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MODEL THEORY OF COMODULES

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The purpose of this paper is to establish some basic points in the model theory of comodules over a coalgebra. It is not even immediately apparent that there is a model theory of comodules since these are not structures in the usual sense of model theory. Let us give the definitions right away so that the reader can see what we mean.

Fix a field k. A k-**coalgebra** C is a k-vector space equipped with a k-linear map $\Delta : C \longrightarrow C \otimes C$, called the **comultiplication** (by $\otimes$ we always mean tensor product over k), and a k-linear map $\epsilon : C \longrightarrow k$, called the **counit**, such that $\Delta \otimes 1_C = 1_C \otimes \Delta$ (coassociativity) and $(1_C \otimes \epsilon)\Delta = 1_C = (\epsilon \otimes 1_C)\Delta$, where we identify C with both $k \otimes C$ and $C \otimes k$. These definitions are literally the duals of those for a k-algebra: express the axioms for C' to be a k-algebra in terms of the multiplication map $\mu : C' \otimes C' \longrightarrow C'$ and the "unit" (embedding of k into C'), $\delta : k \longrightarrow C'$ in the form that certain diagrams commute and then just turn round all the arrows. See [4] or more recent references such as [7] for more.

A (right) **comodule** over the coalgebra C is a k-vector space M equipped with a k-linear map $\rho : M \longrightarrow M \otimes C$ which satisfies $1_M \otimes \Delta = \rho \otimes 1_C$ and $(1_M \otimes \epsilon)\rho = 1_M$, where we identify M and $M \otimes k$ (and, of course, $M \otimes (C \otimes C)$ with $(M \otimes C) \otimes C$). Again, the way to understand this definition is to write the axioms for being a unital module M' over an algebra C' in terms of the structure map $M' \otimes C' \longrightarrow M$ in a diagrammatic way and then reverse all arrows.

The structure on a C-comodule is, therefore, the structure of a k-vector space (which is no problem) together with a morphism from M to $M \otimes C$. Recall that what we do with the structure map $M \otimes C' \longrightarrow M'$ of a module M' is to build each function $- \otimes c : M' \longrightarrow M'$, for $c \in C'$, into the language. It is not so clear how to proceed in the case of comodules. That is, does there exist a language in which one may axiomatise the concept of a C-comodule, where C is a fixed k-coalgebra?

It is not difficult to give plausible reasons as to why this question should have a negative answer. But plausibility is not enough, as we shall see.

A key fact that we use is the equivalence of the category of C-comodules with a subcategory of the category of C^*-modules, where C^* is the **dual algebra** of C. As a vector space, C^* is the dual, $\mathrm{Hom}_k(C, k)$, of k and it is easy to verify that the k-coalgebra structure of C induces, in a natural way, the structure of a k-algebra on C^* (if $f, g \in C^*$, $c \in C$ with $\Delta c = \sum_i c_i' \otimes c_i''$, set $(fg)c = \sum_i f(c_i')g(c_i'')$).

Let M, with structure map $\rho : M \longrightarrow M \otimes C$, be a C-comodule. Then M may be given a left C^*-module structure by defining $fm = (1_M \otimes f)\rho(m)$ (and using the identification of $M \otimes k$ with M via $m \otimes 1 \mapsto m$). This extends to give a functor from the category, Comod-C, of right C-comodules to the category C^*-Mod of left C^*-modules. A C^*-module M is said to be **rational** (the term comes via representations of algebraic groups) if for every $m \in M$ there are $m_1, ..., m_n \in M$ and $c_1, ..., c_n \in C$ such that for every $f \in C^*$ we have $fm = \sum_{i=1}^n f(c_i)m_i$ (equivalently for every $m \in M$ we have that $C^*/\{f \in C^* : fm = 0\}$ is finite-dimensional). We denote by C^*-Rat the full subcategory of C^*-Mod whose objects

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are the rational C^*-modules. Then the theorem is the following (see either of the references above).

Theorem 0.1. *Let C be a k-coalgebra. If M is a C-comodule then, with the induced structure, M is a rational C^*-module. This extends to a functor which is an equivalence between* Comod-C *and* C^*-Rat.

If C is finite-dimensional as a k-vector space then every left C^*- module is rational and in this case we have an equivalence between Comod-C and C^*-Mod. Does this mean that, in this case, the concept of a C-comodule is axiomatisable? If we take the view that an object derives its mathematical structure from the category to which it "naturally" belongs then we conclude that C-comodules are C^*-modules and that, in the case where C is finite-dimensional, the concept of C-comodule is axiomatisable.

So the natural question is then whether, in the general case, the rational C^*-modules form an elementary subclass of the class of all C^*-modules. Even if we do not take such a strongly category-theoretic view then, noting that the C-comodule structure is determined by the C^*-module structure and vice-versa, we are led to the same question.

This question was answered in the thesis of the third author.

Theorem 0.2. [5] *There is a coalgebra C such that the category of rational C^*-modules does not form an elementary subclass of the category of C^*-modules.*

Proof. Let C be a vector space over a field k, with basis $\{c_i\}_{i\in\omega}$. Define $\Delta : C \longrightarrow C \otimes C$ by $\Delta c_k = \sum_{i=0}^{k} c_i \otimes c_{k-i}$ ($k \in \omega$) and define $\epsilon : C \longrightarrow k$ by $\epsilon(c_k) = \delta_{0k}$ (Kronecker delta). With these maps, C is a coalgebra.

For $i \in \omega$ define x_i to be the dual map corresponding to c_i, so $x_i(c_j) = \delta_{ij}$. Note that $x_i^2 = x_i$ and $x_i x_j = 0$ if $i \neq j$. The typical element of the dual algebra C^* is a formal sum, $\sum_{i\in\omega} \lambda_i x_i$ ($\lambda_i \in k$).

Each (two-dimensional) module $M_i = C^*/\langle x_j : j \neq i\rangle$ is clearly rational and so the direct sum, $\bigoplus_i M_i$, of these also is rational. For any modules M_i, $\bigoplus M_i$ is elementarily equivalent to $\prod M_i$. But the element $a \in \prod M_i$ which has the image of $1 \in C^*$ at each coordinate has zero annihilator in C^*: in particular, $C^*/\mathrm{ann}_{C^*}(a)$ is infinite-dimensional and hence $\prod_i M_i$ is not rational. □

We remark that one can see easily that the class of rational C^*-modules is axiomatisable within the class of C^*-modules in an infinitary language $L_{\kappa,\infty}$ for suitably large κ.

That, however, is not the end of the story, for the category C^*-Rat is locally finitely presented. This is noted in [2, 5.5], is shown directly in [5] and can be found in [7] in a more general context (see also [3] for related results). This means that every rational module is a direct limit of finitely presented rational modules, where an object A of a category $\mathcal{A}$ is said to be **finitely presented** if the functor $(A, -)$, that is $\mathrm{Hom}_{\mathcal{A}}(A, -)$, commutes with directed colimits (meaning that if we have a morphism from A to $B = \varinjlim B_\lambda$, where the B_λ form a directed system, then this morphism factors through the canonical map $B_\lambda \longrightarrow B$ for some λ). It is not difficult to see that this is equivalent to the usual definition in terms of generators and relations whenever the latter makes sense. One also requires, for $\mathcal{A}$ to be locally finitely presented, that there be just a set of isomorphism types of finitely presented objects in $\mathcal{A}$.

It is well-known, see [1], that any locally finitely presented category $\mathcal{A}$ is elementary: there is a first-order, finitary language such that every object of $\mathcal{A}$ is a structure for this language and such that $\mathcal{A}$ is an axiomatisable class in this language.

Theorem 0.3. [5] *For any coalgebra C over a field k the class of C-comodules is an elementary class in a suitable first order, finitary, language.*

Proof. The category of C-comodules is equivalent to the category of rational C^*-modules, which is elementary. Therefore (for general or particular reasons, see above) the class of C-comodules is elementary. □

The language of a locally finitely presented abelian category $\mathcal{A}$ is multisorted: it has a sort for each finitely presented object of $\mathcal{A}$ (rather one for each object in a small or even skeletal version of the category, $\mathcal{A}^{\mathrm{fp}}$, of finitely presented objects of $\mathcal{A}$). It has a function symbol from sort B to sort A for each morphism from A to B. If M is an object of $\mathcal{A}$ then the elements of M of sort A are simply the elements of the morphism set (A, M). The action of the function corresponding to a morphism is simply composition with that morphism (note the resulting reversal of arrows). Here we are dealing with additive categories so we add a symbol for 0 and a symbol for addition in each sort. Indeed, since everything is a k-vector space we should also add a function symbol for each scalar multiplication in each sort.

We have reached a conclusion which was to us (in this case meaning the second and third authors) not at all expected: although not axiomatisable in an obvious way, the concept of comodule is an elementary one.

The rest of this paper, which is mainly work of the first and second authors, is devoted to pulling this conclusion back to the original category (of C-comodules). For although we have axiomatisability, we do not have axiomatisability in a "natural" (from the comodule theory point of view) sense. To prove something about the model theory of comodules we would first move to the equivalent category of rational modules, work in the multi-sorted language there, and then pull back our results to the original category. So we believe that there is some value in describing the basic concepts of the model theory of Comod-C in comodule terms. We will describe (systems of) equations and hence pp formulas, pure embeddings and pure-injective comodules. Since Comod-C is a locally finitely presented abelian category we do have pp-elimination of quantifiers and all the usual machinery of the model theory of modules. The concepts that we have mentioned are, therefore, the basic ones and, with them, one can go on to build the rest of the theory (in particular the Ziegler spectrum) in the usual way. We do not do this, being content with laying the foundations, since we do not have any particular applications to comodules in mind. What we have done, we hope, is to clear the path for anyone who does see some interesting goals in this direction.

The first question is what should be an equation, or system of equations, with parameters from a given comodule? Because we are not dealing with structures in the usual equational sense, we start with an algebraic view of (solvability of) systems of equations. Let M be a comodule. By a **system of equations with parameters from** M we mean an embedding, $i : K \longrightarrow P$, of comodules, together with a morphism of comodules $f : K \longrightarrow M$. We say that this system is **solvable in** M if there is a morphism $g : P \longrightarrow M$ such that $gi = f$ (and we call such a morphism a **solution** of the system). In categories of equationally defined structures this is equivalent to the usual notion: if we have a system of equations of the form $s_\lambda(\bar{x}, \bar{a}) = t_\lambda(\bar{x}, \bar{a})$ where the s_λ and t_λ are terms and $\bar{x}, \bar{a}$ are possibly infinite sequences of variables, respectively parameters from M (only finitely many of which appear in each equation), then we take P to be the free structure on variables $\bar{x}, \bar{y}$, with $\bar{y}$ matching $\bar{a}$, subject to the relations $s_\lambda(\bar{x}, \bar{y}) = t_\lambda(\bar{x}, \bar{y})$, take K to be the substructure generated by the variables $\bar{y}$ and let f be the morphism which takes y_i to a_i.

The first question to be resolved is: given comodules P and M and a partial map h from the set P to the set M when does this map extend to a morphism of comodules from the subcomodule of P generated by the domain of h to M? We simplify by assuming that the domain of h generates P. A **subcomodule**, P', of a comodule is a vector subspace which satisfies the condition $\rho_P(P') \subseteq P' \otimes C$ where $\rho_P : P \longrightarrow P \otimes C$ is the comodule structure map. So to say that a subset, X, **generates** P is to say that $\rho_P(P) \subseteq X \otimes C$ (the right-hand side consists of sums of tensors $x_i \otimes c_i$).

Note that if the elements p_i $(i \in I)$ generate P as a comodule then they must generate P as a vector space. For if $y \in P$ then we have $\rho_P y = \sum_i p_i \otimes c_i$ for some $p_i \in P$ and $c_i \in C$. But then, identifying $P \otimes k$ with P, and hence identifying the map $1_P \otimes \epsilon : P \otimes C \longrightarrow P \otimes k$ which takes $p \otimes c$ to $p \otimes \epsilon(c)$ with the map from $P \otimes C$ to P which takes $p \otimes c$ to $p\epsilon(c)$, we have $y = (1_P \otimes \epsilon)\rho_P(y) = (1_P \otimes \epsilon)(\sum p_i \otimes c_i) = \sum p_i.\epsilon(c_i)$ is a k-linear combination of the p_i, as required.

Lemma 0.4. *Let P be a comodule, with generating set p_i $(i \in I)$ and let b_i $(i \in I)$ be elements of a comodule M. Then the map $p_i \mapsto b_i$ extends to a comodule morphism from P to M iff*
(i) $\sum p_i \alpha_i = 0$ implies $\sum b_i \alpha_i = 0$ for all $\alpha_i \in k$ (that is, the map extends to a map of vector spaces) and
(ii) if $\rho_P(p_i) = \sum p_j \otimes e_{ij}$ $(e_{ij} \in C)$ then $\rho_M(b_i) = \sum b_j \otimes e_{ij}$.

Proof. Suppose that we have conditions (i) and (ii). Define $g : P \longrightarrow M$ by sending $y = \sum p_i \alpha_i$ to $\sum b_i \alpha_i$. We must check that this is well-defined and a comodule map.

First, it is well-defined. If $y = \sum p_i \alpha_i = \sum p_i \beta_i$ then $\sum p_i(\alpha_i - b_i) = 0$ so, by assumption (i), $\sum b_i(\alpha_i - \beta_i) = 0$, as required.

That the map is k-linear is direct from the k-linearity of ϵ.

To see that it is a comodule map, let $y \in P$. We must show that $\rho_M g(y) = (g \otimes 1_C)\rho_P(y)$ Suppose that $\rho_P(y) = \sum p_i \otimes c_i$, so $y = \sum p_i \alpha_i$ where $\alpha_i = \epsilon(c_i)$. Then $\rho_M g(y) = \rho_M(\sum b_i \alpha_i) = \sum \rho_M(b_i)\alpha_i = \sum_i \sum_j b_j \otimes e_{ij}\alpha_i$ by (ii). Also $(g \otimes 1_C)\rho_P(y) = (g \otimes 1_C)\sum_i \rho_P(p_i)\alpha_i = (g \otimes 1_C)\sum_i \sum_j p_j \otimes e_{ij}\alpha_i = \sum_i \sum_j b_j \otimes e_{ij}\alpha_i$, which equals $\rho_M g(y)$, as required.

For the converse, if we have a comodule morphism then certainly we have (i) since such a morphism is k-linear and condition (ii) is direct from the fact that this morphism commutes with comultiplication. □

Now we suppose that we have an embedding $i : K \longrightarrow P$ of comodules and a morphism $f : K \longrightarrow M$ of comodules and we look for "equations" with parameters from M, a solution to which provides a factorisation of f through i. Suppose that p_i $(i \in I)$ is a generating set for P. In order to describe P we must describe the k-linear relations between these as well as those which hold in $P \otimes C$. That is, we must describe a generating set of relations of the form $\sum_i \rho_P(p_i)\alpha_i + \sum_i p_i \otimes c_i = 0$ (note that $\sum p_i \beta_i = 0$ is equivalent to $\sum p_i \otimes \beta_i = 0$). By a generating set of relations we mean generating in the sense of vector spaces. (Though, since for any C-comodule N, also $N \otimes C$ is canonically a C-comodule, we could take "generating" in the sense of C-comodules. However, as noted earlier, if a set is generating in this sense then it is in fact generating in the sense of vector spaces.)

Proposition 0.5. *Suppose P is a C-comodule (where C is a k-coalgebra) with generating set p_i $(i \in I)$ and with generating set of relations $\sum_i \rho_P(p_i)\alpha_{i\lambda} + \sum_i p_i \otimes c_{i\lambda} = 0$ $(\lambda \in \Lambda)$ where the $\alpha_{i\lambda}$ are from k and the $c_{i\lambda}$ from C. Suppose also that $i : K \longrightarrow P$ is a subcomodule, with generating set k_l $(l \in L)$. Say $\rho_P(k_l) = \sum_i p_i \otimes d_{li} \in P \otimes C$ $(l \in L)$. Let $f : K \longrightarrow M$ be a morphism of comodules. Then there is a comodule morphism $g : P \longrightarrow M$ such that $gi = f$ iff the following system of equations is solvable in M.*

$\sum_i v_i \otimes d_{li} = \rho_M(fk_l)$ $(l \in L)$

$\sum_i \rho(v_i)\alpha_{i\lambda} + \sum_i v_i \otimes c_{i\lambda} = 0$ $(\lambda \in \Lambda)$

Proof. If there is an extension $g : P \longrightarrow M$ of f then it is quickly checked that gp_i (for v_i) ($i \in I$) is a solution of the system of equations.

If, conversely, m_i ($i \in I$) is a solution of this system of equations in M then we claim that the map from P to M defined by sending p_i to m_i and extending by k-linearity is a well-defined morphism of comodules. It is enough to check the conditions of 0.4.

First, if $\sum p_i\alpha_i = 0$ then $(\sum p_i\alpha_i) \otimes 1 = 0$, that is, $\sum p_i \otimes \alpha_i = 0$. So $\sum v_i \otimes \alpha_i = 0$ is a linear combination of the second set of relations listed and, therefore, $\sum m_i \otimes \alpha_i = 0$ holds, that is, $\sum m_i\alpha_i \otimes 1 = 0$, that is, $\sum m_i\alpha_i = 0$, as required.

Second, if $\rho_P(p_i) = \sum_j p_j \otimes e_{ij}$ then $\rho(v_i) - \sum_j v_j \otimes e_{ij} = 0$ is a linear combination of the second set of equations listed and so $\rho_M(m_i) - \sum m_j \otimes e_{ij} = 0$, as required.

Finally, to check that g does extend f, we have $f(k_l) = (1_M \otimes \epsilon_M)\rho_M(fk_l) = (1 \otimes \epsilon)\sum m_i \otimes d_{li} = \sum m_i\epsilon(d_{li})$. From $\rho_P(k_l) = \sum p_i \otimes d_{li}$ we also have $k_l = \sum p_i\epsilon(d_{li})$ and hence $g(k_l) = \sum m_i\epsilon(d_{li}) = f(k_l)$, as required. $\square$

Our conclusion is that, in the context of C-comodules, an *equation* is an expression of the form $\sum_i \rho_M(v_i)\alpha_i + \sum_i v_i \otimes c_i = 0$ where the $\alpha_i \in k$ and the $c_i \in C$. As usual we go on to define a positive primitive formula to be an existentially quantified conjunction of equations. Then we define an embedding $M \longrightarrow N$ to be pure if every positive primitive formula with parameters from M and a solution in N already has a solution in M and we define a comodule M to be pure-injective if every pure embedding $M \longrightarrow N$ is split (by a comodule morphism). That these coincide with the algebraic / categorical definitions of purity and pure-injectivity is easily checked: 0.5 is the key which links the above definition of purity via positive primitive formulas with those of, for example, [1, p.85] and [6, p.284/5].

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