropy $K < G_{x_e}$, then the corresponding relative equilibrium also has isotropy group K. Analogous to Theorem 6.4, we have the following theorem.

Theorem 7.1 (Montaldi, Roberts [MR99]) *Let x_e be a non-degenerate relative equilibrium with $\mathbf{J}(x_e) = 0$, and K be a subgroup of G_{x_e}. Suppose further that G_{x_e} is finite. Then an isolated point of $\mathrm{Fix}(K, \mathcal{O}_\mu)$ with μ close to zero, corresponds to a relative equilibrium with isotropy containing K.*

For an application of this theorem to point vortices on a sphere, see Section 8.3. In [MR99], the stability of the bifurcating relative equilibria is also calculated using these methods. In the case of a relative equilibrium with a non-zero momentum, one can also give a lower bound of the number of relative equilibria on the nearby reduced spaces (see [Mo97]).

Remark 7.2 Since the function h is defined on $\mathfrak{g}^*$, its differential at any point $dh(\nu) \in \mathfrak{g}$. If ν is a critical point of the restriction of h to $\mathcal{O}_\nu$, so ν is a relative equilibrium, then $dh(\nu)$ is in fact the angular velocity of the relative equilibrium in question, [Mo97].

8 Examples

In this section, we apply the previous work to three symmetric Hamiltonian systems:

- Molecules (as classical mechanical systems)
- Point vortices in the plane
- Point vortices on the sphere

where a *point vortex* is an infinitesimal region of vorticity in a 2-dimensional fluid flow.

The study of molecules is of interest in molecular spectroscopy, while point vortices are of interest in modelling concentrated region of vorticity such as hurricanes. Note that the action of the symmetry group is free and proper in these three examples (if one has more than 2 vortices on the sphere).

8.1 Molecules

We consider a molecule consisting of N atoms. We take advantage of the Born-Oppenheimer approximation, which means essentially that we ignore the movement of the electrons. We obtain a model for the nuclei alone, interacting via a potential energy function which incorporates the effects of the electrons. The configuration space is $\mathbb{R}^{3N}$ and the phase space is $\mathcal{P} = T^*\mathbb{R}^{3N} = \mathbb{R}^{6N}$. After fixing the centre of mass at the origin, the dimension of the phase space becomes $6N - 6$.

Let m_i, q_i and $p_i = m_i\dot{q}_i$ be respectively the mass, the position and the momentum of the i^{th} nucleus. The Hamiltonian of the system is given by:

$$H = \sum_i \frac{1}{2m_i}|p_i|^2 + V(q_1, \ldots, q_N)$$

where V is the potential energy due to the electronic bonding between the nuclei. In the absence of external force, V is $\mathbf{O}(3)$-invariant, and so is the Hamiltonian. Moreover, if some nuclei are identical, then a subgroup Σ of the permutation group S_N acts on the set of the nuclei, and Σ leaves H invariant. As with any classical Hamiltonian system of the form "kinetic + potential", the system is time reversible: H is invariant under the involution $\tau : (p, q) \mapsto (-p, q)$. Finally, the Hamiltonian is $\hat{G}$-invariant where $\hat{G} = \mathbf{O}(3) \times \Sigma \times \mathbb{Z}_2^\tau$. Here we use $\hat{G}$ for a semisymplectic group action, whose symplectic part is G; the temporal character is just the projection $\chi : \mathbf{O}(3) \times \Sigma \times \mathbb{Z}_2^\tau \to \mathbb{Z}_2^\tau$.

The $\mathbf{SO}(3)$-symmetry leads to the following momentum map (see Ratiu's lectures for how to compute a momentum map when $\mathcal{P}$ is a cotangent bundle):

$$\mathbf{J} = \sum_i q_i \times p_i$$

where we identified $\mathfrak{so}(3)^*$ with $\mathbb{R}^3$ (this is of course the usual angular momentum for a collection of particles in $\mathbb{R}^3$). The momentum map is equivariant with the $\mathbf{SO}(3)$-coadjoint action (Theorem 5.2), but it is also $\hat{G}$-equivariant with the following action on $\mathfrak{so}(3)^* \simeq \mathbb{R}^3$:

$$(A, \sigma, \tau^k) \cdot \mu = (-1)^k \det(A) A\mu$$

where $(A, \sigma, \tau^k) \in \hat{G} = \mathbf{O}(3) \times \Sigma \times \mathbb{Z}_2^\tau$, see equation (5.1). The action of $\mathbf{O}(3) \times \Sigma$ is not free, however the action of $\mathbf{SO}(3)$ is free away from collinear configurations. We obtain the following theorem using Section 7. Let the *axis of reflection* be the line through the origin perpendicular to the plane fixed by the reflection.

Theorem 8.1 ([MR99]) *Consider a molecule with a non-degenerate equilibrium with symmetry group $\Gamma < \mathbf{O}(3) \times \Sigma$. There exists $\mu_0 > 0$ such that for all $\mu \in \mathbb{R}^3$ with $|\mu| < \mu_0$ there are at least 6 relative equilibria with momentum μ. Moreover, for each axis l of rotation or reflection in Γ, there are two relative equilibria rotating around the axis l with angular momentum μ, one rotating in each direction.*

The minimum of 6 relative equilibria is a consequence only of time reversal symmetry, and thee relative equilibria are similar to the six occurring for the rigid body.

For example, the methane molecule CH_4 has a tetrahedral symmetry, it has 7 axes of rotation and 6 axes of reflection. By the theorem, there are 26 families of relative equilibria bifurcating from the tetrahedral equilibrium. This result depends only on the tetrahedral symmetry, so it is also true for a molecule such as P_4 (white phosphorous).

However, the stability analysis depends on the molecule in question, this is carried out in [MR99]. For a complete investigation of the very interesting molecule H_3^+, see [KRT99].

8.2 Point vortices in the plane

The literature on planar point vortices is large ([Ar82, Ar83a, Ar83b, AV98, Sa92]), usually treated as a problem in fluid mechanics. Here we present the problem in terms of geometric mechanics.

We consider N point vortices in a planar flow of an ideal fluid. The equations governing the motion of the N point vortices are:

$$\dot{\overline{z_j}} = \frac{1}{2\pi i} \sum_{k, k \neq j} \kappa_k \frac{1}{z_j - z_k}$$

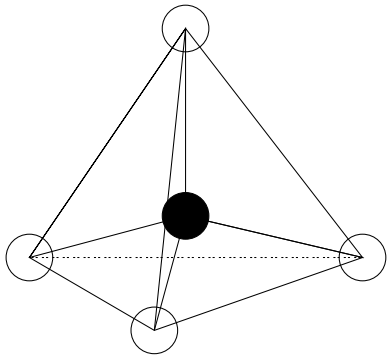

Fig. 8.1. The tetrahedral symmetry of the CH_4 molecule.

where z_j is a complex number representing the position of the j-vortex (we identified the plane with $\mathbb{C}$), and κ_j is the vorticity of the j-vortex. The Hamiltonian for this system is

$$H = -\frac{1}{4\pi}\sum_{i<j}\kappa_i\kappa_j \ln|z_i - z_j|^2$$

and the symmetry group is $\mathbf{SE}(2)$ (which is not compact). After identifying $\mathbf{SE}(2)$ with $\mathbb{C} \rtimes \mathbf{S}^1$ and so $\mathfrak{se}(2)^*$ with $\mathbb{C} \times \mathbb{R}$, the momentum map of the system is

$$\mathbf{J}(z_1, \ldots, z_N) \;=\; \left(i\sum_j \kappa_j z_j,\; -\tfrac{1}{2}\textstyle\sum_j \kappa_j |z_j|^2 \right).$$

This momentum map is interesting because it fails to be equivariant with the coadjoint action on $\mathfrak{se}(2)^*$, and we must use the modified coadjoint action to make it equivariant (see Theorem 5.2):

$$\mathrm{Coad}^{\Lambda}_{(u,A)}(\nu,\psi) \;=\; \big(A\nu,\; \psi + \Im(A\nu\overline{u})\big) \;+\; \Lambda\big(iu, \tfrac{1}{2}|u|^2\big)$$

where $(u, A) \in \mathbb{C}\rtimes\mathbf{S}^1$, $(\nu, \psi) \in \mathbb{C}\times\mathbb{R}$, $\Lambda = \sum_j \kappa_j$, and $\Im(z)$ is the imaginary part of z. The term $\Lambda(iu, |u|^2/2)$ is Souriau's cocycle.

If $\Lambda = 0$, then the coadjoint orbits are of two types: either points on the ψ-axis or cylinders around that axis; if, on the other hand, $\Lambda \neq 0$, the orbits are all paraboloids, with axis equal to the ψ-axis.

2 vortices All configurations are relative equilibria. Indeed, it follows from the equations of motion that if $\Lambda = \kappa_1 + \kappa_2 \neq 0$, then the two vortices rotate

about the fixed point $\frac{1}{\Lambda}(\kappa_1 z_1 + \kappa_2 z_2)$, while if $\Lambda = 0$ they translate together towards infinity in the direction orthogonal to the segment joining them.

These results can also be derived using the previous sections. Indeed, the reduced spaces are just single points, and so correspond to relative equilibria by Proposition 6.1. Moreover if $\Lambda \neq 0$, then for all μ, $G_\mu \simeq \mathbf{SO}(2)$ and the relative equilibria are Lyapunov stable modulo G_μ by Theorem 6.9 (the symplectic slice is trivial).

3 vortices A complete analysis of the motion of three planar vortices was given by Synge [Sy49]. Here we present a modern approach of the problem but restrict attention to non-collinear configurations for brevity and simplicity. Points of the orbit space $\mathcal{P}/G = \mathbb{C}^3/\mathbf{SE}(2)$ correspond to shapes of oriented triangles. Since we are away from collinear configurations, the orientation determines two isomorphic connected components, so we may ignore the orientation. A point in the orbit space is therefore determined by the three lengths l_{12}, l_{13}, l_{23} where $l_{ij} = |z_i - z_j|$. Denote $r_1 = l_{23}^2$, $r_2 = ...$, then the Hamiltonian on $\mathcal{P}/G$ and the orbit momentum map (Section 5.3) are respectively:

$$\begin{aligned} 4\pi H(r_1, r_2, r_3) &= -\kappa_1\kappa_2 \ln(r_3) - \kappa_2\kappa_3 \ln(r_1) - \kappa_1\kappa_3 \ln(r_2) \\ \mathcal{J}(r_1, r_2, r_3) &= p \circ \mathbf{J} = -\kappa_1\kappa_2 r_3 - \kappa_2\kappa_3 r_1 - \kappa_1\kappa_3 r_2 \end{aligned}$$

where the projection $p : \mathfrak{g}^* \simeq \mathbb{C} \times \mathbb{R} \to \mathfrak{g}^*/G \simeq \mathbb{R}$ is given by $p(\nu, \psi) = |\nu|^2 - 2\Lambda\psi$. Recall that the reduced spaces are the fibers of $\mathcal{J}$. The relative equilibria are determined by the critical points of the restriction of H to the reduced spaces, and are therefore critical points of $H - \eta\mathcal{J}$ for some η (η is a Lagrange multiplier). A short computation shows that relative equilibria are all equilateral triangle, of side r say, with $\eta = 1/(4\pi r)$. Then we rely the angular velocity ξ with η:

$$0 = d(H - \eta\mathcal{J})(x_e) = d(H - \xi\mathbf{J})(x_e).$$

Since $\mathcal{J} = p \circ \mathbf{J}$, it follows that $\xi = \eta\, dp(\mu)$ with $\mu = \mathbf{J}(x_e) = (\nu, \psi)$, so

$$\xi = \eta(\nu, -\Lambda).$$

Thus if $\Lambda = 0$, then the motion is rectilinear with constant velocity $\xi = 2\eta\nu = i\sum_j \kappa_j z_j/(2\pi r^2)$.

To determine the stability of these relative equilibria, we use the *reduced energy-momentum method* ([SLM91]); that is, we examine definiteness of the restriction of $d^2(H - \eta\mathcal{J})(x_e)$ to $T_{x_e}\mathcal{P}_\mu$. In fact, this is the energy-momentum method for the reduced Hamiltonian system $(\mathcal{P}_\mu, \omega_\mu, H_\mu)$. The tangent space $T_{x_e}\mathcal{P}_\mu$ is spanned by the two vectors $(\kappa_1, -\kappa_2, 0)$ and $(\kappa_1, 0, -\kappa_3)$, so after some calculus:

$$d^2(H-\eta\mathcal{J})(x_e)_{|T_{x_e}\mathcal{P}_\mu}=\frac{\kappa_1\kappa_2\kappa_3}{4\pi r^2}\begin{pmatrix}\kappa_1+\kappa_2 & \kappa_1\\ \kappa_1 & \kappa_1+\kappa_3\end{pmatrix}$$

This quadratic form is definite if and only if $\sigma_2(\kappa)>0$ where $\sigma_2(\kappa)=\kappa_1\kappa_2+\kappa_1\kappa_3+\kappa_2\kappa_3$. It follows that the equilateral triangles are Lyapunov stable modulo G if and only if $\sigma_2(\kappa)>0$.

The 4-vortex problem has been recently studied by Patrick [Pa00] in the case of zero-momentum configurations. Arrangements involving an arbitrary number of vortices (n-gon/kn-gon) have been considered by Lewis and Ratiu [LR96]. We suggest to check the stability of the relative equilibria of a simulation available on the web[1]. To end, the dynamics of perturbed relative equilibria is studied in [Pa99] both on the sphere and on the plane.

8.3 Point vortices on the sphere

We consider here N point vortices in a spherical layer flow of an ideal fluid. The configuration space is

$$\mathcal{P}=\{(x_1,...,x_N)\in S^2\times\cdots\times S^2 \mid x_i\neq x_j \text{ if } i\neq j\};$$

we do not permit collisions. Let θ_i be the co-latitude, ϕ_i be the longitude, and κ_i be the vorticity of the i-vortex. The equations governing the motion of the N point vortices on the sphere were obtained by Bogomolov [B77]:

$$\dot\theta_i=-\sum_{j=1,j\neq i}^{N}\kappa_j\frac{\sin\theta_j\sin(\phi_i-\phi_j)}{l_{ij}^2}\ ,\ i=1...N$$

$$\sin\theta_i\,\dot\phi_i=\sum_{j,j\neq i}\kappa_j\frac{\sin\theta_i\cos\theta_j-\sin\theta_j\cos\theta_i\cos(\phi_i-\phi_j)}{l_{ij}^2}\ ,\ i=1...N$$

where $l_{ij}^2=2(1-\cos\theta_i\cos\theta_j-\sin\theta_i\sin\theta_j\cos(\phi_i-\phi_j))$ is the square of the Euclidian distance $\|x_i-x_j\|$. The system is Hamiltonian with

$$H=\sum_{i<j}\kappa_i\kappa_j\ln l_{ij}^2,\quad \omega=\bigoplus_{j=1}^{N}\kappa_j\omega_j,$$

where ω_j is the standard area form on the sphere. The dynamical system has full rotational symmetry $G=\mathbf{SO}(3)$, and hence has a 3-component conserved

[1] `http://www.mindspring.com/~brian_tvedt/java.html` and `http://www.ma.umist.ac.uk/jm/vortex.html`

quantity (see Theorem 5.1). We identify $\mathfrak{so}(3)$ with $\mathbb{R}^3$ in the usual way, the momentum map is then given by:

$$\mathbf{J}(x) = \sum_{j=1}^{N} \kappa_j \, x_j.$$

In what follows, we describe some relative equilibria.

2 vortices If the momentum μ of the configuration is non-zero, the two vortices rotate around μ at the same angular velocity, so this is a relative equilibrium. The only possible case for $\mu = 0$ is if $\kappa_1 = \kappa_2$ and $x_1 = -x_2$ which is a fixed equilibrium.

3 vortices The relative equilibria formed of 3 point vortices are completely described in the paper of Kidambi and Newton [KN98], analogous to that of Synge for the planar vortex model. There are two classes of relative equilibria, those lying on a great circle, and those which are equilateral triangle. All equilateral triangle configurations are relative equilibria. The stability of these relative equilibria is computed in [PM98]. One can also use the method described for planar equilateral triangles in the previous section. In this way, one finds that equilateral triangles which do not lie on a great circle are Lyapunov stable modulo $\mathbf{SO}(2)$ if and only if

$$\sigma_2(\kappa) := \kappa_1\kappa_2 + \kappa_1\kappa_3 + \kappa_2\kappa_3 > 0.$$

Note that the stability condition is the same as in the planar case (great circle configurations correspond to collinear configurations, the algebraic volume vanishes there).

Right-angled isosceles triangle lying on a great circle are also relative equilibria, they are Lyapunov stable modulo $\mathbf{SO}(2)$ provided

$$\kappa_2^2 + \kappa_3^2 > 2\sigma_2(\kappa)$$

where x_1 is at the right-angle [PM98].

As in the planar 2-vortex system, one can derive a stability result by a dimension count. Let x_e be a zero-momentum configuration, that is $\mu = \mathbf{J}(x_e) = 0$. Thus x_e lies on a great circle, $\mathbf{SO}(3)_\mu = \mathbf{SO}(3)$ is compact and $\dim \mathcal{P}_0 = 0$. The reduced space $\mathcal{P}_0$ consists of single points, so x_e is a relative equilibrium and is Lyapunov stable modulo $\mathbf{SO}(3)$.

Numerical simulations can be found in [MPS99], and collapse is studied in [KN98],[KN99]. To end, the effect of solid boundaries (such as continents) is taken into account in [KN00].

4 vortices A regular tetrahedron formed of four vortices of arbitrary strength

is always a relative equilibrium [PM98]; the stability of these configurations has not been determined. It is easy to show from the equations of motion that a square lying on a great circle is always a relative equilibrium. Contrary to the 3-vortex case, squares not lying on a great circle are not relative equilibria unless the four vorticities are identical.

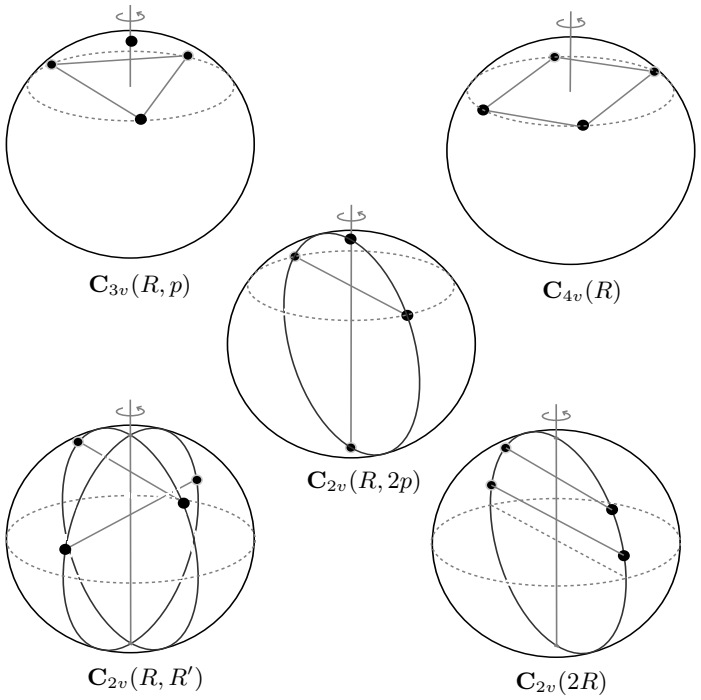

Fig. 8.2. Relative equilibria for 4 identical vortices on the sphere.

N identical vortices In the case of N identical vortices, permutation symmetries arise: the Hamiltonian is $\mathbf{O}(3) \times S_N$-invariant, whereas the vector field is only $\mathbf{SO}(3) \times S_N$-equivariant. The classification of symmetric relative equilibria is carried out in [LMR00]. For example, a regular ring (that is a regular polygon at a fixed latitude) with possibly some vortices at the poles and regular polyhedra, are all relative equilibria. As an exercise, apply Theorem 6.4 to carry out these results.

The linear stability of a regular ring of N identical vortices was studied by Dritschel and Polvani [PD93]. Recently, the results were extended in terms of *Lyapunov stability* in [LMR04], where other configurations of rings and polar vortices are also considered. The result for the single ring is as follows:

Proposition 8.2 *A regular ring of N identical vortices is Lyapunov stable if and only if one of the following assertions is satisfied:*
- $N = 2$ *or* $N = 3$
- $N = 4$ *and* $\cos^2(\theta) > 1/3$
- $N = 5$ *and* $\cos^2(\theta) > 1/2$
- $N = 6$ *and* $\cos^2(\theta) > 4/5$

where θ is the colatitude of the ring.

Note that the rings are "more" stable near the poles than near the equator; while as the number of vortices is increased so the region for which the relative equilibrium is stable diminishes.

2N vortices with opposite vorticities Here we consider N vortices with vorticity $+1$ and N vortices with vorticity -1. The Hamiltonian in this case is $\mathbf{O}(3) \times S_N \times S_N \rtimes \mathbb{Z}_2[\tau]$-invariant, where τ is a permutation of order two which exchanges the $(+1)$-vortices with the (-1)-vortices. The vector field is $\mathbf{SO}(3) \times S_N \times S_N$-equivariant. As before, the relative equilibria are determined using Section 6.2 and stability is computed using Section 6.3, the results can be found in [LP00]. For example, a regular ring formed of the $(+1)$-vortices together with a similar regular ring at the opposite latitude formed of the (-1)-vortices is a relative equilibrium if the offset between the two rings is an integer multiple of π/N.

Bifurcations Changes of stability often involve bifurcations of relative equilibria. Consider the case of a regular ring formed of 4 identical vortices ($\mathbf{C}_{4v}(R)$ in Figure 8.2). By the above proposition, this relative equilibrium is Lyapunov stable if and only if $\cos^2(\theta) > 1/3$ where θ is the colatitude of the ring. In fact, an eigenvalue vanishes when $\cos^2(\theta) = 1/3$, so a bifurcation occurs. As $\cos^2(\theta)$ decreases through $1/3$, there appears a new family of relative equilibria consisting of two rings of two vortices each, the offset between the two rings being equal to $\pi/2$ (denoted $\mathbf{C}_{2v}(R, R')$ in Figure 8.2). These bifurcating relative equilibria are Lyapunov stable: we are in the presence of a supercritical pitchfork bifurcation (see Section 3.2). Others bifurcations such as subcritical pitchfork bifurcation or Hamiltonian-Hopf bifurcation, are described in [LP00].

Bifurcations from zero-momentum state Here we apply Section 7 to our problem. At a zero-momentum configuration, bifurcations occur because the reduced spaces for $\mu = 0$ and $\mu \neq 0$ have different geometry. Consider the arrangement x_e consisting of a regular ring of N identical vortices on the equator. This configuration is a relative equilibrium (in fact a fixed equilibrium)

with momentum zero, and isotropy group isomorphic to $D_{Nh} \simeq D_N \times \mathbb{Z}_2$ (we use the Schönflies-Eyring notation for subgroups of $\mathbf{O}(3)$). Nearby reduced spaces are then locally of the form $\mathcal{P}_\mu \simeq \mathcal{P}_0 \times \mathcal{O}_\mu$ where $\mathcal{O}_\mu$ is the coadjoint orbit through μ, which is a sphere in the present case. It follows from Section 7 that the relative equilibria on $\mathcal{P}_\mu$ near x_e are critical points of a D_{Nh} invariant function $h : \mathcal{O}_\mu \to \mathbb{R}$. The set $\mathrm{Fix}(\mathbf{C}_{Nv}, \mathcal{O}_\mu)$ just consists of the North and South poles, and $\mathbf{C}_{Nv}$ is a maximal isotropy subgroup for $\mu \neq 0$, so by Theorem 7.1 there exist near x_e relative equilibria with isotropy $\mathbf{C}_{Nv}$. These are rotating rings of N vortices at a fixed latitude. The subgroup $\mathbf{C}_{2v}$ with axis of rotations lying in the equatorial plane, is also a maximal isotropy subgroup for $\mu \neq 0$. The same argument holds and provides relative equilibria consisting of m 2-rings with one pole if $N = 2m + 1$, and $(m - 1)$ 2-rings with two poles if $N = 2m$.

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