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# Ideals in mod-$R$ and the $\omega$-radical

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*Dedicated to Claus Michael Ringel on the occasion of his sixtieth birthday*

Let $R$ be an artin algebra and let mod-$R$ denote the category of finitely presented right $R$-modules. The radical rad = rad(mod-$R$) of this category and its finite powers play a major role in the representation theory of $R$. The intersection of these finite powers is denoted $\mathrm{rad}^\omega$ and the nilpotence of this ideal has been investigated in [13], [7] for instance. In [17] arbitrary transfinite powers, $\mathrm{rad}^\alpha$, of rad were defined and linked to the extent to which morphisms in mod-$R$ may be factorised. In particular it was shown that if $R$ is an artin algebra then the transfinite radical, $\mathrm{rad}^\infty$, the intersection of all ordinal powers of rad, is non-zero if and only if there is a 'factorisable system' of morphisms in rad and, in that case, the Krull-Gabriel dimension of mod-$R$, equals $\infty$ (that is, is undefined). More precise results concerning the index of nilpotence of rad for artin algebras have been proved in [14], [20] [24], [25], [26].

If $R$ is an artin algebra then any morphism between indecomposable finitely generated modules which lie in different components of the Auslander-Reiten quiver of $R$ must belong to $\mathrm{rad}^\omega$. In the case that $R$ is tame hereditary it may be observed (see [18], [22]) that such morphisms factor through finite direct sums of infinite length indecomposable pure-injective modules. This leads to the idea that these large pure-injective modules 'glue together' components of the Auslander-Reiten quiver. This is also supported by Ringel's 'sewing' of components, see [21], [23]. I show here that if $R$ is any artin algebra then any morphism in $\mathrm{rad}^\omega$ factors through a finite direct sum of indecomposable infinite length pure-injective modules.

The original proof of this result [16] used ideas (pp formulas and types, free realisations) which come from the model theory of modules and the proof was fairly involved. The proof given here is much simpler and uses results of Krause [14] on ideals in mod-$R$ (from [14] one can deduce only the weaker result that any morphism in $\mathrm{rad}^\omega$ factors through a possibly infinite product of indecomposable, infinite length, pure-injective modules). Indeed our proof applies to arbitrary fp-idempotent ideals of mod-$R$ ($\mathrm{rad}^\omega$ is one such) and we obtain a factorisation result in this generality.

From the point of view of the analysis of [14] the extra ingredient is 4.4 and its corollary 4.5. This account is considerably longer than it would have been if we had simply quoted all needed results from [14] but we feel that our reworking of the relevant results from [14] has some merit, apart from making the paper self-contained, in that the initial part of the analysis is done in complete generality and then we isolate the property of artin algebras which yields the stronger conclusions in that context.

# 1 Serre subcategories of the functor category, definable subcategories of the module category and closed subsets of the Ziegler spectrum

Let $R$ be any ring. Denote by Mod-$R$, respectively mod-$R$, the category of right $R$-modules, respectively finitely presented right $R$-modules. We use $R$-Mod ($R$-mod) to denote left (finitely presented) $R$-modules. Set $D(R) = (R\text{-mod}, \mathbf{Ab})$ to be the category of additive functors from $R$-mod to the category, $\mathbf{Ab}$, of abelian groups. This is a Grothendieck abelian category of global dimension 2 with the representable functors $(L, -)$, as $L$ ranges over (a small version of) $R$-mod, being a generating set of finitely generated projective objects. The full subcategory $C(R) = (R\text{-mod}, \mathbf{Ab})^{\mathrm{fp}}$ of finitely presented functors is an abelian subcategory and the inclusion of $C(R)$ into $D(R)$ is exact. In particular, every finitely presented functor $F$ is a **coherent** object of $D(R)$, meaning that every finitely generated subfunctor of $F$ is again finitely presented. Since the category $D(R)$ is locally finitely presented each object in it is a direct limit of objects in $C(R)$.

We recall, [5], [9], that there is a duality, that is an equivalence $D : C(R^{\mathrm{op}})^{\mathrm{op}} \longrightarrow C(R)$, between $C(R)$ and $C(R^{\mathrm{op}}) = (\text{mod-}R, \mathbf{Ab})^{\mathrm{fp}}$ given by $DF.L = (F, - \otimes_R L)$ for $F \in D(R^{\mathrm{op}})$ and $L \in R$-mod. Here we use the notation $(A, B)$ to denote the set or group of morphisms from $A$ to $B$ when the category to which $A$ and $B$ belong is clear. So $(F, - \otimes_R L)$ denotes the group of natural transformations from the functor $F$ to the functor $- \otimes_R L$. The latter is the object of (mod-$R, \mathbf{Ab}$) which takes any finitely presented right $R$-module $M$ to the group $M \otimes_R L$ and which has the natural effect on morphisms. Since $L$ is finitely presented so is $- \otimes L$ [2]. Writing $D$ also for the duality starting at $C(R)$, we have that $D^2$ is naturally equivalent to the identity functor. From the Yoneda lemma one has $D(X, -) \simeq X \otimes_R -$ for $X \in$ mod-$R$.

A **Serre subcategory** of $C(R)$ is a full subcategory $\mathcal{S}$ which is closed under subobjects, quotients and extensions (so if $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ is exact then $B \in \mathcal{S}$ if and only if $A, C \in \mathcal{S}$). Every Serre subcategory generates a hereditary torsion theory [27] on $D(R)$, with the torsion class being the closure, $\varinjlim \mathcal{S}$, of $\mathcal{S}$ under direct limits in $D(R)$. We refer to objects in this class as **$\mathcal{S}$-torsion**. This hereditary torsion theory is of finite type: given any $F \in C(R)$ the filter $\mathcal{U}_{\mathcal{S}}(F) = \{F' \leq F : F/F' \text{ is } \mathcal{S}\text{-torsion}\}$ of **$\mathcal{S}$-dense** subobjects of $F$ has a cofinal set of finitely generated objects.

If $\mathcal{S}$ is a Serre subcategory of $C(R^{\mathrm{op}}) = (\text{mod-}R, \mathbf{Ab})^{\mathrm{fp}}$ then we let $D\mathcal{S}$ be the full subcategory of $C(R)$ with objects $\{DF : F \in \mathcal{S}\}$. It is easily checked that this is a Serre subcategory of $C(R)$ and we refer to it as the Serre subcategory **dual** to $\mathcal{S}$.

Let $\mathcal{Y}$ be a subset or subcategory of Mod-$R$. Associated to $\mathcal{Y}$ is $\mathcal{S}_{\mathcal{Y}} = \{F \in C(R) : (F, M \otimes -) = 0 \text{ for every } M \in \mathcal{Y}\}$. This is a (typical) Serre subcategory of $C(R)$. Let $\mathcal{S}$ be any Serre subcategory of $C(R)$. Associated to $\mathcal{S}$ is $\mathcal{X}_{\mathcal{S}} = \{M \in \text{Mod-}R : (F, M \otimes -) = 0 \text{ for all } F \in \mathcal{S}\}$. This is a typical **definable** subcategory of Mod-$R$ (that is, a full subcategory of Mod-$R$ which is closed under products, pure submodules and direct limits - equivalently an axiomatisable subcategory of Mod-$R$ which is closed under direct sums and direct summands) and one has that $\mathcal{X}_{\mathcal{S}_{\mathcal{Y}}}$ is the **definable closure**, $\bar{\mathcal{Y}}$, of $\mathcal{Y}$ - the smallest definable subcategory of Mod-$R$ containing $\mathcal{Y}$. In this way we have a bijection between definable subcategories of Mod-$R$ and the Serre subcategories of $C(R)$.

To $\mathcal{Y} \subseteq$ Mod-$R$, we also have the associated Serre subcategory, $D\mathcal{S}_{\mathcal{Y}}$, of $C(R^{\mathrm{op}})$. This can be described directly as follows. Given any functor $F \in D(R^{\mathrm{op}}) = (\text{mod-}R, \mathbf{Ab})$ there is a unique extension of $F$ to a functor $\overrightarrow{F} \in (\text{Mod-}R, \mathbf{Ab})$ which commutes with direct limits [3] (if $M = \varinjlim M_\lambda$ with the $M_\lambda$ finitely presented then set $\overrightarrow{F}M = \varinjlim FM_\lambda$). Then $D\mathcal{S}_{\mathcal{Y}} = \{F \in C(R^{\mathrm{op}}) : \overrightarrow{F}(M) = 0 \text{ for every } M \in \mathcal{Y}\}$.

An embedding $f : M \longrightarrow M'$ of right modules is **pure** if for every (finitely presented) left $R$-module $L$ the map $f \otimes_R 1_L : M \otimes_R L \longrightarrow M' \otimes_R L$ is an embedding. A module $N$ is **pure-injective** (also called **algebraically compact**) if every pure embedding with domain $N$ is split. If $R$ is an artin algebra then every finitely generated module is pure-injective but, unless $R$ is of finite representation type, there will be infinitely generated indecomposable pure-injective modules ([15, 4.66, 13.4]).

The **right Ziegler spectrum** $\mathrm{Zg}_R$ of $R$ [28] is the topological space with points the (isomorphism classes of) indecomposable pure-injective right $R$-

modules and with a basis of compact open sets consisting of those sets of the form $(F) = \{N \in \mathrm{Zg}_R : \overrightarrow{F}(N) \neq 0\}$ as $F$ ranges over $C(R^{\mathrm{op}})$. Using the full and faithful embedding of Mod-$R$ into $D(R)$ which is given on objects by $M \mapsto M \otimes -$, together with the fact that the injective objects of $D(R)$ are, up to isomorphism, exactly those of the form $N \otimes -$ where $N$ is a pure-injective right $R$-module [9], we obtain the following equivalent description of this space. The points are the (isomorphism classes of) indecomposable injective objects of $D(R)$ and a basis of open sets consists of those sets of the form $(G) = \{E : E$ is indecomposable injective and $(G, E) \neq 0\}$ as $G$ ranges over $C(R)$. We write ${}_R\mathrm{Zg}$ for $\mathrm{Zg}_{R^{\mathrm{op}}}$.

Given any definable subclass $\mathcal{X}$ of Mod-$R$ we denote by $\mathcal{X} \cap \mathrm{Zg}_R$ the set of points of $\mathrm{Zg}_R$ which lie in $\mathcal{X}$. This is a typical closed subset of $\mathrm{Zg}_R$ and the correspondence is bijective: if $\mathcal{X}$ and $\mathcal{Y}$ are definable subcategories of Mod-$R$ with $\mathcal{X} \cap \mathrm{Zg}_R = \mathcal{Y} \cap \mathrm{Zg}_R$ then $\mathcal{X} = \mathcal{Y}$ [28].

In summary, we have bijective correspondences between:
closed subsets of $\mathrm{Zg}_R$;
definable subcategories of Mod-$R$;
Serre subcategories of $C(R)$;
Serre subcategories of $C(R^{\mathrm{op}})$.

For future reference, we point out that the torsion theory on $D(R) = (R\text{-mod}, \mathbf{Ab})$ corresponding to the closed subset $C$ of $\mathrm{Zg}_R$ is that which is cogenerated by the set $\{N \otimes - : N \in C\}$ of indecomposable injectives of $D(R)$ and the corresponding torsion class in $D(R^{\mathrm{op}}) = (\text{mod-}R, \mathbf{Ab})$ consists of all those functors $F$ with $\overrightarrow{F}(N) = 0$ for all $N$ in $C$ (in fact it suffices to take $N$ belonging to any dense subset of $C$).

Let $C \in$ mod-$R$ and let $\bar{a}$ be an $n$-tuple of elements from $C$. We also denote by $\bar{a}$ the morphism from $R^n$ to $C$ which takes the $i$-th unit in $R^n$ (under some fixed decomposition) to the $i$-th entry $a_i$ of $\bar{a}$. So we have the induced map $(C, -) \xrightarrow{(\bar{a}, -)} (R^n, -)$. Since $(R^n, -)$ is coherent the image of $(\bar{a}, -)$ is finitely presented.

For more detail see the references cited and also, for example, [6], [10], [11], [15], [19].

## 2 Ideals of mod-$R$ and their annihilators

First we consider subfunctors and quotient functors of representable functors.

**Lemma 2.1** *Let $X \in$ mod-$R$ and suppose that $G \leq (X,-)$. Then $G$ is finitely generated iff it has the form* $\mathrm{im}((f,-) : (Y,-) \longrightarrow (X,-))$ *for some* $X \xrightarrow{f} Y \in$ mod-$R$.

**Proof.** Since $(Y,-)$, when $Y \in$ mod-$R$, is finitely generated any such functor $\mathrm{im}(f,-)$ is finitely generated. Conversely, if $G$ is finitely generated then it is the image of a representable functor, say $(Y,-) \longrightarrow G$ is epi. Compose this morphism with the inclusion $G \longrightarrow (X,-)$ to obtain a morphism $(Y,-) \longrightarrow (X,-)$ which, by the Yoneda Lemma, has the form $(f,-)$ for some $X \xrightarrow{f} Y$, as required. $\square$

Let $F \in C(R^{\mathrm{op}})$. Then (since $F$ is finitely presented) there is a morphism $X \xrightarrow{f} Y \in$ mod-$R$ and an exact sequence $(Y,-) \xrightarrow{(f,-)} (X,-) \longrightarrow F \longrightarrow 0$. We write $F = F_f$, that is $F_f = \mathrm{coker}(f,-)$. We record this for later reference

$$\begin{tikzcd}
(Y,-) \arrow[r, "{(f,-)}"] \arrow[dr] & (X,-) \arrow[r] \arrow[d, dash, "\}F_f"] & F_f \arrow[r] & 0 \\
& \mathrm{im}(f,-) \arrow[d, dash] \arrow[dr] & & \\
& 0 & 0 &
\end{tikzcd}$$

**Lemma 2.2** $F \in (\text{mod-}R, \mathbf{Ab})$ *is finitely presented iff it has the form* $F_f = (X,-)/\mathrm{im}(f,-)$ *for some* $X \xrightarrow{f} Y \in$ mod-$R$.

Of course, $f$ is far from unique but every object of $C(R^{\mathrm{op}})$ may be represented in this way. If $M$ is a (finitely presented) right $R$-module then, evaluating the above exact sequence at $M$, we see that $F_f M$ is the group of morphisms from $X$ to $M$ modulo the subgroup consisting of those which can be factored initially through $f$.

### Annihilators of ideals of mod-$R$

By a **left ideal** of mod-$R$ we mean a collection of morphisms of mod-$R$ closed under addition, where defined, under post-composition by arbitrary morphisms and containing all zero morphisms. If this set is also closed under pre-composition by arbitrary morphisms then it is a (**two-sided**) **ideal**.

Given a morphism $h \in$ mod-$R$, let $\mathcal{L}_h$ denote the left ideal of mod-$R$ generated by $h$. That is, $\mathcal{L}_h = \{gh : g \in \text{mod-}R \text{ and } gh \text{ is defined }\} \cup \{0 \in (C,D) : C, D \in \text{mod-}R\}$. If $\mathcal{L}$ is a left ideal of mod-$R$ and $A, B \in$ mod-$R$, set

$\mathcal{L}(A,-) = \mathcal{L} \cap (A,B)$ - a subgroup of $(A,B)$. For every $A \in$ mod-$R$ this gives a subfunctor , $\mathcal{L}(A,-)$, of $(A,-)$ given on objects by $\mathcal{L}(A,-).B = \mathcal{L}(A,B)$.

We say that a morphism $k$ **initially factors through** the morphism $f$ if $k \in \mathcal{L}_f$.

**Lemma 2.3** *(i) Let* $X \xrightarrow{f} Y \in$ mod-$R$. *Then* $\mathcal{L}_f(X,-) = \mathrm{im}(f,-)$.
*(ii) If* $X \xrightarrow{f} Y$, $X \xrightarrow{g} B$ *are in* mod-$R$ *then* $g \in \mathcal{L}_f(X,B)$ *iff* $g$ *initially factors through* $f$ *iff* $\mathrm{im}(g,-) \leq \mathrm{im}(f,-)$.

**Proof.** (i) We have $\mathcal{L}_f(X,B) = \{g \in (X,B) : g = g'f$ for some $Y \xrightarrow{g'} B\} = \{g \in (X,B) : g = (f,B).g'$ for some $g' \in (Y,B)\} = \mathrm{im}(f,-).B$, establishing (i).

Note that $\mathcal{L}_f(X',-) = 0$ if $X' \neq X$.

(ii) This also follows directly from the definitions. $\square$

The preordering on morphisms with domain $X$ implicit in part (ii) above is considered further in, for example, [20].

Let $\mathcal{I}$ be an ideal of mod-$R$. Set $\mathrm{ann}\mathcal{I} = \{F \in (\text{mod-}R, \mathbf{Ab})^{\mathrm{fp}} : F\mathcal{I} = 0,$ that is $Fh = 0$ for all $h \in \mathcal{I}\}$.

If a functor annihilates a class of morphisms in mod-$R$ then it annihilates the two-sided ideal generated by that class ($Fh = 0$ implies $F(ghf) = 0$ for every $g, f$). So, there is no loss in generality in considering annihilators of two-sided ideals.

**Lemma 2.4** $\mathrm{ann}\mathcal{I}$ *is closed under subfunctors and quotients.*

**Proof.** Say $0 \longrightarrow F' \longrightarrow F$ and $F \longrightarrow G \longrightarrow 0$ are exact and let $F \in \mathrm{ann}\mathcal{I}$. Let $A \xrightarrow{h} B \in \mathcal{I}$. Since $F'B \longrightarrow FB$ is monic it follows directly that $F'h = 0$ and, since $FA \longrightarrow GA$ is epi, that $Gh = 0$.
$\square$

If $\mathrm{ann}\mathcal{I}$ is closed under extensions it then follows that $\mathrm{ann}\mathcal{I}$ is a Serre subcategory of $C(R^{\mathrm{op}})$. In this case Krause, [14, Section 5], says that $\mathcal{I}$ is **fp-idempotent**.

**Lemma 2.5** *[14, Appendix C] Let* $X \xrightarrow{f} Y, A \xrightarrow{h} B \in$ mod-$R$.
*Then* $F_f h = 0$
*iff for all* $X \xrightarrow{g} A \in$ mod-$R$ *there is* $Y \xrightarrow{g'} B \in$ mod-$R$ *with* $hg = g'f$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \arrow[d, dashed, "g'"] \\ A \arrow[r, "h"'] & B \end{tikzcd}$$

**Proof.** Consider

$$\begin{tikzcd} (Y,A) \arrow[r, "{(f,A)}"] \arrow[d, "{(Y,h)}"'] & (X,A) \arrow[r, "\pi_{f,A}"] \arrow[d, "{(X,h)}"] & F_fA \arrow[r] \arrow[d, "F_fh"] & 0 \\ (Y,B) \arrow[r, "{(f,B)}"'] & (X,B) \arrow[r, "\pi_{f,B}"'] & F_fB \arrow[r] & 0 \end{tikzcd}$$

Then $F_fh = 0$ iff $F_fh.\pi_{f,A} = 0$ (since $\pi_{f,A}$ is epi) iff $\pi_{f,B}.(X,h) = 0$ iff $\mathrm{im}(X,h) \leq \mathrm{im}(f,B)$ as required. $\square$

Taking $h = \mathrm{id}_A$ we have that $F_fA = 0$ if and only if every morphism from $X$ to $A$ factors initially through $f$.

**Remark 2.6** *It is easily checked that the natural extension,* $\overrightarrow{F}_f$*, of* $F_f$ *to* $(\text{Mod-}R, \mathbf{Ab})$ *that we discussed earlier also is defined by the exact sequence* $(Y,-) \xrightarrow{(f,-)} (X,-) \longrightarrow \overrightarrow{F}_f \longrightarrow 0$ *where the representable functors are now regarded as belonging to the category* $(\text{Mod-}R, \mathbf{Ab})$ *(for, the cokernel of this map agrees with* $F_f$ *on* $\text{mod-}R$ *and also commutes with direct limits - since* $(X,-)$ *and* $(Y,-)$ *do and by exactness of direct limits in* $\mathbf{Ab}$*). So the criterion of 2.5, modified for* $(\text{Mod-}R, \mathbf{Ab})$*, applies to* $\overrightarrow{F}_f$*.*

**Lemma 2.7** *Let* $X \xrightarrow{f} Y \in \text{mod-}R$ *and let* $\mathcal{I}$ *be an ideal of* $\text{mod-}R$*.*
*Then* $F_f(\mathcal{I}) = 0$ *iff* $\mathcal{I}(X,-) \leq \mathrm{im}(f,-)$*.*

**Proof.** $F_f\mathcal{I} = 0$ iff for all $B \in \text{mod-}R$ and for all $g \in \mathcal{I}(X,B)$, $g$ initially factors through $f$ (by 2.5 and since $\mathcal{I}$ is an ideal) iff for all $B \in \text{mod-}R$ we have $\mathcal{I}(X,-).B \leq \mathrm{im}(f,-).B$ iff $\mathcal{I}(X,-) \leq \mathrm{im}(f,-)$.

$$\begin{tikzcd}
& (X,-) \arrow[dd, dash, "\}F_f"] \\
(Y,-) \arrow[ur, "{(f,-)}"] \arrow[dr] & \\
& \mathrm{im}(f,-) \arrow[d, dash] \\
& \mathcal{I}(X,-)
\end{tikzcd}$$

□

**Annihilators of functors in** mod-$R$

Given any class $\mathcal{A}$ of functors in $C(R^{\mathrm{op}}) = (\text{mod-}R, \mathbf{Ab})^{\mathrm{fp}}$ let $\mathrm{ann}\mathcal{A} = \{h \in \text{mod-}R : Fh = 0 \text{ for all } F \in \mathcal{A}\}$ - a two-sided ideal of mod-$R$.

Following [14, Section 5], for each $F \in C(R^{\mathrm{op}})$ let $r_{\mathcal{A}}F = \bigcap\{F' \leq F :$ ($F'$ is finitely generated and) $F/F' \in \mathcal{A}\}$. So by 2.1 $r_{\mathcal{A}}(X,-) = \bigcap\{\mathrm{im}(f,-) : X \xrightarrow{f} Y \in \text{mod-}R \text{ and } F_f \in \mathcal{A}\}$. It follows that $\bigcup\{r_{\mathcal{A}}(X,B) : X, B \in \text{mod-}R\}$ is a left ideal of mod-$R$. We use the notation $r_{\mathcal{A}}$ for this left ideal.

**Lemma 2.8** *Let $\mathcal{I}$ be an ideal of* mod-$R$ *and set $\mathcal{A} = \mathrm{ann}\mathcal{I}$ and $\bar{\mathcal{I}} = \mathrm{ann}\mathcal{A}$. Then $\bar{\mathcal{I}}$ is an ideal of* mod-$R$ *such that $\mathcal{I} \leq \bar{\mathcal{I}} \leq r_{\mathcal{A}}$ and $\bar{\mathcal{I}}$ is the largest such ideal of* mod-$R$.

**Proof.** Clearly $\mathcal{I} \leq \bar{\mathcal{I}}$. If $g \in \bar{\mathcal{I}}(X,B)$ and if $X \xrightarrow{f} Y \in \text{mod-}R$ is such that $F_f \in \mathcal{A}$ then we have $F_f g = 0$ and so, by 2.7, $g \in \mathrm{im}(f,-).B$. So $\bar{\mathcal{I}}(X,-) \leq \mathrm{im}(f,-)$ for all such $f$, that is $\bar{\mathcal{I}}(X,-) \leq r_{\mathcal{A}}(X,-)$ (for all $X \in \text{mod-}R$).

There is a largest ideal, $\mathcal{I}'$ say, containing $\mathcal{I}$ and contained in $r_{\mathcal{A}}$ (since $r_{\mathcal{A}}$ is a left ideal) and so $\bar{\mathcal{I}}$ is contained in $\mathcal{I}'$. Suppose, conversely, that $A \xrightarrow{h} B \in \mathcal{I}'$ and let $X \xrightarrow{f} Y \in \text{mod-}R$ with $F_f \in \mathcal{A}$. Let $X \xrightarrow{g} A \in \text{mod-}R$. Since $\mathcal{I}'$ is assumed to be a two-sided ideal, $hg \in \mathcal{I}'(X,B) \leq r_{\mathcal{A}}(X,B)$ so $hg \in \mathrm{im}(f,B)$ and hence $hg$ factors initially through $f$ for every such $g$. By 2.5, $F_f h = 0$. Therefore $h \in \bar{\mathcal{I}}$, as required. □

**Remark 2.9** *If $\mathcal{A}$ is any subclass of* $(\text{mod-}R, \mathbf{Ab})^{\mathrm{fp}}$ *and $X \xrightarrow{f} Y \in \text{mod-}R$ then $f \in r_{\mathcal{A}}(X,Y)$ iff $\mathrm{im}(f,-) \leq r_{\mathcal{A}}(X,-)$ (using 2.3(i)).*

**Lemma 2.10** *(cf. [14, 5.4(3)]) Let $R$ be any ring and suppose that $\mathcal{S}$ is a Serre subcategory of* $(\text{mod-}R, \mathbf{Ab})^{\text{fp}}$. *Then $r_{\mathcal{S}}$ is an ideal of* mod-$R$.

*In particular, if $\mathcal{I}$ is any fp-idempotent ideal of* mod-$R$ *then* $\text{annann}\mathcal{I} = r_{\text{ann}\mathcal{I}}$.

**Proof.** We have observed already that $r_{\mathcal{S}}$ is a left ideal.

Let $F \in (\text{mod-}R, \mathbf{Ab})^{\text{fp}}$ and let $G$ be a finitely generated subfunctor of $F$. Then $r_{\mathcal{S}}G \leq r_{\mathcal{S}}F$. For, let $F' \leq F$ be $\mathcal{S}$-dense (that is $F/F' \in \mathcal{S}$) and finitely generated. Then $G/(F' \cap G)$ is a subfunctor of $F/F'$ and hence $F' \cap G$ is $\mathcal{S}$-dense in $G$. Since $F' \cap G$ is also finitely generated (since $(\text{mod-}R, \mathbf{Ab})$ is locally coherent) we have $r_{\mathcal{S}}G \leq F' \cap G \leq F'$. So $r_{\mathcal{S}}G \leq F'$ for all such $F'$ and hence $r_{\mathcal{S}}G \leq r_{\mathcal{S}}F = \bigcap\{F' \leq F : F' \text{ is finitely generated and } \mathcal{S}\text{-dense in } F\}$.

Next, let $X \xrightarrow{f} Y \in r_{\mathcal{S}}(X, Y)$ and let $W \xrightarrow{g} X \in \text{mod-}R$. We show that $fg \in r_{\mathcal{S}}(W, Y)$ by showing that $\text{im}(fg, -) \leq r_{\mathcal{S}}(W, -)$ which, as remarked above, will be enough. By that remark we have $\text{im}(f, -) \leq r_{\mathcal{S}}(X, -)$. Let $F' \leq \text{im}(g, -)$ be finitely generated and $\mathcal{S}$-dense in $\text{im}(g, -)$. Then its full inverse image under $(g, -)$ is $\mathcal{S}$-dense in $(X, -)$ so contains a finitely generated functor which is $\mathcal{S}$-dense in $(X, -)$ (since the torsion theory generated by $\mathcal{S}$ is of finite type), hence contains $r_{\mathcal{S}}(X, -)$ and hence contains $\text{im}(f, -)$. Therefore $(g, -)\text{im}(f, -) \leq F'$. This is so for all such $F'$ and so $\text{im}(fg, -) = (g, -)\text{im}(f, -) \leq r_{\mathcal{S}}(\text{im}(g, -)) \leq r_{\mathcal{S}}(W, -)$ by what we showed above, as required.

The second statement then follows by 2.8. $\square$

# 3 Duality and representation of functors as intersections when $R$ is an artin algebra

What we want from this section is the fact that, if $R$ is an artin algebra, if $F$ is a finitely presented functor from mod-$R$ to $\mathbf{Ab}$ and if $G$ is any subfunctor of $F$ then $G$ is the intersection of the finitely generated subfunctors of $F$ which contain $G$. I do not know a reference for this result, which can be derived from [3] for instance and which seems to be folklore, and so I have included a proof.

Suppose throughout this section that $R$ is an artin $k$-algebra where $k$ is a commutative artinian ring which acts centrally on $R$. Then (because $k$ acts centrally) additive functors to $\mathbf{Ab}$ are $k$-linear functors taking values in the category of $k$-modules. Note that every representable functor has values in mod-$k$, since $R$ has finite length over $k$.

So, following [4, p. 131], let $D_k(R) = (R\text{-mod}, k\text{-mod})$ - where the latter now denotes the category of $k$-linear functors from $R$-mod to the category $k$-mod of *finite length* $k$-modules. Let $C_k(R) = D_k(R)^{\text{fp}}$. (The forgetful functor from $k$-mod to **Ab** induces an embedding of $D_k(R)$ into $D(R)$ and, as a subcategory of $D(R)$, $D_k(R)$ is closed under kernels, cokernels and extensions and hence is an exact, abelian subcategory of $D(R)$.) Note that every subquotient of a functor in $D_k(R)$ also takes values in mod-$k$.

We follow [14] in using the extension of the duality $D$ of finitely presented functors to $D_k(R)$, defined as follows.

Let $F \in D_k(R^{\text{op}})$. Define $F^\star$ ($\in D_k(R)$) by: if $L \in R$-mod set $F^\star L = F(L^*)^*$ where $^*$ on the right-hand side is the duality $(-)^* = \text{Hom}_k(-, E)$, $E$ being a minimal injective cogenerator of Mod-$k$, between mod-$R$ and $R$-mod induced by that between mod-$k$ and $k$-mod $\simeq$ mod-$k$. Note that this extends to a functor which is exact.

**Proposition 3.1** *([14, 5.3] also cf. [4, p. 132]) Let $R$ be an artin $k$-algebra.*
*(a) If $F \in C_k(R^{\text{op}})$ then $F^\star = DF$.*
*(b) If $F$ is a subquotient of a finitely presented functor then so is $F^\star$.*
*(c) If $F \in D_k(R^{\text{op}})$ then $F^\star \in D_k(R)$ and $F^{\star\star} \simeq F$.*

**Proof.** (a) First suppose that $F = (X, -)$, so $DF = X \otimes -$ and hence $DF.L = X \otimes L$ for any finitely presented $L$. Also we have $F^\star L = \text{Hom}_k(\text{Hom}_R(X, \text{Hom}_k(L, E))) \simeq \text{Hom}_k(\text{Hom}_k(X \otimes_R L, E)) \simeq (X \otimes_R L)^{**} \simeq X \otimes_R L$. So $F^\star$ and $DF$ agree on representable functors and so, since both functors $D(-)$ and $(-)^\star$ are exact, they agree on $C_k$.

(b) Say $0 \longrightarrow F \longrightarrow G$ is exact where $F' \longrightarrow G \longrightarrow 0$ is exact and $F'$ is finitely presented. Then we have (since $^\star$ is a duality and exact) exact sequences $G^\star \longrightarrow F^\star \longrightarrow 0$ and $0 \longrightarrow G^\star \longrightarrow F'^\star$ and $F'^\star$ is finitely presented by part (a) and hence $F^\star$ is of the form claimed.

(c) The first part is clear from the definition of $\star$ and, for the second part, we have $F^{\star\star}X = F^\star(X^*)^* = \text{Hom}_k(F^\star(X^*), E) = \text{Hom}_k(F(X^{**})^*, E) \simeq \text{Hom}_k(\text{Hom}_k(FX, E), E) \simeq (FX)^{**} \simeq FX$ since $FX$ is of finite length over $k$. $\square$

So $^\star$ induces a duality between subquotients of finitely presented functors. Let $F$ be any object of $C_k(R^{\text{op}})$ and consider $dF = F^\star$. Denote by Latt$F$ the modular lattice of *all* (not just finitely presented) subfunctors of $F$ in $D_k(R^{\text{op}})$.

**Proposition 3.2** *Let $R$ be an artin $k$-algebra and let $F \in C_k(R^{\mathrm{op}})$. Then the map from* $\mathrm{Latt}F$ *to* $\mathrm{Latt}(dF)$ *which takes a subfunctor $G$ of $F$ to $DG = \ker((G \xrightarrow{i} F)^\star)$ where $i$ is the inclusion of $G$ in $F$, induces a duality* $(\mathrm{Latt}F)^{\mathrm{op}} \simeq \mathrm{Latt}(dF)$ *which commutes with arbitrary intersections and sums:*
$D(\bigcap_\lambda G_\lambda) = \Sigma_\lambda DG_\lambda$;
$D(\Sigma_\lambda G_\lambda) = \bigcap_\lambda DG_\lambda$.

**Proof.** We have the exact sequence $0 \longrightarrow G \longrightarrow F \longrightarrow H \longrightarrow 0$ say, dualising to $0 \longrightarrow H^\star \longrightarrow F^\star \longrightarrow G^\star \longrightarrow 0$, so we have $H^\star = DG = (\mathrm{coker}(G \xrightarrow{i} F))^\star$. Applying the same construction to $H^\star$ and using that ${}^{\star\star} \simeq \mathrm{Id}$ we see that $D^2 = \mathrm{Id}$ (modulo a fixed identification of $F^{\star\star}$ with $F$).

Clearly the map $D$ is order-reversing.

Set $G = \bigcap_\lambda G_\lambda$ so $DG \geq DG_\lambda$ for all $\lambda$ and hence $DG \geq \Sigma_\lambda DG_\lambda \geq DG_\lambda$. Therefore $G_\mu \geq D(\Sigma_\lambda DG_\lambda) \geq D^2G = G$ for all $\mu$ and hence $G = \bigcap_\lambda G_\lambda = D(\Sigma_\lambda DG_\lambda)$. Therefore $DG = D^2(\Sigma_\lambda DG_\lambda) = \Sigma_\lambda DG_\lambda$, that is, $D(\bigcap_\lambda G_\lambda) = \Sigma_\lambda DG_\lambda$, as required.

The proof for the other part is similar. □

Note that if $F' \leq F$ are finitely presented then $DF'$ is finitely presented since (see proof of 3.2) $DF' = d(\mathrm{coker}(F' \longrightarrow F))$ and, since, by 3.1, $\mathrm{coker}(F' \longrightarrow F)$ is finitely presented, so is its dual.

**Corollary 3.3** *Let $R$ be an artin $k$-algebra and let $G \leq F \in C_k(R^{\mathrm{op}})$. Then $G = \bigcap\{F' \leq F : F'$ is finitely presented and $G \leq F'\}$.*

**Proof.** We have $DG = \Sigma\{F'' : F'' \leq DG(\leq dF), F''$ finitely presented$\}$ so $G = D(\Sigma\{F'' : F'' \leq DG(\leq dF), F''$ finitely presented$\}) = \bigcap\{DF'' : DF'' \geq G, F'' \leq F, F''$ finitely presented$\}$ and each $DF''$ is finitely presented (as noted above), as required. □

# 4 Ideals of mod-$R$ and Serre subcategories of $C_k(R)$ when $R$ is an artin algebra

Throughout this section, $R$ is an artin $k$-algebra.

**Proposition 4.1** *[14, 5.7] Let $R$ be an artin algebra and let $\mathcal{I}$ be an ideal of* mod-$R$*. Then $\mathcal{I} = \mathrm{annann}\mathcal{I}$.*

**Proof.** Set $\bar{\mathcal{I}} = \mathrm{annann}\mathcal{I}$. We have $\mathcal{I} \subseteq \bar{\mathcal{I}}$. Let $X \in \text{mod-}R$. Since $R$ is an artin algebra, by 3.3 each of $\mathcal{I}(X,-)$ and $\bar{\mathcal{I}}(X,-)$ is the intersection of the finitely generated subfunctors of $(X,-)$ containing it. So it will suffice to show: if $F \leq (X,-)$ is finitely generated and $F \geq \mathcal{I}(X,-)$ then $F \geq \bar{\mathcal{I}}(X,-)$. We may suppose $F = \mathrm{im}(f,-) \geq \mathcal{I}(X,-)$ where $X \xrightarrow{f} Y \in \text{mod-}R$. Then, by 2.7, $F_f(\mathcal{I}) = 0$ and so $F_f \in \mathrm{ann}\mathcal{I}$. Hence, by definition, $F_f(\bar{\mathcal{I}}) = 0$ and so, again by 2.7, $F = \mathrm{im}(f,-) \geq \bar{\mathcal{I}}(X,-)$, as required. $\square$

Since every Serre subcategory of the functor category is the annihilator of a collection of (identity) morphisms (see Section 1) we have the following.

**Corollary 4.2** *[14, 5.10] Let $R$ be an artin $k$-algebra. Then annihilation induces a bijection between Serre subcategories of* $(\text{mod-}R, \text{mod-}k)^{\mathrm{fp}}$ *and fp-idempotent ideals of* mod-$R$.

In particular if $\mathcal{I}$ is an fp-idempotent ideal of mod-$R$ then, with the notation of Section 2, $r_{\mathrm{ann}\mathcal{I}} = \mathcal{I}$.

**Lemma 4.3** *Let $R$ be any ring and consider a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle k} \\ M & \xrightarrow[i]{} & N \end{array}$$

*where $f$ is a morphism between finitely presented modules and where $i$ is a pure embedding between arbitrary modules.*

*Then there is a morphism $k' : Y \longrightarrow M$ such that $k'f = g$.*

This property, as $f$ ranges over mod-$R$, is equivalent to the morphism $i$ being pure, indeed, it is taken as the definition of purity in [1, p.85].

**Lemma 4.4** *Let $R$ be an artin algebra. Suppose $F \in C_k(R^{\mathrm{op}})$ and let $\overrightarrow{F}$ denote the extension of $F$ to* Mod-$R$ *which commutes with direct limits. Let $M \in$* Mod-$R$*. If $\overrightarrow{F}M \neq 0$ then $Fh \neq 0$ for some $h \in$* mod-$R$ *which factors through $M$.*

**Proof.** Represent $F$ as $F_f$ for some $X \xrightarrow{f} Y \in \text{mod-}R$: so (see 2.6) the same representation serves to define $\overrightarrow{F}$ in the larger functor category. Since $\overrightarrow{F}M \neq 0$ there is $X \xrightarrow{g} M$ which does not factor through $f$.

Since $R$ is an artin algebra there is a pure embedding $M \xrightarrow{i} \prod_\lambda N_\lambda$ with the $N_\lambda$ of finite length (e.g. see [12]). Let $\pi_\mu : \prod_\lambda N_\lambda \longrightarrow N_\mu$ be the projection and set $g_\mu = \pi_\mu i g$.

If, for each $\lambda$, the morphism $g_\lambda$ factors through $f$, say $g_\lambda = k_\lambda f$ for some $Y \xrightarrow{k_\lambda} N_\lambda$, then the product, $Y \xrightarrow{k} \prod_\lambda N_\lambda$, of these morphisms satisfies $ig = kf$. Since $i$ is a pure embedding it follows by 4.3 that there is $Y \xrightarrow{k'} M$ such that $g = k'f$ - contradiction.

So there is $\lambda$ such that $g_\lambda$ does not factor through $f$. Then we have, taking $h = g_\lambda$, a morphism $h \in$ mod-$R$ which factors through $M$ and with $F_f(h) \neq 0$, as required.

□

Suppose that $\mathcal{Y}$ is any subclass of Mod-$R$. Following [14, Section 5] denote by $[\mathcal{Y}]$ the class of morphisms in mod-$R$ which factor through add$\mathcal{Y}$ (the closure of $\mathcal{Y}$ under finite direct sums). Then $[\mathcal{Y}]$ is an ideal of mod-$R$. For clearly $[\mathcal{Y}]$ is closed under right and left composition. Also, if $A \xrightarrow{f} Y \xrightarrow{g} B$, $A \xrightarrow{f'} Y' \xrightarrow{g'} B$ with $Y, Y' \in \mathcal{Y}$ then we have $Y \oplus Y' \in \mathrm{add}\mathcal{Y}$ and $gf + g'f = A \xrightarrow{\binom{f}{f'}} Y \oplus Y' \xrightarrow{(g,g')} B$, so $[\mathcal{Y}]$ is also closed under addition.

**Theorem 4.5** *Let $R$ be an artin algebra and let $\mathcal{Y} \subseteq$ Mod-$R$ be a class of modules. Let $[\mathcal{Y}]$ denote the class of morphisms in* mod-$R$ *which factor through* add$\mathcal{Y}$.

*Then $[\mathcal{Y}] = [\bar{\mathcal{Y}}]$ where $\bar{\mathcal{Y}}$ denotes the definable subcategory of* Mod-$R$ *generated by $\mathcal{Y}$.*

**Proof.**

Let $F \in \mathrm{ann}[\mathcal{Y}]$. By 4.4, $\overrightarrow{F}\mathcal{Y} = 0$ and hence $\overrightarrow{F}\bar{\mathcal{Y}} = 0$ (see Section 1) so, clearly, $F \in \mathrm{ann}[\bar{\mathcal{Y}}]$. Hence $\mathrm{ann}[\mathcal{Y}] \subseteq \mathrm{ann}[\bar{\mathcal{Y}}]$ but, since $\mathcal{Y} \subseteq \bar{\mathcal{Y}}$, the opposite inclusion is clear and hence $\mathrm{ann}[\mathcal{Y}] = \mathrm{ann}[\bar{\mathcal{Y}}]$. But then (both $[\mathcal{Y}]$ and $[\bar{\mathcal{Y}}]$ are ideals) by 4.1 we have $[\mathcal{Y}] = \mathrm{annann}[\mathcal{Y}] = \mathrm{annann}[\bar{\mathcal{Y}}] = [\bar{\mathcal{Y}}]$. □

**Proposition 4.6** *[14, 5.2] Let $R$ be an artin $k$-algebra and let $\mathcal{S}$ be a Serre subcategory of $C_k(R^{\mathrm{op}})$. Then* $\mathrm{ann}\mathcal{S} = [\mathcal{X}_\mathcal{S}]$, *where $\mathcal{X}_\mathcal{S}$ denotes the definable subcategory of* Mod-$R$ *corresponding to $\mathcal{S}$.*

**Proof.** If $h \in [\mathcal{X}_\mathcal{S}]$ then $h = h'' 1_M h'$ for some $M \in \mathcal{X}_\mathcal{S} (= \mathrm{add}\mathcal{X}_\mathcal{S})$ and so, if $F \in \mathcal{S}$ then, since $\overrightarrow{F} 1_M = 1_{\overrightarrow{F}M} = 0$ we have $Fh = \overrightarrow{F}h = \overrightarrow{F}h'' \overrightarrow{F} 1_M \overrightarrow{F}'_h = 0$ and so $h \in \mathrm{ann}\mathcal{S}$. Hence $[\mathcal{X}_\mathcal{S}] \subseteq \mathrm{ann}\mathcal{S}$.

Therefore $\mathrm{ann}[\mathcal{X}_{\mathcal{S}}] \supseteq \mathrm{annann}\mathcal{S} = \mathcal{S}$ by 4.2. But now, if $F \in \mathrm{ann}[\mathcal{X}_{\mathcal{S}}]$ then, by 4.4, $\overrightarrow{F}M = 0$ for every $M \in \mathcal{X}_{\mathcal{S}}$. Hence, by the bijective correspondence between Serre subcategories of $C_k(R^{\mathrm{op}})$ and definable subcategories of Mod-$R$, we deduce that $F \in \mathcal{S}$. So $\mathrm{ann}[\mathcal{X}_{\mathcal{S}}] \subseteq \mathcal{S}$ and these are, therefore, equal.

So $[\mathcal{X}_{\mathcal{S}}] = \mathrm{annann}[\mathcal{X}_{\mathcal{S}}]$ (by 4.1) $= \mathrm{ann}\mathcal{S}$, as required. □

**Corollary 4.7** *Let $R$ be an artin algebra. Let $\mathcal{X}$ be a definable subcategory of* Mod-$R$ *and let* $\mathrm{ann}(\mathcal{S}_{\mathcal{X}})$ *(= $[\mathcal{X}]$ by 4.6) be the associated ideal of* mod-$R$. *Then every morphism in this ideal factors through a finite direct sum of points in (any fixed dense subset of) the Ziegler-closed set corresponding to $\mathcal{X}$.*

**Proof.** If $\mathcal{Y}$ is (any dense subset of) this closed set then the definable closure of $\mathcal{Y}$ is $\mathcal{X}$ so 4.5 applies. □

**Corollary 4.8** *Let $R$ be an artin algebra. Let $\mathcal{I}$ be an fp-idempotent ideal of* mod-$R$ *with associated Serre subcategory $\mathcal{S} = \mathrm{ann}\mathcal{I}$ and let $X$ be a dense subset of the closed subset of* $\mathrm{Zg}_R$ *associated to $\mathcal{S}$. Then each morphism $h \in \mathcal{I}$ factors through a finite direct sum of points of $X$.*

**Proof.** This also is immediate from 4.5 and 4.2. □

Our application is to the omega radical, $\mathrm{rad}^{\omega}$, of the category mod-$R$ where $R$ is an artin algebra. For the transfinite powers of the radical of mod-$R$ see [17] but we recall here that a morphism in mod-$R$ is in the radical, rad, if, when we represent it as a matrix of morphisms between indecomposable modules, no component is an isomorphism. Thus the radical is an ideal of mod-$R$ and the finite powers of this ideal are defined in the obvious way. We define $\mathrm{rad}^{\omega}$ to be the intersection of the finite powers of the radical. It is easily seen (see [14, Section 8.2]) that $\mathrm{rad}^{\omega}$ is an fp-idempotent ideal: the Serre subcategory of the functor category to which it corresponds is the category of all finite length functors and the closed subset of the Ziegler spectrum to which this corresponds is that consisting of all infinite length indecomposable pure-injectives. Therefore we obtain, as a special case of the above, the following conclusion, originally obtained in [16] by a very different proof.

**Theorem 4.9** *Let $R$ be an artin algebra and let $f \in \mathrm{rad}^{\omega}$. Then there is a factorisation of $f$ through a finite direct sum of indecomposable, infinite length, pure-injective modules.*

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