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A class of blocks behaving like blocks of p-solvable groups ☆

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Abstract

We define a class of blocks having similar properties to blocks of p-solvable groups, and show that a version of the Fong–Swan theorem holds for irreducible Brauer characters in such blocks. We also show that the height of an irreducible character in such block is bounded by the exponent of the central quotient of a defect group, which in particular implies that if further the defect groups are abelian, then every irreducible character in the block has height zero.

1. Introduction

Let G be a finite group, p a prime. Let Q be a p-subgroup of G and

$$\mathcal{G} : 1 = G_0 \lhd G_1 \lhd \cdots \lhd G_{n-1} \lhd G_n = G$$

a normal series for G.

We say that $\mathcal{G}$ is *compatible* with Q if for each $i > 0$, G_i/G_{i-1} is a p-group or $Q_i = Q_{i-1}$, where $Q_i = Q \cap G_i$. Note that if $\mathcal{G}$ is compatible with Q, then every refinement of

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$\mathcal{G}$ is compatible with Q, and, in particular, some composition series must be compatible with Q. Another important property is that if $\mathcal{G}$ is compatible with Q, then $\mathcal{G}$ is compatible with every subgroup of Q. A composition series $\mathcal{G}$ is compatible with Q if and only if every composition factor of $\mathcal{G}$ is either avoided by or covered by Q (in the sense of [3]).

In general this definition is problematic. In particular, it is not always the case that if one composition series is compatible with Q, then every composition series must be. To see this, consider $G = H \times \langle y \rangle$, where $|\langle y \rangle| = p$ and H is non-abelian simple with $x \in H$ of order p. Let $Q = \langle xy \rangle$. Then $1 \lhd H \lhd G$ is compatible with Q whilst $1 \lhd \langle y \rangle \lhd G$ is not. Notice however that such a Q cannot occur as a defect group of a p-block of G, since a defect group must contain $O_p(G) = \langle y \rangle$. Remarkably, if D is a defect group for some p-block of G, then it is the case that if some composition series is compatible with D, then so is every composition series for G.

Definition 1.1. Write $\mathcal{C}_p$ for the class of p-blocks B of finite groups G such that there is a normal series for G which is compatible with a defect group of B.

Examples of such blocks are p-blocks whose defect groups are contained in a normal p-solvable group (as studied in [4,6]) and blocks of defect zero. Observe that a group is p-solvable if and only if its principal block is in $\mathcal{C}_p$. It should be noted however that we are not attempting to make a general definition of a 'p-solvable block' in the same way that a nilpotent block is a generalization of a block of a p-nilpotent group. Indeed, there are numerous examples of nilpotent blocks of positive defect of non-abelian simple groups, so nilpotent blocks are not in general in $\mathcal{C}_p$. We should also note that there are numerous examples to show that there are blocks in $\mathcal{C}_p$ which are not blocks of p-solvable groups: for example, we could choose any finite group H with a block of defect zero b_1 and any non-trivial p-group Q, then the block $kQ \otimes b_1$ of $H \times Q$ is in $\mathcal{C}_p$. Rather more interesting examples may be found by considering appropriate wreath products.

The rationale for our definition is as follows: One reason blocks of p-solvable groups are of interest is because we may perform the Fong reductions using $O_{p'}(G)$, effectively replacing it by a central p'-subgroup. This allows induction to be used to prove that a desired property holds for p-solvable groups whenever that property is compatible with the Fong reductions. A classic example of this is in the proof of the Fong–Swan theorem. It is now well known that similar reductions may be applied when we consider blocks covering blocks of defect zero of a normal subgroup. It is therefore sensible to consider p-blocks B of groups such that either $O_p(G) \neq 1$ or there is a non-trivial normal subgroup N such that B covers blocks of defect zero of N. We will see that blocks in $\mathcal{C}_p$ satisfy such a condition.

Throughout, a p-block (or block if p is obvious from the context) B of G is defined with respect to a complete discrete valuation ring $\mathcal{O}$ whose residue field $\mathcal{O}/J(\mathcal{O})$ has characteristic p and is algebraically closed and whose field of fractions K has characteristic zero. We assume that $\mathcal{O}$ contains a primitive $|G|^3$ root of unity. Write $\mathrm{Irr}(G, B)$ for the set of irreducible K-characters of G in B. If $N \lhd G$ and b is a block of N, then write $I_G(b)$ for the stabilizer of b under conjugation in G. If $\mu \in \mathrm{Irr}(N)$, then write $I_G(\mu)$ for the inertia subgroup.

For convenience, when writing composition series we allow repeated terms.

The paper is structured as follows: In Section 2 we present some useful properties of composition series compatible with a defect group, including the crucial property that defect groups of blocks in $\mathcal{C}_p$ are compatible with every composition series of the group. In Section 3 we summarize the Clifford theory necessary for the proof of the version of the Fong–Swan theorem, which we give in Section 4. In Section 5 we give a bound on the height of an irreducible character in a block in $\mathcal{C}_p$.

2. Composition series compatible with a defect group

Lemma 2.1. *Let $B \in \mathcal{C}_p$ be a block of a finite group G with a defect group D. Then there is a composition series for G which is compatible with D.*

Proof. There is a composition series $\mathcal{G}: G_0 \lhd G_1 \lhd \cdots \lhd G_{n-1} \lhd G_n$ compatible with some defect group P of B. Then $D = P^g$ for some $g \in G$, and the composition series $\mathcal{G}^g: G_0^g \lhd G_1^g \lhd \cdots \lhd G_{n-1}^g \lhd G_n^g$ is compatible with D. $\square$

Lemma 2.2.

(a) *If $N \lhd G$ and b is a block of N covered by a block B of G such that $B \in \mathcal{C}_p$, then $b \in \mathcal{C}_p$.*
(b) *If $H \leqslant G$ and b is a block of H with Brauer correspondent B in G such that $B \in \mathcal{C}_p$, then $b \in \mathcal{C}_p$.*

Proof. (a) and (b) follow from Lemma 2.1 and the fact that a defect group of a block of a subgroup is contained in a defect group of a covering block or Brauer correspondent, respectively, and are left to the reader. $\square$

The following are crucial for inductive arguments concerning blocks in $\mathcal{C}_p$, since they enable us to move to quotient groups by allowing us to take a composition series which has the relevant normal subgroup as one of its terms.

Proposition 2.3. *Let B be a p-block of a finite group G with a defect group D. Let $\mathcal{G}: 1 = G_0 \lhd G_1 \lhd \cdots \lhd G_{n-1} \lhd G_n = G$ be a composition series for G. If $\mathcal{G}$ is compatible with D, then every composition series for G is compatible with every defect group of B.*

Proof. Suppose that $\mathcal{G}: G_0 \lhd G_1 \lhd \cdots \lhd G_{n-1} \lhd G_n$ is compatible with D, and that $\mathcal{H}: 1 = H_0 \lhd \cdots \lhd H_n = G$ is another composition series for G. We use induction on $|G|$ to show that $\mathcal{H}$ is also compatible with D. The result then follows by Lemma 2.1. So assume that the result holds for every p-block of every group of order strictly less than $|G|$. We may assume that $n \geqslant 2$.

Write $H = H_{n-1}$, $K = G_{n-1}$ and $N = H \cap K$. Suppose first that $H = K$. By [1, 15.1] there is a block b of H covered by B with defect group $D \cap H$, and by Lemma 2.2, $b \in \mathcal{C}_p$. Since $\mathcal{G}$ is compatible with D, either G/H is a p-group or $D \leqslant H$. By induction

$H_0 \lhd \cdots \lhd H_{n-1}$ is compatible with $D \cap H$, and it follows that $\mathcal{H}$ is compatible with D in this case.

Now suppose that $H \neq K$. Then $H/N \cong G/K$ and $G/H \cong K/N$. By [1, 15.1] there is a block of N covered by B with defect group $D \cap N$ and a block of K covered by B with defect group $D \cap K$. Let $\mathcal{N} : 1 = N_0 \lhd \cdots \lhd N_{n-2}$ be a composition series for N. By Lemma 2.2 and induction, $N_0 \lhd \cdots \lhd N_{n-2} \lhd K$ is a composition series compatible with $D \cap K$. Hence $N_0 \lhd \cdots \lhd N_{n-2} \lhd K \lhd G$ is compatible with D. Similarly $N_0 \lhd \cdots \lhd N_{n-2} \lhd H$ and $H_0 \lhd \cdots \lhd H_{n-2} \lhd H$ are compatible with $D \cap H$. We must show that G/H is a p-group or $D \leqslant H$.

If G/N is a p-group or $D \leqslant N$, then we are done.

If $D \leqslant K$ but $D \nleqslant N$, then K/N is a p-group and again we are done.

Hence we may assume that $D \nleqslant K$ and $D \cap K \leqslant N$. Then $|G/K| = |H/N| = p$, and $|D| = p|D \cap N|$. We may also assume that $G/H \cong K/N = X$, where X is non-abelian simple. Write $H/N = Q$ and $\overline{G} = G/N$. Then $\overline{G} = X \times Q$.

The blocks of N covered by B form a single G-conjugacy class $\{b_1, \ldots, b_t\}$, which we identify with $\{1, \ldots, t\}$. The conjugation action gives a homomorphism $\varphi : \overline{G} \to S_t$. Write $\overline{G}_i$ for the stabilizer in $\overline{G}$ of i under this action. By [1, 15.1], the stabilizer of each b_i contains a defect group of B, hence $\overline{D} := DN/N$ is contained in some $\overline{G}_i$. Without loss of generality assume $\overline{D} \leqslant \overline{G}_1$.

Since H/N is a p-group, by [5, V.3.5] there is a unique block B_H of H covering b_1. By [1, 15.1], there is a defect group P of B_H such that $\overline{P}$ is a Sylow p-subgroup of $\overline{H}_1$. But B_H is covered by B, so P is contained in a defect group of B. Hence if $Q \leqslant \overline{G}_1$, then $\overline{P} = Q$ and $D \leqslant H$ by consideration of the order of D, and we are done. The same argument shows that we are done if Q fixes any point of $\{1, \ldots, t\}$. We assume otherwise, i.e., that Q acts regularly, and derive a contradiction.

Suppose that $X \leqslant \overline{G}_1$. Then $\overline{D} \leqslant \overline{G}_1 = X$, so $D \leqslant K$, a contradiction.

We have shown that φ is faithful and that Q acts regularly (as a product of t/p disjoint p-cycles). Further, $\varphi(Q) \leqslant Z(\varphi(\overline{G}))$. Hence $\varphi(X)$ is isomorphic to a subgroup of $C_p \wr S_{t/p} \leqslant S_t$.

Now $\overline{D}$ has order p and is not Q, so there are $x \in X$ and $y \in Q$, both of order p, such that $\overline{D} = \langle xy \rangle$. Recall that we have $\overline{D} \leqslant \overline{G}_1$, so $\varphi(x^{-1})$ has a p-cycle in common with $\varphi(y)$. Hence $\varphi(X)$ must be a transitive subgroup of S_t, a contradiction since every transitive subgroup of $C_p \wr S_{t/p}$ in S_t has a non-trivial proper normal subgroup. □

Lemma 2.4. *If $B \in \mathcal{C}_p$ is a block of a non-abelian group G, and $O_p(G) \leqslant Z(G)$, then there is a non-trivial normal subgroup N of G, strictly containing $Z(G)$, with $D \cap N = O_p(G)$.*

Proof. Let N be a normal subgroup of G minimal subject to it strictly containing $Z(G)$. Then $N/Z(G)$ is a direct product of simple groups. $N/Z(G)$ cannot be a p-group, otherwise $O_p(G) \nleqslant Z(G)$. If $N/Z(G)$ is a p'-group, then we are done. Otherwise $N/Z(G)$ is a direct product of isomorphic non-abelian simple groups. By Lemma 2.2, if b is a block of N covered by B, then $b \in \mathcal{C}_p$ and hence has defect group $O_p(G)$. □

As a consequence, if $O_p(G) = 1$ and $B \in \mathcal{C}_p$, then with N as in the above lemma, B covers blocks of defect zero of N.

3. Clifford theory

Lemma 3.1. *Let $B \in \mathcal{C}_p$ be a block of G with a defect group D. If $\overline{B}$ is a p-block of $G/O_p(G)$ possessing an irreducible Brauer character which belongs to B when lifted to G, then $\overline{B} \in \mathcal{C}_p$.*

Proof. Choose a composition series $\mathcal{G} : G_0 \lhd \cdots \lhd G_n$ for G which refines $1 \lhd O_p(G) \lhd G$. By Proposition 2.3, $\mathcal{G}$ is compatible with D. By [5, V.4.2], $\overline{B}$ has a defect group P contained in $D/O_p(G)$. But $\overline{\mathcal{G}} : G_0/O_p(G) \lhd \cdots \lhd G_n/O_p(G)$ is compatible with $D/O_p(G)$, and so is also compatible with P. □

Following [2, 3.1] we have the following (see also [7]):

Theorem 3.2. *Let B be a block of positive defect of G and $N \lhd G$. Suppose that B covers a p-block b of defect zero of N. Write $I = I_G(b)$. Then there is a central extension $\tilde{I}$ of I/N by a cyclic p'-group $\widetilde{W}$ and a block $\tilde{b}$ of defect zero of $\widetilde{W}$ such that B is Morita equivalent to a block $\widetilde{B} \in \mathrm{Blk}(\tilde{I})$ covering $\tilde{b}$, with defect groups isomorphic to those of B.*

Further, if $B \in \mathcal{C}_p$, then $\widetilde{B} \in \mathcal{C}_p$.

Proof. Throughout we make use of the proof of [2, 3.1], which gives the first part of the result. Now suppose that $B \in \mathcal{C}_p$.

There is a unique block B_I of I with $B_I^G = B$, and further there is a defect group D of B_I which is also a defect group of B. By Lemma 2.2, $B_I \in \mathcal{C}_p$. By Proposition 2.3 we may choose a composition series $\mathcal{I} : I_0 \lhd \cdots \lhd I_n$ which refines $1 \lhd N \lhd I$ and is compatible with D. Then $\overline{\mathcal{I}} : I_0/N \lhd \cdots \lhd I_n/N$ is compatible with DN/N.

Choose a defect group $\widetilde{D} \leqslant \tilde{I}$ of $\widetilde{B}$ such that $\widetilde{D}\widetilde{W}/\widetilde{W} = DN/N$ and also for each i, $\tilde{I}_i$ containing $\widetilde{W}$ such that $\tilde{I}_i/\widetilde{W} = I_i/N$. Then $\tilde{I}_0 \lhd \cdots \lhd \tilde{I}_n$ is a composition series for $\tilde{I}$ which is compatible with $\widetilde{D}$, and we are done. □

It is well known that if a p-solvable group has trivial intersection (TI) Sylow p-subgroups, then these Sylow p-subgroups must be cyclic or generalized quaternion. Using the classification of finite simple groups, a similar result holds for blocks in $\mathcal{C}_p$:

Proposition 3.3. *Let $B \in \mathcal{C}_p$ be a block of a finite group G, and suppose that B has TI defect groups which are not normal in G. Then the defect groups are cyclic or generalized quaternion.*

Proof. Suppose that B has a defect group which is neither cyclic nor generalized quaternion, i.e., contains a subgroup of the form $C_p \times C_p$. Let b be a block of $O_{p'}(G)$ covered by B. By the proof of [2, 3.1] there is a defect group D of B which is a non-normal TI subgroup of $I_G(b)$. It then follows from [2, 4.1] that $\tilde{I}$ in Theorem 3.2 has non-normal TI defect groups isomorphic to D. Hence we may assume that $O_{p'}(G) \leqslant Z(G)$. Since $B \in \mathcal{C}_p$ we have $D \cap F^*(G) = 1$ (where $F^*(G)$ is the generalized Fitting subgroup), and the result follows by [2, 2.8]. □

4. Lifting Brauer characters

We show that a version of the Fong–Swan theorem holds for blocks in $\mathcal{C}_p$.

Theorem 4.1. *Let $B \in \mathcal{C}_p$ be a block of G and let $\varphi \in \mathrm{IBr}(G, B)$. Then there is $\chi \in \mathrm{Irr}(G, B)$ such that $\varphi = \chi_{p'}$, the restriction of χ to the p-regular elements of G.*

Proof. We use induction first on $[G : Z(G)]$, then on $|G|$. Let φ be an irreducible Brauer character in B. Then φ may be regarded as an irreducible Brauer character belonging to a block, say $\overline{B}$, of $G/O_p(G)$. By Lemma 3.1, $\overline{B} \in \mathcal{C}_p$. If $O_p(G) \neq 1$, then by induction there is an irreducible character χ of $G/O_p(G)$ with $\chi_{p'} = \varphi$. When regarded as a character for G, we again have $\chi_{p'} = \varphi$, so we may assume that $O_p(G) = 1$.

Let D be a defect group for B. By Lemma 2.4 there is a normal subgroup $N \neq 1$ of G such that $D \cap N = 1$ and N strictly contains $Z(G) = O_{p'}(Z(G))$. By Theorem 3.2 there is a block $\widetilde{B} \in \mathcal{C}_p$ of a group $\tilde{I}$ which is Morita equivalent to B, and $[\tilde{I} : Z(\tilde{I})] \leqslant [G : N] < [G : Z(G)]$. By induction, every irreducible Brauer character in $\widetilde{B}$ is given by the restriction of an irreducible character to p-regular elements. Since Morita equivalent blocks have the same decomposition matrices (up to reordering of characters), the result follows. □

Of course as a consequence of this, every irreducible Brauer character in a block of G in $\mathcal{C}_p$ has degree dividing $|G|$ (in fact dividing $[G : A]$ for every abelian normal subgroup A of G).

5. Heights of irreducible characters in blocks in $\mathcal{C}_p$

The defect of an irreducible character χ is the integer $d(\chi)$ such that $p^{d(\chi)}\chi(1)_p = |G|_p$ the height of χ is the difference between $d(\chi)$ and the defect of the block containing χ.

Note from [8] that $\chi \in \mathrm{Irr}(G)$ is called N-projective for $N \lhd G$ if there is a relatively N-projective $\mathcal{O}G$-module affording χ, and that this is equivalent to $\chi(1)_p = [G : N]_p\mu(1)_p$ for each $\mu \in \mathrm{Irr}(N)$ covered by χ.

Theorem 5.1. *Let $B \in \mathcal{C}_p$ be a block of G with a defect group D, and let $[D : Z(D)] = p^c$. If $\chi \in \mathrm{Irr}(G, B)$, then χ has height at most c.*

Proof. Let $\chi \in \mathrm{Irr}(G, B)$ be a counterexample with $|G|$ minimized. Then $|G| > 1$. Write h for the height of χ. Let N be a maximal normal subgroup of G. We choose a block b of N covered by B with defect group $D \cap N$ and $\mu \in \mathrm{Irr}(N, b)$ covered by χ. Then $D \leqslant N$ or $[G : N] = p$. By Lemma 2.2, $b \in \mathcal{C}_p$.

Suppose first that $D \leqslant N$. Then χ is N-projective, and $\chi(1)_p = [G : N]_p\mu(1)_p$. Hence μ has height h in b. This contradicts minimality since $D = D \cap N$ is also a defect group for b.

Hence $D \nleqslant N$, and $[G : N] = p$. Note that $|D| = p|D \cap N|$. Then $I_G(\mu) = N$ or G. If $I_G(\mu) = N$, then $\chi = \mu^G$ and μ has height $h - 1$. By [5, V.1.2], b has Brauer correspondent B, and so by [5, V.1.6], $Z(D) \leqslant D \cap N$. Then $[D \cap N : Z(D \cap N)] =$

$\frac{1}{p}[D : Z(D \cap N)] < [D : Z(D)]$, contradicting minimality. So $I_G(\mu) = G$. Since G/N is cyclic, $\mu = \mathrm{Res}_N^G(\chi)$, and μ has height h. But $[D \cap N : Z(D \cap N)] \leqslant [D : Z(D)]$, again contradicting minimality. □

We immediately have half of Brauer's abelian defect group conjecture for blocks in $\mathcal{C}_p$:

Corollary 5.2. *Let $B \in \mathcal{C}_p$ be a block of G with an abelian defect group D. Then every irreducible character in B has height zero.*

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