

***Two body problem on a sphere in the presence of
a uniform magnetic field***

Balabanova, Nataliya and Montaldi, James

2021

MIMS EPrint: **2021.1**

Manchester Institute for Mathematical Sciences
School of Mathematics

The University of Manchester

Reports available from: <http://eprints.maths.manchester.ac.uk/>

And by contacting: The MIMS Secretary
School of Mathematics
The University of Manchester
Manchester, M13 9PL, UK

ISSN 1749-9097

Two body problem on a sphere in the presence of a uniform magnetic field

Nataliya A. Balabanova & James Montaldi

January 2021

Abstract

We investigate the motion of one and two charged non-relativistic particles on a sphere in the presence of a magnetic field of uniform strength. For one particle, the motion is always circular, and determined by a simple relation between the velocity and the radius of motion. For two identical particles, interacting via a cotangent potential, we show there are two families of relative equilibria, called Type I and Type II. The Type I relative equilibria exist for all strengths of the magnetic field, while those of Type II exist only if the field is sufficiently strong. The same is true if the particles are of equal mass but opposite charge. We also determine the stability of the two families of relative equilibria.

Contents

1	Introduction	2
2	Setup	4
3	Motion of one particle	5
4	Motion of two particles	6
4.1	Relative equilibria	8
5	Identical particles	10
5.1	Energy-Casimir map	12
5.2	Variation of configurations for Type I relative equilibria	14
5.3	Stability of Type I relative equilibria	15
5.4	Stability of Type II relative equilibria	16
5.4.1	Points on the threshold curve	16
5.4.2	Stability of general relative equilibria of Type II	16
5.4.3	Energy-Casimir map revisited	17
5.5	The bifurcation diagram	18
6	Opposite charges	20

1 Introduction

There have been a number of studies on the dynamics of two charged particles in the plane and in space in the presence of a uniform magnetic field [5, 11, 12]. In each of these, the symmetry group consists of rotations and translations in the plane (that is, $SE(2)$), together with translations along the magnetic field direction in the spatial setting.

In this paper we consider a similar problem, but with the particles constrained to move on the surface of a sphere, with the magnetic field vector normal to the surface and of constant magnitude. This ensures the system has spherical symmetry; the symmetry group is therefore the group of rotations $SO(3)$.

Reduction and particle motion

We investigate the motion of one and two particles on a sphere in the presence of a centrally symmetric magnetic field using a Hamiltonian approach.

In the case of a single particle, it is either stationary or has circles as trajectories of motion; we deduce the relation between the velocity of the particle and the radius of the circle, depending on the strength of the magnetic field (Proposition 3.1).

For two particles, we adapt the well-known Hamiltonian approach for the similar non-magnetic problem [2, 3] to our setting and perform the reduction with respect to the $SO(3)$ -action, obtaining the reduced Hamiltonian, Hamilton's equations and the Poisson structure. The latter is degenerate and possesses a Casimir function, the second first integral of motion, which involves the angular momentum and the strength of the magnetic field.

The reduced system is described in terms of the distance q between the particles, its conjugate variable p and the three components of the angular momentum in the body frame (m_1, m_2, m_3) , similar to the analogous problem with no magnetic field [2]. The magnetic field B explicitly enters through the form of the Poisson structure.

The goal of this paper is to find the relative equilibrium states of the system. Solving the equations explicitly in the general case does not appear to be tractable, due to their complexity. Nonetheless, in Theorem 4.5 we establish the general existence of relative equilibria.

Identical particles

For calculations, we choose the explicit potential $V(q) = e_1 e_2 \cot(q)$, where e_1 and e_2 are the charges of the particles. This function arises as solutions of the Laplace-Beltrami equation for the case of a symmetric electric vector field on a three-dimensional sphere S^3 . It also arises as an instance of the solutions of the generalized Bertrand problem of finding potentials depending only on geodesic distance and with closed orbits (see [7]).

Due to the complexity of the general form of the reduced system of equations, in Section 5 we limit our attention to the (tractable) case where the two particles are identical.

Solving the equations explicitly gives four classes of relative equilibria. Two of these—called Type I relative equilibria—exist for all values of (q, B) with $q \neq \pi/2$, while the other two—those of Type II—exist only in a certain region of the (q, B) -plane. The two Type I relative equilibria are related by an exchange of particles (when the particles are identical) but the two Type II relative equilibria are genuinely different.

Reconstructing motion from the solutions of the equations (4.8) requires expressing the vector of angular velocity of the solutions and noticing that it has to be parallel to the value of the

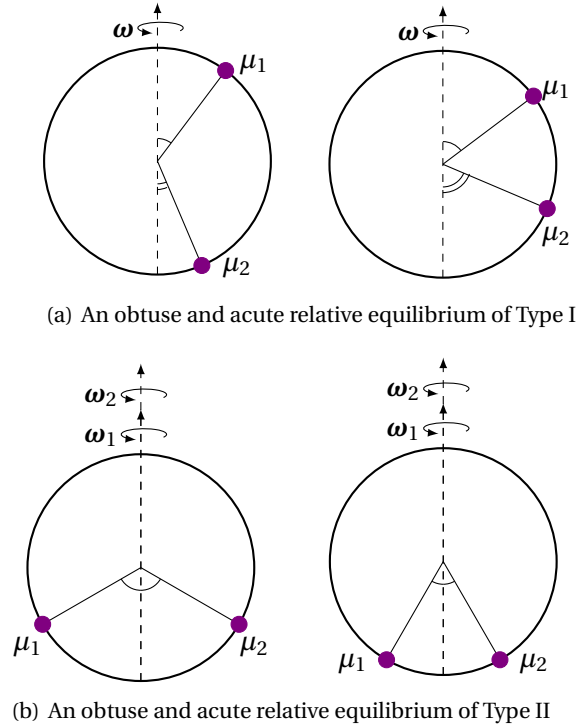


Figure 1.1: The two types of relative equilibrium for equal masses and equal charges, with $V(q) = \cot(q)$.

momentum map, Φ for that configuration.

Type I relative equilibria exist with both acute and obtuse angles between the bodies; in both cases, the motions are *parallel* in the sense that the axis of rotation is located to the side of the two particles.

On the other hand, Type II relative equilibria have *isosceles* configurations: be they obtuse or acute-angled, the axis of rotation is always placed between the bodies, equidistant from both of the particles. They have another interesting property: the same geometric configuration can be occupied by two states with different rates of rotation, and hence different energy levels.

Employing the standard method of investigating linear stability of relative equilibria, we find the regions in the (q, B) -plane for which the equilibria in question are stable or unstable.

Type I relative equilibria allow for an analytic solution of the region of stability. However, this is not the case for Type II relative equilibria, where numerics has to be employed to differentiate between stability and instability.

We finish by calculating the Hessians of different types of relative equilibrium and plotting them on a simplified version of the energy-momentum diagram.

The case of two identical particles with a cotangent potential has a limiting case when $B \rightarrow 0$: the two-body problem with a repelling potential described in [6]. We draw comparisons to it throughout the paper, and in the Appendix we discuss the limiting case of relative equilibria: the right-angle ones. Type I relative equilibria do not exist for $q = \frac{\pi}{2}$ when $B \neq 0$, and Type II relative equilibria are not present for small values of B . It turns out that different right-angled relative equilibria for the gravitational problem appear as limits of the equilibria of Type I when we approach the point $(\frac{\pi}{2}, 0)$ via different curves $B = B(q)$.

We observe, extending a similar observation in [6]), that there is a simple transformation of the phase space that takes the case of identical charges to the case of opposite charges. From the case of identical particles, we can easily obtain the equations of motion with equal masses and opposite charges. Due to the nature of the transformations required for the switch, all the relative equilibria are retained, together with their respective stability properties. The actual motion of the particles will change; we reconstruct it in Section 6.

2 Setup

We begin with recalling some basic facts from electrodynamics.

The force acting on a non-relativistic particle with mass μ and charge e in the presence of an electric field \mathbf{E} and a magnetic field \mathbf{B} is called the Lorentz force and is given by $\mathbf{F} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B})$, where \mathbf{v} is the velocity of the particle. In the absence of any other forces acting on the particle, Newton's second law dictates that

$$\mu \mathbf{a} = e(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \quad (2.1)$$

with \mathbf{a} denoting the acceleration of the particle.

Recall that from Maxwell's equations (see, for example, [8]), $\nabla \cdot \mathbf{B} = 0$.

Using Hodge duality in \mathbb{R}^3 , \mathbf{B} can be represented as a two-form rather than a vector field: $\mathbf{B} = \mathbf{B}_3 dx \wedge dy + \mathbf{B}_2 dz \wedge dx + \mathbf{B}_1 dy \wedge dz$. The divergence-free nature of the vector field is then rephrased as stating that \mathbf{B} is a closed two-form.

Consider now two non-relativistic particles in \mathbb{R}^3 , with respective masses μ_1 and μ_2 , charges e_1 and e_2 , position vectors \mathbf{q}_1 and \mathbf{q}_2 and velocities $\mathbf{v}_1, \mathbf{v}_2$.

We claim that their motion is described by the trajectories of a Hamiltonian system on $T^*\mathbb{R}^3 \times T^*\mathbb{R}^3$, with the Hamiltonian

$$\mathcal{H} = \frac{1}{2\mu_1} |\mathbf{p}_1|^2 + \frac{1}{2\mu_2} |\mathbf{p}_2|^2 + V(\mathbf{q}_1, \mathbf{q}_2) \quad (2.2)$$

and the symplectic form

$$\omega = \left(\begin{array}{cc|c} e_1 \mathfrak{B} & 0 & I_6 \\ 0 & e_2 \mathfrak{B} & \\ \hline & -I_6 & 0 \end{array} \right), \quad (2.3)$$

where \mathfrak{B} is the matrix

$$\mathfrak{B} = \begin{pmatrix} 0 & -B_3 & B_2 \\ B_3 & 0 & -B_1 \\ -B_2 & B_1 & 0 \end{pmatrix}, \quad (2.4)$$

representing the magnetic 2-form, $\mathbf{p}_i = \mu_i \mathbf{v}_i$ is the standard momentum of the i th particle, and $V(\mathbf{q}_1, \mathbf{q}_2)$ is the potential energy of the system, borne out of the interaction between the particles and the effect of any pre-existing electric field.

First, we note that due to closedness of \mathbf{B} , ω is a closed form as well. Together with its obvious non-degeneracy and skew symmetry, this shows (2.3) and (2.2) form a Hamiltonian system indeed.

To prove our claim, we need to demonstrate that the Hamiltonian equations obtained from (2.2) are equivalent to Newton's second law, as written for each of the particles:

$$\begin{cases} \dot{\mathbf{q}}_1 &= \frac{1}{\mu_1} \mathbf{p}_1, \\ \dot{\mathbf{q}}_2 &= \frac{1}{\mu_2} \mathbf{p}_2, \\ \dot{\mathbf{p}}_1 &= -V'(\mathbf{q}_1, \mathbf{q}_2) \mathbf{q}_1 + \frac{e_1}{\mu_1} \mathfrak{B}|_{\mathbf{q}_1}(\mathbf{p}_1), \\ \dot{\mathbf{p}}_2 &= -V'(\mathbf{q}_1, \mathbf{q}_2) \mathbf{q}_2 + \frac{e_2}{\mu_2} \mathfrak{B}|_{\mathbf{q}_2}(\mathbf{p}_2). \end{cases} \quad (2.5)$$

The first two lines in (2.5) are tautological expressions, being equivalent to the definitions of \mathbf{p}_i . The second two are precisely Newton's second law. Thus, Hamiltonian equations on two copies of the cotangent bundle of \mathbb{R}^3 with the symplectic form given above are equivalent to Newton equations for particle motion.

Setup for two particles on the plane and in space

Here we recall briefly the approach to the problem taken by the authors in [5, 11, 12] and [9].

The study by Escobar-Ruiz and Turbinder [5] for two particles in the plane is based on a Lagrangian approach involving a vector potential for the magnetic field (that is, \mathbf{A} satisfying $\mathbf{B} = \nabla \times \mathbf{A}$, or as differential forms, $\mathbf{B} = d\mathbf{A}$).

Littlejohn [9] showed how to avoid the use of a vector potential by incorporating the magnetic field (as a 2-form) into the symplectic form, and then proceed directly with Hamilton's formulation. This was used by him to study the guiding centre problem, and more pertinent for us, was also used by Pinheiro and MacKay [11, 12] in their studies of two particles in the plane and in space.

For a uniform magnetic field on the sphere, there is no vector potential (the existence of a vector potential implies the magnetic field has mean zero, by Stokes' theorem), so we are obliged to use the Hamiltonian approach described above.

All the authors assumed that there is no external electric field, so that the potential is a function of the distance between the particles $V(\mathbf{q}_1, \mathbf{q}_2) = V(\|\mathbf{q}_1 - \mathbf{q}_2\|)$. Hamilton's equations for the motion in the plane are still given by (2.5).

Particles are placed on the horizontal plane in [5] and [11] and in three-dimensional space in [12]. These systems have similarly structured symmetry groups, comprised of translations (in the plane or in space) and rotations around z -axis.

The specific magnetic field studied in all cases is (as a 2-form), $\mathbf{B} = B dx \wedge dy$, which corresponds to a vector field parallel to the z -axis.

3 Motion of one particle

Suppose that one particle of mass μ and charge e is placed on a unit sphere.

We take $V(\mathbf{q}) = 0$ and the magnetic vector field \mathbf{B} orthogonal to S^2 of uniform strength B ; thus, it can be easily seen that the system possesses $SO(3)$ -symmetry.

It can be checked that the restriction of our system to T^*S^2 will be a Hamiltonian system; this is a simple reasoning that we omit here.

Keeping notation as above, we may easily write the symplectic form (on $T^*\mathbb{R}^3$, not restricted to T^*S^2 in this notation), the Hamiltonian and the momentum map as follows:

$$\omega = \begin{pmatrix} e\mathfrak{B} & I \\ -I & 0 \end{pmatrix} \quad (3.1)$$

where \mathfrak{B} is given in Eq. (2.4),

$$\mathcal{H} = \frac{1}{2\mu} |\mathbf{p}|^2 \quad (3.2)$$

$$\Phi : T^* S^2 \rightarrow \mathfrak{so}(3)^*, \quad \Phi(\mathbf{q}, \mathbf{p}) = -eB\mathbf{q} + \mathbf{q} \times \mathbf{p} \quad (3.3)$$

Restricting to the phase space $T^* S^2$, the momentum map Φ is a map from a 4-dimensional space into a three dimensional one and its fibres $\Phi^{-1}(\eta)$, $\eta \in \mathfrak{so}(3)^*$ are one-dimensional.

Additionally, the stabilizer of any non-zero element in $\mathfrak{so}(3)^*$ is isomorphic to $\text{SO}(2)$ and has dimension one. Thus, as quotient spaces, all the reduced spaces $\Phi^{-1}(\eta)/\text{SO}(2)_\eta$ are single points, making any motion a relative (or absolute, in case of a stationary particle) equilibrium.

Explicitly writing the formulae for the preimage of an element in $\mathfrak{so}(3)^*$ and finding the maximal Euclidean distance between any two points therein (keeping in mind that it has to be a circle) gives

Proposition 3.1. *The trajectories of a non-relativistic charged particle on a unit sphere with a uniform magnetic field are circles, whose radius r is related to the velocity \mathbf{v} by the relation*

$$r^2 = \frac{\mu^2 |\mathbf{v}|^2}{B^2 e^2 + \mu^2 |\mathbf{v}|^2} \quad (3.4)$$

Or, in terms of angular velocity $\boldsymbol{\omega}$,

$$r^2 = 1 - \frac{e^2 B^2}{\mu^2 |\boldsymbol{\omega}|^2} \quad (3.5)$$

4 Motion of two particles

Now consider the setup with two particles of respective masses μ_1 and μ_2 and charges e_1 and e_2 , located, once again, on a unit sphere.

As previously, we assume that no external electric fields are present, resulting in $V(\mathbf{q}_1, \mathbf{q}_2) = V(|\mathbf{q}_1 - \mathbf{q}_2|)$, and throughout we assume $V'(q) \neq 0$ for all $q \in (0, \pi)$.

The configuration space of the problem is $\mathcal{Q} = S^2 \times S^2 \setminus \Delta$, where Δ is the union of the diagonal subset of $S^2 \times S^2$ and the subset that contains all the pairs of antipodal points.

The momentum map for the $\text{SO}(3)$ action on the phase space $T^* \mathcal{Q}$ is the sum of those for each particle:

$$\Phi(\mathbf{q}_1, \mathbf{q}_2, \mathbf{p}_1, \mathbf{p}_2) = -B(e_1 \mathbf{q}_1 + e_2 \mathbf{q}_2) + \mathbf{q}_1 \times \mathbf{p}_1 + \mathbf{q}_2 \times \mathbf{p}_2 \quad (4.1)$$

For reduction, we follow along the lines of parametrization used in [2, 3] and [6].

Any matrix in $\text{SO}(3)$ can be written using Euler angles in the form $g(\theta, \phi, \psi)$ as

$$\begin{pmatrix} \cos(\phi) \cos(\psi) - \cos(\theta) \sin(\psi) \sin(\phi) & -\sin(\phi) \cos(\psi) - \cos(\theta) \sin(\psi) \cos(\phi) & \sin(\theta) \sin(\psi) \\ \cos(\phi) \sin(\psi) + \cos(\theta) \cos(\psi) \sin(\phi) & -\sin(\phi) \sin(\psi) + \cos(\theta) \cos(\psi) \cos(\phi) & -\sin(\theta) \cos(\psi) \\ \sin(\theta) \sin(\phi) & \sin(\theta) \cos(\phi) & \cos(\theta) \end{pmatrix} \quad (4.2)$$

and any element of $\mathfrak{so}(3)$ as

$$\xi = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \quad (4.3)$$

Now, consider the two points in $S^2 \subset \mathbb{R}^3$, separated by a geodesic distance q :

$$\mathbf{x}_1 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 0 \\ \sin(q) \\ -\cos(q) \end{pmatrix} \quad (4.4)$$

It is straightforward to see that any configuration can be obtained by placing the two particles at an appropriate distance in the form above and then rotating them. Hence, the tangent bundle can be rewritten in terms of the angles θ, ψ, ϕ , the coordinates on the Lie algebra $\boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3$, the distance q and its rate of change \dot{q} .

Due to the nature of our phase space, we can assume that $q \in I := (0, \pi)$. Left trivialization of $T\text{SO}(3) = \text{SO}(3) \times \mathfrak{so}(3)$ allows us to write the parametrization

$$TI \times T\text{SO}(3) \rightarrow T\mathcal{Q} \\ (q, \dot{q}, \theta, \phi, \psi, \boldsymbol{\omega}_1, \boldsymbol{\omega}_2, \boldsymbol{\omega}_3) \mapsto (g \cdot \mathbf{x}_1, g \cdot \mathbf{x}_2, g\xi \cdot \mathbf{x}_1, g\xi \cdot \mathbf{x}_2 + g\mathbf{x}_2' \dot{q}), \quad (4.5)$$

where g is given in (4.2). This gives ξ the physical meaning of angular velocity in the body frame [10].

Since the problem is not determined by a Lagrangian, to transfer to a set of conjugate variables we use a slightly modified method from the usual one (compare with the Lagrangian approach in [2]).

Let T denote the kinetic energy of the system (as rewritten in our reduced coordinates): $T = \frac{\mu_1}{2}(\boldsymbol{\omega}_1^2 + \boldsymbol{\omega}_2^2) + \frac{\mu_2}{2}((\boldsymbol{\omega}_1 + \dot{q})^2 + (\sin(q)\boldsymbol{\omega}_3 + \cos(q)\boldsymbol{\omega}_2)^2)$ and introduce $m_i = \frac{\partial T}{\partial \boldsymbol{\omega}_i}$ in place of the $\boldsymbol{\omega}_i$.

Moving from TI to T^*I can be accomplished by taking $p := \frac{\partial T}{\partial \dot{q}}$. The relations between the variables are as follows:

$$\begin{cases} \boldsymbol{\omega}_1 &= \frac{1}{\mu_1}(m_1 - p), \\ \boldsymbol{\omega}_2 &= \frac{1}{\mu_1}(m_2 - m_3 \cot(q)) \\ \boldsymbol{\omega}_3 &= \frac{1}{\mu_1}(\cot(q)(m_3 \cot(q) - m_2)) + \frac{1}{\mu_2}(m_3 \csc^2(q)) \\ \dot{q} &= \frac{1}{\mu_1 \mu_2}(p(\mu_1 + \mu_2) - \mu_2 m_1) \end{cases} \quad (4.6)$$

The momentum map Φ is, in the new set of variables,

$$\Phi(g, m_1, m_2, m_3, q, p) = g \cdot \begin{pmatrix} m_1 \\ m_2 - B e_2 \sin(q) \\ m_3 + B e_1 + B e_2 \cos(q) \end{pmatrix}. \quad (4.7)$$

Since the symplectic form is not standard, we need to rewrite it according to the general rule: if the change of coordinates is given by a Jacobian matrix J , the new symplectic structure will be $\tilde{\omega} = (J)^T \omega J$. Keeping our choice of signs in line with [9], we write the Poisson structure as $\tilde{\sigma} = -((J)^T \omega J)^{-1}$, and the Hamiltonian equations are consequently given by $\tilde{\sigma} \cdot \text{grad } \mathcal{H}$, with $\text{grad } \mathcal{H}$ rewritten in the new set of variables.

After performing the calculations, it can be observed that the last five equations form an independent subsystem:

$$\left\{ \begin{array}{lcl} \dot{m}_1 & = & -\frac{1}{\mu_1\mu_2} (\mu_2(m_2 - m_3 \cot q)(Be_1 + m_2 \cot q + m_3) + Be_2\mu_1 m_3 \csc q - \mu_1 m_2 m_3 \csc^2 q) \\ \dot{m}_2 & = & \frac{1}{\mu_1\mu_2} (\mu_2(m_1 - p)(Be_1 + m_3) + Be_2\mu_1 p \cos(q) + \\ & & + \mu_2 m_1 \cot q(m_2 - m_3 \cot q) - \mu_1 m_1 m_3 \csc^2 q) \\ \dot{m}_3 & = & \frac{1}{\mu_1\mu_2} (\mu_1(Be_2 p \sin q) + \mu_2(m_2 p - m_1 m_3 \cot q)) \\ \dot{q} & = & \frac{1}{\mu_1\mu_2} (p(\mu_1 + \mu_2) - \mu_2 m_1) \\ \dot{p} & = & -\frac{1}{\mu_1\mu_2} (m_3 \csc q(Be_2\mu_1 + \csc q(\mu_2 m_2 - m_3(\mu_1 + \mu_2) \cot q)) + \mu_1\mu_2 V'(q)). \end{array} \right. \quad (4.8)$$

The quintuple of coordinates (m_1, m_2, m_3, q, p) describes the system reduced with respect to the $SO(3)$ -action. From here on, we restrict our attention exclusively to these reduced equations.

The Hamiltonian is rotationally invariant, so its reduced form is obtained just by substitution of the new set of variables:

$$\begin{aligned} \mathcal{H} = & \frac{1}{2\mu_1\mu_2} \left(\mu_2((m_1 - p)^2 + m_2^2) + m_3(-2\mu_2 m_2 \cot(q) + \mu_1 m_3 \csc^2 q + \mu_2 m_3 \cot^2 q) \right) + \\ & + \frac{p^2}{2\mu_2} + V(q). \end{aligned} \quad (4.9)$$

The non-zero Poisson brackets in the reduced variables are given by

$$\begin{aligned} \{m_1, m_2\} &= -m_3 - B(e_1 + e_2 \cos(q)), & \{m_2, m_3\} &= -m_1, \\ \{m_1, m_3\} &= m_2 - Be_2 \sin(q), & \{m_2, p\} &= Be_2 \cos(q), \\ \{m_3, p\} &= Be_2 \sin(q), & \{q, p\} &= 1. \end{aligned} \quad (4.10)$$

It can easily be seen that the Poisson structure (4.10) is generically of rank 4, and has Casimir function given by the square of the momentum map:

$$\mathcal{C} = m_1^2 + (m_2 - Be_2 \sin q)^2 + (m_3 + B(e_1 + e_2 \cos q))^2. \quad (4.11)$$

Remark 4.1. The limiting case of $B = 0$ takes these expressions to the reduced equations of motion for the two body problem on a sphere from [2, 6] (for details, see Appendix).

Remark 4.2. Note that the simultaneous change in the signs of B, m_1, m_2, m_3, p is a time reversing symmetry, for it takes the Poisson structure to its opposite, while leaving \mathcal{H} invariant.

4.1 Relative equilibria

Our primary goal is describing and locating relative equilibria of the system. The condition for that is the right hand side of the system (4.8) equals 0.

Solving (4.8) gives $p = m_1 = 0$ from the second and fourth equations. Assuming otherwise leads to a linear relation between m_2 and m_3 , which ultimately yields $V'(q) = 0$, contradicting our initial assumptions about the potential.

Substituting zero values of m_1 and p into the equations, we obtain a system for m_2 and m_3 , consisting of the first and the fifth equation from (4.8).

Solving the first equation for m_2 yields

$$m_2 = \pm \frac{1}{2\mu_2} \tan(q) \left(-\sqrt{A} - Be_1\mu_2 - \mu_2 m_3 + \mu_1 m_3 \csc^2 q + \mu_2 m_3 \cot^2 q \right) \quad (4.12)$$

where

$$A = 4\mu_2 m_3 \cot(q) \csc(q) (\mu_2 \cos(q) (Be_1 + m_3) - Be_2 \mu_1) + \left(-\mu_2 (Be_1 + m_3) + \mu_1 m_3 \csc^2 q + \mu_2 m_3 \cot^2 q \right)^2. \quad (4.13)$$

Firstly, we note that the expression for m_2 is indeterminate when $q = \frac{\pi}{2}$, separating this into a special case.

Secondly, due to a square root being present in the expressions, the right hand side of (4.13) is not defined for all m_3 . The inequality

$$A \geq 0 \quad (4.14)$$

describes the permitted values of the variable.

A little rearranging of the polynomial (4.13) shows that the coefficient of m_3^2 is

$$\csc^4(q) (\mu_1^2 + \mu_2^2 + 2\mu_1\mu_2 \cos(2q)).$$

Note that since this expression is always positive, all the “bad” values of m_3 in (4.14) always lie between the two solutions of the equation $A = 0$.

Substituting (4.12) into the fifth equation of (4.8), we obtain, for m_3 :

$$m_3 \csc(q) \sec(q) (Be_1\mu_2 - 2Be_2\mu_1 \cos(q) - 2\mu_1 m_3 + m_3(\mu_1 + \mu_2) \csc^2 q) \pm \pm m_3 \csc(q) \sec(q) \sqrt{A} - 2\mu_1\mu_2 V'(q) = 0. \quad (4.15)$$

In order to prove the existence of roots for (4.15), we employ the following observation: the value of A at $m_3 = 0$ is $B^2 e_1^2 \mu_2^2$, therefore, A is always positive at $m_3 = 0$. Thus, the interval in m_3 where the function on the left hand side of (4.15) is not defined lies entirely to the left or to the right of $m_3 = 0$.

Consider the two equations in (4.15). With appropriate arrangements, they both square to a fourth degree polynomial in m_3 :

$$\begin{aligned} & -4\mu_1 m_3^4 \csc^4(q) + 4Bm_3^3 \csc^4(q) (e_1\mu_2 - e_2\mu_1 \cos(2q) \sec(q)) + \\ & + 2m_3^2 \csc^3(q) \sec(q) (2B^2 e_2 \sin(q) (e_2\mu_1 \cos(q) - e_1\mu_2) - 2\mu_2 V'(q) (\mu_2 + \mu_1 \cos(2q))) + \\ & + 4B\mu_2 m_3 \csc(q) V'(q) (2e_2\mu_1 - e_1\mu_2 \sec(q)) + 4\mu_1 \mu_2^2 V'(q)^2 = 0 \end{aligned} \quad (4.16)$$

At this point we employ a classical theorem:

Theorem 4.3 (Descartes's rule of signs, [4]). *The number of positive roots of a polynomial with real coefficients is equal to or less by an even number than the number of changes of sign of the coefficients of the polynomial in question, when written in the order of descending degree of the variable.*

By looking at (4.16), it can be easily observed that since $\mu_1 > 0, \mu_2 > 0$, the highest coefficient is always negative, and the lowest one is positive. Since the degree of the polynomial is 4, Descartes's rule of signs, as applied to the polynomial of m_3 and then $-m_3$, gives that the first equation in (4.16) must have at least one positive and one negative root. But then at least one of them does not lie in the interval between the roots of (4.13). Hence, for (4.15), at least one solution always exists.

Now, we fill in the gap by taking $q = \frac{\pi}{2}$ where we can give a more precise statement depending on the value of $V'(\pi/2)$. Taking $m_1 = 0$, $p = 0$, $q = \frac{\pi}{2}$ in the system (4.8) gives us only two nontrivial equations:

$$\begin{aligned}\mu_2 m_2 (B e_1 + m_3) + B e_2 \mu_1 m_3 - \mu_1 m_2 m_3 &= 0, \\ m_3 (B e_2 \mu_1 + \mu_2 m_2) + \mu_1 \mu_2 V' \left(\frac{\pi}{2} \right) &= 0\end{aligned}\tag{4.17}$$

By analytically solving (4.17), one can check that it has real solutions if and only if

$$B^4 e_1^2 e_2^2 + 2B^2 e_1 e_2 (\mu_1 + \mu_2) V' \left(\frac{\pi}{2} \right) + (\mu_1 - \mu_2)^2 V' \left(\frac{\pi}{2} \right)^2 \geq 0,\tag{4.18}$$

with two solutions if the left hand side of (4.18) is strictly greater than 0, one if it is equal to 0 and none when it is less than zero.

Remark 4.4. From the explicit form of the solutions of (4.17) (we omit the computations here), it can be easily seen that the limiting case $B = 0$ agrees with the results for the two body problem on a sphere from [2] and [6]: no relative equilibria exist when $q = \frac{\pi}{2}$ unless $\mu_1 = \mu_2$.

Thus, we have demonstrated the

Theorem 4.5. 1. For each $q \in (0, \pi) \setminus \{\pi/2\}$, and for all non-zero values of B , e_1 , e_2 , μ_1 , μ_2 and any smooth function $V(q)$ there exists at least one relative equilibrium;

2. for $q = \pi/2$, there are precisely 0, 1 or 2 relative equilibria accordingly as the discriminant in (4.18) is negative, zero or positive.

5 Identical particles

The most natural case to investigate closely is the case of two identical particles, i.e. the one with $e_1 = e_2 = 1$, $\mu_1 = \mu_2 = 1$. Without loss of generality we may assume $B > 0$, as $B < 0$ can be reduced to this via the time-reversing symmetry described in Remark 4.2.

Our choice of the potential is $V(q) = \cot(q)$. This gives $V'(q) = -\csc^2(q) < 0$, so describes a repelling force in accordance with the physics. As already explained more generally, the reduced equations of motion (4.8) lead to $p = m_1 = 0$ and 2 equations for m_2, m_3 , namely

The reduced equations of motion then read:

$$\begin{cases} \dot{m}_1 &= -(m_2 - m_3 \cot(q))(B + m_2 \cot(q) + m_3) - m_3 \csc(q)(B - m_2 \csc(q)) \\ \dot{m}_2 &= m_1(B + 2m_3) - p(2B \sin^2(\frac{q}{2}) + m_3) + m_1 m_2 \cot(q) - 2m_1 m_3 \csc^2 q \\ \dot{m}_3 &= p(B \sin(q) + m_2) - m_1 m_3 \cot(q) \\ \dot{q} &= 2p - m_1 \\ \dot{p} &= \csc(q) (\csc(q) (-m_2 m_3 + 2m_3^2 \cot(q) + 1) - B m_3) \end{cases}\tag{5.1}$$

We proceed to find the stationary points of this system, which correspond to relative equilibria of the original system. The conditions $\dot{m}_2 = \dot{m}_3 = \dot{q} = 0$ imply $p = m_1 = 0$ (as pointed out more generally above). If we then solve $\dot{m}_1 = 0$ for m_2 and substitute the two solutions of this into $\dot{p} = 0$ we obtain two quadratic equations (since A from (4.13) becomes a square) for m_3 . The four solutions can be split into 2 pairs leading to the following solutions (the analytic expressions were found using MATHEMATICA).

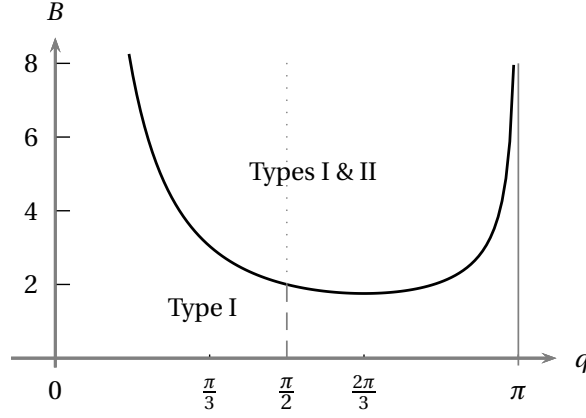


Figure 5.1: The threshold curve: only Type I solutions exist below the curve

$$\text{Type I: } \begin{cases} m_1 = 0, \\ p = 0, \\ m_2^\pm = \frac{2B \sin^3(\frac{q}{2}) \sin(q) \pm \cos^{\frac{3}{2}}(\frac{q}{2}) \cos(q) \sqrt{4 \csc(\frac{q}{2}) + B^2 \sec(\frac{q}{2}) \sin^2(q) \tan^2(q)}}{\sin(\frac{q}{2}) - \sin(\frac{3q}{2})}, \\ m_3^\pm = \frac{1}{2} \left(B \sin(q) \tan(q) \mp \sqrt{\cos(\frac{q}{2})} \sqrt{4 \csc(\frac{q}{2}) + B^2 \sec(\frac{q}{2}) \sin^2(q) \tan^2(q)} \right); \end{cases} \quad (5.2)$$

$$\text{Type II: } \begin{cases} m_1 = 0, \\ p = 0, \\ m_2^\pm = -2 \sin^4(\frac{q}{2}) \csc(q) \left(B \pm \sqrt{B^2 - 2 \csc^2(\frac{q}{2}) \csc(q)} \right), \\ m_3^\pm = \sin^2(\frac{q}{2}) \left(B \pm \sqrt{B^2 - 2 \csc^2(\frac{q}{2}) \csc(q)} \right). \end{cases} \quad (5.3)$$

It is convenient to express the existence and stability of all the solutions in (5.2) and (5.3) through the two ‘parameters’, B and q .

Observe that the two solutions in (5.2) exist for all values of B and q , except for $q = \frac{\pi}{2}$ — we refer to these as relative equilibria of Type I.

For solutions (5.3) to exist, the expression under the square root needs to be positive. It is clear that for each value of $q \in (0, \pi)$ when B is sufficiently large, there is a solution. We call these relative equilibria of Type II. More precisely, relative equilibria of Type II exist when $B^2 \geq 2 \csc^2(q/2) \csc(q)$. Since we are assuming that $B > 0$, the threshold value of B is,

$$B = 2 \sqrt{\frac{\csc(q)}{1 - \cos(q)}}. \quad (5.4)$$

We will refer to the graph of (5.4), given in Figure 5.1, as “the threshold curve”

The minimal value that the function above assumes is $B = \frac{4}{3} 3^{1/4}$ (which is approximately 1.755) at $q = \frac{2\pi}{3}$. The value of (5.4) at $q = \frac{\pi}{2}$ is 2 (this fact will come useful later).

Note that for the values of q and B on the threshold curve the two solutions from (5.3) coincide.

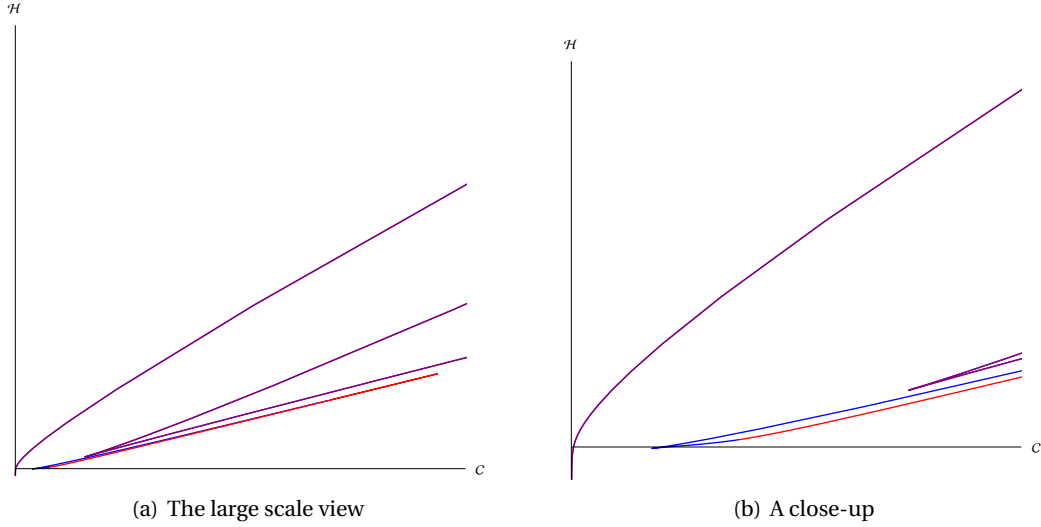


Figure 5.2: Energy-momentum bifurcation diagram with $B = 2.5$; the red and blue curve corresponds to Type II RE, the others to Type I. See also Fig. 5.7

Remark 5.1. For an arbitrary choice of the potential $V(q)$ with $V'(q) \neq 0$ there are four solutions as well, with one pair existing for all values of B and q , and the second pair for the values above some threshold curve.

5.1 Energy-Casimir map

Having the solutions of (5.1), we want to approach the problem from a more physical angle: that of the energy-Casimir (or energy-momentum) map.

The first natural question to ask is that of the form of its image: namely, what does the set of values of $(\mathcal{C}, \mathcal{H})$ look like? Since $\mathcal{C} \geq 0$, the entirety of it lies on the right half-plane, including the boundary $\mathcal{C} = 0$.

To determine which values \mathcal{H} can assume with a fixed $\mathcal{C} = C_0$, we assign specific values to our variables: $m_1 = \sqrt{C_0}$, $m_2 = B \sin(q)$, $m_3 = -B(1 + \cos(q))$, $p = 0$. At all of these points \mathcal{C} is indeed equal to C_0 , and \mathcal{H} is a function of q , reduced to

$$\mathcal{H}(q) = \frac{C_0}{2} + B^2 + \frac{1}{2} \cot\left(\frac{q}{2}\right) + B^2 \csc^2\left(\frac{q}{2}\right) - \frac{1}{2} \tan\left(\frac{q}{2}\right), \quad (5.5)$$

This is a simple monotonic decreasing function of q , with limits $+\infty$ when $q \rightarrow 0$ and $-\infty$ when $q \rightarrow \pi$. Therefore, *the image of the energy-momentum map is the entire right half-plane in $(\mathcal{C}, \mathcal{H})$.*

The singular values of the energy-Casimir map form the ‘bifurcation diagram’, and are the images of the set of relative equilibria. The large scale view and the close-up of the bifurcation diagram are given in Figure 5.2. For now, we will concentrate only on the values of the Casimir and the Hamiltonian, and discuss the details later on.

The blue and red lines of Type II relative equilibria, respectively given by (m_2^-, m_3^-) and (m_2^+, m_3^+) from (5.3), form a closed loop with two cusps, and the solutions (m_2^\pm, m_3^\pm) of Type I from (5.2) give the same values of \mathcal{H} and \mathcal{C} , comprising the purple line in the figure. This suggests that there should be some relation between the two, which we now describe.

Lemma 5.2. *The \mathbb{Z}_2 -action of swapping the two particles is a symmetry of the system in our case and induces a coordinate change on the reduced phase space which leaves q invariant and multiplies $(m_1, m_2, m_3, p)^T$ by the matrix*

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -\cos(q) & -\sin(q) & 0 \\ 0 & -\sin(q) & \cos(q) & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \quad (5.6)$$

Proof. Without loss of generality we can assume that the initial placement of our particles, and the one at which the exchange happens, is at the two points \mathbf{x}_1 and \mathbf{x}_2 from (4.4).

The matrix exchanging said points is given by

$$\mathrm{SO}(3) \ni A_0 := \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\cos(q) & -\sin(q) \\ 0 & -\sin(q) & \cos(q) \end{pmatrix}$$

With this in mind, the swapping can be written as $g\mathbf{x}_1 \mapsto gA_0\mathbf{x}_1$, with a similar relation for the second particle. Therefore, under this \mathbb{Z}_2 -action we have for \dot{g} and ξ

$$\begin{aligned} g &\mapsto gA_0, \\ \dot{g} &\mapsto \dot{g}A_0 + g\dot{A}_0 = g\xi A_0 + gA_0\xi_0 = gA_0(A_0^{-1}\xi A_0 + \xi_0), \end{aligned}$$

where ξ_0 is the tangent element of the Lie algebra $\mathfrak{so}(3)$ to the one-parameter subgroup A_0 with varying q .

Hence,

$$\xi \mapsto A_0^{-1}\xi A_0 + \xi_0$$

or, explicitly,

$$\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & \omega_2 \sin(q) - \omega_3 \cos(q) & -\omega_2 \cos(q) - \omega_3 \sin(q) \\ \omega_3 \cos(q) - \omega_2 \sin(q) & 0 & \dot{q} + \omega_1 \\ \omega_2 \cos(q) + \omega_3 \sin(q) & -\dot{q} - \omega_1 & 0 \end{pmatrix}$$

The Jacobian matrix \mathcal{J}_1 of this change in the variables $(\omega_1, \omega_2, \omega_3, \dot{q})$ is then

$$\begin{pmatrix} -1 & 0 & 0 & -1 \\ 0 & -\cos q & -\sin q & 0 \\ 0 & -\sin q & \cos q & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

while the Jacobian \mathcal{J}_2 of the transfer from $(\omega_1, \omega_2, \omega_3, \dot{q})$ to (m_1, m_2, m_3, p) in the case of identical particles is given by

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -\cot(q) & 0 \\ 0 & -\cot(q) & \cot^2 q + \csc^2 q & 0 \\ -1 & 0 & 0 & 2 \end{pmatrix}$$

Finally, the final transformation matrix for the reduced conjugate coordinates is $\mathcal{J}_2^{-1} \mathcal{J}_1 \mathcal{J}_2$ and is explicitly given by (5.6), which is straightforwardly an involutory matrix. \square

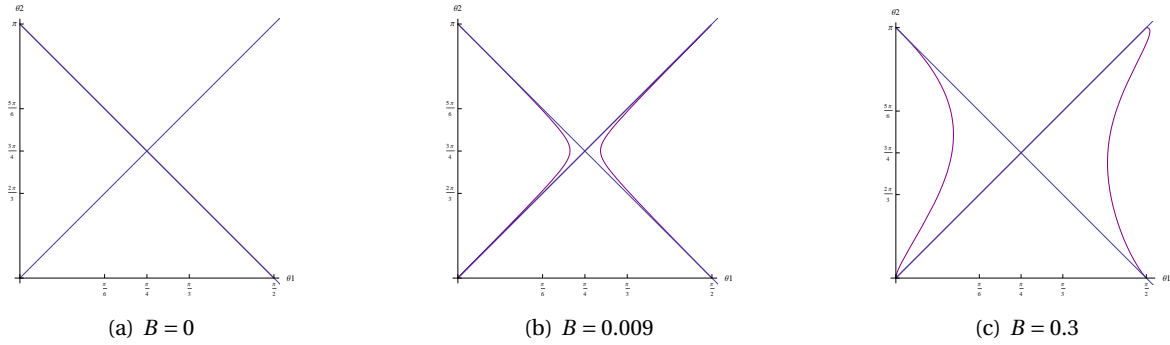


Figure 5.3: Relative equilibria of Type I, plotted on (θ_1, θ_2) -plane for various values of B . The plot of RE for $B = 0$ is provided in blue on the second and third picture for comparison.

We apply Lemma 5.2 to the Type I RE, and multiply the vector $(0, m_2^+, m_3^+, 0)$ by (5.6), which maps it to $(0, m_2^-, m_3^-, 0)$, and vice versa. Thus, the two solutions are the same up to changing the labelling of the particles.

We arrange our observations into

Theorem 5.3. *For the system described above, with two identical particles, let $B > 0$.*

1. *For every $q \in (0, \pi) \setminus \{\frac{\pi}{2}\}$ there is a unique relative equilibrium of Type I, up to particle exchange, and none exists for $q = \frac{\pi}{2}$.*
2. *For the relative equilibria (RE) of Type II, for a fixed value of $B > \frac{4}{3}3^{1/4}$, let us denote the two solutions of $B = 2\sqrt{\frac{\csc(q)}{1-\cos(q)}}$ by q_0 and q_1 , $q_0 \leq q_1$.*
 - *For $0 < B < \frac{4}{3}3^{1/4}$ there are no RE of Type II.*
 - *Let $B = \frac{4}{3}3^{1/4}$. Then for $q = \frac{2\pi}{3}$ there is one RE of Type II. For $q \neq 2\pi/3$ there are none.*
 - *Let $B > \frac{4}{3}3^{1/4}$. Then for $q \in (q_0, q_1)$ there are two distinct RE of Type II, for $q \in \{q_0, q_1\}$ there is just one while for $q \notin [q_0, q_1]$ there are none.*

5.2 Variation of configurations for Type I relative equilibria

Here we briefly comment on how the relative equilibria of the non-magnetic 2 body problem on the sphere with a repelling potential [6] deform into the RE of Type I in the current problem, as mentioned in Remark 4.4.

Let θ_1 be the angle between the directed axis of rotation and the first particle and θ_2 the angle between the axis and the second particle. For given q we have $\theta_2 = \theta_1 + q$.

With $B = 0$ there are two families of RE, the isosceles ones with $\theta_1 + \theta_2 = \pi$ and the right angled ones with $\theta_2 - \theta_1 = q = \pi/2$ [6, Theorem 2.1]. These are the two lines shown in Figure 5.3(a).

Now with $B > 0$, for every $q \neq \frac{\pi}{2}$ there will be a single relative equilibrium of Type I (as mentioned previously, up to a particle exchange). Therefore, $\theta_1 = \theta_1(q)$ and $\theta_2 = \theta_1(q) + q$. This relation is plotted on (θ_1, θ_2) -plane in Figure 5.3(b,c).

For negative values of B , the two branches of the curve will lie in the remaining upper and lower quadrants formed by the lines in Figure 5.3(a).

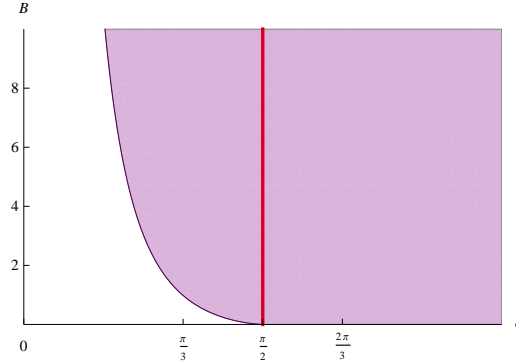


Figure 5.4: Stability for the relative equilibria of Type I: the coloured region is where the RE are linearly stable

5.3 Stability of Type I relative equilibria

A lengthy calculation shows that the characteristic polynomial of the linearized matrix of the system (5.1) is of the form $x(-x^4 + ax^2 + b)$ (as it would be for every 4 dimensional Hamiltonian system: the factor of x is due to the Casimir).

Substituting $y = x^2$, we obtain a quadratic equation: $-y^2 + ay + b = 0$. Thus, for linear stability, three conditions need to be fulfilled:

- the top of the parabola must be to the left of $y = 0$,
- the value of $-y^2 + ay + b = 0$ at the top must be greater than 0,
- the value of $-y^2 + ay + b = 0$ at $y = 0$ must be less than 0.

In this case, the conditions for stability can be written analytically.

Since the two RE of Type I are related by particle exchange (and the particles are identical) we only need perform calculations for one of the explicit solutions. By doing so, we obtain Figure 5.4.

Relative equilibria are linearly stable in the region coloured lilac and linearly unstable in the white coloured one. The thick red line is where $q = \frac{\pi}{2}$ for which there is no RE.

In Figure 5.4, the transition curve is obtained from the third condition for stability from Section 5.0.3 and is explicitly given by the expression $B = \sqrt{\frac{\cos^3(q)(2+\cos(q))}{2\sin^3(q)\sin^2(\frac{q}{2})}}$. Note that the graph meets the horizontal axis $B = 0$ at $q = \frac{\pi}{2}$.

We have demonstrated

Proposition 5.4. *For the system with identical particles, the linear stability of the RE of Type I (solutions of (5.2)), is as follows:*

- when (q, B) is to the left of the graph $B = \sqrt{\frac{\cos^3(q)(2+\cos(q))}{2\sin^3(q)\sin^2(\frac{q}{2})}}$, the relative equilibrium is linearly unstable;
- when (q, B) is to the right of the graph $B = \sqrt{\frac{\cos^3(q)(2+\cos(q))}{2\sin^3(q)\sin^2(\frac{q}{2})}}$, $q \neq \frac{\pi}{2}$, the relative equilibrium is linearly stable.

See Figure 5.4.

Remark 5.5. Almost all of the linearly stable Type I RE will probably be nonlinearly (Lyapunov) stable, by KAM theory; however there are non-degeneracy conditions to check in the higher order terms of the Hamiltonian near each RE. For the non-magnetic 2-body problem these conditions are checked numerically for many of the RE [2, Sec. 4.2.2]. We do not pursue this here.

5.4 Stability of Type II relative equilibria

Here, we employ the same method as the one from the previous section. However, for these relative equilibria, obtaining analytic results for general values of q and B is not computationally possible.

5.4.1 Points on the threshold curve

First, we investigate the stability on the line $B = 2\sqrt{\frac{\csc(q)}{1-\cos(q)}}$, where one can obtain analytic results. Here, as previously mentioned, the two relative equilibria of Type II, namely (m_2^\pm, m_3^\pm) , coincide, and we are looking to determine whether the resulting one is stable.

We employ the standard method for establishing linear stability: linearizing the system and taking $B = 2\sqrt{\frac{\csc(q_0)}{1-\cos(q_0)}}$ for some fixed q_0 .

Computing the characteristic polynomial of the matrix of the linearized system at any relative equilibrium on the threshold curve gives us

$$2x(x^2 + 2\csc^3(q_0))(x^2 + 2(1 + 2\cos(q_0))\csc^3(q_0)) = 0$$

with the solutions

$$x = 0, x = \pm i\sqrt{2}\csc^{\frac{3}{2}}(q_0), x = \pm \sqrt{-2(2\cot(q_0)\csc^2(q_0) + \csc^3(q_0))}$$

The sign of the expression under the root determines the linear stability of the equilibrium.

It is greater than zero (rendering the equilibrium linearly unstable) when $\frac{2\pi}{3} < q_0 < \pi$ and less than 0 with a linearly stable equilibrium when $0 < q_0 < \frac{2\pi}{3}$

We have demonstrated

Proposition 5.6. *For the values of (q, B) on the threshold curve, the Type II equilibrium is*

- *linearly stable if (q, B) is to the left of $(\frac{2\pi}{3}, \frac{4}{3}3^{1/4})$,*
- *linearly unstable if (q, B) is to the right of $(\frac{2\pi}{3}, \frac{4}{3}3^{1/4})$.*

5.4.2 Stability of general relative equilibria of Type II

Now we position ourselves in the region strictly above the threshold curve. Again, we linearize the system at the equilibrium point and look at the zeros of the characteristic polynomial.

However, due to the complexity of the equations we have to employ numerical methods.

We have performed the numerics for values of B less than 100, and found that up to this value the properties are always as shown in Figure 5.5.

As calculations demonstrate, the curves that separate the regions of stability from those of instability consist of degenerate relative equilibria; these are, in fact, the only degenerate relative equilibria of Type II. As will be discussed later, for a fixed value of B the Casimir function assumes its minima and maxima on the relative equilibria lying on the curves where stability changes.

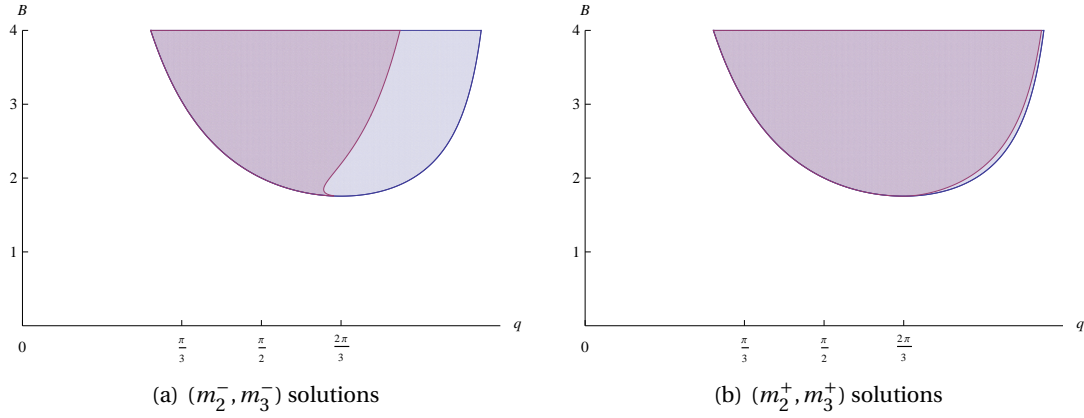


Figure 5.5: Stability of relative equilibria of Type II in terms of B and q : in the dark purple regions, the corresponding RE is linearly stable, while the blue region denotes linear instability.

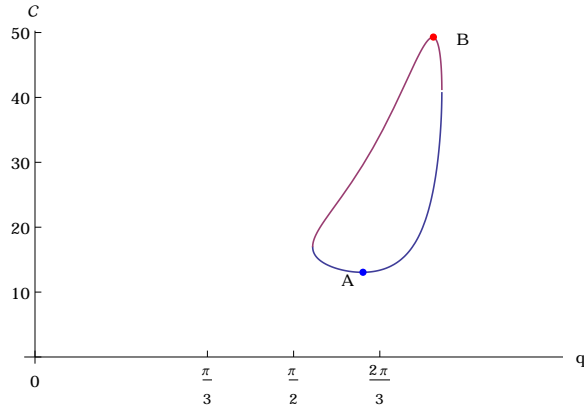


Figure 5.6: The values of \mathcal{C} for relative equilibria of Type II with a fixed $B = 1.9$. The blue part of the curve is given by (m_2^-, m_3^-) , the red by (m_2^+, m_3^+) .

In Figure 5.6, the points A and B on the curve are the points where the Casimir is minimal and maximal respectively, on the family of RE for that fixed value of the magnetic field strength B . Stable relative equilibria lie to the left of these points and unstable ones to the right. These two points represent saddle-node bifurcations of the RE of Type II as the value of the Casimir is varied.

For both (m_2^+, m_3^+) and (m_2^-, m_3^-) the curves dividing the stability region from that of instability separate from the threshold curve at the point $(q, B) = (\frac{2\pi}{3}, \frac{4}{3}3^{1/4})$

5.4.3 Energy-Casimir map revisited

With the newly acquired information about the stability, we cast another look at the Energy-Casimir map. Figure 5.7 (a) and (b) shows the set of singular values of this map, which are the images of the set of relative equilibria. The figure in (b) is schematic, as the two branches are very close in reality. The cusps on the curves, emphasized by dots, are the configurations at which the transition between stability and instability occurs. They represent saddle-node bifurcations when using the Casimir as a parameter. The cusps in (b) correspond to the points marked A and B in Figure 5.6.

The topmost line in Figure 5.7(a) corresponds to the values of $q > \pi/2$, the part of the bottom

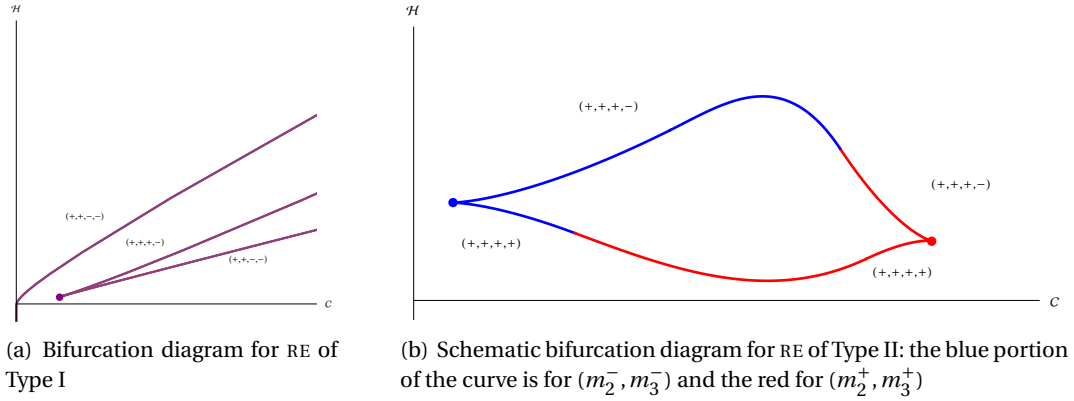


Figure 5.7: Energy-Casimir bifurcation diagrams with cusp points and signatures of the Hessian

line above the cusp to the values of q less than the root (solved for q) of $B = \sqrt{\frac{\cos^3(q)(2+\cos(q))}{2\sin^3(q)\sin^2(\frac{q}{2})}}$ for a fixed B , and the bottom half to the rest of the interval between said root and $\pi/2$.

We have simplified the form of the bifurcation diagram in Figure 5.7(b), but the essential features remain: the two solutions, one of which is linearly stable and the other linearly unstable, merge together at the two cusps in saddle-node bifurcations.

Figure 5.7 has the signatures of the Hessian of \mathcal{H} , as restricted to the level sets of \mathcal{C} next to each part of the bifurcation diagram. Since the eigenvalues of the Hessian depend smoothly on B and q and the only points where the Hessian matrix has a zero eigenvalue are the cusp points (the only points where the matrix of the linearized system has a zero eigenvalue), it is sufficient to calculate the signature of the Hessian on each part of the diagram for a single value of B and q . By continuity, the signatures will remain the same throughout the changes in B and q .

It follows that the linearly stable (relative) equilibria in Proposition 5.6 are in fact nonlinearly stable.

Remark 5.7. It is curious that for the Type I RE, the unstable configurations occur when the particles are closer together, but for the Type II RE, the unstable ones occur when they are further apart. However, in both cases, the unstable ones occur when the particles are further from the axis of rotation (see Figure 1.1).

5.5 The bifurcation diagram

So far, we have described the existence of relative equilibria in terms of B and q . This is a reasonable approach for presenting the results, but carries no physical meaning in terms of bifurcations. Indeed, for each fixed value of B , there is a 1-parameter family of reduced systems parametrized naturally by the Casimir. We therefore have two parameters: the Casimir (an ‘internal’ parameter) and the magnetic field strength (an ‘external’ parameter).

For relative equilibria of Type I, parametrized by B and q , the explicit expression for the Casimir is

$$\frac{1}{2} \cot\left(\frac{q}{2}\right) \csc^2\left(\frac{q}{2}\right) \sec^2(q) (1 + \cos(2q) + B^2 \sin^3(q)),$$

which for all values of B tends to $+\infty$ if $q \rightarrow 0$ and to 0 if $q \rightarrow \pi$, spanning all the values in between.

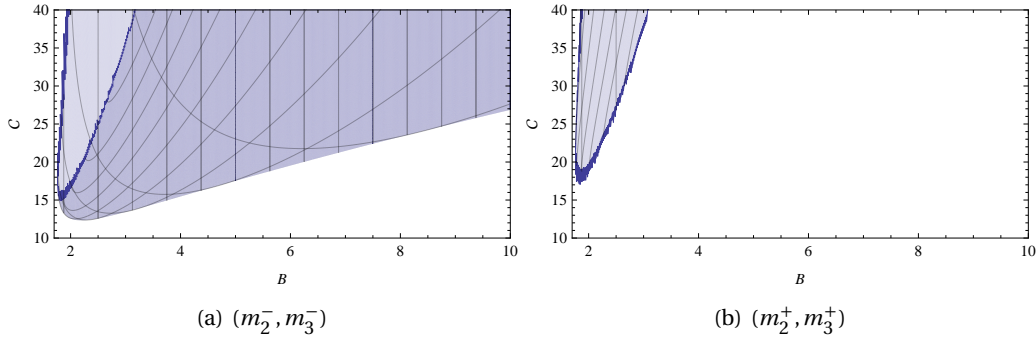


Figure 5.8: Possible values of the Casimir for the two classes of relative equilibria of Type II, as a function of B

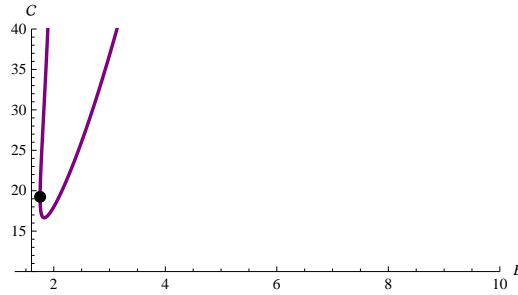


Figure 5.9: Values of \mathcal{C} on the threshold curve

Therefore, the region spanned by all possible Casimir values as B varies is the entirety of the first quadrant in the (B, \mathcal{C}) -plane.

The situation is more complicated when we pose the same problem, but for relative equilibria of Type II: once again, we have to resort to numerics due to the complexity of the computations. The image of the set of RE for (m_2^-, m_3^-) is depicted in Figure 5.8(a). The darker blue area on the diagram denotes the locus of the points in the (B, \mathcal{C}) plane that are images of two relative equilibria.

For (m_2^+, m_3^+) (Type II) the possible values of the Casimir lie inside the set depicted in Figure 5.8(b).

As can be noted, the region in the second diagram fits in the lighter area of the first one, which is clear since every point strictly inside the union of the two sets corresponds to two values of q and, therefore, to two relative equilibria.

The image of the threshold curve in the (B, \mathcal{C}) -plane is the transition between the two regions in Figure 5.8 and is shown in Figure 5.9.

Since the values of the Casimir for the RE of Type II, as plotted against q (see Figure 5.6) form a closed curve, the set of possible values of \mathcal{C} on the set of relative equilibria is bounded for every value of B . Saddle-node bifurcations arise at the extreme points of \mathcal{C} on the curve.

These relative equilibria are degenerate; they coincide with the set of cusp points for Type II relative equilibria in Figures 5.2 and 5.7 and, as the only degenerate relative equilibria of Type II, with the curves in Figure 5.2 that separate the regions of stability and instability. Therefore, the bifurcation curve is the image of said curves on the (B, \mathcal{C}) plane.

Indeed, seeing as the relative equilibria on the bifurcation curve are degenerate, the only point where it can meet the threshold curve is the only degenerate relative equilibrium on the threshold

curve: $B = 4/3^{3/4}$, $\mathcal{C} = 20/(3\sqrt{3})$, the black dot in Figure 5.9.

From the discussion above it can be easily seen that for every fixed value of B and \mathcal{C} for which there are 2 RE of Type II, one of these is stable the other (linearly) unstable.

6 Opposite charges

After describing in some detail the relative equilibria with equal charges, the next logical step is to investigate the case with opposite charges, thereby replacing the repelling potential by an attracting one. The following observation is based on [6, Lemma 2.4], which there is for a Lagrangian system and here includes the magnetic field.

On the configuration space $\mathcal{Q} = S^2 \times S^2 \setminus \Delta$ (see Sec. 4), the involution $(\mathbf{q}_1, \mathbf{q}_2) \mapsto (\mathbf{q}_1, -\mathbf{q}_2)$ is antisymplectic on the second component. If this is combined with a change of sign of charge $e_2 \mapsto -e_2$ of the second particle, then the symplectic form in (2.3) is unchanged (we are not changing the sign of the magnetic field).

Write $V_1(\mathbf{q}_1, \mathbf{q}_2)$ for the potential energy obtained from $V(\mathbf{q}_1, \mathbf{q}_2)$ after changing e_2 to $-e_2$. Then if $V(\mathbf{q}_1, \mathbf{q}_2) = V_1(\mathbf{q}_1, -\mathbf{q}_2)$ then the involution transforms the Hamiltonian to itself. Under this condition, the involution will map the Hamiltonian system with potential energy V to the one with potential energy V_1 .

A case in point is $V(\mathbf{q}_1, \mathbf{q}_2) = e_1 e_2 \cot(q)$ where q is the geodesic distance between \mathbf{q}_1 and \mathbf{q}_2 on the sphere. For then $V_1(\mathbf{q}_1, \mathbf{q}_2) = e_1(-e_2) \cot(q)$, and $V(\mathbf{q}_1, -\mathbf{q}_2) = e_1 e_2 \cot(\pi - q) = -e_1 e_2 \cot(q)$.

The effect of this involution on the reduced coordinates is

$$(m_1, m_2, m_3, q, p) \longmapsto (-m_1, -m_2, m_3, \pi - q, -p)$$

(notice that m_3 is unchanged). The Casimir is invariant, and so is the Hamiltonian provided the potential changes as discussed above. Each relative equilibrium, as well as its stability properties, for charges (e_1, e_2) is therefore mapped to a relative equilibrium for charges $(e_1, -e_2)$, together with its stability properties, by this involution.

Now consider the specific potential $V(q) = e_1 e_2 \cot q$ (repelling for like charges, attracting for charges of opposite sign). All the conclusions about the relative equilibria found in Section 5 carry over here, up to reflection of all the graphs with respect to the line $q = \frac{\pi}{2}$. For example, the threshold curve becomes $B = 2\sqrt{\frac{\csc(q)}{1+\cos(q)}}$. In the same way, the RE are divided into two types, I and II, depending on domains of existence. We refer to them accordingly, depending on the one they coincide with via the involution described above.

The geometric differences with the case of identical particles is illustrated in Figure 6.1. In particular, the new Type I relative equilibria have the axis between the particles, and the configuration is, still, not symmetric. (Compare with the analogous transformation in [6].)

Type II relative equilibria are no longer isosceles configurations in the sense we have used before; however, they retain a symmetry, with the axis of rotation lying to the side of the particle pair. As previously, the same geometric arrangement can be occupied by systems with two rates of rotation, and hence two different energy levels.

The plots of the regions of stability and instability shown in Figures 5.4 and 5.5 remain the same, except for a reflection in the line $q = \pi/2$.

In particular (cf. Remark 5.7) RE with configurations that are closer to the axis of rotation are now more likely to be stable.

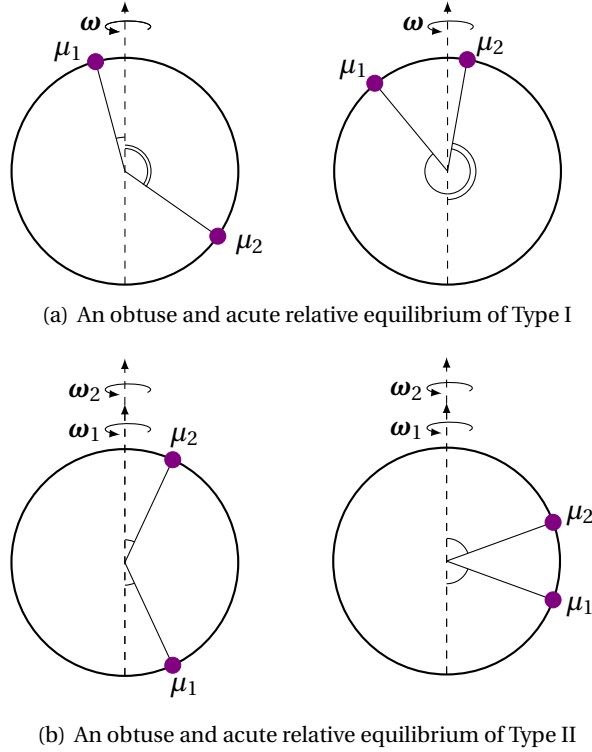


Figure 6.1: The two types of relative equilibrium for equal masses and opposite charges, with $V(q) = -\cot(q)$.

Appendix

Here we elaborate on the behaviour of the solutions in (5.2) when $q \rightarrow \frac{\pi}{2}$, $B \rightarrow 0$.

In the case of equal masses and a repelling potential, right angle equilibria exist for the two-body problem on a sphere [6]. However, they are not defined for the case when $B \neq 0$ and is less than the minimum value of the threshold curve. On the other hand, seeing that the equal masses with a repelling potential gravitational two-body problem is a limiting case with $B \rightarrow 0$ for two identical particles, these equilibria should arise from the ones that we have described.

When $B = 0$, the system (5.1) is reduced to one equation

$$m_2 m_3 = 1, \tag{A.1}$$

giving a family of right angle equilibria in accordance with [6]. We have already mentioned that setting B equal to 0 and taking the Taylor series at $q = \frac{\pi}{2}$ gives finite limits in the cases of the solutions (5.2). Indeed, we get

$$m_2^\pm = \mp 1 \pm \frac{3}{2} \left(q - \frac{\pi}{2} \right) + \bar{O} \left(q - \frac{\pi}{2} \right)^2 \tag{A.2}$$

$$m_3^\pm = \mp 1 \pm \frac{1}{2} \left(q - \frac{\pi}{2} \right) + \bar{O} \left(q - \frac{\pi}{2} \right)^2 \tag{A.3}$$

However, setting $q = \frac{\pi}{2}$ and **then** taking $B = 0$ results in an indefinite expression. To explain this, let us consider the expression $m_2^\pm m_3^\pm$.

$$m_2^\pm m_3^\pm = \pm \frac{1}{4} \cot\left(\frac{q}{2}\right) \left(\sqrt{B^2 \sin^2(q) \tan^2(q) + 4 \cot\left(\frac{q}{2}\right) - B \sin(q) \tan(q)} \right) * \\ * \left(\sqrt{B^2 \sin^2(q) \tan^2(q) + 4 \cot\left(\frac{q}{2}\right) + B(\cos(q) + \sec(q) - 2)} \right) \quad (\text{A.4})$$

When $B \rightarrow 0$,

$$m_2^\pm m_3^\pm \rightarrow \cot^2\left(\frac{q}{2}\right), \quad (\text{A.5})$$

but it can easily be seen that $m_2^\pm m_3^\pm$ does not converge uniformly to $\cot^2\left(\frac{q}{2}\right)$ in the neighbourhood of $q = \frac{\pi}{2}$ with $B \rightarrow 0$ (see Part III, Chapter XVI of [1] for definitions). Thus, the order of the limits can't be changed.

However, we can take B as a function of q , demand that $B\left(\frac{\pi}{2}\right) = 0$ and see whether a limit exists when $q \rightarrow \frac{\pi}{2}$.

Calculation of Taylor series of m_2^\pm and m_3^\pm shows that only the linear approximation of $B(q)$ (i.e. $B'\left(\frac{\pi}{2}\right)$) plays a role in the behaviour at $q = \frac{\pi}{2}$. In fact, if $B'\left(\frac{\pi}{2}\right) = a$, we have

$$m_2^\pm = \frac{1}{2} \left(a \mp \sqrt{a^2 + 4} \right) + \bar{O}\left(q - \frac{\pi}{2}\right) \\ m_3^\pm = \frac{1}{2} \left(\mp \sqrt{a^2 + 4} - a \right) + \bar{O}\left(q - \frac{\pi}{2}\right), \quad (\text{A.6})$$

resulting in

$$m_2^\pm m_3^\pm = 1 + \bar{O}\left(q - \frac{\pi}{2}\right), \quad (\text{A.7})$$

with different directions of approaching the point $(q, B) = \left(\frac{\pi}{2}, 0\right)$ giving us different instances of right angle relative equilibria for the two body problem.

Note that the set of right-angle relative equilibria is "wrapped" into one point on (q, B) -plane. It is precisely the non-uniform convergence of the solutions that allows us to approximate the whole family of right-angle relative equilibria rather than just one: indeed, for every small value of B we can find a stalk of functions $B(q)$ such that respective relative equilibria approximate the given right-angle one.

But what happens to right-angle RE?

As was described above, right-angled equilibria do not exist until the value of B reaches a certain threshold.

Suppose now that the particles are in a right angle equilibrium state, with $B = 0$, and we "switch on" the magnetic field. What happens to the particle motion?

When the newly appeared magnetic field is weak ($B < \frac{4}{3^{\frac{3}{4}}}$) for a right angle relative equilibrium, we land in an initial state with a non-zero B and $q = \frac{\pi}{2}$, which can not be a relative equilibrium, and neither it can turn into one with the passage of time, since our system is deterministic.

For a stronger magnetic field, we theoretically might achieve a Type II relative equilibrium. As mentioned above, from the equations (5.1) with $B = 0$ and $q = \frac{\pi}{2}$, the conditions for RE are $p = 0$, $m_1 = 0$, $m_2 m_3 = 1$. On the other hand, the product of m_2 and m_3 for any Type II relative equilibrium is always negative.

Thus, for any change in the strength of the magnetic field, the right angle relative equilibria do not persist.

Remark It would be interesting to analyze this 2-body problem from a control theoretic perspective, where B is the control parameter.

References

- [1] G. Arkhipov, V. Sadovnichii, and V. Chubarikov. Lectures on the calculus. *Vysshaya Shkola, Moscow*, 1999.
- [2] A. Borisov, L. García-Naranjo, I. Mamaev, and J. Montaldi. Reduction and relative equilibria for the two-body problem on spaces of constant curvature. *Celestial Mech. and Dyn. Astr.*, 130(6):43, 2018.
- [3] A. V. Borisov, I. S. Mamaev, and A. A. Kilin. Two-body problem on a sphere. reduction, stochasticity, periodic orbits. *Regular and Chaotic Dynamics*, 9(3):265–279, 2004.
- [4] R. Descartes. *La géométrie*. Hermann, 1886.
- [5] M. Escobar-Ruiz and A. Turbiner. Two charges on a plane in a magnetic field: II. moving neutral quantum system across a magnetic field. *Annals of Physics*, 359:405–418, 2015.
- [6] L. C. García-Naranjo and J. Montaldi. Attracting and repelling 2-body problems on a family of surfaces of constant curvature. *J. Dyn. and Diff. Equ.*, pages 1–25, 2020.
- [7] V. V. Kozlov. Dynamics in spaces of constant curvature. *Vestnik Moskovskogo Universiteta. Seriya 1. Matematika. Mekhanika*, (2):28–35, 1994.
- [8] L. D. Landau and E. M. Lifshitz. *The classical theory of fields*. 1971.
- [9] R. G. Littlejohn. A guiding center hamiltonian: A new approach. *J. of Math. Phys.*, 20(12):2445–2458, 1979.
- [10] J. E. Marsden and T. S. Ratiu. *Introduction to mechanics and symmetry: a basic exposition of classical mechanical systems*, volume 17. Springer, 2013.
- [11] D. Pinheiro and R. S. MacKay. Interaction of two charges in a uniform magnetic field: I. Planar problem. *Nonlinearity*, 19(8):1713, 2006.
- [12] D. Pinheiro and R. S. MacKay. Interaction of two charges in a uniform magnetic field: II. Spatial problem. *J. Nonlinear Sci.*, 18(6):615, 2008.

N. Balabanova & J. Montaldi
 Department of Mathematics,
 University of Manchester
 Manchester M13 9PL, UK
nataliya.balabanova@manchester.ac.uk
j.montaldi@manchester.ac.uk