Semidirect Products and Applications to Geometric Mechanics

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2019

MIMS EPrint: **2020.1**

Manchester Institute for Mathematical Sciences
School of Mathematics

The University of Manchester

ISSN 1749-9097

SEMIDIRECT PRODUCTS AND APPLICATIONS TO GEOMETRIC MECHANICS

A thesis submitted to the University of Manchester
for the degree of Doctor of Philosophy
in the Faculty of Science and Engineering

2019

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Word Count: 30915

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Abstract

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Semidirect Products and Applications to Geometric Mechanics
27th of September, 2019

In this thesis we provide an overview of themes in geometric mechanics and apply them to the study of adjoint and coadjoint orbits of a semidirect product, and to the two-body problem on a sphere.

Firstly, we show the existence of a geometrically defined bijection between the sets of adjoint and coadjoint orbits for a particular class of semidirect product. We demonstrate the bijection for the examples of the affine linear group and the Poincaré group. Additionally, we prove that any two orbits paired between this bijection are homotopy equivalent.

Secondly, a correspondence is found between the two-body problem on a three-dimensional sphere and the four-dimensional Lagrange top. This correspondence establishes an equivalence between the two problems after reduction, and allows us to treat both reduced problems simultaneously. We implement a semidirect product reduction by stages to exhibit the reduced spaces as coadjoint orbits of a special Euclidean group, and then reduce by a further symmetry to obtain a full reduced system. This allows us to fully classify the relative equilibria for both problems and describe their stability.

Declaration

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

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Acknowledgements

I would like to extend my sincerest thanks to James Montaldi for his wisdom and guidance throughout this PhD. Whenever I have walked into his office James has always generously provided me with his time and full attention. It has been a real pleasure to have learned such a great deal from his advice, his knowledge, and from his style of mathematics.

This thesis marks the culmination of twenty-two years of education, the last nine of which have been in mathematics. As such, I am compelled to thank those who have helped sculpt my mathematical development and lead me to where I am now. As a sixth-form student in the summer of 2009 I had my first encounter with serious mathematics thanks to a research placement with Ian Porteous at the University of Liverpool. Ian was a generous and thoughtful mentor who encouraged me to pursue mathematics further. I am grateful for his brief but lasting influence at the start of my academic journey.

At Cambridge I had the enormous privilege to be a student of Tadashi Tokieda. His inspirational teachings and almost mystical approach to science has had a profound impact on how I view and practise mathematics. I also had the good fortune to befriend Gavin Ball, William Cook, and David Nesbitt. We don't talk about tensors or bundles as much as we used to, but I am grateful to each of them for a friendship which has become more than just mathematics.

Lastly, I would like to thank my parents, Brenda and Wayne Arathoon. For your continual and unlimited support, and for being at my side throughout all of my endeavours. It is thanks to you that I am where I am today.

Introduction

Geometric mechanics lies within the intersection of geometry and physics. Many problems in classical dynamics may be formulated as Hamiltonian systems defined on symplectic manifolds. The necessary framework therefore to describe these classical systems belongs to the arena of symplectic geometry. By combining these two pillars of mathematics and physics we obtain a rich and broad theory whose geometric foundations lead to insights in physics, and where the physical understanding of a problem often advances its mathematical underpinnings. Owing to the breadth of this intersection of ideas there is no strict definition for what geometric mechanics actually is. Its study includes the main focus of this thesis, that of classical dynamics, but its reach also includes quantum mechanics through the theory of geometric quantization, along with infinite dimensional systems which, for example, include fluid flow equations, various lattice models, and certain partial differential equations.

It is often the case that some dynamical problem is invariant with respect to a certain symmetry. Take for example the motion of a planet around a star. Imagine pausing the motion at an instant and moving both bodies to another point in space whilst keeping their distance from each other constant. What we have applied is a so-called rigid motion. If we now resume the dynamics we will find that the resulting motion of the planet and star is the same as it was originally, albeit translated by this rigid motion. More technically, this invariant property is described by a certain action of a group on the phase space. The necessary language used to describe such actions is that of group theory, particularly that of Lie groups. Among such groups a fairly general collection is given by those which are semidirect products between a reductive group with an abelian group. Indeed, the rigid motions from this example arise from the action of such a group.

In this thesis we will primarily be interested in applying ideas from the theory of semidirect products to systems in classical dynamics which are invariant under an action of a group of this kind. This explains the rationale behind the choice of title for this thesis. Applications of semidirect products to geometric mechanics is not however a tightly focused aim to which we will unwaveringly adhere. Instead, our title is designed to be sufficiently broad to allow for self-contained excursions into questions which pertain solely to the theory of semidirect products, or to geometric mechanics.

Central to this thesis is the theory of reduction for Hamiltonian systems. The concept of reduction, that being the process of removing the redundancy in a system which arises from a symmetry, has its roots in the works of Euler, Lagrange, Poisson, Liouville, Hamilton, Routh, Noether, Poincaré, and many others. The theory in its modern form was initiated by Arnold [Arn66], Smale [Sma70], Meyer [Mey73], and Marsden and Weinstein [MW74]. The works of Arnold in particular are especially relevant to the content of this thesis. It was Arnold who generalised Euler's equations for the motion of a rigid body and exhibited them as a simple example of a more general theory concerning geodesics on a Lie group with an invariant metric. These equations, sometimes referred to as the Euler-Arnold equations, are defined on the coalgebra of the group. Although Arnold was the first to notice the connection with geodesics, a general system of equations defined on a Lie algebra was first introduced by Poincaré, and even earlier still by Lagrange in a few restricted examples [HMR98]. For this reason, a more general system of equations on a Lie algebra are referred to as Euler-Poincare equations. These equations are all obtained from a variational principle using the Euler-Lagrange equations. By applying the Hamiltonian formulation instead, we obtain a set of equations on the coalgebra known as Lie-Poisson equations.

A few years after Arnold's paper in 1966, the modern theory of reduction would be introduced in [MW74] and would continue to flourish until present day. By using this theory, specifically that of Poisson reduction developed later, the Lie-Poisson equations on the dual of a Lie algebra can be seen as the reduced system obtained from an invariant Hamiltonian system defined on the cotangent bundle of the group. This idea would prove to be enormously fruitful. Not only is the motion of a rigid body an example of such a system, but so too is the infinite dimensional example for the motion of a perfect fluid, geodesics of invariant metrics, the Kirchhoff equations for a body moving through a fluid, and the dynamics of a heavy top, to name just a few.

The Lie-Poisson equations corresponding to the heavy top are especially curious as they are defined on the coalgebra of the group of rigid motions in space. This group is a semidirect product as we have already mentioned, but it is strange that it should apply to the heavy top since such rigid motions seemingly have no significance; the heavy top is after all constrained to move about a fixed point. These equations of motion were known to Poincaré, however it took almost seventy years for a geometric answer to explain why the coalgebra should be a semidirect product. This answer came from the works of Guillemin and Sternberg [GS80], and Marsden, Ratiu and Weinstein [MW74, MRW84c]. They present a theory of semidirect product reduction by stages which shows that, under a certain kind of symmetry breaking, a Hamiltonian system defined on a cotangent bundle of a group reduces to a system on a coalgebra of a semidirect product of that group. For the example of the heavy top, it is the introduction of a gravitational field which breaks the symmetry.

In a general sense, the theory of semidirect product reduction by stages and the corresponding Lie-Poisson systems defined on a coalgebra is the unifying theme which runs throughout this thesis. As we will now explain, it is this central idea which motivates us to look more closely at the structure of the coadjoint orbits of a semidirect product, and to look for new ways to apply the theory to interesting physical examples.

The structure of this thesis is in three main parts. The first is a large introductory chapter which establishes all of the necessary background for the next two chapters. Additionally, this chapter also serves to function as a self-contained and readable exposition to the ideas from dynamics that we have just mentioned. The next two chapters are both individual papers written solely by the author which have been published in peer-reviewed journals [Ara19a, Ara19b]. As such, this thesis is understood to be submitted in 'Alternative Format'. The content of these chapters does not differ from that found in the respective publications, with the exception of some minor stylistic and formatting changes to better accommodate this thesis submission. As these two chapters both represent individual papers, they are entirely self-contained within the thesis. In particular, there are no references in and out of these chapters.

The introductory background chapter begins with some Lie theory. The adjoint and coadjoint representations for a Lie group are defined and presented alongside a few guiding examples. The theory of symplectic reduction is given next. The presentation is fairly non-standard, and differs from the usual program of introducing Hamiltonian actions, then a momentum map, and then the reduced spaces. Instead we begin in greater generality and present what is essentially Poisson reduction. This has the principal advantage of quickly demonstrating the significance of the entire coalgebra, not just the coadjoint orbits alone. Indeed, the first example developed of Poisson reduction is that for the action of a group on its cotangent bundle. In doing so, the Poisson structure on a coalgebra is obtained along with the corresponding Lie-Poisson equations which a Hamiltonian system defines. The section is concluded with the usual introduction of momentum maps, Hamiltonian group actions, and orbit-reduced spaces. The discussion concerning adjoint and coadjoint orbits is then resumed for the case of semidirect products and culminates in an original proof of the Semidirect Product Reduction by Stages Theorem. This theorem finds its application both later in the background chapter, but also crucially in the third chapter where it is once again presented, except without proof. The chapter ends with a tour through classical dynamics exhibiting many well-known problems as examples of the general theory presented earlier. The rigid body is shown to correspond to the Euler-Arnold equations for the special orthogonal group; the Kirchhoff equations for the special Euclidean group; and the heavy top is shown to define a system of Lie-Poisson equations on the coalgebra of the special Euclidean group. After this chapter the reader is well-equipped for the next two, which are inevitably less pedestrian in tone.

The second chapter concerns a classification of orbit types for the adjoint and coadjoint representations of a semidirect product. Cushman and Van Der Kallen in [CVDK06a] noticed that the orbit types for the Poincaré group exhibit a "curious bijection". As the two representations are not isomorphic such a bijection is, as the authors note, "unusual". In this chapter we present an explanation for this bijection for a more general class of semidirect products and demonstrate it explicitly for the case of the Poincaré group and the group of affine linear transformations. We make essential use of a classification of coadjoint orbits introduced by Rawnsley in [Raw75] in terms of bundles of little-group orbits, and generalise this to the adjoint orbits. Even more unusually, we also show that any two orbits which are paired in the bijection are homotopy equivalent.

In the third and final chapter we explore the problem of two gravitating bodies constrained to move on a sphere. It suffices to consider the problem defined on the three-dimensional sphere. By taking this sphere to be the set of unit length quaternions we are able to leverage a great deal of geometry and apply it to the problem. In particular, owing to this special geometric circumstance we are able to identify the phase space of the problem with a double cover of the phase space for the four-dimensional heavy top. For when this heavy top is the symmetric Lagrange top, the symmetries for both problems coincide and both descend to give equivalent reduced spaces. Therefore, we are able to treat the symplectic reduction of the two-body problem and the four-dimensional Lagrange top simultaneously. The quaternionic formulation of the problem leads to a very pleasant simplification of much of the algebra involved, and it is without too much strain that we are able to describe the reduced spaces and provide a full classification of the relative equilibria.

Conventions and notation

Regrettably, there is a persistent sign-error which plagues Hamiltonian dynamics owing to a number of alternative conventions different authors choose to take. Speaking from experience, although seemingly innocuous, this sign-error is often infuriating, and therefore, we will here set in stone our definitions and notation which shall be used throughout.

– The pairing between a (real) vector space V and its dual V^* will be denoted by

$$\langle\ ,\ \rangle \colon V^* \times V \longrightarrow \mathbb{R}.$$

– For a diffeomorphism $\phi\colon M \to N$, the pullback ϕ^* of a one-form η on N is the one-form $\phi^*\eta$ on M which satisfies

$$\langle \phi^*\eta, X\rangle = \langle \eta, \phi_* X\rangle$$

for all tangent vectors X and where ϕ_* denotes the pushforward. If M and N are vector spaces and ϕ a linear map, then ϕ^* is elsewhere referred to in the literature as the 'adjoint' to ϕ. For reasons which will be clearer later, we actively avoid this terminology to avoid a sign confusion.

– If π is a representation of G on V then the contragredient representation π^* is defined by

$$\langle \pi_g^*\eta, X\rangle = \langle \eta, \pi_g^{-1} X\rangle$$

for all $X \in V$. Correspondingly, if Π is a representation of the Lie algebra $\mathfrak{g}$ on V then the contragredient representation Π^* is defined by

$$\langle \Pi^*_\omega \eta, X \rangle = -\langle \eta, \Pi_\omega X \rangle$$

for all $X \in V$.

- The standard Euclidean inner product between vectors in $\mathbb{R}^n$ will be written as
$$x \cdot y = x^T y.$$

- The Lie algebra $\mathfrak{g}$ of a group G has a Lie-bracket defined by either taking the algebra of left- or right-invariant vector fields. For when we write $\mathfrak{g}$ it assumed that we are using the algebra of left-invariant vector fields. In situations for when it is not clear, we write $\mathfrak{g}_\mp$ where the minus sign indicates left- and the plus sign indicates right-invariant vector fields.

- If λ is the canonical one-form on a cotangent bundle, the canonical symplectic form is taken to be $\omega = -d\lambda$. The Hamiltonian vector field V_f of a function f is defined to satisfy

$$\omega(V_f, \cdot) = df.$$

The Poisson bracket between two functions is defined as

$$\{f, g\} = \omega(V_f, V_g).$$

Be advised that the change in a function g along the flow of a Hamiltonian vector field V_f is then given by

$$\dot{g} = V_f(g) = \{g, f\}.$$

Chapter 1

Background Material

1.1 Adjoint and Coadjoint representations

1.1.1 Lie algebras and trivializations

Let G be a Lie group and denote by $\mathcal{L}$ and $\mathcal{R}$ the *left* and *right* group multiplications

$$\mathcal{L}_a g = ag, \quad \text{and} \quad \mathcal{R}_a g = ga$$

for a and g in G. A property enjoyed by a Lie group is that the left and right multiplications each provide a free and transitive group action of G on itself. As these actions are free and transitive they allow us to unambiguously push forward any tangent vector defined at any point of the group, and push it around to globally define a left- or right-invariant vector field.

To demonstrate this more precisely, let X be an element of the tangent space of G at the identity e. In the case of left translation we may define a global vector field on G by setting

$$X(g) = (\mathcal{L}_g)_* X$$

where $(\mathcal{L}_g)_*$ denotes the push-forward. This vector field is left-invariant by construction, and it is clear that every left-invariant vector field arises in this way. Therefore, we may identify the tangent space T_eG with the space of left-invariant vector fields on G. As the space of left-invariant vector fields is closed under the Lie bracket of vector fields, we obtain by restriction to the identity a Lie bracket defined on T_eG. This bracket equips T_eG with the structure of a Lie algebra which we will from now on denote by $\mathfrak{g}$.

Remark 1.1.1. We can instead use right-invariant vector fields to define a different Lie algebra on T_eG. As we will see later in Proposition 1.1.1, these two structures only differ by sign. Therefore, in situations where we need to distinguish between the two algebras we will write $\mathfrak{g}_{\mp}$ where the minus sign indicates left-invariant, and the plus sign indicates right-invariant vector fields.

Apart from endowing the tangent space to the identity with a Lie algebra structure, an additional use of the left and right translations is in showing that the tangent bundle TG is trivial. The trivializations are given explicitly by the left and right multiplications below

$$\mathcal{L}\colon TG \longrightarrow G \times \mathfrak{g}; \quad X_g \longmapsto (g,\ (\mathcal{L}_{g^{-1}})_* X_g)$$
$$\mathcal{R}\colon TG \longrightarrow G \times \mathfrak{g}; \quad X_g \longmapsto (g,\ (\mathcal{R}_{g^{-1}})_* X_g)$$

where $X_g \in T_gG$. Moreover, by taking the pullback instead of the pushforward we can also construct trivializations for the cotangent bundle too

$$\mathcal{L}^*\colon T^*G \longrightarrow G \times \mathfrak{g}^*; \quad \eta_g \longmapsto (g,\ (\mathcal{L}_g)^* \eta_g)$$
$$\mathcal{R}^*\colon T^*G \longrightarrow G \times \mathfrak{g}^*; \quad \eta_g \longmapsto (g,\ (\mathcal{R}_g)^* \eta_g)$$

for $\eta_g \in T_g^*G$.

1.1.2 The Adjoint representation

What is the difference between the two choices of trivialization, either by the left or right? The answer to this question is given by the Adjoint and Coadjoint representations. The left and right trivializations are both vector bundle isomorphisms over G and the composition $\mathcal{R} \circ \mathcal{L}^{-1} = \mathcal{L}^{-1} \circ \mathcal{R}$, which may be thought of as measuring the difference between the two, is a bundle endomorphism of $G \times \mathfrak{g}$ given by

$$(g, X) \longmapsto (g, (\mathcal{L}_g \circ \mathcal{R}_{g^{-1}})_* X)\,.$$

We denote the fibrewise isomorphisms $(\mathcal{L}_g \circ \mathcal{R}_{g^{-1}})_*$ on $\mathfrak{g}$ by Ad_g. Observe that $\mathrm{Ad}_{gh} = \mathrm{Ad}_g \circ \mathrm{Ad}_h$ and that Ad_e is the identity. It follows that $\mathrm{Ad}\colon G \to GL(\mathfrak{g})$ is a group homomorphism, and thus, Ad defines a representation of G on $\mathfrak{g}$; the *Adjoint representation*. In a vague sense, this is a measure of how non-commutative the group is. If G is commutative then $\mathcal{L}_g \circ \mathcal{R}_{g^{-1}}$ is always trivial for all g and hence the Adjoint representation is trivial.

Example 1.1.1 (Adjoint orbits of $SO(3)$)**.** Many, but not all, Lie groups G may be realised as (closed) subgroups of some $GL(n)$; a so-called *matrix group*. For these groups the group product is given by matrix multiplication which, as this is linear, imply that the Adjoint representation is given by conjugation. As an explicit example we will present the case of $SO(3)$ whose Lie algebra is

$$\mathfrak{so}(3) = \left\{ \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix},\ \omega_1, \omega_2, \omega_3 \in \mathbb{R} \right\}.$$

We may identify such elements with vectors $(\omega_1, \omega_2, \omega_3)$ and in doing so identify $\mathfrak{so}(3)$ with $\mathbb{R}^3$. Note that the squared norm of this vector is equal to

$$\frac{1}{2}\,\mathrm{Trace}(\omega^T\omega), \tag{1.1}$$

and that the Adjoint action $\mathrm{Ad}_g\,\omega = g\omega g^{-1}$ preserves this norm since $gg^T = I$ for $g \in SO(3)$. It follows that the Adjoint orbits must be contained to spheres of constant radius centred at the origin in $\mathbb{R}^3 \cong \mathfrak{so}(3)$. In fact, it can be shown that the identification $\mathfrak{so}(3) \cong \mathbb{R}^3$ is an interwining map between the Adjoint representation of $SO(3)$ and the standard representation on $\mathbb{R}^3$. The orbits of the Adjoint action therefore coincide with those of $SO(3)$ on $\mathbb{R}^3$ which we have included in Figure 1.1(a).

Example 1.1.2 (Adjoint orbits of $SL(2;\mathbb{R})$)**.** Another example is given by the group $SL(2;\mathbb{R})$ with Lie algebra

$$\mathfrak{sl}(2;\mathbb{R}) = \left\{ \begin{pmatrix} \xi_2 & \xi_1 + \xi_3 \\ \xi_1 - \xi_3 & -\xi_2 \end{pmatrix},\ \xi_1, \xi_2, \xi_3 \in \mathbb{R} \right\}.$$

Such elements ξ may be identified with vectors $(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$ which again identifies $\mathfrak{sl}(2;\mathbb{R})$ with $\mathbb{R}^3$. Observe that

$$\det \xi = -\xi_1^2 - \xi_2^2 + \xi_3^2. \tag{1.2}$$

is an invariant of the Adjoint action since conjugation preserves the determinant. The orbits are therefore contained to the surfaces of constant $-\xi_1^2 - \xi_2^2 + \xi_3^2$. For

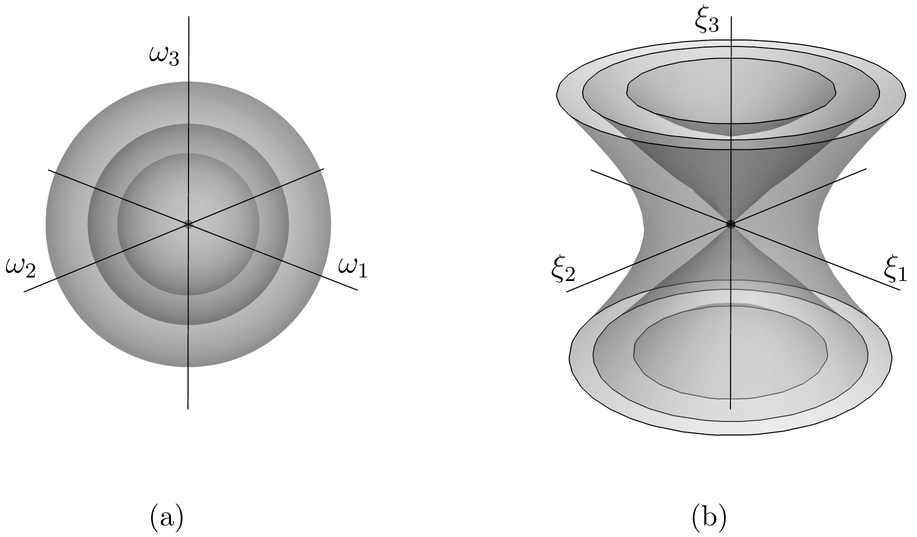


Figure 1.1: Adjoint orbits of $SO(3)$ and $SL(2;\mathbb{R})$.

$\det \xi \neq 0$ the orbits are the connected components of these hyperboloids in $\mathbb{R}^3$. For when $\det \xi = 0$ there are three orbits: the origin $\xi = 0$, and the two conical surfaces minus the vertex. These orbits are illustrated in Figure 1.1(b).

Example 1.1.3 (Adjoint orbits of $SE(2)$)**.** Unlike the last two examples the example we now consider is not immediately given to us as a matrix group. We consider the group $SE(2)$ of rigid motions of the plane. Such a motion is determined by a pair (r, d) where $r \in SO(2)$ represents a rotation and $d \in \mathbb{R}^2$ a translation. The composition of two such motions defines the group multiplication

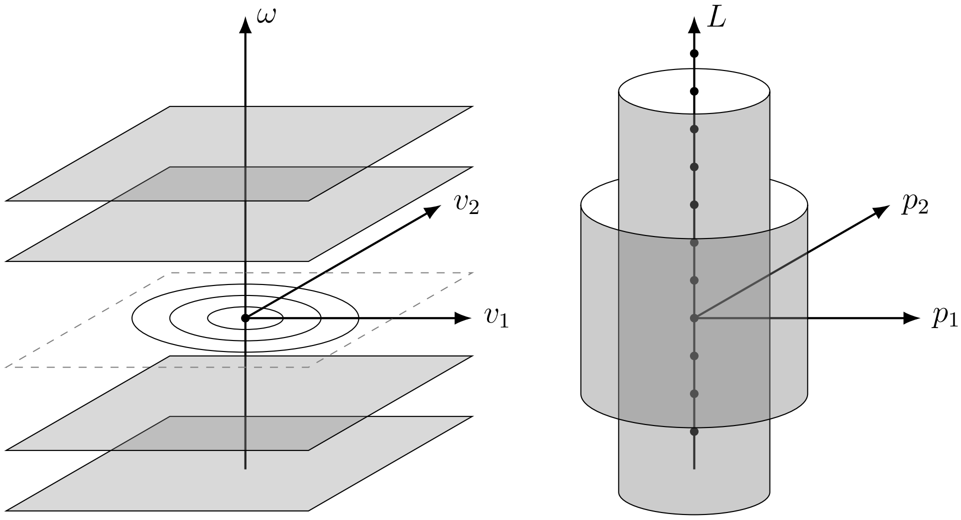
$$(r_1, d_1) \cdot (r_2, d_2) = (r_1 r_2, r_1 d_2 + d_1).$$

The following map defines a faithful representation of $SE(3)$ on $\mathbb{R}^3$, and thus, we may proceed to consider $SE(2)$ as a matrix group

$$(r, d) \longmapsto \begin{pmatrix} r & d \\ 0 & 1 \end{pmatrix}.$$

By differentiating a curve in $SE(2)$ through the identity we may identify the Lie algebra of $SE(2)$ with

$$\mathfrak{se}(2) = \left\{ \begin{pmatrix} \omega\mathbb{J} & v \\ 0 & 0 \end{pmatrix}, \omega \in \mathbb{R}, v \in \mathbb{R}^2 \right\} \quad \text{where} \quad \mathbb{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$



(a) (b)

Figure 1.2: Adjoint and Coadjoint orbits of $SE(2)$

As everything is now written in terms of matrices the Adjoint action is simply given by conjugation. It is then only a matter of calculation to show that

$$\mathrm{Ad}_{(r,d)}(\omega, v) = (\omega, rv - \omega \mathbb{J} d) = \begin{pmatrix} 1 & 0 \\ -\mathbb{J}d & r \end{pmatrix} \begin{pmatrix} \omega \\ v \end{pmatrix}. \tag{1.3}$$

The orbits are illustrated in Figure 1.2(a).

1.1.3 The adjoint representation and some Lie theory

Consider the Adjoint representation $\mathrm{Ad}\colon G \to GL(\mathfrak{g})$. If we differentiate this map at the identity we obtain

$$\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$$

which is also, confusingly, called the *adjoint representation*. In this chapter we will distinguish between the two representations with the use of an upper and lowercase letter 'a'. The linear map $\mathrm{ad}(X)$ will be denoted by ad_X.

Proposition 1.1.1. *The adjoint representation on* $\mathfrak{g}_\mp$ *satisfies* $\mathrm{ad}_X Y = \pm[X, Y]$ *where* $[X, Y]$ *denotes the Lie bracket on the algebra of left- and right-invariant* *vector fields, $\mathfrak{g}_-$ and $\mathfrak{g}_+$, respectively. Incidentally, this shows that the Lie bracket on $\mathfrak{g}_\mp$ differs only in sign.*

Proof. For elements X and Y in $\mathfrak{g}$ let X and Y also denote the corresponding left-invariant vector fields on G. These vector fields generate locally defined flows $x(t)$ and $y(t)$ through the origin at $t = 0$. For any $g \in G$ consider the flow through this point generated by X. As $X(g) = (\mathcal{L}_g)_* X$ the vector $X(g)$ is the tangent vector to the curve $\mathcal{L}_g x(t)$ at $t = 0$. The flow through g generated by X is therefore right-translation by $x(t)$. The same argument applies for Y. We can then use this to find the Lie bracket of vector fields $[X, Y]$ at the identity by using the Lie derivative formula

$$(\mathcal{L}_X Y)_e = \lim_{t\to 0} \left\{ \frac{(\mathcal{R}_{x(t)^{-1}})_* Y(x(t)) - Y}{t} \right\}.$$

As $Y(x(t)) = (\mathcal{L}_{x(t)})_* Y$ we may rewrite the above as

$$\lim_{t\to 0} \left\{ \frac{(\mathcal{L}_{x(t)} \circ \mathcal{R}_{x(t)^{-1}})_* Y - Y}{t} \right\}.$$

By definition $(\mathcal{L}_{x(t)} \circ \mathcal{R}_{x(t)^{-1}})_*$ is equal to $\mathrm{Ad}_{x(t)}$ and so

$$(\mathcal{L}_X Y)_e = \lim_{t\to 0} \left\{ \frac{\mathrm{Ad}_{x(t)} Y - Y}{t} \right\} = \mathrm{ad}_X Y.$$

If we were using right-invariant vector fields the flow would be generated by left translation of $x(t)$ instead of right, and we would have $Ad_{x(t)^{-1}}$ in the last line above. Upon differentiation this yields $-\mathrm{ad}_X Y$. □

Example 1.1.4 (The adjoint representation of $\mathfrak{so}(3)$)**.** As the Adjoint action for matrix groups is given by conjugation, the Lie bracket is simply the commutator of matrices $[\omega, \xi] = \omega\xi - \xi\omega$. With respect to the identification $\mathfrak{so}(3) \cong \mathbb{R}^3$ established in Example 1.1.1, one may verify with a calculation that the Lie bracket $[\omega, \xi]$ between elements in $\mathfrak{so}(3)$ corresponds to the vector cross-product $\omega \times \xi$.

We mentioned earlier that the Adjoint representation may be thought of as a measure of how non-commutative the group G is. In fact, the entire structure of the group is, in a sense, encoded by the Lie-bracket on $\mathfrak{g}$ and thus by the adjoint representation. The necessary disclaimer here is that the claim applies to a group up to covering, and for more detail we refer the reader to the Campbell-Hausdorff formula which expresses the product on the group purely in terms of the Lie bracket [FH13][8.3]. We will now outline some of the ways that the adjoint representation plays a key role in determining structural aspects of the group G and its Lie algebra.

A normal subgroup N of G has the defining property that $gNg^{-1} = N$ for any g. From this it follows that the Adjoint representation leaves the subalgebra $\mathfrak{n}$ in $\mathfrak{g}$ invariant. Consequently, by differentiating $\mathrm{Ad}_G\,\mathfrak{n} = \mathfrak{n}$ we obtain $[\mathfrak{g}, \mathfrak{n}] \subset \mathfrak{n}$, and thus $\mathfrak{n}$ is an ideal in $\mathfrak{g}$. Conversely, by exponentiating any ideal in $\mathfrak{g}$ we find that any normal subgroup (up to covering) arises in this way. In summary, invariant subspaces of the adjoint representation are by definition ideals of $\mathfrak{g}$ and correspond to normal subgroups of G.

A non-abelian Lie algebra which possesses no proper ideals is called a *simple Lie algebra*. Equivalently, a simple algebra is one whose adjoint representation is irreducible. The simple Lie algebras over $\mathbb{C}$ were first classified by Killing [Kil88] and then refined a few years later by Cartan in his PhD thesis [Car94]. Cartan would later go on to classify all of the simple algebras over $\mathbb{R}$. As one can see from Figure 1.1, the Adjoint representations for $\mathfrak{so}(3)$ and $\mathfrak{sl}(2;\mathbb{R})$ have no invariant proper subspaces and are therefore simple, unlike the case for $\mathfrak{se}(2)$ where the $\omega = 0$ subspace is invariant; this corresponds to the normal subgroup of translations $\mathbb{R}^2 \subset SE(2)$.

Suppose $\mathfrak{h}$ is an invariant subspace in $\mathfrak{g}$ with respect to the adjoint representation. A nice scenario is when $\mathfrak{h}$ admits an invariant complement $\mathfrak{k}$. We then have two subalgebras $\mathfrak{h}$ and $\mathfrak{k}$ with $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$, $[\mathfrak{h}, \mathfrak{g}] \subset \mathfrak{h}$, and $[\mathfrak{k}, \mathfrak{g}] \subset \mathfrak{k}$. It follows that $[\mathfrak{h}, \mathfrak{k}] = \{0\}$ and consequently the two algebras commute. Therefore, as Lie algebras, $\mathfrak{g}$ is the direct product of $\mathfrak{h}$ and $\mathfrak{k}$, and hence, the same is true of G, at least up to covering.

Concerning the structure of a Lie algebra, one first checks if $\mathfrak{g}$ is abelian or simple, and if not, looks for the irreducible invariant subspaces. In the nicest possible scenario these subspaces all admit invariant complements. An equivalent way of saying this is to say that the adjoint representation is completely reducible, and in which case, the algebra is called *reductive*. In this case one can decompose the Lie algebra into the direct sum of simple and abelian algebras. If all of these algebras are simple then the Lie algebra is called *semisimple*. Sadly, there exist many Lie algebras with invariant subspaces which do not possess an invariant complement, see for example $\mathfrak{se}(2)$ in Figure 1.2(a). A classification of all Lie algebras is a very distant hope, and as such, these Lie algebras are generally less well understood; they include the solvable and nilpotent algebras.

1.1.4 The Coadjoint and coadjoint representations

The *Coadjoint representation* $Ad^*\colon G \to GL(\mathfrak{g}^*)$ is the contragredient of the Adjoint representation. It is defined by satisfying

$$\langle \operatorname{Ad}^*_g \eta, X\rangle = \langle \eta, \operatorname{Ad}_{g^{-1}} X\rangle \tag{1.4}$$

for all $X \in \mathfrak{g}$ where $\langle\ ,\ \rangle$ denotes the pairing between $\mathfrak{g}$ and its dual. One may verify that Ad^*_g is the fibrewise isomorphism in the bundle endomorphism $\mathcal{R}^* \circ \mathcal{L}^{*-1}$ of $G \times \mathfrak{g}^*$. The *coadjoint representation* is the infinitesimal Coadjoint representation $\operatorname{ad}^*\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}^*)$. This representation may equivalently be defined as the contragredient to the adjoint representation as it satisfies

$$\langle \operatorname{ad}^*_X \eta, Y\rangle = -\langle \eta, \operatorname{ad}_X Y\rangle \tag{1.5}$$

for all $Y \in \mathfrak{g}$. As with the Adjoint and adjoint representations we will in this chapter distinguish between the Coadjoint and coadjoint representations with an upper and lowercase 'c'.

Proposition 1.1.2. *Let V be a representation of G and V^* the contragredient. The two representations are isomorphic if and only if there exists a non-degenerate, G-invariant bilinear form on V.*

Proof. If $\phi\colon V \to V^*$ is a G-equivariant isomorphism the form $K(x, y) = \langle \phi(x), y\rangle$ is non-degenerate and invariant. Conversely suppose K is non-degenerate and invariant. The map $\phi\colon V \to V^*$ given by defining $K(x, y) = \langle \phi(x), y\rangle$ for all $y \in V$ is a G-equivariant isomorphism as desired. □

The examples given earlier for the Adjoint representations of $\mathfrak{so}(3)$ and $\mathfrak{sl}(2;\mathbb{R})$ both possess invariant, non-degenerate bilinear forms: for $\mathfrak{so}(3)$ it is the trace form in (1.1), and for $\mathfrak{sl}(2;\mathbb{R})$ it is the determinant given in (1.2). According to the proposition we can therefore conclude that the Adjoint and Coadjoint representations are identical. This is part of a more general consequence concerning semisimple Lie algebras, as we now explain.

To every Lie algebra there is a canonically defined symmetric form called the *Killing form*

$$K(X, Y) = \mathrm{Trace}(\mathrm{ad}_X \circ \mathrm{ad}_Y).$$

Cartan's criterion states that this form is non-degenerate if and only if the Lie algebra $\mathfrak{g}$ is semisimple [FH13][C.10]. It follows from the proposition above that the Adjoint and Coadjoint representations of semisimple algebras are isomorphic. In fact, more can be said. If the algebra is reductive then $\mathfrak{g}$ is a direct product of a semisimple Lie algebra with an abelian algebra. As the Adjoint action fixes the abelian part, we may extend the invariant form on the semisimple part to the abelian component arbitrarily. The two representations are therefore isomorphic for all reductive Lie algebras. We note that this condition is not necessary, and that there exist Lie algebras which are not reductive but still have isomorphic Adjoint and Coadjoint representations as we will see later in the example of $SE(3)$.

Example 1.1.5 (Coadjoint orbits of $SE(2)$)**.** Here we present our first example of a Coadjoint representation which is not isomorphic to the Adjoint representation. As we have already remarked, the algebra $\mathfrak{se}(2)$ is not semisimple, and we shall now describe the Coadjoint action explicitly and show that it differs to the Adjoint action. As in Example 1.1.3 we will identify elements of the Lie algebra $\mathfrak{se}(2)$ with those $(\omega, v) \in \mathbb{R} \times \mathbb{R}^2$. We shall also identify the algebra with its dual using the pairing

$$\langle (L, p), (\omega, v) \rangle = L\omega + p \cdot v \tag{1.6}$$

between elements in $\mathfrak{se}(2)$. By definition

$$\langle \mathrm{Ad}^*_{(r,d)}(L, p), (\omega, v) \rangle = \langle (L, p), \mathrm{Ad}_{(r,d)^{-1}}(\omega, v) \rangle$$

Using the expression for the Adjoint action in (1.3) and the fact that $(r, d)^{-1} = (r^T, -r^T d)$ we can write the right-hand side above as

$$\langle (L, p), (\omega, r^T v + \omega \mathbb{J} r^T d) \rangle$$

from which it follows that

$$\mathrm{Ad}^*_{(r,d)}(L, p) = (L + (rp)^T \mathbb{J} d, rp) = \begin{pmatrix} 1 & -d^T \mathbb{J} r \\ 0 & r \end{pmatrix} \begin{pmatrix} L \\ p \end{pmatrix}. \tag{1.7}$$

The orbits for this action are illustrated in Figure 1.2(b), from which it is clear that it is not isomorphic to the Adjoint representation.

1.2 Symplectic reduction

1.2.1 The problem setting

The motivation for symplectic reduction comes from a scenario where we have a Hamiltonian system on M whose Hamiltonian H is invariant with respect to a group of symmetries G acting on M. The desire is to remove the redundancy in the dynamical system which arises from the G-symmetry and obtain a 'smaller' Hamiltonian system instead.

An obvious way of removing this redundancy is to pass to the orbit quotient

$$\pi \colon M \longrightarrow M/G$$

and ask whether the dynamics project to a dynamical system on M/G. This idea faces a few immediate challenges.

1. The projection π may not give a well-defined map from trajectories in M to those in M/G. That is, suppose $x(t)$ and $y(t)$ are two solutions in M. Then if $\pi(x(t)) = \pi(y(t))$ for $t = 0$, there is no immediate reason why this should hold for all t.

2. The topology on M/G could be disastrously unhelpful. It is quite common for the quotient topology to not even be Hausdorff, as it is for example in Figure 1.1(b): the origin orbit cannot be separated from the conical orbits by open sets.

3. Even if M/G is topologically 'nice' and there is a well-defined projection of dynamics, the symplectic geometry and Hamiltonian dynamics is seemingly abandoned.

By decreasing the generality of the original scenario and imposing sufficiently many hypotheses we may remedy all of these concerns, to varying degrees.

1.2.2 Poisson reduction

We begin by addressing the first concern in the list above. To resolve this we must necessarily suppose that the G-action on M is by symplectomorphisms. We now claim that if $x(t)$ is a solution in M then so is $gx(t)$. It suffices to show that the Hamiltonian vector field at gx is given by the pushforward by g of the Hamiltonian vector field V_H at x. Let $Y \in T_{gx}M$ be arbitrary and consider the following chain of equalities

$$\omega(g_* V_H, Y) = \omega(V_H, g_*^{-1} Y) = \langle dH, g_*^{-1} Y\rangle = \langle dH, Y\rangle = \omega(V_H, Y). \tag{1.8}$$

In the first step we have used the invariance of the symplectic form by G, then the definition of a Hamiltonian vector field in the next equality, and finally the G-invariance of H along with the property that pullbacks commute with the exterior derivative. As Y was arbitrary we conclude that $g_* V_H(x) = V_H(gx)$ as desired. We therefore have a well-defined dynamical system on M/G. In fact we have shown even more: the argument applies to any G-invariant function on M, and therefore we see that any such function on M generates a flow on M/G.

The second concern involving the topology of M/G is resolved by asking for the action of G on M to be sufficiently nice. For instance, if the action is free and proper the quotient has the structure of a smooth manifold [DK12]. This is the best we could hope for, but often this restriction is too severe as we will be interested in actions which are not free and where the quotient is not always a smooth manifold. A standard, although rather subversive, way around this is to simply restrict attention to the subset of M at which the action is free. However, it is sometimes the case that although the quotient is not a smooth manifold, it is still a reasonable topological space, albeit with a few defects. Instead of being smooth it might be smooth in a large set and possess singular regions. Although it is beyond my intention to discuss these ideas in detail, the resulting spaces might be described as orbifolds, manifolds with corners, stratified spaces, differential spaces, etc. [Pfl01a, Pfl01b, Ś13, SL91]. We will briefly discuss some aspects of this later.

Finally we turn to the third issue which is that the resulting dynamical system on M/G seems to have nothing to do with Hamiltonian dynamics. Indeed, even if M/G is smooth, it is rarely a symplectic manifold. However, what we will now show is that it is a Poisson space. That is to say, the ring of smooth functions

$C^\infty(M/G) \equiv C^\infty(M)^G$ is equipped with a Poisson bracket $\{\ ,\ \}_{M/G}$. This bracket is determined by

$$\{f, g\}_{M/G} \circ \pi = \{f \circ \pi, g \circ \pi\}_M \tag{1.9}$$

for all $f, g \in C^\infty(M/G)$. Here it should be highlighted that we have made essential use of the fact that G preserves the symplectic form on M. It is a consequence of this that the Poisson bracket between two G-invariant functions is again G-invariant, and that the expression above is well defined. Equivalently, Equation (1.9) is the condition for π to be Poisson.

As M/G possesses a Poisson bracket, if we additionally suppose M/G is smooth (or at least restrict attention to those points which are) then to every function g on M/G we can define the associated Hamiltonian vector field V_g by

$$V_g(f)_p = \{f, g\}_{M/G}(p) \tag{1.10}$$

for all functions f defined in a neighbourhood of $p \in M/G$. Consider the push-forward $\pi_* V_{g\circ\pi}$ of the Hamiltonian vector field $V_{g\circ\pi}$ on M and observe that

$$\pi_* V_{g\circ\pi}(f)_{\pi(x)} = \langle df_{\pi(x)}, \pi_* V_{g\circ\pi}\rangle = \langle d(f \circ \pi)_x, V_{g\circ\pi}\rangle = \{f \circ \pi, g \circ \pi\}_M(x).$$

From (1.10) and (1.9) we therefore conclude that $\pi_* V_{g\circ\pi}$ coincides with V_g. As we have already seen, the Hamiltonian flow of a G-invariant function $g \circ \pi$ on M may be projected to give a flow on M/G. The calculation above tells us that this projected flow is equivalently the Hamiltonian flow of g. This result is actually a direct consequence of π being a Poisson map.

To summarise this section, we have shown how a symplectic G-action on M can be used to define a Poisson structure on the algebra $C^\infty(M)^G$ and that, if M/G is sufficiently nice topologically (which may be imposed by insisting that G acts properly and freely) this turns M/G into a Poisson manifold where the projection π is a Poisson map. Consequently, Hamiltonian flows in M of G-invariant functions project to Hamiltonian flows in M/G. The original motivation concerned a Hamiltonian system on M with a G-invariant Hamiltonian. We now obtain a 'smaller' Hamiltonian system on the Poisson space M/G.

Remark 1.2.1. With some slight adjustments everything in this subsection applies equally well when we replace the symplectic manifold M with a Poisson manifold, and the symplectomorphic G-action with a Poisson action. Insisting that the orbit map π is Poisson defines a unique Poisson structure on the algebra of invariant functions as before. This more general construction belongs to the theory of *Poisson reduction*, see [MR86, Vai96, OR98]

1.2.3 Cotangent bundle reduction of a Lie group

Consider the action of left-multiplication by a Lie group G on itself, lifted to its cotangent bundle. For $\eta_g \in T_g^*G$ the action by $a \in G$ sends this form to $(\mathcal{L}_{a^{-1}})^*\eta_g$ in T_{ag}^*G. If we use the left-trivialization $\mathcal{L}^*\colon T^*G \to G \times \mathfrak{g}^*$ then this left-action by G is pushed forward to a rather simple action on $G \times \mathfrak{g}^*$ merely given by

$$a \cdot (g, \Omega) = (ag, \Omega). \tag{1.11}$$

The group quotient T^*G/G may therefore be identified with $\mathfrak{g}^*$ and can be realised by sending the orbit through η_g to $(\mathcal{L}_{a^{-1}})^*\eta_g$. We now have a smooth manifold as the group quotient, which is unsurprising given that the group action is free and proper. After having described the reduced space we now turn to describing its Poisson structure defined by (1.9). In order to do this we must first describe the Poisson structure on T^*G.

Proposition 1.2.1. *The left-trivialization $\mathcal{L}^*\colon T^*G \to G \times \mathfrak{g}^*$ pushes forward the canonical symplectic form on T^*G to the form ω on $G \times \mathfrak{g}^*$ which satisfies*

$$\omega_{(g,\eta)}\left((X_g, \Omega_1), (Y_g, \Omega_2)\right) = \langle \Omega_2, (\mathcal{L}_{g^{-1}})_* X_g \rangle - \langle \Omega_1, (\mathcal{L}_{g^{-1}})_* Y_g \rangle + \langle \eta, [(\mathcal{L}_{g^{-1}})_* X_g, (\mathcal{L}_{g^{-1}})_* Y_g] \rangle. \tag{1.12}$$

Proof. The canonical one-form λ on T^*G is given by

$$\lambda_\eta(X) = \langle \eta, \pi_* X \rangle$$

where $\pi\colon T^*G \to G$ is the bundle projection and X a tangent vector to $\eta \in T^*G$. We will proceed to use the left-trivialization and work with $G \times \mathfrak{g}^*$ instead of T^*G. In this case, it is fairly straightforward to show that the canonical one-form pushes forward to the form θ where

$$\theta_{(g,\eta)}(X_g, L) = \langle \eta, (\mathcal{L}_{g^{-1}})_* X_g \rangle$$

The canonical symplectic form is then pushed forward to $-d\theta$. We will be able to calculate this form using the coordinate-invariant formula for the exterior derivative

$$-d\theta(V,W) = -V(\langle\theta,W\rangle) + W(\langle\theta,V\rangle) + \langle\theta,[V,W]\rangle \tag{1.13}$$

for vector fields V and W on $G\times\mathfrak{g}^*$. Let's evaluate this expression at (g,η) in $G\times\mathfrak{g}^*$ and write $V=(X_g,\Omega_1)$ and $W=(Y_g,\Omega_2)$ where X_g and Y_g are tangent vectors to G at g and $\Omega_1,\Omega_2\in\mathfrak{g}^*$. Furthermore, we shall suppose that X and Y are left-invariant vector fields on G with $X(g)=X_g$ and $Y(g)=Y_g$. We begin by noting that the function $\langle\theta,W\rangle$ is that which assigns to (g,η) the value

$$\langle\eta,(\mathcal{L}_{g^{-1}})_*Y_g\rangle = \langle\eta,Y_e\rangle$$

where we have used the left-invariance of Y. The function $\langle\theta,W\rangle$ therefore only depends on η, and hence, the expression $V(\langle\theta,W\rangle)$ is given by $\langle\Omega_1,(\mathcal{L}_{g^{-1}})_*Y_g\rangle$. Similarly, one finds that $W(\langle\theta,V\rangle)$ is equal to $\langle\Omega_2,(\mathcal{L}_{g^{-1}})_*X_g\rangle$. Substituting all of this into (1.13) and using the definition of the Lie bracket on $\mathfrak{g}$ gives the equation for the symplectic form in (1.12). □

Now that we have an expression for the symplectic form we can use it to write the Poisson bracket between two left-invariant functions on T^*G.

Theorem 1.2.2. *Left-invariant functions on T^*G are sent via the left-trivialization $\mathcal{L}^*\colon T^*G\to G\times\mathfrak{g}^*$ to functions which are constant in G. Given two such functions let f and g denote their restriction to the $\mathfrak{g}^*$ component. The Poisson bracket between these two functions on T^*G gives another G-invariant function whose restriction to the $\mathfrak{g}^*$ component gives the reduced Poisson bracket*

$$\{f,g\}(\eta) = -\langle\eta,[\delta f,\delta g]\rangle \tag{1.14}$$

on $\mathfrak{g}^*$. Here we have introduced the directional derivative *δf of a function f. This is the unique element $\delta f\in\mathfrak{g}$ which satisfies*

$$\langle df,\Omega\rangle = \langle\Omega,\delta f\rangle \tag{1.15}$$

for all $\Omega\in\mathfrak{g}^*$.

Proof. From the group action in (1.11) on the left-trivialization we see that G-invariant functions f and g may be considered as functions on $G\times\mathfrak{g}^*$ which are constant in G. We will abuse notation and write f and g to mean both the functions on $G \times \mathfrak{g}^*$ and their restrictions to $\mathfrak{g}^*$. Using the left-invariance we may without any loss of generality consider the Hamiltonian vector field V_f evaluated at the fibre over the identity $e \in G$. This vector field is defined by

$$-d\theta_{(e,\eta)}(V_f, W) = \langle df, W \rangle$$

for all tangent vectors W. Write V_f as (X, Ω_1) and W as (Y, Ω_2) for $X, Y \in \mathfrak{g}$ and $\Omega_1, \Omega_2 \in \mathfrak{g}^*$. The right hand side in the equation above may be written as $\langle df, \Omega_2 \rangle$ since f is constant in G. By using the expression in (1.12) for the symplectic form we must therefore have

$$\langle \Omega_2, X \rangle - \langle \Omega_1, Y \rangle + \langle \eta, [X, Y] \rangle = \langle df, \Omega_2 \rangle$$

for all $(Y, \Omega_2) \in \mathfrak{g} \times \mathfrak{g}^*$. From this we deduce that X must be equal to δf and $\Omega_1 = -\operatorname{ad}^*_{\delta f} \eta$. We have now shown that $V_f = (\delta f, -\operatorname{ad}^*_{\delta f} \eta)$ and $V_g = (\delta g, -\operatorname{ad}^*_{\delta g} \eta)$ are the Hamiltonian vector fields of f and g at $(e, \eta) \in G \times \mathfrak{g}^*$. Upon substituting these into the symplectic form in (1.12) we obtain (1.14). □

Remark 1.2.2. Were we instead to reduce T^*G by the cotangent lift of right multiplication of G on itself, the arguments above would proceed almost exactly the same with every instance of left replaced with right. The one exception would be in (1.13) where the final term involves the Lie-bracket of two right-invariant vector fields. From Proposition 1.1.1 we know that this corresponds to the negative Lie-bracket of the corresponding left-invariant vector fields. If we identify $\mathfrak{g}^*_+$ with $\mathfrak{g}^* = \mathfrak{g}^*_-$ as sets, not Lie algebras, the overall effect will be a change of sign in (1.14).

1.2.4 Poisson manifolds and the foliation into symplectic leaves

For a Poisson manifold P there is a natural subspace defined on the tangent space at every point p given by the span of all locally defined Hamiltonian vector fields

$$D(p) = \operatorname{Span}\{V_f(p) \mid f \in C^\infty(U),\ p \in U \text{ open}\} \subset T_pP. \tag{1.16}$$

This defines the *characteristic distribution* on P. This is a generalised distribution in the sense that the dimension is allowed to vary from point to point. We will not trouble ourselves with the technicalities of such distributions, instead referring the reader to [Ste74a, Ste74b, Sus73]. An integral leaf of D is an immersed, connected submanifold of P whose tangent space at every point is given by the distribution. We say that D is *integrable* if there is such a leaf through every point of P. The resulting partition of P into leaves defines a *generalised foliation*. We will simply claim the Symplectic Foliation Theorem [OR13][4.1.27] which states that the characteristic distribution is integrable, and that the integral leaves $\mathcal{L}$ are all immersed symplectic submanifolds equipped with the unique symplectic form which makes the inclusion $\mathcal{L} \hookrightarrow P$ a Poisson map.

The integral leaves of the characteristic distribution may equivalently be defined as the *accessible sets* of D. One defines an equivalence relation on P by saying that two points are related if their exists a finite composition of flows generated by locally defined Hamiltonian vector fields which sends one point to the other. The resulting equivalence classes are the accessible sets and these coincide with the integral leaves. Clearly then, any Hamiltonian flow on P must preserve these leaves. Therefore, if we return to our earlier motivation concerning a Hamiltonian system on M and the resulting reduced dynamics on M/G, we can obtain a Hamiltonian system on an even smaller space by considering the dynamics on these symplectic leaves of M/G. To demonstrate this we will find the symplectic leaves on the Poisson space $\mathfrak{g}^*$ introduced earlier.

Theorem 1.2.3. *The connected integral leaves of the characteristic distribution defined on $\mathfrak{g}^*$ are equal to the connected components of the Coadjoint orbits. For the Coadjoint orbit $\mathcal{O}$ through $\eta \in \mathfrak{g}^*$ the symplectic form at $T_\eta\mathcal{O}$ is*

$$\omega_\eta(\operatorname{ad}^*_X \eta, \operatorname{ad}^*_Y \eta) = -\langle \eta, [X, Y] \rangle \tag{1.17}$$

for all $X, Y \in \mathfrak{g}$. This form is often referred to as the Kostant-Kirillov-Souriau (KKS) symplectic form.

Proof. Let f and g be functions on $\mathfrak{g}^*$ perhaps only locally defined in a neighbourhood of $\eta \in \mathfrak{g}^*$ and consider the Hamiltonian vector field V_f. By (1.14) we have

$$\langle dg, V_f \rangle(\eta) = \{g, f\}(\eta) = \langle \eta, [\delta f, \delta g] \rangle = -\langle \operatorname{ad}^*_{\delta f} \eta, \delta g \rangle.$$

By the definition of δg the expression on the far right is equal to $-\langle dg, \operatorname{ad}^*_{\delta f} \eta \rangle$ from which it follows that $V_f(\eta) = -\operatorname{ad}^*_{\delta f} \eta$. The characteristic distribution is therefore given by

$$D(\eta) = \{\operatorname{ad}^*_X \eta \mid \forall X \in \mathfrak{g}\}$$

which is clearly equal to the tangent space of the Coadjoint orbit through η. □

Remark 1.2.3. As left and right multiplications by a group on itself commute, it follows that the right cotangent-lifted action of G on T^*G descends to give a well-defined action on the quotient of T^*G by left multiplication. By realising this quotient as $\mathfrak{g}^*$ using the left-trivialization we see that this action descends to give the Coadjoint action $\mathcal{R}^*_g \circ \mathcal{L}^{*-1}_g = \operatorname{Ad}^*_g$. Since right multiplication lifted to T^*G is a symplectomorphism the Coadjoint action on $\mathfrak{g}^*$ is Poisson, and therefore restricts to a transitive symplectic group action on each Coadjoint orbit. Moreover, from the proof above we see that this group action is generated by Hamiltonian vector fields: the infinitesimal vector field V_X generated by $X \in \mathfrak{g}$ is the Hamiltonian vector field V_f generated by the linear function $f(\eta) = -\langle \eta, X \rangle$.

Remark 1.2.4. It is a rather special result that all Coadjoint orbits are symplectic manifolds. It provides a helpful method for providing examples of symplectic manifolds, particularly symplectic manifolds which possess transitive Hamiltonian group actions, such as complex flag manifolds in the case for Coadjoint orbits of $U(n)$. In fact, all symplectic manifolds which admit transitive symplectic actions by some group are, up to coverings, Coadjoint orbits [Kir74, Kos70, Sou70]. As the group of symplectomorphisms acts transitively on a connected symplectic manifold, an interesting generalisation of this result is the claim that every symplectic manifold is, in a sense, a Coadjoint orbit of its own group of symplectomorphisms [IZ16].

1.2.5 Hamiltonian actions and momentum maps

There appears to be no immediate way to describe the symplectic leaves of M/G given an arbitrary symplectic group action of G on M. The purpose of this and the next subsection is to show that these leaves can be described when the symplectic group action is a Hamiltonian group action, which we shall now describe.

For the group action of G on M there is also the infinitesimal action which assigns to every X in $\mathfrak{g}$ a vector field V_X on M. We will denote this correspondence $X \mapsto V_X$ by κ and note that $\kappa\colon \mathfrak{g} \to \operatorname{Vect}(M)$ is a Lie algebra homomorphism. The correspondence $f \mapsto V_f$ sending functions to Hamiltonian vector fields also defines a Lie algebra homomorphism which we shall denote by $\rho\colon C^\infty(M) \to \mathrm{Vect}(M)$. The group action will be called *Hamiltonian* if there exists a map $\lambda\colon \mathfrak{g} \to C^\infty(M)$ for which $\kappa = \rho \circ \lambda$. In other words, so that the triangle below commutes.

$$\begin{array}{ccc} C^\infty(M) & \xrightarrow{\ \rho\ } & \mathrm{Vect}(M) \\ & {}_{\lambda}\nwarrow & \uparrow{\scriptstyle \kappa} \\ & & \mathfrak{g} \end{array}$$

If such a λ exists we emphasize that it is not unique since Hamiltonian functions which differ by locally constant functions give rise to identical Hamiltonian vector fields. The aim now is to find whether, for a given Hamiltonian action, there exists a λ which is also a Lie algebra homomorphism.

We may suppose λ is always linear: choose a basis of $\mathfrak{g}$ and extend the definition of λ on this basis by linearity. For λ to be a Lie algebra homomorphism we would like the following function on M to be zero

$$\Sigma(X,Y) = \lambda([X,Y]) - \{\lambda(X), \lambda(Y)\}.$$

The vector field

$$\rho(\Sigma(X,Y)) = \rho \circ \lambda([X,Y]) - \rho\left(\{\lambda(X), \lambda(Y)\}\right)$$

generated by this function equals $\kappa([X,Y]) - [\kappa(X), \kappa(Y)] = 0$ as κ and ρ are both Lie algebra homomorphisms and $\rho \circ \lambda = \kappa$. If we suppose that the manifold M is connected, then $\rho(\Sigma(X,Y)) = 0$ shows that the Hamiltonian vector field generated by the function $\Sigma(X,Y)$ for any given X and Y is zero, and hence, the function is constant on M. The function Σ therefore defines a two-form on $\mathfrak{g}$. Although we won't perform the calculation here, one can show that this form is closed as a 2-form in the chain complex of forms on $\mathfrak{g}$, and consequently, it defines a cohomology class $[\Sigma]$ in $H^2(\mathfrak{g})$.

Suppose that λ' is an alternative choice for λ. As $\rho \circ (\lambda - \lambda') = 0$ the two maps must differ by a constant function on M which by linearity must be a linear functional β on $\mathfrak{g}$. For $\lambda' = \lambda + \beta$ one can show that the corresponding 2-form Σ'

is

$$\begin{aligned}\Sigma'(X,Y) &= \lambda([X,Y]) + \beta([X,Y]) - \{\lambda(X)+\beta(X), \lambda(Y)+\beta(Y)\}\\ &= \Sigma(X,Y) + \beta([X,Y]).\end{aligned}$$

In the language of Lie algebra cohomology $\beta([X,Y])$ is the closed 2-form $-d\beta$, and hence, Σ' and Σ both define the same cohomology class. This class is therefore an obstruction to finding such a homomorphism λ. Moreover, if the one-form β is closed, then $\Sigma = \Sigma'$, and thus, if such a λ exists as desired, it is unique up to additions of closed one-forms; or in other words, objects in $H^1(\mathfrak{g})$ [GS90][24.1].

Proposition 1.2.4. *If a group action of G on a connected symplectic manifold M is Hamiltonian, then there exists a well-defined cohomology class $[\Sigma] \in H^2(\mathfrak{g})$ which measures the obstruction to finding a Lie algebra homomorphism $\lambda\colon \mathfrak{g} \to C^\infty(M)$ for which $\rho \circ \lambda = \kappa$. Such a λ only exists if $[\Sigma] = 0$, and if so is parametrised by $H^1(\mathfrak{g})$.*

Remark 1.2.5. The first cohomology group $H^1(\mathfrak{g})$ is the space of all forms $\beta \in \mathfrak{g}^*$ with $d\beta(X,Y) = -\beta([X,Y]) = 0$ for all $X, Y \in \mathfrak{g}$. If this group is trivial then it implies that $\mathfrak{g} = [\mathfrak{g},\mathfrak{g}]$ which is itself the definition for a Lie algebra to be *perfect*. All semisimple Lie algebras are perfect. Moreover, from the Whitehead Lemmas we have $H^1(\mathfrak{g}) = H^2(\mathfrak{g}) = 0$ for $\mathfrak{g}$ semisimple [Jac79]. Consequently, any Hamiltonian action of a semisimple group admits a unique lift λ with $\Sigma = 0$.

Let's proceed to consider the case for when M is connected and G is a Hamiltonian group action with cocyle Σ. As λ is linear, for every $p \in M$ there is a one-form $\mu(p) \in \mathfrak{g}^*$ for which

$$\langle \mu(p), X\rangle = H_X(p)$$

where we will now also write H_X to denote the Hamiltonian function $\lambda(X)$. This defines a map $\mu\colon M \to \mathfrak{g}^*$ which we call the *momentum map*. A rather finnicky calculation reveals that the momentum map is G-equivariant if and only if the cocycle Σ vanishes, see for instance [MS17][5.16]. We will now present a few classical examples of the momentum map.

Example 1.2.1 (Momentum map for a subgroup action)**.** Suppose G has a Hamiltonian action on M with a momentum map μ_G. For a subgroup K of

G consider the restricted action of K on M. For $X \in \mathfrak{k}$ the Hamiltonian H_X is equal to $\langle \mu_G(p), X\rangle$ and so the momentum map $\mu_K \colon M \to \mathfrak{k}^*$ is given by restricting the domain of the linear forms $\mu_G(p) \in \mathfrak{g}^*$ to $\mathfrak{k} \subset \mathfrak{g}$. We therefore have $\mu_K = \iota^* \circ \mu_G$ where $\iota\colon \mathfrak{k} \hookrightarrow \mathfrak{g}$ is the inclusion.

Example 1.2.2 (Momentum map for a Coadjoint orbit)**.** Following Remark 1.2.3 the Coadjoint action on a Coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$ is Hamiltonian. We also saw that the vector field V_X is the Hamiltonian vector field generated by the function $f(\eta) = -\langle \eta, X\rangle$ from which it immediately follows that the momentum map is the negative of the inclusion $\mu\colon \mathcal{O} \hookrightarrow \mathfrak{g}^*$.

Example 1.2.3 (Momentum map for a group acting on its cotangent bundle)**.** Consider a group action of G on some manifold Q and the cotangent-lift of this action to T^*Q. This cotangent-lift preserves the canonical one-form λ. If we let V_X denote the infinitesimal vector field on T^*G generated by this action for $X \in \mathfrak{g}$ then by Cartan's magic formula

$$0 = \mathcal{L}_{V_X}\lambda = \iota_{V_X} d\lambda + d(\iota_{V_X}\lambda)$$

we must have

$$\omega(V_X, \cdot) = d(\langle \lambda, V_X\rangle).$$

This tells us that the vector field V_X is Hamiltonian with Hamiltonian function

$$H_X(\eta_q) = \langle \lambda, V_X(\eta_q)\rangle = \langle \eta_q, X_* q\rangle \tag{1.18}$$

where X_*q is the vector at $q \in Q$ generated by the infinitesimal group action of X. The cotangent-lift is therefore a Hamiltonian group action. If we now suppose Q is a Lie group G and that G is acting by left-multiplication on itself, then the tangent vector X_*g is $(\mathcal{R}_g)_*X$. It follows from (1.18) above that there exists an equivariant momentum map $\mu\colon T^*G \to \mathfrak{g}^*$ given by

$$\mu(\eta_g) = (\mathcal{R}_g)^*\eta_g.$$

In other words, after projection onto the $\mathfrak{g}^*$ component, the right-trivialization $\mathcal{R}^*$ of T^*G gives the momentum map for the left cotangent-lifted action. Conversely, one can show that the negative of the left-trivialization is the momentum map for the right cotangent-lifted action.

Remark 1.2.6. It's not a huge problem if a Hamiltonian action does not admit a lift λ which is a homomorphism. One can redefine the Poisson structure on $\mathfrak{g}^*$ and the Coadjoint action to obtain a momentum map which is equivariant. One first shows that the function

$$\sigma(g) = \mathrm{Ad}_g^* \mu(p) - \mu(gp)$$

does not depend on M and therefore defines a *non-equivariant one-cocycle* $\sigma \colon G \to \mathfrak{g}^*$. Then one can define the *affine Coadjoint action*

$$g \cdot \eta = \mathrm{Ad}_g^* \eta + \sigma(g)$$

of G on $\mathfrak{g}^*$ and the altered Poisson bracket

$$\{f, g\}(\eta) = -\langle \eta, [\delta f, \delta g] \rangle + \Sigma(\delta f, \delta g).$$

This Poisson structure on $\mathfrak{g}^*$ can also be seen as the reduced Poisson structure on T^*G/G which arises when we alter the canonical symplectic form on T^*G with a magnetic term $\pi^*\Sigma$. With this new Poisson structure and group action μ becomes an equivariant momentum map [OR13][4.5].

1.2.6 Ordinary symplectic reduction

Suppose we have a connected symplectic manifold M and a Hamiltonian group action by G with an equivariant momentum map μ. We will now show how this information allows us to describe (at least partially) the symplectic leaves of M/G. These spaces coincide with the ordinary Marsden-Weinstein reduced spaces found more often in the literature [MW74]. We begin with the important result of Noether which states that the momentum of a Hamiltonian system with a G-invariant Hamiltonian is conserved.

Theorem 1.2.5 (Noether's Theorem)**.** *Let f be a G-invariant function on a connected symplectic manifold M and suppose the action of G is Hamiltonian with momentum map μ. The flow of the Hamiltonian vector field V_f preserves the fibres of μ.*

Proof. As f is G-invariant we have $\{f, H_X\} = 0$ for all $X \in \mathfrak{g}$. As $H_X(p) = \langle \mu(p), X \rangle$ this implies that μ is constant along the flow of V_f. □

The symplectic leaves of the characteristic distribution in M/G are equivalently defined as the accessible sets obtained by flowing along the composition of a finite number of locally defined Hamiltonian vector fields. The result above demonstrates that any of these corresponding flows of G-invariant functions on M preserve the fibres of μ. If we consider the commutative diagram below

$$\begin{array}{ccc} M & \xrightarrow{\mu} & \mathfrak{g}^* \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle /G} \\ M/G & \xrightarrow{\widetilde{\mu}} & \mathfrak{g}^*/G \end{array} \tag{1.19}$$

then this states that, for any Coadjoint orbit $\mathcal{O}$ in $\mathfrak{g}^*$ the pre-image $\mu^{-1}(\mathcal{O})$ is always an invariant set for the flows of G-invariant functions, and hence descends under the projection to give a set

$$\pi(\mu^{-1}(\mathcal{O})) = \widetilde{\mu}^{-1}([\mathcal{O}])$$

in M/G which is always invariant under any Hamiltonian flow. This set, which we shall denote by $M_{\mathcal{O}}$ and refer to as the *symplectic orbit-reduced space* of M at $\mathcal{O}$ is consequently a union of symplectic leaves in M/G. Although we will not prove it here, if the action of G on $\mu^{-1}(\mathcal{O})$ is free and proper, then $M_{\mathcal{O}}$ is precisely a single symplectic leaf in M/G and has the structure of a smooth symplectic manifold [OR13][6.3.1].

If the action is not free then the space $M_{\mathcal{O}}$ will not, in general, be a symplectic manifold, but instead, a *stratified symplectic space*. The precise definition of these objects belongs to the theory of *singular reduction*. More detail can be found in the seminal work of [SL91] and also in [OR13] for a comprehensive introduction. For our purposes we are content to state that this means $M_{\mathcal{O}}$ is a union of symplectic manifolds, called the strata of $M_{\mathcal{O}}$, and that these strata correspond to the different isotropy types of points in $\mu^{-1}(\mathcal{O})$. To shed some light on the plausibility of this statement consider the subsets M_H of M consisting of those points for which the subgroup of G fixing each point is given exactly by the subgroup H of G. We claim that for any G-invariant function f the flow of the Hamiltonian vector field V_f must leave these sets M_H invariant. Let $x(t)$ be a solution curve defined locally where $x(0) \in M_H$. Then by (1.8) the curve $hx(t)$ is also a solution for any $h \in H$. As $hx(0) = x(0)$ it follows that $hx(t) = x(t)$ for all t that $x(t)$ is defined for. Therefore $x(t) \in M_H$ for all t as was claimed. The set

$$M_{(H)} = \bigcup_{g \in G} gM_H,$$

which is equivalently defined to be the set of all points whose isotropy subgroup is conjugate to H, is a G-invariant set in M. We may consider the corresponding subset $\pi(M_{(H)})$ in M/G. As the flows of G-invariant functions preserve every M_H and therefore $M_{(H)}$, it follows that $\pi(M_{(H)})$ is invariant under all possible Hamiltonian flows, and hence, is a union of symplectic leaves in M/G. We therefore obtain a finer decomposition of reduced spaces into symplectic leaves by considering the stratum

$$M_{\mathcal{O}} \cap \pi(M_{(H)})$$

for any conjugacy class of subgroups (H). Curiously, it is apparently the case that these strata do not necessarily give the finest possible stratification into symplectic leaves [OR13][8.3.3].

Remark 1.2.7. The definition we have presented here is that of an orbit-reduced space $M_{\mathcal{O}}$ corresponding to a Coadjoint orbit in $\mathcal{O} \subset \mathfrak{g}^*$. More frequently one encounters *point-reduced spaces.* Given $\eta \in \mathfrak{g}^*$ one considers the preimage $\mu^{-1}(\eta)$ in M. As μ is equivariant the isotropy group G_η acts on this set, and so we may define the point-reduced space M_η to be the quotient $\mu^{-1}(\eta)/G_\eta$. To make things simpler, if we suppose η is a regular value of μ then $\mu^{-1}(\eta)$ is a smooth submanifold of M, and if moreover, we suppose G acts freely and properly then we can safely talk of the reduced space M_η as a smooth manifold. The standard theory of symplectic reduction establishes that M_η may be equipped with a unique symplectic form which pulls back under the quotient map to give the restriction of the symplectic form on M to $\mu^{-1}(\eta)$. It turns out that the orbit- and point-reduced spaces are equivalent [OR13][6.4.1]: if $\mathcal{O}$ is the Coadjoint orbit through η then $M_{\mathcal{O}} \cong M_\eta$.

Example 1.2.4 (A particle in a space with rotational symmetry)**.** Consider the standard action of $O(3)$ on $\mathbb{R}^3$ and its cotangent lift to $T^*\mathbb{R}^3$. By using the standard inner product to identify cotangent spaces to points in $\mathbb{R}^3$ with $\mathbb{R}^3$ itself we may identify $T^*\mathbb{R}^3$ with the set of pairs $(q, p) \in \mathbb{R}^3 \times \mathbb{R}^3$. The lifted action of $O(3)$ now becomes the standard diagonal action on $\mathbb{R}^3 \times \mathbb{R}^3$. By (1.18) the Hamiltonian for $\xi \in \mathfrak{so}(3)$ is given by

$$H_\xi(q, p) = p \cdot \xi q = \mathrm{Trace}(p^T \xi q).$$

This may be manipulated to equal

$$\frac{1}{2}\,\mathrm{Trace}((pq^T - qp^T)^T\xi).$$

The pairing between $\mathfrak{so}(3)$ and its dual given by the trace form in (1.1) allows us to write this as $\langle pq^T - qp^T, \xi\rangle$. The equivariant momentum map $\mu\colon T^*\mathbb{R}^3 \to \mathfrak{so}(3)^*$ is therefore

$$\mu(p,q) = pq^T - qp^T.$$

Upon identifying $\mathfrak{so}(3)$ with $\mathbb{R}^3$ as in Example 1.1.1 a quick calculation shows that $\mu(p,q)$ is equal to the vector cross product $q \times p$. This coincides with the standard physical quantity of angular momentum, and hence, we have established the familiar result that angular momentum is conserved in systems with rotational symmetry. As the Adjoint orbits in $\mathfrak{so}(3) \cong \mathbb{R}^3$ are given by spheres centred at the origin, the sets $\mu^{-1}(\mathcal{O})$ are therefore equal to the subsets of $\mathbb{R}^3 \times \mathbb{R}^3$ with $|q \times p|^2$ constant.

We highlight that the $O(3)$-action on $\mathbb{R}^3 \times \mathbb{R}^3$ is nowhere free. For generically independent q and p the isotropy subgroup is $\mathbb{Z}_2$. If q and p are colinear but not all zero, the isotropy subgroup is isomorphic to $O(2)$, and if both are zero then the entire group $O(3)$ is the stabiliser. Due to this action not being free we cannot expect the group quotient $T^*\mathbb{R}^3/O(3)$ to be a smooth manifold, and indeed it isn't. Nevertheless, it is a nice topological space. Following the theory of universal reduction in [ACG91] and emulating the method of invariants used in [CB97, LMS93] we may identify the group quotient with the image of the Hilbert map which sends points to the values taken by generators of the ring of $O(3)$-invariant functions on $\mathbb{R}^3 \times \mathbb{R}^3$. These generators may be taken to be the pairwise inner products between the vectors q and p. Consequently, we may realise the Hilbert map as the map sending (q,p) to

$$\xi = \begin{pmatrix} q\cdot p & -|p|^2 \\ |q|^2 & -q\cdot p \end{pmatrix}.$$

This might seem like an awkward way of expressing the quotient as a subset of $\mathfrak{sl}(2;\mathbb{R})$, however, we shall now explain our reasons. Observe that the determinant of the matrix is equal to

$$|q|^2|p|^2 - (q\cdot p)^2 = |q \times p|^2.$$

As the reduced spaces $M_{\mathcal{O}}$ are the images of the invariant sets $\mu^{-1}(\mathcal{O})$ under the orbit map, and since these sets are those subsets of constant $|q \times p|^2$, we see that these reduced spaces coincide with the surfaces of constant $\det \xi$ given in (1.2). In fact, it turns out that the Poisson structure on the group quotient coincides exactly with the Poisson structure on $\mathfrak{sl}(2;\mathbb{R})^*$ and hence, the connected components of the Coadjoint orbits are the connected components of the reduced spaces. In the commutative diagrams below we rewrite that given in (1.19) for this specific example.

$$\begin{array}{ccc} T^*\mathbb{R}^3 & \xrightarrow{\mu} & \mathfrak{so}(3)^* \cong \mathbb{R}^3 \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ \mathfrak{sl}(2;\mathbb{R})^* & \longrightarrow & \mathbb{R}^{\geq 0} \end{array} \qquad \begin{array}{ccc} (q,p) & \longmapsto & q \times p \\ \downarrow & & \downarrow \\ \xi & \longmapsto & \det \xi = |q \times p|^2 \end{array}$$

The orbit quotient is not the whole of $\mathfrak{sl}(2;\mathbb{R})^*$. It is clear that $\det \xi = |q \times p|^2$ must always be non-negative as well as the entries $|q|^2$ and $|p|^2$. The quotient is therefore identified with the lower cone in Figure 1.1(b) and the union of all the hyperboloids below it.

For $|q \times p|^2$ greater than zero these reduced spaces are the connected components of the two-sheeted hyperboloid, and are thus diffeomorphic to the plane. On the other hand, for when $|q \times p|^2$ is zero the reduced space $\pi(\mu^{-1}(0))$ is the entire bottom cone together with its vertex. This is not a smooth manifold, and is instead an example of a stratified symplectic space. The two strata in question are the open conical surface, and the origin at the vertex. As we briefly explained earlier, these strata correspond to the different isotropy types of the points in the preimage $\mu^{-1}(0)$. For points with q and p colinear but not all zero, the isotropy subgroup is isomorphic to $O(2)$ and the stratum corresponds to the open conical surface. For the case when both q and p are zero, the isotropy group is the whole of $O(3)$ and the corresponding stratum is the origin. For the generic reduced spaces q and p are not colinear and the isotropy type for all of these points is conjugate to $\mathbb{Z}_2$, and thus, they consist of a single stratum.

1.3 Semidirect products

1.3.1 Definitions and split exact sequences

Suppose H and N are groups and $\varphi\colon H \to \mathrm{Aut}(N)$ a homomorphism. The automorphism group $\mathrm{Aut}(N)$ is the set of isomorphisms from N into itself. We define the *semidirect product* $G = H \ltimes_\varphi N$ to be the set $H \times N$ together with the product

$$(h_1, n_1)(h_2, n_2) = (h_1 h_2, n_1 \varphi_{h_1}(n_2))$$

whose inverses are then given by

$$(h, n)^{-1} = \left(h^{-1}, \varphi_{h^{-1}}(n^{-1})\right).$$

Observe that $H \cong H \times \{e\}$ is a subgroup of G and that $N \cong \{e\} \times N$ is a normal subgroup of G. This observation motivates a separate definition which will turn out to be equivalent. We say that an exact sequence of groups

$$\{e\} \longrightarrow N \xrightarrow{\iota} G \xrightarrow{\pi} H \longrightarrow \{e\}$$

is a *split exact sequence* if there exists a homorphism $\sigma \colon H \to G$ called a *splitting map* with $\pi \circ \sigma = \mathrm{Id}_H$. In such a situation we also have that $H \cong \sigma(H)$ is a subgroup of G and that $N \cong \iota(N)$ is a normal subgroup of G since it is the kernel of the homomorphism π.

It is clear that the semidirect product $H \ltimes_\varphi N$ defines a split exact sequence in the obvious way. We now show that the converse is true.

Proposition 1.3.1 (The Splitting Lemma)**.** *Consider a split exact sequence*

$$\{e\} \longrightarrow N \longrightarrow G \longrightarrow H \longrightarrow \{e\}.$$

For the sake of easing notation identify N with the normal subgroup of G given by the inclusion above, and H with the subgroup of G given by the image of the splitting map. The group G is isomorphic to the semidirect product $H \ltimes_\varphi N$ where the automorphism φ is defined by

$$\varphi_h(n) = hnh^{-1} \tag{1.20}$$

for all $n \in N$ and each $h \in H$.

Proof. Define a map $G \to H \times N$ which sends g to $(h(g), n(g))$ where $h(g)$ is the projection $G \to H$ in the exact sequence above and where $n(g)$ is the unique element in N satisfying

$$n(g)h(g) = g, \tag{1.21}$$

which is well defined since the kernel of the projection $G \to H$ is N. Consider the image $(h(g_1g_2), n(g_1g_2))$ of the product g_1g_2 between two elements g_1 and g_2 in G. As the projection map is a homomorphism we have $h(g_1g_2) = h(g_1)h(g_2)$. By the definition in (1.21) we have

$$n(g_1g_2) = g_1g_2h(g_2)^{-1}h(g_1)^{-1} = g_1n(g_2)h(g_1)^{-1} = n(g_1)\left[h(g_1)n(g_2)h(g_1)^{-1}\right]$$

from which it follows that $g \mapsto (h(g), n(g))$ defines a homomorphism from G into $H \ltimes_\varphi N$ with φ as defined in (1.20). Injectivity and surjectivity of this map is fairly clear, and thus, this map defines an isomorphism as desired. □

The easiest example of a semidirect product is when the homomorphism φ is trivial. In this case the semidirect product is the direct product between groups. The most common semidirect products we encounter in this thesis are presented in the next example.

Example 1.3.1 (Semidirect product with a vector space)**.** Suppose H has a representation ρ on a vector space V over a field $\mathbb{F}$. This defines the semidirect product $G = H \ltimes_\rho V$. This group has a natural action on V

$$(h, d) \cdot v = \rho(h)v + d.$$

The group G can therefore be thought of as the affine version of H acting on V. Specific examples of groups of this kind include the special Euclidean group $SE(n) = SO(n) \ltimes \mathbb{R}^n$ of rigid motions of $\mathbb{R}^n$, and the Poincaré group of affine transformations preserving Minkowski space. We can define a representation of G on $V \times \mathbb{F}$ by sending (h, d) to the matrix

$$\begin{pmatrix} \rho(h) & d \\ 0 & 1 \end{pmatrix}$$

in $GL(V \times \mathbb{F})$. If the representation ρ is faithful then so too is this representation, and so G may be identified with the matrix subgroup of elements above.

Remark 1.3.1. If we differentiate a split exact sequence at the identity we obtain a split exact sequence of Lie algebras. In precisely the same way as we define a semidirect product as the group in the middle of such a sequence, we also define a semidirect product algebra to be the middle algebra in such a split sequence of

Lie algebras. Any Lie algebra $\mathfrak{g}$ fits into an exact sequence

$$0 \longrightarrow \mathrm{Rad}(\mathfrak{g}) \longrightarrow \mathfrak{g} \longrightarrow \underbrace{\mathfrak{g}/\mathrm{Rad}(\mathfrak{g})}_{\mathfrak{s}}$$

where $\mathrm{Rad}(\mathfrak{g})$ denotes the *radical* of $\mathfrak{g}$ which is the maximum solvable ideal in $\mathfrak{g}$. The quotient algebra $\mathfrak{s}$ is always semisimple. The Levi Decomposition states that this sequence is a split exact sequence and therefore, that every Lie algebra is isomorphic to the semidirect product algebra $\mathfrak{s} \ltimes \mathrm{Rad}(\mathfrak{g})$ [FH13][9.1].

1.3.2 The Adjoint representation of a semidirect product

Semidirect products are not typically reductive Lie algebras and therefore there is no reason to suspect that the Adjoint and Coadjoint representations might be isomorphic. The purpose of this and the next subsection is to explicitly find expressions for the Adjoint and Coadjoint actions of a semidirect product with a vector space.

Let H be a Lie group and V a representation of H. We will not bother to introduce notation for the representation and instead write rd to denote the action of $r \in H$ on $d \in V$. The group multiplication in $G = H \ltimes V$ is

$$(r_1, d_1)(r_2, d_2) = (r_1 r_2, d_1 + r_1 d_2)$$

and the inverses are

$$(r, d)^{-1} = (r^{-1}, -r^{-1}d).$$

For any $(a, x) \in G$ the conjugation by (r, d) is then

$$(r, d)(a, x)(r, d)^{-1} = \left(rar^{-1}, d + rx - (rar^{-1})d\right).$$

Recall that the Adjoint action is given by the infinitesimal conjugation evaluated at the identity. By identifying the tangent space $\mathfrak{g}$ with $\mathfrak{h} \times V$ in the obvious sense, we may differentiate (a, x) in the expression above at the identity to obtain the Adjoint action of a semidirect product

$$\mathrm{Ad}_{(r,d)}(\omega, v) = (\mathrm{Ad}_r\, \omega, rv - (\mathrm{Ad}_r\, \omega)d) \tag{1.22}$$

for $(\omega, v) \in \mathfrak{g}$. By differentiating (r, d) in the Adjoint action at the identity we also obtain an expression for the adjoint action and Lie bracket of a semidirect product

$$\mathrm{ad}_{(\omega_1,v_1)}(\omega_2,v_2) = [(\omega_1,v_1),(\omega_2,v_2)] = ([\omega_1,\omega_2],\omega_1 v_2 - \omega_2 v_1)\,. \tag{1.23}$$

Again, we are being lazy with our notation and denoting the infinitesimal action of $\omega \in \mathfrak{h}$ on $v \in V$ by ωv.

We will now describe the isotropy subgroups $G_{\omega,v}$ of the Adjoint action. Observe that for (r,d) to belong to $G_{\omega,v}$ we require Ad_r to fix ω. It follows that r must belong to the isotropy subgroup H_ω of H. It now remains to check that $rv - \omega d$ remains equal to v. The group H_ω acts by restriction on V and since it preserves the subspace $\mathrm{Im}\,\omega$ it descends to give an action on $V/\,\mathrm{Im}\,\omega$. It follows that for $rv - \omega d$ to equal v we require exactly for r to fix $[v]$ in $V/\,\mathrm{Im}\,\omega$. That is, we require $r \in H_{[v]}$ which then determines d up to $\ker\omega$. We can then define an exact sequence

$$0 \longrightarrow \ker\omega \overset{\iota}{\longrightarrow} G_{\omega,v} \overset{\pi}{\longrightarrow} H_\omega \cap H_{[v]} \longrightarrow \{e\}$$

where $\iota(d) = (e,d)$ and $\pi(r,d) = r$. In the language of groups we call the isotropy subgroup $G_{\omega,v}$ a *group extension* of $H_\omega \cap H_{[v]}$ by $\ker\omega$. We note in passing that this exact sequence is not necessarily a split exact sequence.

Example 1.3.2 (Adjoint orbits of the special Euclidean group $SE(3)$)**.** Following Example 1.1.1 we may identify $\mathfrak{so}(3)$ with $\mathbb{R}^3$ which interwines both the adjoint and vector representations. Furthermore, from Example 1.1.4 this identifies the Lie bracket with the cross product, and so, upon identifying $\mathfrak{se}(3)$ with $\mathbb{R}^3 \times \mathbb{R}^3$ the Adjoint action in (1.22) becomes

$$\mathrm{Ad}_{(r,d)}(\omega,v) = (r\omega, rv - (r\omega)\times d)\,.$$

By observation one can see that the generic orbits are given by the sets

$$\left\{(\omega,v) \mid |\omega|^2 = C_1^2,\ v\cdot\omega = C_2\right\}$$

for constants $C_1, C_2 \in \mathbb{R}$ where $C_1 \neq 0$. These orbits are diffeomorphic to the tangent bundle T^*S^2 of a sphere of radius $|C_1|$. The degenerate Adjoint orbits are not given by sets of the above form. These are the orbits through points of the form $(0, v)$ and are diffeomorphic to spheres of radius $|v|$.

1.3.3 The Coadjoint representation of a semidirect product

We may identify the dual of $\mathfrak{g}$ with $\mathfrak{h}^* \times V^*$ via the pairing

$$\langle (L, p), (\omega, v) \rangle = \langle L, \omega \rangle + \langle p, v \rangle \tag{1.24}$$

where $\langle\ ,\ \rangle$ denotes the pairing between a space and its dual. By using the expression fo the Adjoint action in (1.22) we have

$$\begin{aligned} \langle \operatorname{Ad}^*_{(r,d)}(L, p), (\omega, v) \rangle &= \langle (L, p), \operatorname{Ad}_{(r,d)^{-1}}(\omega, v) \rangle \\ &= \langle (L, p), \big(\operatorname{Ad}_{r^{-1}} \omega, r^{-1}v + (\operatorname{Ad}_{r^{-1}} \omega) rd\big) \rangle \\ &= \langle L, \operatorname{Ad}_{r^{-1}} \omega \rangle + \langle p, r^{-1}v \rangle + \langle p, r^{-1}\omega d \rangle. \end{aligned}$$

By the very definition of the Coadjoint action on $\mathfrak{h}^*$ the first term in the final expression is equal to $\langle \operatorname{Ad}^*_r L, \omega \rangle$ and the second term may be rewritten as $\langle rp, v \rangle$, where once again we are being lazy and writing rp to denote the contragredient action of H on V^*. The final expression may then be written as $\langle rp, \omega d \rangle$ which we will rewrite as $\langle \mu(rp, d), \omega \rangle$ where we have introduced a map $\mu \colon V^* \times V \to \mathfrak{h}^*$ which satisfies

$$\langle \mu(p, v), \xi \rangle = \langle p, \xi v \rangle$$

for all $\xi \in \mathfrak{h}$. With reference to (1.18) one recognises that the map μ is actually the equivariant momentum map for the cotangent-lifted action of H to $T^*V \cong V^* \times V$. Putting this all together, the final line in the expression above may be arranged to give

$$\langle \mu(rp, d), \omega \rangle$$

from which we conclude that the Coadjoint action is given by

$$\operatorname{Ad}^*_{(r,d)}(L, p) = \left(\operatorname{Ad}^*_r L + \mu(rp, d), rp\right). \tag{1.25}$$

By differentiating this expression at (r, d) through the identity we obtain the coadjoint action

$$\operatorname{ad}^*_{(\omega,v)}(L, p) = \left(\operatorname{ad}^*_\omega L + \mu(p, v), \omega p\right). \tag{1.26}$$

We now consider the isotropy subgroups $G_{L,p}$ of this Coadjoint action. Any element $(r,d) \in G_{L,p}$ requires r to fix p and thus $r \in H_p$. Secondly, we require $\operatorname{Ad}^*_r L + \mu(p,d)$ to be equal to L. Similarly to the Adjoint action from earlier, we wish to consider the quotient action of H_p on $\mathfrak{h}^*/\mu(p,V)$. It turns out that this quotient has a very special structure which we will now pause to prove.

Lemma 1.3.2. *The subspace $\mu(p,V)$ in $\mathfrak{h}^*$ is equal to the annihilator*

$$\mathfrak{h}_p^\circ = \{L \in \mathfrak{h}^* \mid \langle L, \omega \rangle = 0,\ \forall \omega \in \mathfrak{h}_p\}$$

for any $p \in V^*$ where $\mathfrak{h}_p$ denotes the Lie algebra of the group $H_p = \{r \in H \mid rp = p\}$.

Proof. Denote by τ_p the map $\mu(p,\cdot)\colon V \to \mathfrak{h}^*$ for a fixed p. The kernel $\ker \tau_p$ consists of all $v \in V$ which satisfy

$$0 = \langle \mu(p,v), \xi \rangle = \langle p, \xi v \rangle = -\langle \xi p, v \rangle$$

for all $\xi \in \mathfrak{h}$. This is equivalent to v belonging to $(\mathfrak{h} \cdot p)^\circ$ where $\mathfrak{h} \cdot p$ is the tangent space to p at the H-orbit through p, which has dimension $\dim \mathfrak{h} - \dim \mathfrak{h}_p$. We therefore have $\dim(\mathfrak{h} \cdot p)^\circ = \dim V + \dim \mathfrak{h}_p - \dim \mathfrak{h}$. By rank-nullity $\dim \operatorname{Im} \tau_p = \dim V - \dim \ker \tau_p = \dim \mathfrak{h} - \dim \mathfrak{h}_p = \dim \mathfrak{h}_p^\circ$. Finally, it is clear that $\operatorname{Im} \tau_p$ belongs to $\mathfrak{h}_p^\circ$ since

$$\langle \mu(p,v), \xi \rangle = \langle p, \xi v \rangle = -\langle \xi p, v \rangle = 0$$

for all $\xi \in \mathfrak{h}_p$ from which it follows that $\operatorname{Im} \tau_p = \mathfrak{h}_p^\circ$ as desired. □

By exactness of the sequence

$$\mathfrak{h}_p^\circ \longrightarrow \mathfrak{h}^* \xrightarrow{\iota_p^*} \mathfrak{h}_p^*$$

we have that $\mathfrak{h}^*/\mathfrak{h}_p^\circ$ is isomorphic to $\mathfrak{h}_p^*$. Moreover, as all of the maps in this exact sequence are equivariant with respect to the action of H_p it follows that the quotient action of H_p on $\mathfrak{h}^*/\mathfrak{h}_p^\circ$ is canonically isomorphic to the Coadjoint action of H_p on $\mathfrak{h}_p^*$. Returning to our discussion above and using the fact that $\mu(p,V) = \mathfrak{h}_p^\circ$ it follows that for r to belong to $G_{L,p}$ we must have $r \in (H_p)_{\iota_p^* L}$. That is, the Coadjoint action of $r \in H_p$ on $\mathfrak{h}_p^*$ must fix $\iota_p^* L$. In order to ensure

$\mathrm{Ad}^*_r L + \mu(p,d)$ remains equal to L the vector d is then uniquely defined up to translates of $\ker \tau_p$. The isotropy subgroup fits into an exact sequence

$$0 \longrightarrow \ker \tau_p \stackrel{\iota}{\longrightarrow} G_{L,p} \stackrel{\pi}{\longrightarrow} (H_p)_{\iota_p^* L} \longrightarrow \{e\}$$

where $\iota(d) = (e,d)$ and $\pi(r,d) = r$. Once again, as with the Adjoint action we remark that this exact sequence is not necessarily a split exact sequence [Bag98]. This exact sequence expressing the isotropy group as an extension of the group $(H_p)_{\iota_p^* L}$ by $\ker \tau_p$ first appears in [Raw75].

Example 1.3.3 (Coadjoint orbits of the special Euclidean group $SE(3)$)**.** For the Coadjoint action we may identity $\mathfrak{so}(3)^*$ with $\mathfrak{so}(3)$ using the invariant form in (1.1) and then identify this again with $\mathbb{R}^3$. We noted earlier that the map μ is the momentum map for the cotangent lifted action of $SO(3)$ to $\mathbb{R}^3 \times \mathbb{R}^3$. We have already calculated this momentum map in Example 1.2.4 and have seen that it is $\mu(p,q) = q \times p$. The Coadjoint action in (1.25) may then be written as

$$\mathrm{Ad}^*_{(r,d)}(L,p) = (rL - (rp) \times d, rp)\,. \tag{1.27}$$

The generic orbits can also be seen to be equal to the sets

$$\left\{(L,p) \mid |p|^2 = C_1^2,\ L \cdot p = C_2\right\}$$

for constants $C_1, C_2 \in \mathbb{R}$ with $C_1 \neq 0$. These orbits are diffeomorphic to tangent bundles over spheres. The degenerate orbits which are not given by level sets of the Casimirs are those through points of the form $(L,0)$ and are diffeomorphic to spheres of radius $|L|$.

Remark 1.3.2. The Adjoint and Coadjoint orbits of $SE(3)$ are quite unusual as the orbit types for both coincide. See for example the case for $SE(2)$ in Figure 1.2 where this is not the case. In general, the Adjoint and Coadjoint representations for a semidirect product are not isomophic, however for $SE(3)$ it turns out that they are. Following Proposition 1.1.2 this fact may be deduced from the existence of the invariant non-degenerate form

$$B((\omega_1, v_1), (\omega_2, v_2)) = \omega_1 \cdot v_2 + \omega_2 \cdot v_1$$

on $\mathfrak{se}(3)$ which establishes an equivariant isomorphism between $\mathfrak{se}(3)$ and its dual.

Note that this form is not the Killing form as the algebra is not semisimple. More generally this can be seen as a special instance of a semidirect product of the form $G \ltimes_{\mathrm{Ad}} \mathfrak{g}$. For groups of this kind there is always an isomorphism between the Adjoint and Coadjoint representations.

For the Adjoint and Coadjoint orbits of $SE(n)$ for general n we would like to recommend our (unpublished) work in [AM18] together with J. Montaldi, where we describe the geometry of the orbits as special kinds of affine flag manifolds.

Proposition 1.3.3. *Let H be a Lie group with a representation V and consider the semidirect product $G = H \ltimes V$ with Lie algebra $\mathfrak{g} = \mathfrak{h} \times V$. For functions f and g on the dual $\mathfrak{g}^*_-$ the Poisson bracket is given by*

$$\{f, g\}(L, p) = -\left\langle L, \left[\frac{\delta f}{\delta L}, \frac{\delta g}{\delta L}\right]\right\rangle - \left\langle p, \frac{\delta f}{\delta L}\frac{\delta g}{\delta p} - \frac{\delta g}{\delta L}\frac{\delta f}{\delta p}\right\rangle \tag{1.28}$$

where the elements $\delta f/\delta L \in \mathfrak{h}$ and $\delta f/\delta p \in V$ are directional derivatives uniquely defined by satisfying

$$\langle df, (\Omega, 0)\rangle = \left\langle \Omega, \frac{\delta f}{\delta L}\right\rangle \quad \textit{and} \quad \langle df, (0, \alpha)\rangle = \left\langle \alpha, \frac{\delta f}{\delta p}\right\rangle$$

for all $\Omega \in \mathfrak{h}^*$ and $\alpha \in V^*$.

Proof. From Theorem 1.2.2 we first need the directional derivatives of a function f. As we are identifying $\mathfrak{g}^*$ with $\mathfrak{h}^* \times V^*$ as in (1.24) δf is equal to $(\delta f/\delta L, \delta f/\delta p)$. Substituting this into (1.14) and using the adjoint action of a semidirect product given in (1.23) yields (1.28). □

1.3.4 The Semidirect Product Reduction by Stages theorem

As we have already seen, the reduced space obtained by reducing a cotangent bundle of a Lie group T^*H by the group acting on itself is the coalgebra $\mathfrak{h}^*$. In this subsection we will show that if we reduce T^*H by certain subgroups of H acting once again by left- or right-multiplication, the reduced space is a subset of the coalgebra of a larger group G which is a semidirect product of H with some vector space. In order to best present this result we must first conduct a more detailed analysis of the Coadjoint orbits of a semidirect product.

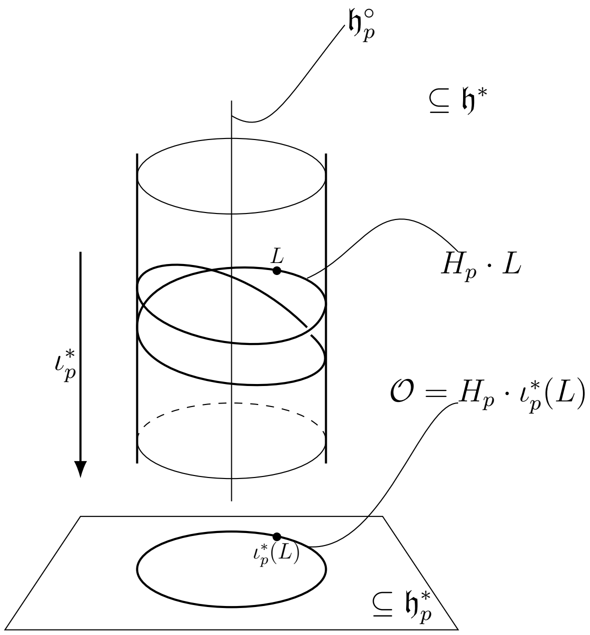


Figure 1.3: The Coadjoint orbit of G through (L, p) is a bundle over the orbit of H through $p \in V^*$. The fibre over p is illustrated in this diagram and shown to be $\mathfrak{h}_p^\circ \times \mathcal{O}$ where $\mathcal{O}$ is the Coadjoint orbit of H_p through $\iota_p^*(L)$.

Let $G = H \ltimes V$ be as before in the previous subsection and consider the Coadjoint action (1.25). The orbits fibre over the H-orbit through $p \in V^*$ but what are the fibres? For a given p we have seen that the translation subgroup V acts by adding $\mathfrak{h}_p^\circ$ to the $\mathfrak{h}^*$ component. We may factor out this subspace using the H_p-equivariant projection $\iota_p^*\colon \mathfrak{h}^* \to \mathfrak{h}_p^*$. It follows that once we factor out by $\mathfrak{h}_p^\circ$ the fibre over p is a Coadjoint orbit of H_p. Figure 1.3 gives a schematic picture of the fibres of such an orbit. For a given p and Coadjoint orbit $\mathcal{O}$ in $\mathfrak{h}_p^*$ we have a corresponding Coadjoint orbit in $\mathfrak{g}^*$ equal to the set

$$\left\{(\operatorname{Ad}_r^* L, rp) \mid \text{for all } r \in H \text{ and } L \text{ with } \iota_p^* L \in \mathcal{O}\right\}. \tag{1.29}$$

All Coadjoint orbits in $\mathfrak{g}^*$ are equal to sets of this form. Using this idea Rawnsley establishes a one-to-one correspondence between Coadjoint orbits and bundles over orbits in V^* whose fibres are Coadjoint orbits of the so-called *little-group* orbits H_p [Raw75].

We now turn to consider the cotangent-lifted action of G on T^*G by left/right multiplication. Fix some $p \in V^*$ and consider the diagram below where a map π_K denotes the projection from a space upon which K acts to the space of K-orbits.

$$\begin{array}{ccc} T^*G & & \\ \downarrow{\scriptstyle \pi_V} & \searrow^{\pi_G} & \\ T^*H \times V^* & \xrightarrow{\pi_H} & T^*G/G \\ \uparrow & \nearrow_{\pi_{H_p}} & \\ T^*H \times \{p\} & & \end{array}$$

If we naturally identify T^*V with $V \times V^*$ then the translation subgroup $V = \{e\} \times V$ only acts on the base space of T^*V in $T^*G = T^*H \times T^*V$. Consequently, we have identified the orbit space T^*G/V with $T^*H \times V^*$ and π_V with the map which projects away the V component. The action of $H = H \times \{0\}$ on T^*G descends though π_V to give the diagonal action of H on $T^*H \times V^*$, and since any G-orbit is an orbit of V-orbits under the action of H, the top triangle commutes. The H-orbits through $T^*H \times \{p\} \subset T^*H \times V^*$ precisely correspond to H_p-orbits through T^*H and this gives commutativity of the bottom triangle.

Theorem 1.3.4 (Semidirect Product Reduction by Stages)**.** *Let H_p act on T^*H by cotangent lift on the left/right. There is a smooth Poisson map from T^*H into $\mathfrak{g}^*_\mp = \mathfrak{h}^* \times V^*$ whose fibres are precisely the H_p-orbits given by sending $\eta \in T^*_r H$ to*

$$\left(\mathcal{L}^*_{r^{-1}}\eta, r^{-1}p\right) \in \mathfrak{g}^*_- \quad or \quad (\mathcal{R}^*_{r^{-1}}\eta, rp) \in \mathfrak{g}^*_+ \tag{1.30}$$

for the left/right case respectively. Moreover, if we let $\mathcal{O}$ denote a Coadjoint orbit in $\mathfrak{h}^*_p$ and $\mu\colon T^*H \to \mathfrak{h}^*_p$ the equivariant momentum map for the action of H_p on T^*H, then $\mu^{-1}(\mathcal{O})$ is sent to the Coadjoint orbit in $\mathfrak{g}^*$ through (L,p) with $\iota^*_p L \in \mathcal{O}$. As the fibres of this map are the H_p-orbits it follows that this Coadjoint orbit in $\mathfrak{g}^*_\mp$ is symplectomorphic to the orbit-reduced space $M_\mathcal{O}$.

Proof. Equip $T^*H \times V^*$ and T^*G/G in the diagram with Poisson structures induced by the maps π_V and π_G together with the canonical structure on T^*G. Translation invariant functions f and g are constant in the V-component in T^*V. Therefore, as the product $T^*G = T^*H \times T^*V$ is symplectically orthogonal, the Poisson bracket between two such functions restricts to the canonical Poisson structure on T^*H. If we identity $T^*H \times \{p\}$ with T^*H endowed with the canonical Poisson structure this then implies the inclusion into $T^*H \times V^*$ is Poisson, and hence, π_H restricted to $T^*H \times \{p\}$ is a Poisson map π_{H_p} into T^*G/G.

The projection π_G may be taken to be left/right translation to the identity and this identifies T^*G/G with $\mathfrak{g}^*_\mp$ (where the $\mp$ indicates that the Poisson structure differs up to sign depending on whether we use left or right translation as explained in Remark 1.2.2). The map π_{H_p} must necessarily be that given in (1.30) for the top triangle to commute. The map is clearly smooth and the fibres are exactly the H_p-orbits. Note that group multiplication on the right is only a group action if it is right-inverse multiplication; hence the difference in $r^{-1}p$ and rp between the two expressions in (1.30).

From Example 1.2.3 the momentum map on T^*H for left/right multiplication by H is given by right/left translation to the identity. As the H_p-action is given by restriction of this action, from Example 1.2.1 the momentum map is given by $\mu(\eta) = \iota_p^* \circ \mathcal{L}_{r^{-1}}^*(\eta)$ for $\eta \in T_r^*H$ (we are here dealing with the case of the right-action for notative ease, although the left-case is entirely similar). The set $\mu^{-1}(\mathcal{O})$ is then equal to all such η with $\iota_p^* \circ \mathcal{L}_{r^{-1}}^*(\eta) \in \mathcal{O}$. By noting that $\mathrm{Ad}_r^* = \mathcal{L}_r^* \circ \mathcal{R}_{r^{-1}}^*$ and writing $L = \mathcal{L}_{r^{-1}}^*(\eta)$ observe that the image of $\mu^{-1}(\mathcal{O})$ under the Poisson map in (1.30) is precisely the Coadjoint orbit described in (1.29). □

Remark 1.3.3. Suppose one wishes to reduce T^*H by the action of a subgroup K of H. The theorem above tells us that we can realise the reduced spaces as Coadjoint orbits of some semidirect product if we can find a representation of H for which K is the isotropy subgroup of some vector. For an arbitrary subgroup K it is not clear that such a representation should always exist. In this direction we have the Mostow-Palais embedding theorem which states that, if a manifold M admits a smooth transitive action by a compact group H then it can be equivariantly embedded into a unitary representation V of H [Mos57, Pal57]. In particular, a homogeneous space H/K which is a manifold may be equivariantly embedded in V. This means that for some vector in V the isotropy subgroup will be K. Therefore, as a corollary we see that for H compact and K any subgroup for which H/K is a manifold, the reduced spaces of T^*H/K are Coadjoint orbits of the semidirect product $H \ltimes V$.

1.4 Applications to mechanics

1.4.1 The Legendre transform

In this section we consider Hamiltonian systems defined on cotangent bundles which arise from problems in classical dynamics. The purpose of this subsection is to explain the choices for the Hamiltonians given later. In order to understand why a particular problem in classical dynamics has a particular Hamiltonian we must first review the passage from the Lagrangian to the Hamiltonian formulation of mechanics; this is accomplished by way of the Legendre transform.

A mechanical system is described by a configuration manifold Q which is the set of all possible instantaneous configurations of the system. Any solution curve $q(t)$ parametrised by time canonically lifts to a curve $\dot{q}(t)$ in the tangent bundle TQ. An apparent feature of the universe is that all physical evolution is second-order in time, meaning that any motion is uniquely determined by its position and velocity at any instant. We therefore take the dynamical system to be defined on TQ. The dynamics will be generated by a *Lagrangian* L which is a smooth function on TQ. By the Principal of Least Action the dynamically realisable paths $\dot{q}(t)$ in TQ are those extremals which are local minimums of the *action functional*

$$\mathcal{A}[\dot{q}(t)] = \int_{t_0}^{t_1} L \circ \dot{q} \; dt.$$

Taking the first variation of $\mathcal{A}$ and solving for when this is zero give the standard Euler-Lagrange equations from Lagrangian dynamics.

Given a vector $v \in T_q Q$ we can define a tangent vector F_v in TQ to any point w in the fibre $T_q Q$ by differentiating the curve $w + tv$ at $t = 0$. If we let π denote the projection $TQ \to Q$ then we may define a one-form θ_L on TQ by setting

$$\langle \theta_L, Y \rangle = \langle dL, F_{\pi_* Y} \rangle$$

for any tangent vector Y to TQ. One can then define a two-form $\omega_L = -d\theta_L$ which, depending on some mild regularity condition on L which we shall call *regular*, is non-degenerate. For any solution $\dot{q}(t)$ in TQ of the Euler-Lagrange equations, one can show with a fairly lengthy calculation in a local coordinate chart, see [Woo97][2.1], that the tangent vector X to $\dot{q}(t)$ is the unique vector which satisfies

$$\omega_L(X, \cdot) = d\hat{H}$$

where $\hat{H}$ is a smooth function on TQ which sends $v \in T_q Q$ to

$$\hat{H}(v) = \langle dL, F_v \rangle - L.$$

It follows that, provided L is regular, (TQ, ω_L) is a symplectic manifold and solutions in TQ are the Hamiltonian flows of the function $\hat{H}$. An unsatisfactory feature of this formulation is that the symplectic form on the tangent bundle depends on the choice of Lagrangian. This is the reason why we introduce the *Legendre transform* $\mathbb{I}\colon TQ \to T^*Q$ which establishes a symplectomorphism between (TQ, ω_L) and T^*Q equipped with the canonical symplectic form. For any $v \in T_qQ$ the element $\mathbb{I}(v) \in T_q^*Q$ is the unique one-form which satisfies

$$\langle \mathbb{I}(v), w\rangle = \langle dL, F_w\rangle \tag{1.31}$$

for all $w \in T_qQ$. This defines a smooth fibrewise-linear map, and thus a bundle morphism, and it is a straightforward exercise to show that it pulls back the canonical one-form on T^*Q to θ_L. It follows immediately from the definition of $\mathbb{I}$ and $\hat{H}$ that the Hamiltonian on TQ is pushed forward to the Hamiltonian

$$H(p) = \langle p, v\rangle - L \tag{1.32}$$

on T^*Q for $v \in T_qQ$ and where we write $p = \mathbb{I}(v)$. The Hamiltonian formulation of mechanics considers this Hamiltonian system defined on T^*Q with the Hamiltonian given above, defined by the Lagrangian L on TQ.

Example 1.4.1 (Geodesic flow as a Hamiltonian system)**.** As an application we will show how the geodesic flow on a manifold Q may be cast as a Hamiltonian system on T^*Q. Recall that a (pseudo-)Riemannian metric on Q is given by a smoothly varying, non-degenerate, symmetric bilinear form B_q defined on the fibres T_qQ for each $q \in Q$. The geodesics are those curves $\dot{q}(t)$ in TQ which minimise the length

$$\int_{t_0}^{t_1} \frac{1}{2} B_q(\dot{q}, \dot{q})\ dt.$$

This is identical to the problem in Lagrangian mechanics whose Lagrangian L on TQ is

$$L(v) = \frac{1}{2} B_q(v, v)$$

for $v \in T_qQ$. For any vector $w \in T_qQ$ recall that F_w is the tangent vector to any $v \in T_qQ$ given by differentiating the curve $v + tw$ at $t = 0$. The function $\langle dL, F_w\rangle$ evaluated at v is then

$$\left.\frac{d}{dt}\right|_{t=0} \frac{1}{2} B_q(v + tw, v + tw) = B_q(v, w).$$

By definition of the Legendre transform we see that $\mathbb{I}(v)$ is the unique form in

T_q^*Q which satisfies

$$\langle \mathbb{I}(v), w \rangle = B_q(v, w). \tag{1.33}$$

Finally, from (1.32) the Hamiltonian at $p = \mathbb{I}(v) \in T_q^*Q$ is given by

$$H(p) = \langle p, v \rangle - L(v) = B_q(v, v) - \frac{1}{2}B_q(v, v) = \frac{1}{2}B_q(v, v) = \frac{1}{2}\langle p, \mathbb{I}^{-1}(p) \rangle. \tag{1.34}$$

1.4.2 Left-invariant geodesics on a Lie group

Consider a Lie group G equipped with a Riemannian metric. On each fibre of TG there is a smoothly varying inner product

$$B_g(v, w) = \langle \mathbb{I}_g(v), w \rangle \tag{1.35}$$

for any two vectors $v, w \in T_gG$ where $\mathbb{I} \colon TG \to T^*G$ is a smooth fibrewise diffeomorphism which we know from the last subsection to be the Legendre transform. The geodesic flows on G thus define a Hamiltonian system on T^*G with Hamiltonian

$$H(\eta) = \frac{1}{2}\langle \eta, \mathbb{I}_g^{-1}(\eta) \rangle \tag{1.36}$$

for $\eta \in T_g^*G$. Suppose that the metric is left-invariant on G, that is to say that $\mathcal{L}_g^*B = B$ for all $g \in G$. One can check that this is equivalent to

$$(\mathcal{L}_g)^* \circ \mathbb{I}_g \circ (\mathcal{L}_g)_* = \mathbb{I}_e \tag{1.37}$$

for all $g \in G$. This implies that the Hamiltonian in (1.36) is also a left-invariant function on T^*G. As we have a left-invariant Hamiltonian on T^*G we may apply the methods of symplectic reduction introduced earlier. Before jumping into this we pause to introduce some more terminology which will be useful later for the example of the rigid body.

Given a tangent vector $\dot{g} \in T_gG$ we will refer to the the resulting vectors in $\mathfrak{g}$ obtained by left and right trivializations as the *body* and *spatial angular velocity*, ω_s and ω_b, respectively. To be specific

$$\omega_b = (\mathcal{L}_{g^{-1}})_*\dot{g},$$
$$\omega_s = (\mathcal{R}_{g^{-1}})_*\dot{g}.$$

The Legendre transform sends $\dot{g}$ to a one-form $\eta = \mathbb{I}_g(\dot{g}) \in T_g^*G$ which may

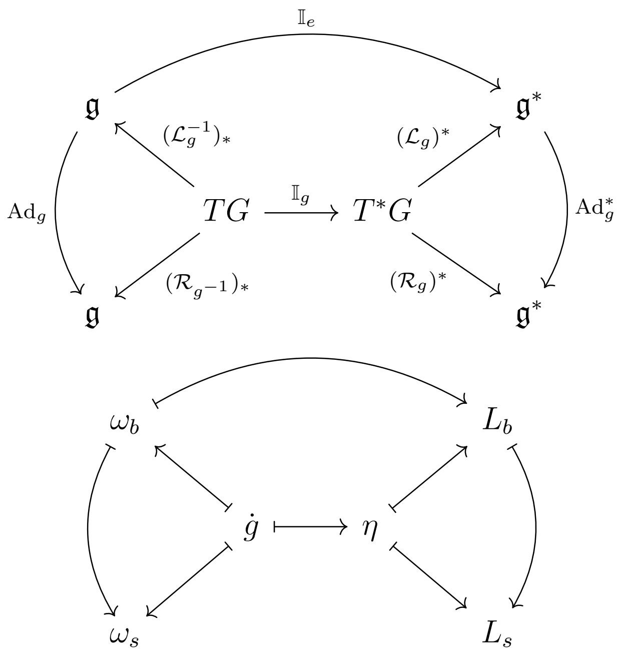


Figure 1.4: How left and right trivializations on the tangent and cotangent bundles of G define the spatial and body angular momentum/velocity vectors. As the metric is left-invariant we have from (1.37) that the top square commutes and is independent of g.

also be sent to an element in $\mathfrak{g}^*$ by either the left or right trivialization. We refer to these L_s and L_b as the *body* and *spatial angular momentum*, respectively. Precisely these are

$$L_b = (\mathcal{L}_g)^* \eta,$$
$$L_s = (\mathcal{R}_g)^* \eta.$$

The spatial and body vectors are related to each other by the Adjoint and Coadjoint representations

$$\omega_s = \mathrm{Ad}_g \, \omega_b, \tag{1.38}$$
$$L_s = \mathrm{Ad}_g^* \, L_b.$$

All of this notation is more compactly illustrated in Figure 1.4. We now return to our discussion of the Hamiltonian system on T^*G with the left-invariant Hamiltonian (1.36). The following theorem is due to Arnold [Arn66].

Theorem 1.4.1 (Euler-Arnold equations)**.** *Let G be a Lie group equipped with a left-invariant Riemannian metric on G. The body and spatial angular momentum vectors satisfy the following differential equations*

$$\dot{L}_b = -\operatorname{ad}^*_{\omega_b} L_b, \tag{1.39}$$
$$\dot{L}_s = 0.$$

Proof. From Example 1.2.3 the momentum map for the left action of G lifted to T^*G is right-translation to $\mathfrak{g}^*$. The momentum is therefore the spatial angular momentum L_s which by Noether's Theorem 1.2.5 is constant. Secondly, following Theorem 1.2.2 the orbit quotient T^*G/G is obtained by left trivialization and may be identified with the space of body angular momentum vectors in $\mathfrak{g}^*$. The left-invariant Hamiltonian in (1.36) descends to give

$$H(L_b) = \frac{1}{2}\langle L_b, \omega_b \rangle.$$

From the proof of Theorem 1.2.3 the Hamiltonian vector field $V_H(L_b)$ is $-\operatorname{ad}^*_{\delta H} L_b$. From the definition of the directional derivative in (1.15) and using the property that $\mathbb{I}$ defines a symmetric form on $\mathfrak{g}$ we see that δH is ω_b from which (1.39) follows. □

1.4.3 The rigid body with a fixed point

Consider a rigid body in $\mathbb{R}^n$ free to move about a fixed point which we take to be the origin. Fix the standard orthonormal frame in $\mathbb{R}^n$ and refer to this as the *space frame*. For a given initial configuration of the body attach an orthonormal frame fixed to the body, which we may suppose in this initial configuration coincides with the space frame. This frame will be called the *body frame*, and relative to an observer in the space frame will vary as the body moves. In this way we may identify the space of all possible configurations of the body with the group $SO(n)$ where a given configuration of the body shall correspond to the unique transformation $r \in SO(n)$ which sends the space frame to the body frame.

Let x denote a point within the body at $t = 0$ when the body is in its initial configuration. For a motion $r(t) \in SO(n)$ of the body consider the corresponding motion $x(t) = r(t)x$ of this material point. Using the definitions of the body and spatial angular velocities given earlier observe that the velocity of this point may be given by the following equivalent expressions

$$\dot{x}(t) = \dot{r}(t)x = r(t)\omega_b x = \omega_s x(t).$$

For an observer in the space frame, the velocity of any point at $x(t)$ is therefore given by multiplying the vector by the spatial angular velocity. Conversely, for an observer in the body frame, the velocity of the point x, which to an observer in the space frame is given by $\dot{x}(t)$ and which therefore transforms to the vector $r(t)^{-1}\dot{x}(t)$ as seen in the body frame, is given by multiplying the position by the body angular velocity. This explains the choice of terminology for the angular velocity vectors.

In the Lagrangian formulation this describes a mechanical problem defined on $TSO(n)$ whose Lagrangian is given purely by the kinetic energy. The kinetic energy as a function of $\dot{r}(t) \in T_{r(t)}SO(n)$ is

$$\frac{1}{2}\int_B \rho|\dot{r}(t)x|^2 \; dx$$

where the integral is taken over the body $B \subset \mathbb{R}^n$ in its initial configuration, and where ρ is the density of the body. This defines an inner product on each of the fibres of $TSO(n)$ and therefore we have a Riemannian metric on $SO(n)$ whose geodesics correspond to motions of the rigid body. Furthermore, the kinetic energy above may be rewritten as

$$\frac{1}{2}\rho\int_B |\omega_b x|^2 \; dx \tag{1.40}$$

which is manifestly left-invariant, and hence we have a left-invariant metric on $SO(n)$. We are now able to apply the results of Theorem 1.4.1 directly, but before doing so we will express the Legendre transform $\mathbb{I}$ explicitly.

The kinetic energy above defines an inner product on the space of body angular velocity vectors in $\mathfrak{g}$ which may be written as

$$B(\omega, \xi) = \frac{1}{2}\int_B \rho(\omega x \cdot \xi x) \; dx = \frac{1}{2}\int_B \rho \, \mathrm{Trace}(x^T\omega^T\xi x) \; dx = \frac{1}{2}\mathrm{Trace}(M\omega^T\xi)$$

where we have introduced the symmetric *mass matrix*

$$M = \int_B \rho x x^T \; dx.$$

By the definition of the Legendre transform we must have that $L = \mathbb{I}_e(\omega)$ satisfies $\langle L, \xi \rangle = B(\omega, \xi)$ for all $\xi \in \mathfrak{so}(n)$. Upon identifying $\mathfrak{so}(n)$ with its dual using the trace form in (1.1) we may rewrite $B(\omega, \xi)$ as

$$\frac{1}{2}\,\mathrm{Trace}(M\omega^T\xi) = \frac{1}{4}\,\mathrm{Trace}\left((\omega M + M\omega)^T\xi\right) = \frac{1}{2}\langle \omega M + M\omega, \xi \rangle$$

from which we conclude that

$$L = \omega M + M\omega.$$

One may now directly apply (1.39) to obtain the standard equations of motion for a free rigid body in $\mathbb{R}^n$. As a specific example we will now present the most familiar case for when $n = 3$.

Example 1.4.2 (The free rigid body in $\mathbb{R}^3$)**.** The case in three dimensions is particularly special as the adjoint and vector representations of $SO(3)$ are, as we have already seen, isomorphic. Upon identifying $\mathfrak{so}(3)^* \cong \mathfrak{so}(3)$ with $\mathbb{R}^3$ we see that in a suitable orthonormal basis the inner product defined by $\mathbb{I}$ corresponds to a symmetrix matrix $\mathrm{diag}(I_1, I_2, I_3)$ whose entries are referred to as the *moments of inertia*. The angular momentum is then related to the angular velocity by $L = \mathbb{I}\omega$. From Example 1.1.4 $\mathrm{ad}_\omega\,\xi$ is identified with $\omega \times \xi$. Equation (1.39) becomes

$$\dot{L}_b = -\omega_b \times L_b.$$

If we write the vectors L_b and ω_b in components as (L_1, L_2, L_3) and $(\omega_1, \omega_2, \omega_3)$, respectively, then using $L_b = \mathbb{I}\omega_b$ we may expand this equation into components and express them in terms of the body-velocity components to give the familiar standard set of Euler equations

$$\begin{aligned} I_1\dot{\omega}_1 &= \omega_2\omega_3(I_2 - I_3), \\ I_2\dot{\omega}_2 &= \omega_1\omega_3(I_3 - I_1), \\ I_3\dot{\omega}_3 &= \omega_1\omega_2(I_1 - I_2). \end{aligned}$$

Finally, as we have a Hamiltonian system defined on $\mathfrak{so}(3)^*$ with Hamiltonian

$$H(L) = \frac{1}{2}\langle L, \mathbb{I}^{-1}L \rangle = \frac{1}{2}\left(\frac{L_1^2}{I_1} + \frac{L_2^2}{I_2} + \frac{L_3^2}{I_3}\right)$$

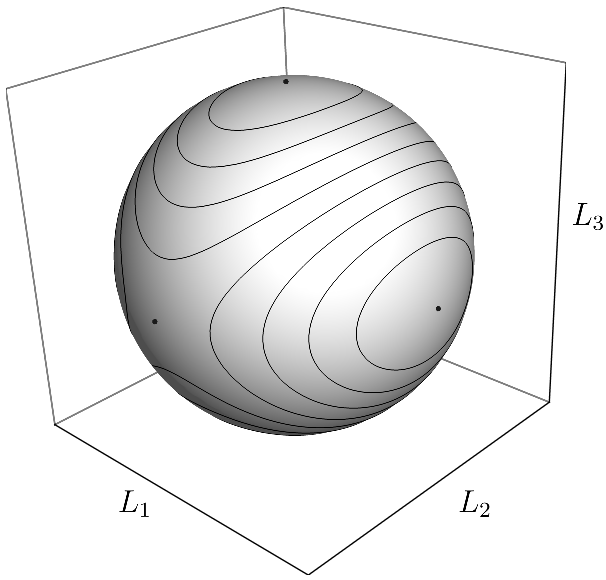


Figure 1.5: Solution curves on a Coadjoint orbit in $\mathfrak{so}(3)^*$ for the three-dimensional rigid body with a fixed point.

the solution curves are given by the intersections of level sets of H with Coadjoint orbits in $\mathfrak{so}(3)^*$. Recall that these orbits are spheres centred at the origin. A sphere of radius $|L|$ corresponds to a motion whose angular momentum has magnitude $|L|$. The solutions curves on this sphere for $|L| > 0$ are given in Figure 1.5 by intersecting the orbit with the ellipsoids of constant H, where we have taken $I_1 > I_2 > I_3 > 0$.

Remark 1.4.1. The theory we have presented concerning geodesics of a left-invariant metric on G can be altered with a few changes to give a description for right-invariant metrics. It was Arnold who originally showed that just as the rigid body can be described by left-invariant geodesics on $SO(3)$, the motion of an incompressible fluid on some manifold can be described as geodesics on the infinite-dimensional group of volume-preserving diffeomorphisms equipped with a right-invariant metric [Arn66, Arn13, AK99]. It is quite remarkable that both sets of corresponding reduced equations were known to Euler, and now both referred to as *Euler's equations*.

1.4.4 The Kirchhoff equations

If the rigid body is not fixed at a point and is instead free to move around in $\mathbb{R}^n$ by any rigid motion we obtain Euler-Arnold equations for the semidirect product $SE(n) = SO(n) \ltimes \mathbb{R}^n$. If the motion is free then the results are not too interesting as the motion can be decomposed into the independent motions of the centre of mass together with the motion of the body about this point. This decomposition is a consequence of the Galilean invariance of the problem. Therefore, in this subsection we will instead consider the problem of the motion of a rigid body which is immersed in an incompressible fluid. This system lacks Galilean invariance and consequently the motion of the body about its centre of mass and the motion of the centre of mass itself are inextricably coupled.

By introducing a fluid into the problem we have seemingly increased the complexity enormously. After all, a fluid's motion is specified by a vector field defined at every point which has infinitely many degrees of freedom. Rather surprisingly, it turns out, as we will show shortly, that the state of the fluid is completely determined by the position and velocity of the body, therefore allowing us to retain $SE(n)$ as the configuration space.

As before, we identify a configuration of the body with an element $g = (r, d)$ in $SE(n)$ which is the transformation sending the body from some initial configuration to its current one. The Lagrangian on the tangent bundle is equal to the total kinetic energy of both the body and the fluid

$$L(\dot{g}) = KE_B(\dot{g}) + KE_F(\dot{g}). \tag{1.41}$$

We describe the kinetic energy of the body first. For a point x initially within the body at $t = 0$ its position at time t is $r(t)x + d(t)$ where $g(t) = (r(t), d(t))$ is a curve in $SE(n)$. The velocity is obtained by differentiating this expression and hence one may write

$$\begin{aligned} KE_B(\dot{g}) &= \frac{1}{2}\int_B \rho |\dot{r}x + \dot{d}|^2 \, dx \\ &= \frac{1}{2}\int_B \rho |\dot{r}x|^2 \, dx + M(\dot{r}c_0 \cdot \dot{d}) + \frac{M}{2}|\dot{d}|^2. \end{aligned}$$

Here we have introduced the total mass $M = \int_B \rho dx$ of the body and its initial centre of mass $c_0 = M^{-1}\int_B \rho x dx$. One may show that the left-invariant body-velocity vector $g(t)^{-1}\dot{g}(t)$ obtained by left translation to the identity is

$$(\omega, v) = \left(r(t)^{-1}\dot{r}(t), r(t)^{-1}\dot{d}(t) \right).$$

To remove some clutter we are dropping the letter ‘b’ and denoting the *body angular-velocity* by ω and the *body linear-velocity* by v. The kinetic energy of the body may now be written as

$$\frac{1}{2}\rho \int_B |\omega x|^2 \, dx + M(\omega c_0 \cdot v) + \frac{M}{2}|v|^2 \tag{1.42}$$

which manifestly defines a left-invariant metric on $SE(n)$. This Hamiltonian may be simplified by choosing the initial position of the body to be such that its centre of mass coincides with the origin. In this way the vector c_0 becomes zero and the second term above vanishes. This decision to alter the initial position of the body deserves more comment. The following discussion concerning the representation of $\mathrm{Sym}^2\mathfrak{g}^*$ might seem unnecessarily complicated compared to our reasoning to simply move the origin, however, the material developed here is interesting and will be useful later on.

Imagine that we fix a frame in $\mathbb{R}^n$ and define this to be both the fixed space frame and the initial body frame, but leave the initial position of the body undetermined. As soon as we choose an initial position of the body we determine the location of the body frame fixed relative to the body. Note that the origin of the body frame does not need to be inside the body. Suppose that one choice of initial body position is given by $B \subset \mathbb{R}^n$. The configuration space of the body is then identified with transformations $g\colon B \hookrightarrow \mathbb{R}^n$ for $g \in SE(n)$. If B' is an alternative initial position of the body then there exists a unique $a \in SE(n)$ with $B = aB'$. The configuration $g\colon B \hookrightarrow \mathbb{R}^n$ corresponds to the configuration ga relative to B'. Therefore, altering the initial body position corresponds to the transformation $\mathcal{R}_a$ on the group. As right and left multiplications commute, this pulls back the left-invariant metric to another left-invariant metric. If we evaluate this at the identity we have

$$\begin{aligned}(\mathcal{R}^*_{a^{-1}}B)_e(X,Y) &= B_{a^{-1}}\left((\mathcal{R}_{a^{-1}})_*X, (\mathcal{R}_{a^{-1}})_*Y\right)\\ &= B_e(\mathrm{Ad}_a X, \mathrm{Ad}_a Y)\end{aligned}$$

where in the last line we have used the left-invariance of B and $\mathrm{Ad}_a = (\mathcal{L}_a \circ \mathcal{R}_a)_*$. As the metric B is determined by a map $\mathbb{I}\colon \mathfrak{se}(n) \to \mathfrak{se}(n)^*$ as in (1.35), we see that the new metric is defined by the altered inertia operator

$$\mathrm{Ad}^*_{a^{-1}} \circ \mathbb{I} \circ \mathrm{Ad}_a\,. \tag{1.43}$$

For a any Lie group G this action is the canonical representation of G on the space $\mathrm{Sym}^2\mathfrak{g}^*$ of symmetric bilinear forms on $\mathfrak{g}$, or equivalently, of left-invariant symmetric bilinear forms on G. In summary, the effect of changing the initial body position corresponds to a right-multiplication on the configuration group, and that the resulting effect on the metric $\mathbb{I}$ is via the standard representation on $\mathrm{Sym}^2\mathfrak{g}^*$.

This is all a very long way of saying what was said earlier: that we may eliminate the term involving c_0 by changing the initial body position so that its centre of mass is at the origin. The quadratic form on $\mathfrak{se}(n) = \mathfrak{so}(n) \times \mathbb{R}^n$ in (1.42) contains a cross-term coupling the $\mathfrak{so}(n)$ and $\mathbb{R}^n$ components for when c_0 is non-zero. Equivalently then, we can state that the representation of $SE(n)$ on the space of such quadratic forms removes this coupling and separates the form into two independent forms on $\mathfrak{so}(n)$ and $\mathbb{R}^n$ corresponding to the rotational energy of the body and the energy from its linear velocity, respectively. It is because of this decoupling that the motion decomposes into the independent motions of the body about its centre of mass, and the centre of mass itself. For when we introduce new terms into the quadratic form corresponding to the energy of the fluid the representation of $SE(n)$ cannot separate the form into angular and linear components. It is because of this irremovable coupling that linear motions may induce rotational motions and vice versa. This however should not be surprising as it is precisely due to this coupling that a rotating propeller propels a ship and that wind though a turbine induces it to spin.

We now turn our attention to the kinetic energy of the fluid. To avoid the misery of having to talk about Hodge stars and using the metric to alternate between forms and vectors, we will focus on the case $n = 3$ so that we may use the familiar operators of vector calculus to describe the fluid dynamics.

The motion of the fluid at time t is specified by a velocity vector field u defined on the complement $B_t^c \subset \mathbb{R}^3$ of the body's position. We shall suppose that the fluid flow is potential and incompressible. Therefore, there exists a function ψ defined on B_t^c with $\nabla\psi = u$ and $\nabla^2\psi = 0$. The fluid must always surround the body, it cannot enter it (penetration), and it cannot separate from the surface of the body creating a vacuum (cavitation). Let x be a point on the boundary of the body and n_x the normal vector. If v is the velocity of the point x then relative to this point the relative fluid velocity is $u - v$. The no penetration/cavitation condition is then expressed as

$$(n - v) \cdot n_x = 0$$

or equivalently as

$$\nabla\psi \cdot n_x = v \cdot n_x. \tag{1.44}$$

Let x_0 denote the initial position of the point on the boundary at $t = 0$. The transformation $g \in SE(3)$ preserves the standard inner product on $\mathbb{R}^3$ and so $\nabla\psi(gx_0) = g\nabla\psi(x_0)$. Furthermore, as $n_x = gn_{x_0}$ the left hand side in the expression above may be written in terms of the initial body position x_0 as

$$\nabla\psi(x_0) \cdot n_{x_0}.$$

Similarly, if we consider the right hand side in (1.44) we may write the velocity $v = \dot{x}(t) = \dot{r}(t)x_0 + \dot{d}(t)$ of the point on the boundary as $(r, d) \cdot (\omega x_0 + v)$ where (ω, v) are the left-invariant body-velocity vectors as before. We may therefore write $v \cdot n_x$ as

$$(\omega x_0 + v) \cdot n_{x_0}.$$

Since this expression is linear in (ω, v) there exists a map $\beta \colon \partial B \to \mathfrak{se}(3)^*$ defined on the boundary of the body given in its initial configuration which satisfies

$$\langle \beta(x_0), (\omega, v) \rangle = (\omega x_0 + v) \cdot n_{x_0} \tag{1.45}$$

for all $(\omega, v) \in \mathfrak{se}(3)$. Equation (1.44) defines a Neumann initial value problem on B^c. It is known that the solution ψ is unique up to a constant. If we suppose that $|\nabla\psi|$ decays sufficiently quickly at infinity, then we may suppose that the integral of ψ taken over B^c is finite, and hence may solve for ψ uniquely by insisting that this integral is zero. Furthermore, as the right hand side in (1.44) is linear in (ω, v) it follows that there exists a unique map $\Psi \colon \partial B \to \mathfrak{se}(3)^*$ which satisfies

$$\langle \Psi(x_0), (\omega, v) \rangle = \psi(x_0) \tag{1.46}$$

for all $(\omega, v) \in \mathfrak{se}(3)$ and where ψ is the unique solution to (1.44) for the initial configuration of the body.

The total kinetic energy of the fluid is given by

$$KE_F(\dot{g}) = \frac{1}{2}\int_{B_t^c} |u|^2 \; dV = \frac{1}{2}\int_{B_t^c} |\nabla\psi(x)|^2 \; dV.$$

Using the fact that $\nabla\psi(gx_0) = g\nabla\psi(x_0)$ we may rewrite this as an integral over the complement of body in its initial configuration and then use the divergence theorem to write this as an integral over its boundary

$$\frac{1}{2}\int_{B_t^c} |\nabla\psi(x_0)|^2 \; dV = \frac{1}{2}\int_{\partial B} \psi(x_0)\nabla\psi(x_0)\cdot n_{x_0} \; dA.$$

From (1.44) and the definitions of β and Ψ we thus have

$$KE_F(\dot{g}) = \frac{1}{2}\int_{\partial B} \langle \Psi(x_0), (\omega, v)\rangle \langle \beta(x_0), (\omega, v)\rangle \; dA.$$

This expression only depends on the left-invariant velocity vectors (ω, v) and clearly defines a quadratic form on $\mathfrak{se}(3)$. By combining the kinetic energy of the body and fluid in (1.41) we obtain a left-invariant metric on $SE(3)$. The reduced equations on the coalgebra were first found by Kirchhoff in [KH83].

Theorem 1.4.2 (Kirchhoff equations)**.** *Consider a non-degenerate quadratic form on* $(\omega, v) \in \mathfrak{se}(3) \cong \mathbb{R}^3 \times \mathbb{R}^3$ *given by*

$$\frac{1}{2}(\omega \cdot A\omega) + (\omega \cdot Bv) + \frac{1}{2}(v \cdot Cv)$$

where A, B *and* C *are symmetric* 3×3 *matrices. Let* $\mathbb{I}\colon \mathfrak{se}(3) \to \mathfrak{se}(3)^*$ *denote the corresponding non-degenerate symmetric tensor in* $Sym^2\mathfrak{se}(3)^*$*. For the Hamiltonian*

$$H(L, p) = \frac{1}{2}\langle (L, p), \mathbb{I}^{-1}(L, p)\rangle$$

on $\mathfrak{se}(3)^*$ *the equations of motion are given by*

$$\begin{aligned} \dot{L} &= L \times \omega + p \times v \\ \dot{p} &= p \times \omega \end{aligned}$$

where $L = A\omega + Bv$ *and* $p = B\omega + Cv$*.*

Proof. This is a straightforward implementation of Theorem 1.4.1. We use the expression for the coadjoint action of a semidirect product in (1.26) and substitute it into (1.39). It is assumed that we are identifying $\mathfrak{se}(3)$ with its dual as in Example 1.3.3. □

Example 1.4.3 (The motion of a rigid body in a two-dimensional fluid)**.** The case for when $n = 2$ is peculiar for a few reasons. Firstly, as the vector calculus on $\mathbb{R}^2$ is different to that on $\mathbb{R}^3$ the solution ψ is unique up to a choice of circulation of the fluid around the body [Lam93, Cha56]. Secondly, as we will now show, unlike the case for $n > 2$, the representation of $SE(2)$ is able to diagonalise any inner product on $\mathfrak{se}(2)$. The angular and linear velocities can therefore always be decoupled by choosing an appropriate body frame. In a sense, one can imagine that this statement shows that there cannot exist propellers in a two-dimensional fluid.

Write elements in $\mathfrak{se}(2) = \mathfrak{so}(2) \times \mathbb{R}^2$ as vectors (ω, v) and identify $\mathfrak{se}(2)$ with its dual as in (1.6). Any quadratic form on $\mathfrak{se}(2)$ determined by a map $\mathbb{I}\colon \mathfrak{se}(2) \to \mathfrak{se}(2)^*$ may then be identified with a symmetric matrix

$$\mathbb{I} = \begin{pmatrix} a & b^T \\ b & C \end{pmatrix}$$

for $a \in \mathbb{R}$, $b \in \mathbb{R}^2$, and C a 2×2 symmetric matrix. Using the expressions in (1.3) and (1.7) one can write the action of $(r, d) \in SE(2)$ on $\mathbb{I}$ given in (1.43) using matrix multiplication

$$\underbrace{\begin{pmatrix} 1 & d^T r \mathbb{J} r \\ 0 & r^T \end{pmatrix}}_{\mathrm{Ad}^*_{(r,d)^{-1}}} \underbrace{\begin{pmatrix} a & b^T \\ b & C \end{pmatrix}}_{\mathbb{I}} \underbrace{\begin{pmatrix} 1 & 0 \\ -\mathbb{J} d & r \end{pmatrix}}_{\mathrm{Ad}_{(r,d)}} = \begin{pmatrix} a + d^T \mathbb{J} b - d^T \mathbb{J} C \mathbb{J} d & b^T r + d^T \mathbb{J} C r \\ r^T b - r^T C \mathbb{J} d & r^T C r \end{pmatrix} \tag{1.47}$$

As the form $\mathbb{I}$ is non-degenerate, so too is C. Therefore there exists a unique $d \in \mathbb{R}^2$ with

$$b = C \mathbb{J} d.$$

It follows from (1.47) that transforming the initial body frame by (I, d) removes the coupling term b. Furthermore, we can also act by an appropriate r to diagonalise C and put $\mathbb{I}$ into a purely diagonal form. We have thus shown that for an appropriate choice of body frame the inertia tensor is diagonal, say

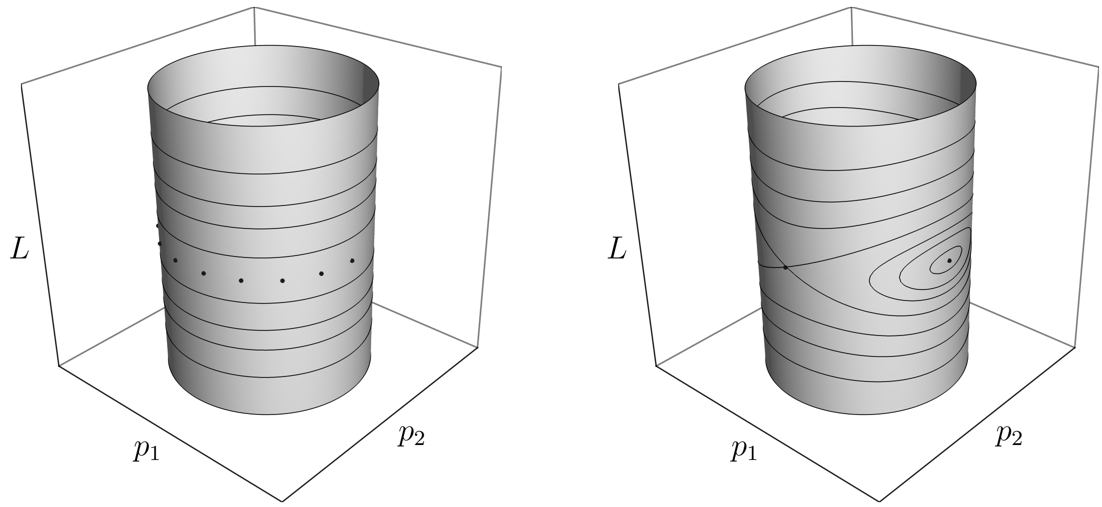


Figure 1.6: Solution curves in $\mathfrak{se}(2)^*$ for a two-dimensional rigid body. On the left we have a free rigid body and on the right a rigid body immersed in a fluid. Notice that for the free rigid body with zero angular momentum the fixed point solutions correspond to the body moving uniformly along a straight line without rotating. On the other hand, for the body in a fluid there are only two principal directions along which the body moves in this way, one stable, and one unstable. Any linear motion not aligned along these principal directions induces the body to spin.

$\mathbb{I} = \mathrm{diag}(\Omega, m_1, m_2)$. From (1.36) the Hamiltonian on $\mathfrak{se}(2)^*$ is

$$H(L, p_1, p_2) = \frac{1}{2}\left(\frac{L^2}{\Omega} + \frac{p_1^2}{m_1} + \frac{p_2^2}{m_2}\right).$$

The intersection of the Coadjoint orbits with these ellipsoids of constant H give the solution curves in $\mathfrak{se}(2)^*$ and are illustrated in Figure 1.6. We also include the degenerate case for $m_1 = m_2$ which describes the situation for when the fluid is absent. Differentiating (1.7) to find the coadjoint action and substituting this into the Euler-Arnold equations in (1.39) yields the equations of motion

$$\begin{aligned}
\dot{L} &= p_1 p_2 (m_2^{-1} - m_1^{-1}) \\
\dot{p}_1 &= +L p_2/\Omega \\
\dot{p}_2 &= -L p_1/\Omega
\end{aligned}$$

1.4.5 The heavy top

We now return to consider a rigid body in $\mathbb{R}^n$ with a fixed point at the origin but shall no longer assume that the body is free. Instead, we shall suppose that the body moves under the influence of a constant gravitational field. We shall take the acceleration due to gravity to be the vector $-\gamma v_0$ for $\gamma > 0$ a constant and v_0 a fixed unit-vector in $\mathbb{R}^n$ which we shall call the vertical. The configuration space is once again $SO(n)$ and the problem defines a mechanical system on the tangent bundle with Lagrangian

$$L(\dot{r}) = \frac{1}{2}\int_B \rho|\dot{r}(t)x|^2 \ dx - \gamma \int_B \rho(r(t)x \cdot v_0) \ dx$$

where we are taking the integral over the initial body configuration B. We may rewrite this as

$$\frac{1}{2}\int_B \rho|\omega x|^2 \ dx - \gamma M(c_0 \cdot r(t)^{-1}v_0)$$

where ω is an abbreviation for the body angular velocity ω_b from earlier. This Lagrangian coincides with that for the free rigid body in (1.40) except for the final term involving $r(t)$. It is due to the presence of this term that the Lagrangian is no longer left-invariant and now depends additionally on the configuration $r(t)$ in the group. The tangent vector F_w involved in the definition of the Legendre transform in (1.31) is tangent to the fibres of the tangent bundle. The Legendre transform is therefore unaffected when the Lagrangian is altered by a function constant along these fibres. Consequently, the Legendre transform $\mathbb{I}$ on $TSO(n)$ for the heavy top is identical to that for the free rigid body as before. The resulting Hamiltonian on the cotangent bundle is then

$$H(\eta) = \frac{1}{2}\langle \eta, \mathbb{I}^{-1}(\eta)\rangle + \gamma M(c_0 \cdot r^{-1}v_0) \tag{1.48}$$

for $\eta \in T_r^*SO(n)$.

Although this Hamiltonian is no longer left-invariant it is invariant under the left-multiplication of the subgroup $SO(n)_{v_0} \cong SO(n-1)$. We are therefore in a position to apply the Semidirect Product Reduction by Stages Theorem 1.3.4. To apply the theorem we take H to be $SO(n)$ and V to be the standard representation on $\mathbb{R}^n$. As this representation is self dual we shall proceed to identify V with V^*. We then take p to be the unit vertical v_0 and hence, H_p is $SO(n)_{v_0}$. Implementing the theorem shows that the map sending $\eta \in T_r^* SO(n)$ to the element

$$(\mathcal{L}_r^* \eta, r^{-1} v_0)$$

in $\mathfrak{se}(n)^*$ is the orbit quotient $T^* SO(n)/SO(n)_{v_0}$. We recognise $\mathcal{L}_r^* \eta$ to be equal to the body angular momentum L and $r^{-1} v_0$ to be the vertical vector viewed from within the body frame, which we shall denote by p. The Hamiltonian in (1.48) descends through this quotient to

$$H(L, p) = \frac{1}{2}\langle L, \omega\rangle + \gamma M(c_0 \cdot p). \tag{1.49}$$

We may use the expression in (1.28) for the Poisson structure on a semidirect product to find Hamilton's equations of motion. To do this we must first find the directional derivatives

$$\frac{\delta H}{\delta L} = \omega \quad \text{and} \quad \frac{\delta H}{\delta p} = \gamma M c_0.$$

It is then a straightforward calculation to derive the following reduced equations in $\mathfrak{se}(n)^*$ for the heavy top in $\mathbb{R}^n$

$$\begin{aligned} \dot{L} &= -\operatorname{ad}^*_\omega L - \gamma M \mu(p, c_0) \\ \dot{p} &= -\omega p \end{aligned} \tag{1.50}$$

Example 1.4.4. [The heavy top in three dimensions] Upon identifying $\mathfrak{se}(3)^*$ with $\mathbb{R}^3 \times \mathbb{R}^3$ the set of equations in (1.50) becomes

$$\begin{aligned} \dot{L} &= L \times \omega + M\gamma(p \times c_0) \\ \dot{p} &= p \times \omega \end{aligned} \tag{1.51}$$

Recall that L is the body angular momentum and p is the location of the vertical viewed from within the body frame. From Example 1.3.3 the Coadjoint orbits are the level sets of the functions $|p|^2$ and $L \cdot p$. These functions are therefore constants of motion which, as we now explain, correspond to classical conserved quantities. If we resurrect our old notation from (1.38) we have $L_b = \operatorname{Ad}^*_{r^{-1}} L_s \equiv r^{-1} L_s$. The conserved quantity $L \cdot p = (r^{-1} L_s) \cdot (r^{-1} v_0)$ therefore corresponds to $L_s \cdot v_0$, the component of angular momentum along the vertical in the space frame. The conserved quantity $|p|^2 = |v_0|^2$ is equal to 1 since we assumed the vertical vector to be normal.

1.4.6 The Lagrange top

The Lagrange top is the three-dimensional heavy top introduced in the last subsection but with an additional symmetry. Suppose there exists an axis within the body which passes through the origin about which the body is invariant with respect to rotations. This additional symmetry corresponds to the action of an $SO(2)$-subgroup acting on $SO(3)$ by right-multiplication. This extra symmetry occurs when two of the diagonal entries in the inertia matrix are equal. Therefore, without any loss of generality we shall suppose that $\mathbb{I} = \mathrm{diag}(1, 1, I_3)$. Furthermore, we shall write $\{e_x, e_y, e_z\}$ as the standard basis on $\mathbb{R}^3$ and suppose that the vertical vector is $v = e_z$. We shall also suppose that the initial centre of mass vector c_0 is aligned along the vertical $c_0 = |c_0|e_z$.

As we are identifying $\mathfrak{so}(3) \cong \mathfrak{so}(3)^*$ with $\mathbb{R}^3$ we may write the body angular momentum as the vector $L = (L_1, L_2, L_3)$ and the body angular velocity as $\omega = (\omega_1, \omega_2, \omega_3)$. They are related to each other by $L = \mathbb{I}\omega$. The Hamiltonian (1.49) on $\mathfrak{se}(3)^* \cong \mathbb{R}^3 \times \mathbb{R}^3$ becomes

$$H(L, p) = \frac{1}{2}\left(L_1^2 + L_2^2 + \frac{L_3^2}{I_3}\right) + M|c_0|\gamma p_3.$$

Recall the Coadjoint action given in (1.27) and notice that this Hamiltonian is invariant under the action of the subgroup $SO(3)_{e_z} \cong SO(2)$. This extra $SO(2)$-symmetry is descended from the original right $SO(2)$-symmetry on $T^*SO(3)$.

Combining the results from Examples 1.2.1 and 1.2.2, the momentum map $\mu\colon \mathfrak{se}(3)^* \to \mathfrak{so}(2)^* \cong \mathbb{R}$ for the action of $SO(2)$ on the Coadjoint orbits of $\mathfrak{se}(3)^*$ projects (L, p) onto the component $-L_3$. Since this action is not free we do not expect the quotient $\mathfrak{se}(3)^*/SO(2)$ to be a smooth manifold, and so we realise this quotient via the Hilbert map as before in Example 1.2.4 by taking generators of the $SO(2)$-invariant ring on $\mathfrak{se}(3)^*$. This ring has 6 generators

$$L_3,\ p_3,\ \underbrace{L_1^2 + L_2^2}_{k_{11}},\ \underbrace{L_1p_2 + L_2p_2}_{k_{12}},\ \underbrace{p_1^2 + p_2^2}_{k_{22}},\ \underbrace{L_1p_2 - L_2p_1}_{\delta}.$$

which satisfy the algebraic relation $\delta^2 = k_{11}k_{22} - k_{12}^2$. We may therefore identify

$\mathfrak{se}(3)^*/SO(2)$ with the 5-dimensional semialgebraic variety of points

$$(L_3, P_3, k_{11}, k_{12}, k_{22}, \delta) \in \mathbb{R}^6$$

which satisfy $\delta^2 = k_{11}k_{22} - k_{12}^2$ along with the inequalities $k_{11}, k_{22} \geq 0$. In this reduced space the Hamiltonian descends to

$$H = \frac{1}{2}\left(k_{11} + \frac{L_3^2}{I_3}\right) + M|c_0|\gamma p_3. \tag{1.52}$$

As we mentioned in Example 1.4.4, the functions $L \cdot p$ and $|p|^2$ are constant along any Hamiltonian flow in $\mathfrak{se}(3)^*$. By Noether's theorem the momentum L_3 is also constant along the flow of any $SO(2)$-invariant function. It follows that the symplectic leaves in this reduced space are contained to the connected components defined by the level sets of three Casimirs:

angular momentum about the body axis,	L_3
angular momentum about the vertical,	$L_z = k_{12} + L_3 p_3$
modulus of p,	$k_{22} + p_3^2$

The modulus of p is set to 1 since v_0 is a unit vector. The reduced spaces are then parametrised by selecting values for the angular momentum L_3 about the body axis, and L_z about the vertical. For such a selection we may eliminate the variables k_{22} and k_{12} and write the resulting space as the variety

$$\delta^2 = k_{11}(1 - p_3^2) - (L_z - L_3 p_3)^2 \tag{1.53}$$

for fixed values of L_z and L_3.

The reduced spaces $\mu^{-1}(\mathcal{O})/G$ are smooth symplectic manifolds when the action of G on $\mu^{-1}(\mathcal{O})$ is free. Otherwise, we expect a stratified space with symplectic strata. The action of $SO(2)$ on $(L, p) \in \mathfrak{se}(3)^*$ is free everywhere except for when L and p are both colinear to the e_3-axis. In this case one sees from the definitions above that this must imply $L_z = \pm L_3$. Therefore, for when $L_z \neq \pm L_3$ we expect smooth 2-dimensional manifolds, and indeed, in such instances the variety given in (1.53) is the graph of the function

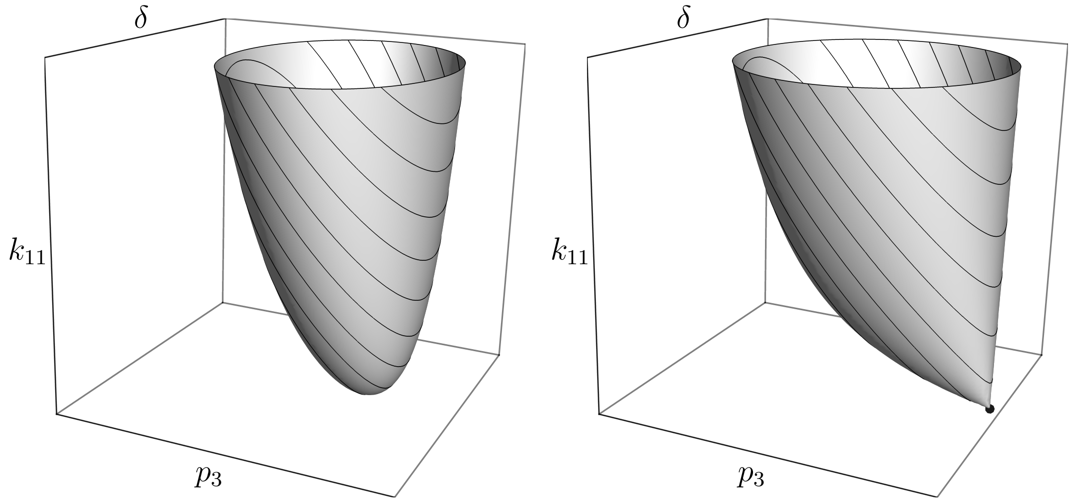
$$(\delta, p_3) \longmapsto k_{11} = \frac{\delta^2 + (L_z - L_3 p_3)^2}{1 - p_3^2}$$

which is diffeomorphic to $\mathbb{R}^2$ and illustrated in Figure 1.7(a). For $L_z = +L_3 \neq 0$ we obtain a singular variety as shown in Figure 1.7(b), and similarly in Figure 1.7(c) we have the corresponding picture for $L_z = -L_3 \neq 0$. Observe that these stratified spaces both consist of the the point-strata at $p_3 = \pm 1$ corresponding to the non-free situation we described earlier where L and p are non-zero and colinear to the e_3-axis. In the most singular situation possible we have $L = 0$ and consequently $L_3 = L_z = 0$. The corresponding variety is given in Figure 1.7(d). Once again, the singular stratum are the two points $p_3 = \pm 1$.

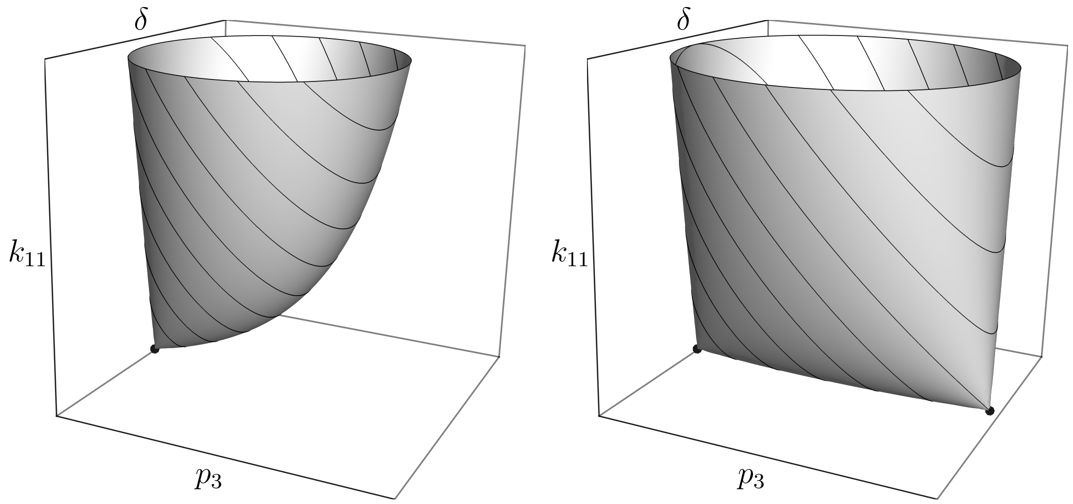
Finally, by substituting the equations of motion from (1.51) into the definitions for the algebraic invariants, we may derive the equations of motion on this full reduced space.

$$
\begin{aligned}
\dot{L}_3 &= 0 & \dot{k}_{11} &= 2M|c_0|\gamma\delta \\
\dot{p}_3 &= -\delta & \dot{k}_{12} &= L_3\delta \\
\dot{\delta} &= M|c_0|\gamma k_{22} + p_3 k_{11} - L_3 k_{12} & \dot{k}_{22} &= 2p_3\delta
\end{aligned}
\tag{1.54}
$$

One may use these equations to verify that the three Casimirs and algebraic relation are constant, as expected.



(a) $L_3 \neq L_z$ (b) $L_3 = +L_z \neq 0$



(c) $L_3 = -L_z \neq 0$ (d) $L_3 = L_z = 0$

Figure 1.7: Reduced spaces for the Lagrange top with respect to the left and right $SO(2)$-symmetry about the vertical and body axes. The reduced spaces are parametrised by the angular momentum of the body about its body axis L_3, and the angular momentum in space about the vertical L_z. The contours are the level sets of the Hamiltonian H given in (1.52).

1.5 The next two chapters

In the next chapter we will use the expressions in (1.22) and (1.25) for the Adjoint and Coadjoint actions of a semidirect product to classify the orbits. From hereon we will no longer use the upper and lowercase letters to distinguish between the group action and its infinitesimal action as it shall be clear from the context. From the example of $SE(2)$ in Figure 1.2 we saw that although the orbits are different, there appears to be a bijection between them which also preserves the homotopy type. In [AM18] together with J. Montaldi we prove that this property holds in $SE(n)$ for all n. In the next chapter we obtain a stronger result by showing that it holds for a much larger class of semidirect products, and exhibit the Poincaré group as a particular example.

The final chapter has much more in common with the previous background chapter. We will connect with the material developed earlier concerning the heavy top to show that the phase space for two bodies on the 3-sphere is a double cover over the phase space of a 4-dimensional heavy top. We can therefore apply the Semidirect Product Reduction by Stages Theorem 1.3.4 just as we did for the heavy top and show that the reduced spaces are coadjoint orbits of $SE(4)$. Analogously to the ordinary 3-dimensional Lagrange top, we then reduce the space $\mathfrak{se}(4)^*$ once again by the action of $SO(3)$. In exactly the same way as before, we shall use the method of invariants to express these reduced spaces as semialgebraic varieties and explicitly exhibit the reduced spaces and the equations of motion.

Chapter 2

A Bijection Between the Adjoint and Coadjoint Orbits of a Semidirect Product

We prove that there exists a geometric bijection between the sets of adjoint and coadjoint orbits of a semidirect product, provided a similar bijection holds for particular subgroups. We also show that under certain conditions the homotopy types of any two orbits in bijection with each other are the same. We apply our theory to the examples of the affine group and the Poincaré group, and discuss the limitations and extent of this result to other groups.

2.1 Background and outline

For a reductive Lie algebra the adjoint and coadjoint representations are isomorphic; consequently, the orbits are identical. In [BC77] a method is devised to obtain normal forms for the adjoint orbits of any semisimple Lie algebra, real or complex. These methods are then extended in [CVDK06a] and applied to the Poincaré group. The Poincaré group is an example of a non-reductive group, and consequently there is no reason, in general, to expect any relation between the adjoint and coadjoint representations. Nevertheless, in [CVDK06a] a "curious bijection" is found between the normal forms of both representations.

Before proceeding any further with the details, it might be pertinent to exhibit a hands-on example of what we mean by such a bijection. In Figure 2.1 we illustrate the orbits of the group Aff(1) of affine isomorphisms of the real line.

The adjoint and coadjoint representations are not isomorphic, indeed, the orbits are different; and yet, there seems to be some sort of bijection between the two. In our work in [AM18] we explore in detail the orbits of the Euclidean group and prove such a bijection result. However, for other semidirect products, such as the Poincaré group, our methods no longer apply. Thus, the purpose of this chapter is to prove a bijection result for a wider class of semidirect products.

The study of coadjoint orbits, particularly those of a semidirect product, is a large and venerable subject; and one which we will mostly be able to sidestep. For a greater insight into the physical significance and applications of this study, consult [Bag98, MMO$^+$07, CVDK06b, MRW84a, GS90], to name but a few.

In the first section we prove our central result: that their exists a geometric bijection between the sets of adjoint and coadjoint orbits of a semidirect product provided a similar bijection holds for particular subgroups. To be precise, for a semidirect product $G = H \ltimes V$ the particular subgroups in question are: the Wigner little groups $H_p \subset H$ which are the stabilisers of a vector $p \in V^*$; and the centralizer subgroups $H_\omega \subset H$ which are the stabilisers of an element ω in the Lie algebra of H. We prove that the bijection result holds if: there is a bijection between the sets of adjoint and coadjoint orbits of the groups H_p; and a bijection between the sets of orbits of H_ω with respect to a particular representation and its contragredient. Thus, the task of establishing an orbit bijection is reduced to a similar task, albeit for a collection of 'smaller' groups.

In the second and third sections we demonstrate this bijection for the examples of the affine group of isomorphims of affine space, and the Poincaré group of affine linear maps preserving Minkowski space. The methods used in both examples are the same, however the exposition is more straightforward for the affine group. Therefore, the affine group is presented first and the Poincaré group second, following closely the template laid out by the affine group's example. In both cases the hardest part of the proof is proving the bijection result for the centralizer subgroups.

Our fourth section demonstrates a method for proving that two orbits in bijection with each other are homotopy equivalent. This relies on showing that the bijected orbits corresponding to the little subgroups and centralizer group orbits satisfy a property which we call being *zigzag related*. This is an equivalence relation defined on homogeneous spaces which is stronger than that of being homotopy equivalent. Using this method we show that an adjoint and coadjoint orbit of the Poincaré group corresponding under the bijection are homotopy equivalent to each other.

We end with some remarks concerning the wider applicability of our methods.

2.2 A bijection between orbits

2.2.1 The coadjoint orbits

Let H be some Lie group, V a representation of this group, and G the semidirect product $H \ltimes V$. The dual $\mathfrak{g}^*$ is canonically isomorphic to $\mathfrak{h}^* \times V^*$ and the coadjoint action given by [Raw75]

$$\operatorname{Ad}^*_{(r,d)}(L,p) = (\operatorname{Ad}^*_r L + \mu(r^*p, d), r^*p)\,. \tag{2.1}$$

Here $(r,d) \in G$ and $(L,p) \in \mathfrak{h}^* \times V^* = \mathfrak{g}^*$. The map $\mu\colon V^* \times V \longrightarrow \mathfrak{h}^*$ is defined by

$$\langle \mu(p,v), \omega \rangle = \langle p, \omega v \rangle$$

for all $\omega \in \mathfrak{h}$. The subgroup $H_p = \{r \mid r^*p = p\}$ is referred to in the literature as the *little group*. The Lie algebra $\mathfrak{h}_p$ of this group satisfies $\mu(p,V) = \mathfrak{h}_p^\circ$, where $\mathfrak{h}_p^\circ$ denotes the annihilator of the subalgebra. We now reproduce the result given in [Raw75] which effectively parametrises the orbits in $\mathfrak{g}^*$ by an orbit, say through p in V^*, together with a *little-group orbit* in $\mathfrak{h}_p^*$.

Consider the set

$$\Pi = \left\{(l,p) \mid l \in \mathfrak{h}_p^*,\ p \in V^*\right\}, \tag{2.2}$$

and a coadjoint orbit $\mathcal{O}^*$ in $\mathfrak{g}^*$. There is a map $\mathfrak{g}^* \longrightarrow \Pi$ given by sending (L,p) to $(\iota_p^* L, p)$, where ι_p^* is the canonical projection of $\mathfrak{h}^*$ onto $\mathfrak{h}_p^*$. Let Y denote the image of $\mathcal{O}^*$ under this map. We may define an action of H on Π by setting $r(l,p) = (rl, r^*p)$, where rl is the form in $\mathfrak{h}_{rp}^*$ given by satisfying

$$\langle rl, \omega \rangle = \langle l, \operatorname{Ad}_{r^{-1}} \omega \rangle$$

for all $\omega \in \mathfrak{h}_{rp}$. Since $\iota_p^*\colon \mathfrak{h}^* \longrightarrow \mathfrak{h}_p^*$ commutes with the action of H_p and has kernel $\ker \iota_p^* = \mathfrak{h}_p^\circ$, the space Y is an orbit of H in Π, and the map $\mathcal{O}^* \longrightarrow Y$ becomes an H-equivariant bundle with affine fibres $\mathfrak{h}_p^\circ$. The space Y is also an H-equivariant bundle over an orbit in V^* given by projecting (l,p) onto the second factor. The fibre of this projection above p is the coadjoint orbit through $l \in \mathfrak{h}_p^*$, the so-called *little-group orbit*. Conversely, given an orbit Y in Π there exists a coadjoint orbit $\mathcal{O}^* \subset \mathfrak{g}^*$ which is mapped to Y. For $(l, p) \in Y$ the corresponding coadjoint orbit is that through (L, p), where L is any element with $\iota_p^* L = l$. In the literature the orbits of Π are referred to as *bundles of little-group orbits*.

Theorem 2.2.1 ([Raw75])**.** *There is a bijection between the set of coadjoint orbits of G and the set of orbits of Π. Given an orbit $\mathcal{O}^*$ and corresponding bundle Y, there is a G-equivariant affine bundle $\mathcal{O}^* \to Y$.*

2.2.2 The adjoint orbits

We can adapt the bundle-of-little-group-orbits construction to the adjoint action, and obtain an analogous classification of orbits in $\mathfrak{g}$. For $(\omega, v) \in \mathfrak{g}$ the adjoint action is [GS90, Section 19]

$$\mathrm{Ad}_{(r,d)}(\omega, v) = (\mathrm{Ad}_r\, \omega, rv - (\mathrm{Ad}_r\, \omega)\, d)\,. \tag{2.3}$$

The isotropy subgroup $H_\omega = \{r \mid \mathrm{Ad}_r\, \omega = \omega\}$ is called the *centralizer group.*

Lemma 2.2.2. *There is a canonical isomorphism between the quotient space $V/\operatorname{Im}\omega$ and $(\ker \omega^*)^*$ (here ω^* denotes the adjoint of the linear map ω). Furthermore, this is an intertwining map for the representations of H_ω on these spaces.*

Proof. The result follows by dualizing the exact sequence, $\ker \omega^* \hookrightarrow V^* \xrightarrow{\omega^*} V^*$, whose arrows all commute with H_ω. □

Consider the set

$$\Sigma = \{(\omega, x) \mid \omega \in \mathfrak{h},\ x \in (\ker \omega^*)^*\}\,. \tag{2.4}$$

There is a map $\mathfrak{g} \longrightarrow \Sigma$ which sends (ω, v) to (ω, x), where x is the element mapped from $[v] \in V/\operatorname{Im}\omega$ under the isomorphism in Lemma 2.2.2. Now let $\mathcal{O}$ be an adjoint orbit and let X denote the image of this orbit under the map into Σ. In the same way as we did for the coadjoint orbits, we can equip Σ with an H-action and establish $\mathcal{O} \longrightarrow X$ as an H-equivariant bundle with fibres $\operatorname{Im}\omega$. Specifically, we define $r \cdot (\omega, x)$ to be $(\mathrm{Ad}_r\, \omega, r \cdot x)$ where $r \cdot x$ is the element in $(\ker \mathrm{Ad}_r\, \omega^*)^* = (r \ker \omega^*)^*$ which satisfies

$$\langle r \cdot x, p\rangle = \langle x, r^{-1} p\rangle$$

for all $p \in \ker \mathrm{Ad}_r \, \omega^* = r \ker \omega^*$. The space X is itself an H-equivariant bundle over an adjoint orbit through, say ω in $\mathfrak{h}$, with fibre above ω equal to an orbit of H_ω in $(\ker \omega^*)^*$, what we shall call a *centralizer group orbit*. In the same way that Theorem 2.2.1 is proven, we can establish an analogous theorem.

Theorem 2.2.3. *There is a bijection between the set of adjoint orbits of G and the set of orbits of Σ. Given an orbit $\mathcal{O}$ and corresponding bundle X, there is a G-equivariant affine bundle $\mathcal{O} \to X$.*

2.2.3 Constructing the bijection

Suppose we have an action of a group G on X, and of H on Y. We will not give a precise meaning to the existence of a 'geometric orbit bijection' between the two actions. For us an *orbit bijection* will merely mean a bijection between the sets of orbits in X with orbits in Y. This is a weak notion and does not capture the 'geometric' sense of an orbit bijection as that given in the example from Figure 2.1. To justify more rigorously what a 'geometric' orbit bijection might mean would be a digressive and ultimately unnecessary exercise; the geometric nature of our bijections (whatever that may mean) will be clear from the construction we now give and from the examples to follow.

As we have seen, there is a bijection between the set of adjoint orbits of G with the set of H-orbits in Σ, and a bijection between the coadjoint orbits of G with the H-orbits in Π. Our strategy for showing an adjoint and coadjoint orbit bijection will be to exhibit a space Δ equipped with an action of H for which there is an orbit bijection with both Π and Σ.

Consider the diagonal action of H on the product $\mathfrak{h} \times V^*$. We introduce the H-invariant subset $\Delta \subset \mathfrak{h} \times V^*$ given by the three equivalent definitions

$$\Delta \colon = \{(\omega, p) \mid \omega^* p = 0\} = \{(\omega, p) \mid \omega \in \mathfrak{h}_p\} = \{(\omega, p) \mid p \in \ker \omega^*\} . \qquad (2.5)$$

Observe that any orbit in Δ is two different H-equivariant bundles given by projecting onto either the first or second factor. On the one hand, an orbit is a bundle over an orbit through $p \in V^*$ with fibre equal to the adjoint orbit through $\omega \in \mathfrak{h}_p$. On the other hand, it is also a bundle over the adjoint orbit through $\omega \in \mathfrak{h}$ with fibre equal to the H_ω-orbit through $p \in \ker \omega^*$.

Theorem 2.2.4 (Orbit bijection)**.** *Suppose that for any $p \in V^*$ there exists a bijection between the set of adjoint and coadjoint orbits of H_p. Additionally,* *suppose there is a bijection between the set of H_ω-orbits on* $\ker\omega^*$ *with the set of H_ω-orbits of the contragredient representation on* $(\ker\omega^*)^*$. *Then there exists a bijection between the set of orbits in Δ with each of the sets of adjoint and coadjoint orbits of G.*

Proof. Any coadjoint orbit of G uniquely determines an orbit Y through, say (l,p) in Π. The space Y is a bundle over the orbit through $p \in V^*$ whose fibre over p is the coadjoint orbit through $l \in \mathfrak{h}_p^*$. Contrast this with an orbit through (ω,p) in Δ: a bundle over the orbit through $p \in V^*$ whose fibre over p is an adjoint orbit in $\mathfrak{h}_p$. Since there is a bijection between adjoint and coadjoint orbits of H_p, let ω be any element belonging to the adjoint orbit which is in bijection with the coadjoint orbit through l. We designate the orbit Z through (ω,p) to correspond to the coadjoint orbit we selected at the beginning.

This correspondence currently depends on which point p we select in the orbit through V^*. For instance, had we taken the point $(rl, rp) \in Y$ instead, then the bijection between adjoint and coadjoint orbits of H_{rp} may not result in the same choice of designated orbit in Δ as it did for H_p. To ward against this we make an additional assumption about our bijections. For a given $p \in V^*$ let $\mathcal{O}$ and $\mathcal{O}^*$ be orbits in $\mathfrak{h}_p$ and $\mathfrak{h}_p^*$ respectively which are in bijection with each other. We insist that for any $r \in H$ the bijection between adjoint and coadjoint orbits of H_{rp} is given by bijecting the adjoint orbit $\mathrm{Ad}_r \mathcal{O}$ with the coadjoint orbit $\mathrm{Ad}^*_{r^{-1}} \mathcal{O}^*$ (note that $\mathrm{Ad}_r \colon \mathfrak{h}_p \longrightarrow \mathfrak{h}_{rp}$ is an isomorphism).

With this assumption on the 'consistency' of the H_p-orbit bijections, the correspondence we described above no longer depends on the point $(l,p) \in Y$, but instead only depends on the orbit Y itself. In this way we define a bijection between the orbits of Π with the orbits of Δ.

The proof for the adjoint orbits is analogous so we will merely sketch it. It suffices to define a bijection between orbits in Σ with those in Δ. Orbits in both these spaces are bundles over adjoint orbits in $\mathfrak{h}$ but whose fibres are either H_ω-orbits in $\ker\omega^*$ or its dual. Making similar assumptions about the H_ω-bijections being 'consistent' we may replicate the construction given for the coadjoint orbits to define an orbit-to-orbit bijection between Σ and Δ. $\square$

2.3 The affine group

2.3.1 Preliminaries

The *affine group* $\mathrm{Aff}(V)$ of a vector space V is the group of affine linear transformations of V. It is a semidirect product

$$\mathrm{Aff}(V) = GL(V) \ltimes V$$

with respect to the defining representation of $GL(V)$. If we choose a basis for V then this group is isomorphic to the matrix group

$$\left\{ \begin{pmatrix} r & d \\ 0 & 1 \end{pmatrix} \;\middle|\; r \in GL(V),\ d \in V \right\}. \tag{2.6}$$

If we identify V with its dual using the standard inner product corresponding to our choice of basis, then the group $\mathrm{Aff}(V^*)$ is isomorphic to

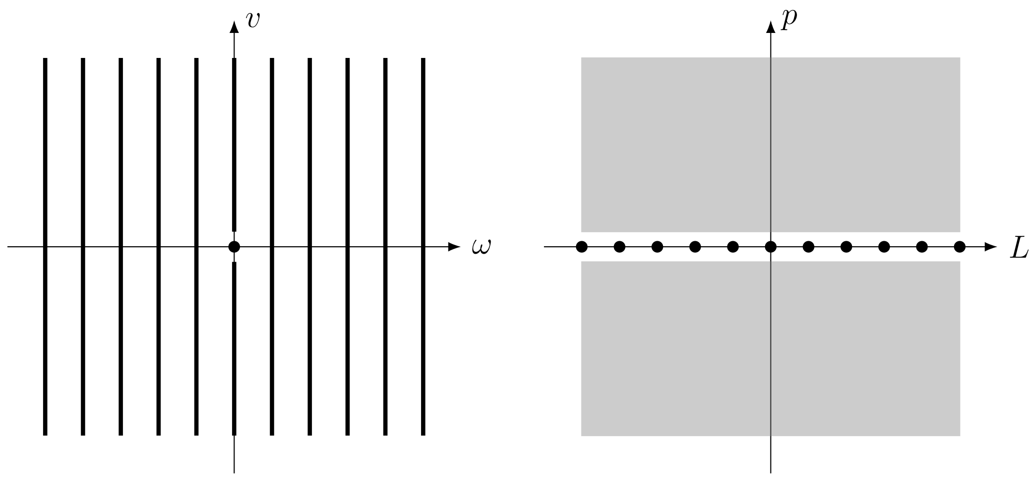
$$\left\{ \begin{pmatrix} r^{-1} & 0 \\ d^T & 1 \end{pmatrix} \;\middle|\; r \in GL(V),\ d \in V \right\}. \tag{2.7}$$

For when $V = \mathbb{R}^n$ we write $\mathrm{Aff}(V) = \mathrm{Aff}(n)$ and $\mathrm{Aff}(V^*) = \mathrm{Aff}(n^*)$. Observe that the Lie algebra $\mathfrak{aff}(n)$ is isomorphic to the set of $(n+1)$ square matrices whose final row consists of zeros, and $\mathfrak{aff}(n^*)$ to the set of $(n+1)$ square matrices whose final column consists of zeros.

Proposition 2.3.1. *For $v \in \mathbb{R}^{n+1}$ non-zero we have that the isotropy subgroup $GL(n+1)_v$ is isomorphic to $\mathrm{Aff}(n^*)$. Similarly, for p a non-zero linear functional on $\mathbb{R}^{n+1}$, the subgroup $GL(n+1)_p$ is isomorphic to $\mathrm{Aff}(n)$.*

Proof. Since $GL(n+1)$ acts transitively on non-zero vectors we may suppose without loss of generality that v is equal to the final basis vector for our choice of basis. The subgroup which preserves this vector is then precisely that given in Equation (2.7). The dual action on $\mathbb{R}^{n+1}$ is given by right multiplication on row vectors. The stabiliser of the final basis element now corresponds to matrices as in Equation (2.6). □

Example 2.3.1 (Orbits of $\mathrm{Aff}(1)$)**.** Denote group elements by $(r, d) \in GL(1) \ltimes \mathbb{R}^1 = \mathrm{Aff}(1)$ and Lie algebra elements by $(\omega, v) \in \mathfrak{gl}(1) \times \mathbb{R}^1 = \mathbb{R} \times \mathbb{R}$. We will



(a) Adjoint orbits (b) Coadjoint orbits

Figure 2.1: Adjoint and coadjoint orbits of Aff(1).

identify $\mathfrak{aff}(1)$ with its dual by setting $\langle (L,p),(\omega,v)\rangle = L\omega + pv$. It can be shown that the adjoint action (see (2.3)) is given by

$$\mathrm{Ad}_{(r,d)}(\omega, v) = (\omega, rv - \omega d)$$

and the coadjoint action (see (2.1)) by

$$\mathrm{Ad}^*_{(r,d)}(L,p) = \left(L + r^{-1}pd, r^{-1}p\right).$$

In Figure 2.1 we illustrate these orbits. Observe that there is indeed a 'geometric' bijection between the orbits: both origins to each other, the full-line adjoint orbits to the remaining coadjoint point orbits, and the two-half-line adjoint orbit to the open, dense coadjoint orbit.

2.3.2 The centralizer group representation

For Theorem 2.2.4 to hold we require a bijection result for the H_p- and H_ω-orbits. We will begin with the representation of H_ω on $\ker \omega^*$ and its dual.

Let Φ denote the representation

$$\Phi\colon H_\omega \longrightarrow GL(\ker \omega^*)$$

given by restriction of $r \in H_\omega$ to $\ker\omega^*$. The group $H_\omega = GL(n)_\omega$ is the subgroup of all isomorphisms of V which commute with ω. Equivalently, it is the subgroup of all isomorphisms of V^* which commute with ω^*. Observe that H_ω must therefore preserve the flag

$$F_{\omega^*} = \left(\ker\omega^* \supset \operatorname{Im}\omega^* \cap \ker\omega^* \supset \operatorname{Im}\omega^{*2} \cap \ker\omega^* \supset \dots\right). \tag{2.8}$$

Proposition 2.3.2. *The group $\Phi(H_\omega)$ is the group of all isomorphisms of* $\ker\omega^*$ *which preserve the flag F_{ω^*}.*

Proof. Define the *length* $l = l(F_{\omega^*})$ of the flag F_{ω^*} to be the least positive integer with $\operatorname{Im}\omega^{*l} \cap \ker\omega^* = \{0\}$. The length of F_{ω^*} is finite and the proposition is trivially true for when $l = 0$. For when $l = 1$ we have the direct sum decomposition $V^* = \operatorname{Im}\omega^* \oplus \ker\omega^*$ as $\operatorname{Im}\omega^* \cap \ker\omega^* = \{0\}$. The proposition is true for this case as we may set r to be the identity on $\operatorname{Im}\omega^*$ and equal to any isomorphism on $\ker\omega^*$. Suppose for induction that the result is true for all $\overline{\omega}^*$ with $l(F_{\overline{\omega}^*}) < l(F_{\omega^*})$. For a given isomorphism r of $\ker\omega^*$ preserving F_{ω^*} it suffices to show that it may be extended to an isomorphism of V^* which commutes with ω^*.

Let $\overline{\omega}^*$ denote the restriction of ω^* to $\operatorname{Im}\omega^*$. The flag $F_{\overline{\omega}^*}$ is equal to the subflag

$$\operatorname{Im}\omega^* \cap \ker\omega^* \supset \operatorname{Im}\omega^{*2} \cap \ker\omega^* \supset \dots$$

of F_{ω^*}, and therefore $l(F_{\overline{\omega}^*}) < l(F_{\omega^*})$. By the induction hypothesis, we may extend the definition of r to an isomorphism of $\operatorname{Im}\omega^*$ in such a way that it commutes with ω^*.

Now that we have defined r on $\operatorname{Im}\omega^* + \ker\omega^*$, let $y_1, \dots, y_m$ be vectors in V^* such that $\{[y_1], \dots, [y_m]\}$ is a basis of $V^*/(\operatorname{Im}\omega^* + \ker\omega^*)$. We define each ry_i to be equal to any element of V^* which satisfies $\omega^* r y_i = r\omega^* y_i$, and extend the definition linearly over the y_is. Observe that r commutes with the isomorphism

$$V^*/(\operatorname{Im}\omega^* + \ker\omega^*) \xrightarrow{\cong} \operatorname{Im}\omega^*/\operatorname{Im}\omega^{*2} \tag{2.9}$$

given by sending the class $[v]$ to $[\omega^* v]$. We therefore have extended r to an isomorphism over all of V^* which commutes with ω^* as desired. $\square$

2.3.3 Establishing the orbit bijection

We are now in a position to show that there is a bijection between the set of orbits of H_ω on $\ker \omega^*$ with the set of H_ω-orbits on $(\ker \omega^*)^*$. We begin by rewriting the flag F_{ω^*} in (2.8) as a strictly descending sequence of subspaces starting at $\ker \omega^* \subset V^*$ and terminating at $\{0\}$

$$F_{\omega^*} = (\ker \omega^* = E_0 \supsetneq E_1 \supsetneq E_2 \supsetneq \cdots \supsetneq E_{k+1} = \{0\})\,.$$

The previous proposition tells us that H_ω acts on $\ker \omega^*$ by all isomorphisms preserving this flag. Therefore, there are precisely $(k+2)$ orbits given by the sets

$$E_0 \setminus E_1,\; E_1 \setminus E_2,\; \ldots,\; E_k \setminus E_{k+1},\; E_{k+1} = \{0\}. \tag{2.10}$$

The contragredient representation of H_ω on $(\ker \omega^*)^*$ must act by all isomorphisms which preserve the *dual flag* $F^\circ_{\omega^*}$ given by the ascending sequence of annihilators

$$F^\circ_{\omega^*} = \big(\{0\} \subsetneq E^\circ_1 \subsetneq E^\circ_2, \ldots, E^\circ_k \subsetneq E^\circ_{k+1} = (\ker \omega^*)^*\big)\,. \tag{2.11}$$

There are thus $(k+2)$ orbits in $(\ker \omega^*)^*$ given by the sets

$$E^\circ_0 = \{0\},\; E^\circ_1 \setminus E^\circ_0,\; ,\; E^\circ_2 \setminus E^\circ_1,\; \ldots,\; E^\circ_{k+1} \setminus E^\circ_k. \tag{2.12}$$

We define the bijection between the sets of orbits in (2.10) and (2.12) to be given by

$$E_{k+1} \longleftrightarrow E^\circ_0, \text{ and } (E_j \setminus E_{j+1}) \longleftrightarrow (E^\circ_{j+1} \setminus E^\circ_j) \text{ for } 0 \le j \le k. \tag{2.13}$$

Theorem 2.3.3 (Affine-group orbit bijection)**.** *There is a bijection between the set of $GL(n)$-orbits through Δ with each of the sets of adjoint and coadjoint orbits of* $\mathrm{Aff}(n)$.

Proof. The groups H_p are isomorphic to either $GL(n)$ if $p = 0$ or $\mathrm{Aff}(n-1)$ for $p \neq 0$ (see Proposition 2.3.1). For $GL(n)$ the adjoint and coadjoint representations are isomorphic, therefore there is trivially a bijection between the two sets of orbits. For $\mathrm{Aff}(n-1)$ we suppose for induction that the result is true, noting from Figure 2.1 that this is true for $\mathrm{Aff}(1)$. The H_ω-orbit bijection is given in (2.13) and thus the theorem follows by a direct application of Theorem 2.2.4

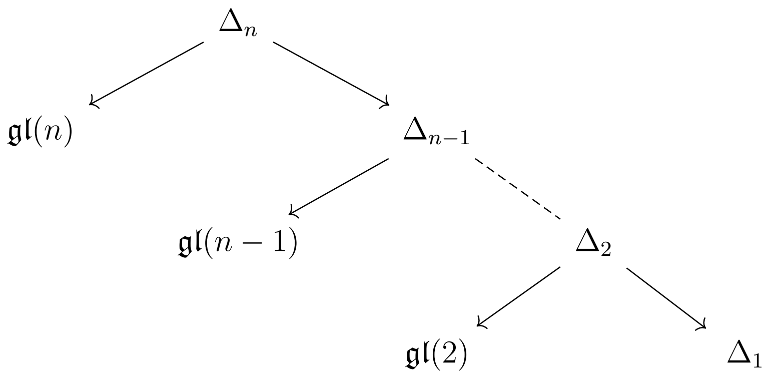


Figure 2.2: Hierarchy of orbit types for $\mathrm{Aff}(n)$.

together with induction on n. □

2.3.4 An iterative method for obtaining orbit types

Consider the set of orbits in $\Delta = \Delta_n$ for $G = \mathrm{Aff}(n)$. The set of orbits through $(\omega, 0) \in \Delta$ is in bijection with the set of adjoint orbits of $\mathfrak{h} = \mathfrak{gl}(n)$. Since the action of H on non-zero vectors in V^* is transitive, the remaining orbits are those through the points (ω, p), for some fixed non-zero p. The set of all such orbits is in bijection with the set of adjoint orbits of $\mathfrak{h}_p$, which by Proposition 2.3.1 is isomorphic to $\mathfrak{aff}(n-1)$. By Theorem 2.3.3, the set of adjoint orbits in $\mathfrak{aff}(n-1)$ is itself in bijection with the set of $GL(n-1)$-orbits through Δ_{n-1}. We may apply the same argument iteratively to obtain a hierarchy of orbit types as demonstrated in Figure 2.2

Theorem 2.3.4. *The set of orbits in Δ for $G = \mathrm{Aff}(n)$ with $n > 1$ is in bijection with the union of the sets of adjoint orbits for the groups $GL(k)$ for $2 \leq k \leq n$ and* $\mathrm{Aff}(1)$.

2.4 The Poincaré group

2.4.1 Preliminaries

Let V be a real n-dimensional vector space equipped with a non-degenerate, symmetric bilinear form Q of signature (m, n). Vectors v are partitioned into three sets depending on the value of $Q(v, v)$: if it is positive v is said to be *timelike*, *spacelike* if negative, and *null* if it is zero. All such vectors spaces are isomorphic to *Minkowski space* $\mathbb{R}^{m+n}$. Letting $\{e_1, \ldots, e_m, f_1, \ldots, f_n\}$ denote the standard basis of $\mathbb{R}^{m+n}$ the bilinear form is given by $Q(e_i, f_j) = 0$, $Q(e_i, e_j) = +\delta_{ij}$, and $Q(f_i, f_j) = -\delta_{ij}$.

The *indefinite orthogonal group* or *Lorentz group* $O(V; Q)$ is the group of isomorphisms of V which preserve Q. The *Poincaré* group is the semidirect product $E(V; Q) = O(V; Q) \ltimes V$. For Minkowski space we write the Lorentz and Poincaré groups as $O(m, n)$ and $E(m, n)$, and their Lie algebras by $\mathfrak{so}(m, n)$ and $\mathfrak{se}(m, n)$ respectively.

Proposition 2.4.1. *Let τ, σ and ν be non-zero vectors in $\mathbb{R}^{m+n}$ which are timelike, spacelike and null respectively. Then we have isomorphisms: $O(m, n)_\tau \cong O(m-1, n)$, $O(m, n)_\sigma \cong O(m, n-1)$ and $O(m, n)_\nu \cong E(m-1, n-1)$.*

Proof. The case for the timelike and spacelike vectors follows from the fact that the orthogonal complement to these vectors has signature $(m-1, n)$ and $(m, n-1)$ respectively and is an invariant subspace. For the non-zero null vector, this is proven in [CVDK06a, Section 2]. □

Example 2.4.1 (Orbits of $E(1, 1)$)**.** The group $O(1, 1)$ is the group of *Lorentz boosts*, which we write as r_ψ (where ψ denotes the *rapidity*). Elements in $E(1, 1)$ will be denoted by $(r_\psi, d) \in O(1, 1) \ltimes \mathbb{R}^{1+1}$ and in the Lie algebra by $(\omega, v) \in \mathfrak{so}(1, 1) \times \mathbb{R}^{1+1}$. The adjoint action can be shown to be (see (2.3))

$$\mathrm{Ad}_{(r_\psi, d)}(\omega, v) = (\omega, r_\psi v - \omega d) .$$

Identify $\mathfrak{se}(1, 1)$ with its dual by setting $\langle (L, p), (\omega, v) \rangle = \mathrm{Trace}(L^T \omega) + p^T Q v$ where now Q denotes the matrix $\mathrm{diag}(1, -1)$. The coadjoint action may be shown to be (see (2.1))

$$\mathrm{Ad}^*_{(r_\psi, d)}(L, p) = (L + \mu(r_\psi p, d), r_\psi p) ,$$

where $\mu(p, v) = (pv^T - vp^T)Q$. Here there exists the following orbit bijection: both origins to each other; adjoint orbits through (ω, v) for $\omega \neq 0$ to coadjoint orbits through $(L, p) = (\omega, 0)$; and coadjoint orbits through (L, p) for $p \neq 0$ to adjoint orbits through $(\omega, v) = (0, p)$.

2.4.2 The centralizer group representation

As with the affine group, the hardest part of our method will be finding an orbit bijection for the centralizer orbits of H_ω on $\ker \omega^*$ and its dual. From now on, a *form* on a vector space will mean a symmetric or skew-symmetric bilinear form.

Let H be the group $O(V;Q)$ of all isomorphisms of V which preserve a given non-degenerate form Q. For a given element ω in the Lie algebra $\mathfrak{h}$ of H we consider the centralizer subgroup $H_\omega = \{r \in H \mid r\omega = \omega r\}$ and the representation

$$\Phi\colon H_\omega \longrightarrow GL(\ker \omega^*)$$

given by restricting r to $\ker \omega^*$. As ω is a skew-self-adjoint operator with respect to Q we may identify V with V^* and ω with $-\omega^*$ and from now on consider the action of H_ω on $\ker \omega$. The group H_ω is the subgroup of all isomorphisms preserving Q which commute with ω and therefore the following flag is preserved.

$$F_\omega = \big(\ker \omega \supset \operatorname{Im} \omega \cap \ker \omega \supset \operatorname{Im} \omega^2 \cap \ker \omega \supset \dots\big)\,. \tag{2.14}$$

Proposition 2.4.2. *In addition to preserving the flag F_ω the group $\Phi(H_\omega)$ also preserves non-degenerate forms defined on the quotient spaces*

$$\operatorname{Im} \omega^m \cap \ker \omega / \operatorname{Im} \omega^{m+1} \cap \ker \omega \tag{2.15}$$

for $m \geq 0$ and where ω^0 is the identity. For $\omega^m x$ and $\omega^m y$ belonging to $\operatorname{Im} \omega^m \cap \ker \omega$, the form is given on the quotient space in Equation (2.15) by

$$\langle [\omega^m x], [\omega^m y] \rangle \coloneqq Q(\omega^m x, y). \tag{2.16}$$

Proof. Suppose for induction that $\overline{V} = \operatorname{Im} \omega^k$ is equipped with a non-degenerate form given by $\overline{Q}(\omega^k x, \omega^k y) = Q(\omega^k x, y)$; this is true for $k = 0$. Let $\overline{\omega}$ denote the restriction of ω to $\overline{V}$.

Observe that $\overline{\omega}$ is skew-self-adjoint with respect to $\overline{Q}$ and so $(\ker \overline{\omega})^\perp = \operatorname{Im} \overline{\omega}$. Therefore $\operatorname{Im} \overline{\omega} \cap \ker \overline{\omega}$ is the null space for $\ker \overline{\omega}$. It follows that the quotient $\ker \overline{\omega} / \operatorname{Im} \overline{\omega} \cap \ker \overline{\omega}$ inherits a non-degenerate form by restriction of $\overline{Q}$. By noting that $\operatorname{Im} \overline{\omega} = \operatorname{Im} \omega^{k+1}$ and $\ker \overline{\omega} = \operatorname{Im} \omega^k \cap \ker \omega$, we see that we have proved the result for $m = k + 1$.

The image $\operatorname{Im} \overline{\omega}$ is also equipped with a non-degenerate form given by

$$\overline{\overline{Q}}(\overline{\omega} a, \overline{\omega} b) = \overline{Q}(\overline{\omega} a, b).$$

Recalling that $\operatorname{Im} \overline{\omega} = \operatorname{Im} \omega^{k+1}$ and writing $a = \omega^k x$ and $b = \omega^k y$, we therefore see that $\operatorname{Im}\omega^{k+1}$ is equipped with a non-degenerate form given by $\overline{\overline{Q}}(\omega^{k+1}x, \omega^{k+1}y) = Q(\omega^{k+1}x, y)$, and thus the result is proven by induction on k. □

Theorem 2.4.3. *The subgroup $\Phi(H_\omega) \subset GL(\ker\omega)$ is precisely equal to the group of all isomorphisms of $\ker\omega$ which preserve the flag F_ω together with all the forms given in Equation (2.16) on the quotient spaces.*

Proof. For when $l(F_\omega) = 1$ we have $\operatorname{Im}\omega \cap \ker\omega = \{0\}$ and thus we have the orthogonal, direct sum decomposition $V = \operatorname{Im}\omega \oplus \ker\omega$. The theorem is true in this case as we may define r to be the identity on $\operatorname{Im}\omega$, and any isomorphism on $\ker\omega$ which preserves Q. Suppose for induction that the theorem is true for all triples $(\overline{V}, \overline{Q}, \overline{\omega})$ with $l(F_{\overline{\omega}}) < l(F_\omega)$. We begin by fixing an isomorphism r of $\ker\omega$ which preserves F_ω together with all of its forms. It suffices to show that r may be extended to an isomorphism of V which preserves Q and commutes with ω.

Let $\overline{\omega}$ denote the restriction of ω to $\overline{V} = \operatorname{Im}\omega$ and recall that this is equipped with the non-degenerate form $\overline{Q}(\omega x, \omega y) = Q(\omega x, y)$. We leave it as an exercise to show that the flag $F_{\overline{\omega}}$ is equal to the subflag

$$\operatorname{Im}\omega \cap \ker\omega \supset \operatorname{Im}\omega^2 \cap \ker\omega \supset \dots$$

of F_ω and that moreover, the forms on the quotient spaces coincide. By the induction hypothesis, since $l(F_{\overline{\omega}}) < l(F_\omega)$, we can extend r to an isomorphism of $\operatorname{Im}\omega$ which commutes with ω and preserves $\overline{Q}$.

Now that we have defined r on $\operatorname{Im}\omega + \ker\omega$, let $y_1, \dots, y_m$ be vectors in V such that $\{[y_1], \dots, [y_m]\}$ is a basis of $V/\operatorname{Im}\omega + \ker\omega$. We define each ry_i to be equal to any element of V which satisfies $\omega r y_i = r\omega y_i$, and extend the definition linearly over the y_is. Observe that r then commutes with the isomorphism

$$V/(\operatorname{Im}\omega + \ker\omega) \xrightarrow{\cong} \operatorname{Im}\omega/\operatorname{Im}\omega^2 \tag{2.17}$$

given by sending the class $[v]$ to $[\omega v]$. We therefore have extended r to an isomorphism over all of V which additionally preserves the form $\overline{Q}$ on $\operatorname{Im}\omega$, and commutes with ω. This implies that $Q(r\omega x, ry) = Q(\omega x, y)$ for all $\omega x \in \operatorname{Im}\omega$ and $y \in V$.

It remains then to show that $Q(rx, ry) = Q(x, y)$ for x and y not in $\operatorname{Im}\omega$. We will apply a Gram-Schmidt style procedure to alter the definition of each ry_i to force this to hold. Let $x_1, \dots, x_m$ be vectors in $\ker\omega$ such that $\{[x_1], \dots, [x_l]\}$ is a basis of $\ker\omega / \operatorname{Im}\omega \cap \ker\omega$. Observe that $\{[x_1], \dots, [x_l], [y_1], \dots, [y_m]\}$ is then a basis of $V/\operatorname{Im}\omega$. It therefore suffices to show that r preserves Q when restricted to the elements $x_1, \dots, x_l, y_1, \dots, y_m$.

We begin by claiming that $Q(rx_i, rx_j) = Q(x_i, x_j)$; this follows from the fact that r preserves the form Q restricted to the quotient $\ker\omega/\operatorname{Im}\omega \cap \ker\omega$. As the pairing between $\ker\omega$ and $V/\operatorname{Im}\omega$ is non-degenerate, it follows that there exist $k_i \in \ker\omega$ for each $1 \le i \le m$ which satisfy

$$Q(k_i, rx_j) = Q(y_i, x_j) - Q(ry_i, rx_j)$$

for all $1 \le j \le l$. Redefine each ry_i to now equal $ry_i + k_i$ and extend linearly over the y_is. Verify that $Q(ry_i, rx_j) = Q(y_i, x_j)$ and $\omega r y_i = r\omega y_i$ for all pairs of i and j.

As the isomorphism in Equation (2.17) commutes with r it follows that the elements $[\omega r y_1], \dots, [\omega r y_m]$ form a basis of $\operatorname{Im}\omega/\operatorname{Im}\omega^2 = \overline{V}/\operatorname{Im}\overline{\omega}$. As the pairing between $\ker\overline{\omega} = \operatorname{Im}\omega \cap \ker\omega$ and $\overline{V}/\operatorname{Im}\overline{\omega}$ is non-degenerate, we may sequentially construct $\omega z_i \in \ker\overline{\omega}$, for each $1 \le i \le m$, which satisfy

$$\overline{Q}(\omega z_i, \omega r y_j) = Q(y_i, y_j) - Q(ry_i, ry_j) - Q(ry_i, \omega z_j)$$

for all $j < i$; and, if Q is symmetric, $2\overline{Q}(\omega z_i, \omega r y_i) = Q(y_i, y_i) - Q(ry_i, ry_i)$. We once again alter the definition of ry_i and change it to equal $ry_i + \omega z_i$. One can now verify that $Q(ry_i, rx_j) = Q(y_i, x_j)$, $Q(ry_i, ry_j) = Q(y_i, y_j)$, $Q(ry_i, r\omega x) = Q(y_i, \omega x)$, and that $r\omega y_i = \omega r y_i$ for all pairs of i and j, and all $\omega x \in \operatorname{Im}\omega$. By extending the definition of r linearly over the y_is we thus obtain an isomorphism of V which commutes with ω and preserves Q as desired. □

2.4.3 Establishing the orbit bijection

To apply Theorem 2.2.4 and prove an orbit bijection result for the Poincaré group we need to demonstrate an orbit bijection for the little subgroups H_p and the centralizer groups H_ω. We begin by considering the representation of H_ω on $\ker\omega$ for where $H = O(m, n)$. Rewrite the flag in Equation (2.14) as a strictly descending sequence of subspaces

$$F_\omega = (\ker\omega = E_0 \supsetneq E_1 \supsetneq E_2 \supsetneq \cdots \supsetneq E_{k+1} = \{0\}). \tag{2.18}$$

Let Q_j denote the non-degenerate form defined on each quotient E_j/E_{j+1} given by Equation (2.16) in Proposition 2.4.2. From Theorem 2.4.3, the representation of H_ω acts on $\ker\omega$ by all isomorphisms which preserve F_ω together with all of the forms Q_j. We therefore see that any orbit which is not $E_k = \{0\}$ is contained to a set of the form $E_j \setminus E_{j+1}$ for $0 \le j \le k$. Suppose the element p belongs to such an orbit. We remark that any other element $\widetilde{p}$ belonging to the same non-zero equivalence class of $[p] \in E_j/E_{j+1}$ also belongs to the same orbit. As H_ω must preserve the form Q_j it follows that the orbit through p uniquely defines an orbit through $[p]$ of $O(E_j/E_{j+1}; Q_j)$. In summary then, the H_ω-orbits correspond to the origin $\{0\}$, and an integer $0 \le j \le k$ together with an orbit of $O(E_j/E_{j+1}; Q_j)$ in E_j/E_{j+1}.

The contragredient representation of H_ω on $(\ker\omega)^*$ must act by the group of all isomorphisms which preserve the dual flag

$$F_\omega^\circ = \left(\{0\} \subsetneq E_1^\circ \subsetneq E_2^\circ \subsetneq \cdots \subsetneq E_{k+1}^\circ = (\ker\omega)^*\right)$$

together with the non-degenerate co-forms Q_j^* defined on each quotient $E_{j+1}^\circ/E_j^\circ \cong (E_j/E_{j+1})^*$. Repeating the argument in the previous paragraph, we see that the orbits are either the origin, or contained to the sets of the form $E_{j+1}^\circ \setminus E_j^\circ$ for each $0 \le j \le k$, and that such an orbit determines an orbit of $O(E_{j+1}^\circ/E_j^\circ; Q_j^*)$ in E_{j+1}°/E_j°.

For any j with $0 \le j \le k$ consider the map $\varphi\colon E_j \to E_{j+1}^\circ$ given by satisfying

$$\langle \varphi(p), q\rangle = Q_j([p],[q]) \tag{2.19}$$

for all $p, q \in E_j$. The map φ descends to the quotient spaces to give an isomorphism $\varphi\colon E_j/E_{j+1} \to E_{j+1}^\circ/E_j^\circ$ which preserves the forms; that is $\varphi^* Q_j^* = Q_j$. This map therefore establishes the bijection.

Proposition 2.4.4. *There is a bijection between the set of H_ω-orbits in* $\ker\omega$ *with the set of H_ω-orbits in* $(\ker\omega)^*$. *Any orbit through a non-zero $p \in \ker\omega$ is contained to a set $E_j \setminus E_{j+1}$ with respect to the flag F_ω given in* (2.18) *and for $0 \le j \le k$. The corresponding bijected orbit is that through $\varphi(p) \in E_{j+1}^\circ \setminus E_j^\circ$ where φ is given in* (2.19). *The orbits equal to the origin are both bijected with each other.*

Theorem 2.4.5 (Poincaré-group orbit bijection)**.** *There is a bijection between* *the set of $O(m,n)$-orbits through Δ with each of the sets of adjoint and coadjoint orbits of $E(m,n)$.*

Proof. To begin with, the H_ω-orbit bijection follows from Proposition 2.4.4. Therefore, in order to apply Theorem 2.2.4 it remains to show that there is a bijection between the adjoint and coadjoint orbits of the H_p little groups.

Since the vector representation of $O(m,n)$ is isomorphic to its dual, the subgroups H_p for $p \in V^*$ are isomorphic to the stabiliser subgroups for different vectors in V. From Proposition 2.4.1 these groups are isomorphic to $O(m,n)$, $O(m-1,n)$, $O(m,n-1)$ and $E(m-1,n-1)$ for when p is zero, timelike, spacelike, and non-zero and null respectively (and for whenever these groups are defined). For the first three cases, these groups are semisimple, and thus the adjoint and coadjoint representations are isomorphic; consequently they trivially exhibit an orbit bijection. Therefore the theorem is true for when G is a Euclidean group $E(m,0)$ or $E(0,n)$. For when $H_p \cong E(m-1,n-1)$ we proceed by induction, assuming that the orbit bijection is already true for this group; our base cases being the groups $E(m,0)$ and $E(0,n)$, and $E(1,1)$ which we verified in Example 2.4.1. □

2.4.4 An iterative method for obtaining orbit types

Consider the set of orbits in $\Delta = \Delta_{m,n}$ for $G = E(m,n)$. Identify V with its dual using the form Q and recall that the orbits of V correspond to the sets: $Q(v,v)$ equal to a non-zero constant, $v = 0$, and the set of non-zero null vectors. It follows that every orbit of $\Delta_{m,n}$ contains a point of the form: $(\omega, 0)$, $(\omega, t\tau)$, $(\omega, s\sigma)$, and (ω, ν); for τ, σ, and ν fixed timelike, spacelike, and non-zero and null vectors respectively, and scalars $t, s > 0$. From Proposition 2.4.1, it follows that the set of orbits through these points is in bijection with the set of adjoint orbits of $\mathfrak{so}(m,n)$, $\mathfrak{so}(m-1,n) \times \mathbb{R}^{>0}_t$, $\mathfrak{so}(m,n-1) \times \mathbb{R}^{>0}_s$, and $\mathfrak{se}(m-1,n-1)$ respectively. By Theorem 2.4.5, the set of adjoint orbits of $\mathfrak{se}(m-1,n-1)$ is itself also in bijection with the set of $O(m-1,n-1)$-orbits through $\Delta_{m-1,n-1}$. We may apply the same argument iteratively to obtain a hierarchy of orbit types, as demonstrated in Figure 2.3.

Theorem 2.4.6. *The set of orbits in Δ for $G = E(m,n)$ is in bijection with the union of the sets of adjoint orbits of $O(m-k,n-k)$ for $0 \leq k \leq \min\{m,n\}$,*

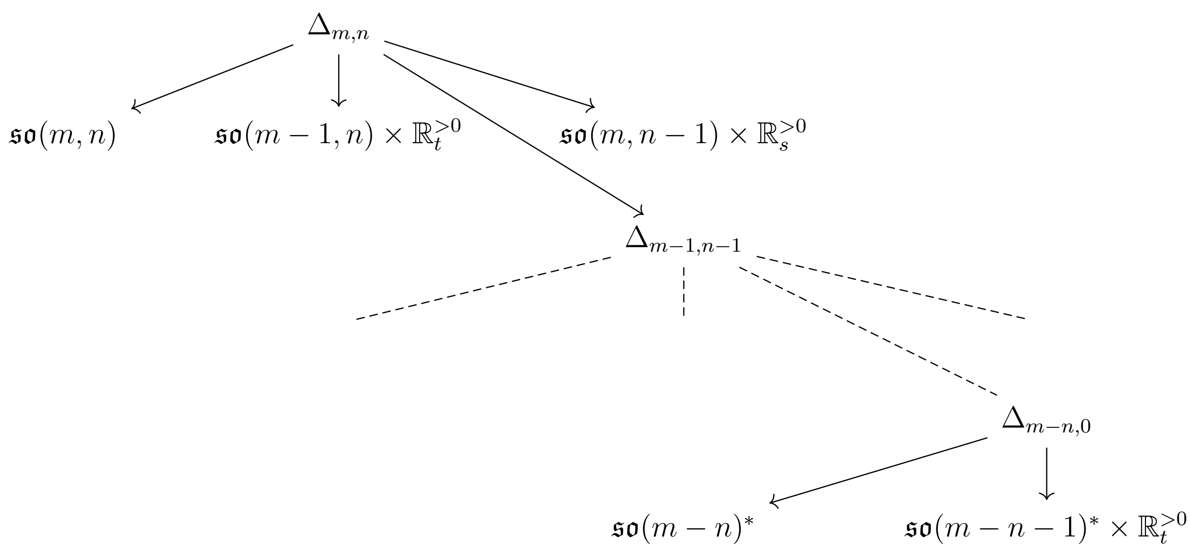


Figure 2.3: Hierarchy of orbit types for $E(m,n)$ with $m > n$.

$O(m-k-1,n-k)\times\mathbb{R}_t^{>0}$ for $0 \le k \le \min\{m-1,n\}$, and $O(m-k,n-k-1)\times\mathbb{R}_s^{>0}$ for $0 \le k \le \min\{m,n-1\}$.

Example 2.4.2 (Orbit types for $E(1,3)$)**.** The set of orbits in $\Delta_{1,3}$ for $G = E(1,3)$ is in bijection with the set of adjoint orbits of $O(1,3)$, $O(0,3)\times\mathbb{R}_t^{>0}$, $O(1,2)\times\mathbb{R}_s^{>0}$, $O(0,2)$, and $O(0,1)\times\mathbb{R}_s^{>0}$. We may find explicit normal forms for these orbits.

- The group $O(1,3)$ is isomorphic to $SL(2;\mathbb{C})$ viewed as a real Lie group. The adjoint orbits of this group are equal to the sets of 2×2 traceless, complex matrices ξ with a fixed non-zero determinant $\zeta = \det\xi$, along with the origin $\xi = 0$, and the nilpotent orbit through the nilpotent Jordan block $\xi = N_2$.

- The adjoint orbits of $O(3)$ are spheres parametrised by their radius $\rho \ge 0$.

- The adjoint representation of $O(1,2)$ is isomorphic to the vector representation on $\mathbb{R}^{1+2}$. The orbits are equal to the sets of $v \in \mathbb{R}^{1+2}$ with $Q(v,v) = c$ for $c \neq 0$, along with the origin $v = 0$, and the set of non-zero null vectors.

- The adjoint orbit of $O(2)$ through $x \in \mathfrak{so}(2) \cong \mathbb{R}$ is equal to the set $\{x,-x\}$. The orbits are therefore parametrised by $x \ge 0$.

- The group $O(1)$ has a single point orbit.

We enumerate a list of orbit types for $E(1,3)$ demonstrated in Table 2.1. We count fourteen orbit types collected into five groups. This coincides with that given in [CVDK06a] in Tables 3 and 6.

	Adjoint orbit normal forms
$O(1,3)$	$\zeta = x + iy = \det \xi \in \mathbb{C} \setminus \{0\}$
	$\zeta = 0$
	$\xi = N_2$
$O(3) \times \mathbb{R}_t^{>0}$	$(t, \rho),\ t > 0,\ \rho > 0$
	$(t, 0),\ t > 0,\ \rho = 0$
$O(1,2) \times \mathbb{R}_s^{>0}$	$(s, c),\ s > 0,\ c = Q(v, v) \in \mathbb{R} \setminus \{0\}$
	$(s, 0),\ s > 0,\ c = 0$
	$s > 0,$ for v a non-zero null vector in $\mathbb{R}^{1+2}$
$O(2)$	$x > 0$
	$x = 0$
$O(1) \times \mathbb{R}_s^{>0}$	$s > 0$

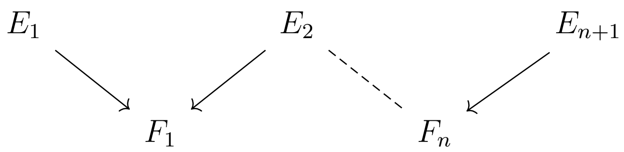
Table 2.1: Orbit types for $E(1,3)$.

2.5 A homotopy equivalence between orbits

2.5.1 Showing bijected orbits are homotopy equivalent

Consider two bijected orbits $\mathcal{O} \subset \mathfrak{g}$ and $\mathcal{O}^* \subset \mathfrak{g}^*$. Recall that they are both affine bundles over their corresponding orbits $X \subset \Sigma$ and $Y \subset \Pi$ respectively and hence each share the same homotopy type as them. In Theorem 2.2.4 the orbit bijection is established using an intermediate orbit Z in Δ to which both X and Y correspond. Our strategy will be to show that both X and Y are homotopy equivalent to Z and therefore so too are $\mathcal{O}$ and $\mathcal{O}^*$.

For a group H suppose we have a finite collection of H-spaces together with H-equivariant bundle maps connecting them in the sense below.



(2.20)

If the fibres of all these bundles are contractible then we will say that any two of these spaces are *zigzag related*. There are now two things to note: that being zigzag-related is an equivalence relation on H-spaces; and, that being zigzag related also means that the two spaces have the same homotopy type.

Now let B be a G-space and for some $b \in B$ let H denote the isotropy subgroup G_b. The map $G \to B$ given by sending g to gb defines a principal H-bundle over B. Now suppose that E is some H-space and recall the definition of the associated fibre bundle $B_E \coloneqq (G \times E)/H$; a bundle over B with fibre E. This is given by the group quotient of $G \times E$ with respect to the H-action $h(g, x) = (gh, h^{-1}x)$. The space B_E also inherits a transitive G-action given by $\tilde{g}[(g, x)] = [(\tilde{g}g, x)]$ which commutes with the bundle projection $B_E \to B$.

Proposition 2.5.1. *Let X be a G-space together with a G-equivariant bundle map $X \to B$ with fibre F above b. Suppose there is an H-equivariant bundle map $\phi\colon E \longrightarrow F$ with fibre D. Then there is a G-equivariant bundle map $B_E \to X$ with fibre D such that the following diagram commutes:*

$$\begin{array}{ccc} & & B_E \\ & \swarrow & \downarrow \\ X & \longrightarrow & B \end{array} \tag{2.21}$$

Proof. Fix some $x_0 \in E$ and observe that any point in B_E may be represented by a class of the form $[(g, x_0)]$. The bundle map in question is given by sending $[(g, x_0)]$ to $g\phi(x_0)$. This is readily seen to be well defined and G-equivariant. $\square$

Corollary 2.5.2. *Any G-space X with equivariant fibre bundle $X \longrightarrow B$ and fibres H-equivariantly diffeomorphic to E is itself G-equivariantly diffeomorphic to the associated bundle B_E.*

Lemma 2.5.3 (Zigzag Lemma)**.** *Consider a collection of H-spaces as in Equation* (2.20) *which are zigzag related. Then the corresponding associated fibre bundles over B are also zigzag related.*

Proof. A direct application of Proposition 2.5.1 shows that we may lift the zigzag of bundle maps in Equation (2.20) to

$$\begin{array}{ccccccc} B_{E_1} & & & B_{E_2} & & & B_{E_{n+1}} \\ & \searrow & \swarrow & & \ddots & \swarrow & \\ & & B_{F_1} & & B_{F_n} & & \end{array} \tag{2.22}$$

whose fibres are all contractible. □

Theorem 2.5.4 (Homotopy-type preserving bijection)**.** *In addition to the hypotheses in Theorem 2.2.4, suppose further that the bijected H_ω- and H_p-orbits are zigzag related. Then the adjoint and coadjoint orbits of G which are in bijection with each other are also zigzag related; in particular, they are homotopy equivalent.*

Proof. Let Z be an orbit in Δ through some (ω, p), and X and Y the corresponding orbits in Σ and Π which are both in bijection with Z. Both X and Z are H-equivariant bundles over the adjoint orbit $H \cdot \omega$ through $\omega \in \mathfrak{h}$ whose fibres are respectively given by bijected orbits in $\ker \omega^*$ and its dual. By Corollary 2.5.2, both X and Y are associated fibre bundles to the principal bundle $H \to H \cdot \omega$. As the fibres are assumed to be zigzag related, it follows from the Zigzag Lemma that X and Z are also zigzag related; let's write this as $X \sim Z$. A similar argument with Y also establishes that $Y \sim Z$. Now let $\mathcal{O}$ and $\mathcal{O}^*$ denote the adjoint and coadjoint orbits corresponding to X and Y respectively. As we have shown, $\mathcal{O}$ is an equivariant bundle over X with affine fibre $\operatorname{Im} \omega$, and $\mathcal{O}^*$ an equivariant bundle over Y with affine fibre $\mathfrak{h}_p^\circ$. Therefore $\mathcal{O} \sim X$, $\mathcal{O}^* \sim Y$ and thus $\mathcal{O} \sim \mathcal{O}^*$. □

2.5.2 The case for the Poincaré group

Proposition 2.5.5. *For $H = O(m, n)$ and some $\omega \in \mathfrak{so}(n, m)$, the H_ω-orbit bijection given in Proposition 2.4.4 has the property that two orbits in bijection with each other are zigzag related.*

Proof. For when the orbit in question is the origin, this is clear. Consider then the orbit through a non-zero $p \in \ker \omega$ contained to a set $E_j \setminus E_{j+1}$ with respect to the flag given in (2.18) for $0 \leq j \leq k$, and the corresponding bijected orbit through $\varphi(p) \in E_{j+1}^\circ \setminus E_j^\circ$. There is an equivariant bundle map from the orbit through p to the orbit of $O(E_j/E_{j+1}; Q_j)$ through $[p]$ whose fibres are translates of E_{j+1}; thus the fibres are contractible. The orbit through $\varphi(p)$ is likewise a bundle over the $O(E_{j+1}^\circ/E_j^\circ; Q_j^*)$-orbit through $[\varphi(p)]$ with contractible fibres equal to translates of E_j°. Since the group H_ω preserves the form Q_j, the isomorphism $\varphi\colon E_j/E_{j+1} \to E_{j+1}^\circ/E_j^\circ$ is equivariant with respect to H_ω, and therefore the orbits through $[p]$ and $[\varphi(p)]$ are H_ω-equivariantly diffeomorphic via the map φ. Thus the orbits through p and $\varphi(p)$ are zigzag related. □

Theorem 2.5.6. *For $G = E(m, n)$, consider the orbit bijection given in Theorem 2.4.5. Take an adjoint and coadjoint orbit both in bijection with each other (via a bijected orbit in Δ). These two orbits are zigzag related; in particular, bijected adjoint and coadjoint orbits are homotopy equivalent.*

Proof. To apply Theorem 2.5.4 we need to show that bijected H_p- and H_ω-orbits are zig-zag related. The proposition above demonstrates that this is true for the centralizer group orbits. It remains to show that it is true for the orbits of H_p. From Proposition 2.4.1, these groups are isomorphic to $O(m, n)$, $O(m-1, n)$, $O(m, n-1)$ and $E(m-1, n-1)$ for when p is zero, timelike, spacelike, and non-zero and null respectively. For the first three cases, these groups are semisimple, and thus the adjoint and coadjoint representations are isomorphic; consequently the trivial orbit bijection is equivariant and bijected orbits are zigzag related. Therefore the theorem is true for when G is a Euclidean group $E(m, 0)$ or $E(0, n)$. For when $H_p \cong E(m-1, n-1)$ we proceed by induction, assuming that bijected orbits are zigzag related; our base cases being the groups $E(m, 0)$ and $E(0, n)$, and $E(1, 1)$ which is verified from Example 2.4.1. □

2.5.3 The case for the affine group

Frustratingly, Theorem 2.5.4 cannot be directly applied to the affine group without some modification. We will here explain the problem and briefly sketch its resolution. It can indeed be shown that bijected orbits are homotopic, however we shall be consciously light on the details, leaving a rigorous proof as an exercise for the interested reader.

The attempted proof proceeds analogously to the case of the Poincaré group. Consider the H_ω-orbit $E_j \setminus E_{j+1}$ together with the corresponding orbit $E^\circ_{j+1} \setminus E^\circ_j$ as given in (2.13). Each of these orbits is an equivariant bundle with contractible fibres over the non-zero vector orbits of $GL(E_j/E_{j+1})$ and $GL(E^\circ_{j+1}/E^\circ_j)$ respectively. The problem now lies with the fact that, with respect to the canonical isomorphism $E^\circ_{j+1}/E^\circ_j \cong (E_j/E_{j+1})^*$, these two orbits, although identical, are not equivariantly isomorphic. In particular, the bijected H_ω-orbits are not in general zig-zag related.

A remedy to this problem is to define a notion of being 'pseudo-equivariant', whereby a map φ satisfies $\varphi(rp) = r^{-T}\varphi(p)$ for all $r \in GL(n)$. One then weakens the definition of being zigzag related in (2.20) to allow pseudo-equivariant bundle maps between the spaces. After proving the zigzag lemma for this new weakened definition, the proof of Theorem 2.5.4 follows verbatim and may be applied to the affine group.

2.6 Conclusions

To what extent we have found a geometric explanation for the orbit bijection found in [CVDK06a] is debatable. Although we have demonstrated a framework and strategy for proving such a result, the problem has now shifted into a similar bijection question concerning the little-groups and centralizer subgroups.

In practice, a key step in establishing the orbit bijections for the affine and Poincaré groups was an induction argument which took advantage of the fact that the subgroups H_p were either reductive, or equal to an affine or Poincaré group defined on a space of lower dimension. Using the same idea, it is possible to prove the orbit bijection for other semidirect products, including the special and connected versions of the affine and Poincaré groups, and even the Galilean group. It is worth noting the limitations however of this inductive argument. Consider for example the semidirect product

$$\mathrm{Symp}(2n; \mathbb{R}) \ltimes \mathbb{R}^{2n}$$

of the symplectic group with its defining vector representation. The little subgroup $\mathrm{Symp}(2n; \mathbb{R})_p$ fixing a non-zero p is called the *odd symplectic group*, and for $n > 1$ is isomorphic to the semidirect product $\mathrm{Symp}(2n-2; \mathbb{R}) \ltimes H_{2n-2}$ [Cus07]. Here H_{2n-2} is the Heisenberg group corresponding to the symplectic vector space $\mathbb{R}^{2n-2}$. For this example, our inductive argument no longer applies. However, fortunately the bijection result may still be rescued by realising that this odd symplectic group (which may be found in the literature alternatively by the names affine extended symplectic group or Schrödinger group) is a one-dimensional central extension of the original semidirect product defined for a smaller dimension. As central elements are unaffected by the adjoint representation, the bijection result still holds for this example.

The obvious question is to ask: to what extent does such an orbit bijection result hold for other groups? This author, at the time of writing, has not encountered a single example of a group which does not exhibit a geometric bijection between the sets of adjoint and coadjoint orbits, together with the property that bijected orbits share the same homotopy type. It is tempting then to conjecture that perhaps this result is true, if not for all groups, but for a large class of groups.

A next step could be to consider the general case of a Lie algebra $\mathfrak{g}$ containing some ideal. In [Myk12], they generalise the bundle-of-little-group-orbits construction given in [Raw75] to any $\mathfrak{g}$, and obtain an analogous classification of the coadjoint orbits of $\mathfrak{g}$ with respect to a given ideal. It might then be possible to expand on this work, and derive a set analogous to our set Δ; the set of orbits through which might be shown to be in bijection with each of the sets of adjoint and coadjoint orbits of $\mathfrak{g}$.

Chapter 3

Singular Reduction of the 2-Body Problem on the 3-Sphere and the 4-Dimensional Spinning Top

We consider the dynamics and symplectic reduction of the 2-body problem on a sphere of arbitrary dimension. It suffices to consider the case for when the sphere is 3-dimensional. As the 3-sphere is a group it acts on itself by left and right multiplication, which together generate the action of the $SO(4)$ symmetry. This gives rise to a notion of left and right momenta for the problem, and allows for a reduction in stages, first by the left and then the right, or vice versa. The intermediate reduced spaces obtained by left or right reduction are shown to be coadjoint orbits of the special Euclidean group $SE(4)$. The full reduced spaces are generically 4-dimensional and we describe these spaces and their singular strata.

The dynamics of the 2-body problem descend through a double cover to give a dynamical system on $SO(4)$ which, after reduction and for a particular choice of Hamiltonian, coincides with that of a 4-dimensional spinning top with symmetry. This connection allows us to 'hit two birds with one stone' and derive results about both the spinning top and the 2-body problem simultaneously. We provide the equations of motion on the reduced spaces and fully classify the relative equilibria and discuss their stability.

3.1 Background and outline

The 2-body problem in ordinary flat Euclidean space enjoys not only the symmetries of rotation and translation, but also the larger group of Galilean transformations. It is through such a transformation into a centre of mass frame that the problem reduces to the ordinary Kepler problem. For quite some time now people have been interested in the generalisation of the 2-body problem to spaces of constant non-zero curvature, where it is no longer the case that it reduces to the problem of one body. Principal contributions in this area include, but are not limited to, the numerous works of Diacu, see in particular the book [Dia12], the papers of Borisov, Mamaev and others in [BMK04, BM06, BMB16], and the work of Shchepetilov in [Shc98]. Recently the case for the 2-dimensional surfaces of constant non-zero curvature, the sphere and the hyperbolic/Lobachevsky plane, has been comprehensively treated in [BGNMM18]. Therein they perform Poisson reduction restricted to the subset where the action is free, and fully classify the relative equilibria. This paper is written in response to open problems presented at the end of that work, in particular we aim to address the generalisation of their results to the 3-sphere.

Typically, when discussing the 2-body problem in spaces with non-zero curvature one begins by highlighting that, unlike in the Euclidean case, the symmetry of the problem no longer includes translations. For positive curvatures this is indeed true for all but two cases for when the sphere is itself a group; that is, for when it is the circle or the 3-sphere. Consequently, for this very special case translations do exist, and in a sense there are more than in the flat case. As the group is non-abelian, there is a difference between translations given by group multiplication on the left and right. The entire group $SO(4)$ of symmetries is generated in this way. This establishes the well-known double cover of $SO(4)$ and allows us to identify both the configuration space for the 2-body problem on the 3-sphere with its group of symmetries. It is this curious aspect of the problem which underlies the work contained in this paper, an outline of which we now provide.

We begin by casting the problem entirely in terms of quaternions. This is a natural setting for the 3-sphere which may be taken to be the set of elements of unit-length, and where group multiplication is simply given by multiplication of quaternions. As left and right multiplications commute, we set out a plan to reduce the problem in stages: first reducing by either the left or right translations to obtain an intermediate reduced space, and then by the other to obtain the full reduced space. We conclude the introduction by demonstrating how, for a particular choice of Hamiltonian, the dynamics project under the $SO(4)$ double cover to give the symmetric heavy top in 4-dimensional space. To be precise, the Hamiltonian is that for two particles of equal mass, and for a potential proportional to $\cos\theta$ where θ is the angle subtended by the two particles.

By drawing an analogy with the reduction of the Lagrange top, where reduction is also done in stages by first reducing in the body frame and then in the space frame about the axes of symmetry, we invoke the Semidirect Product Reduction by Stages Theorem to express the left and right reduced spaces as coadjoint orbits of $SE(4)$. This is entirely analogous to the situation for the Lagrange top, whose intermediate reduced spaces are coadjoint orbits of $SE(3)$. As the actions of left and right translation are free, these reduced spaces are well-behaved smooth manifolds. However, to complete the full reduction by the residual left or right action we necessarily have to handle non-free and singular points of the momentum. We employ the methods of singular and universal reduction through the use of some invariant theory to describe these reduced spaces, which are generically 4-dimensional. We give the corresponding equations of motion on the full reduced space for both the 2-body problem and the spinning top, and explicitly exhibit an additional integral for the symmetric spinning top demonstrating complete integrability.

We then turn our attention to the relative equilibria. Instead of classifying these by finding fixed points in the reduced space directly, we instead find solutions in the intermediate left and right reduced spaces which are the orbits of one-parameter subgroups. This pleasantly turns out to be comparatively easy, amounting to an entirely linear problem in Euclidean geometry. Having classified the solutions in the left reduced space, it is then only a matter of reconstruction to obtain the full classification of relative equilibria on the original space. We then explore the stability of the corresponding fixed points in the full reduced space by linearising the flow at these points. In this way, for the Hamiltonians corresponding to the 2-body problem and the Lagrange top, we derive the linear stability results for the relative equilibria. We also provide the images of the energy-momentum map, and in doing so, obtain a picture for the bifurcations of the relative equilibria. Finally, in an effort to strengthen the stability results, we give the signature of the Hessian at the fixed points for the relative equilibria for the 2-body problem, and obtain the strongest possible stability result, that of Lyapunov stability, for linearly stable points of the Lagrange top.

3.2 Introduction

3.2.1 The problem setting

Consider the motion of two interacting particles of mass m_1 and m_2 constrained to move on the unit sphere $S^n \subset \mathbb{R}^{n+1}$. The interaction is governed by a potential V which is a function of the distance between the two particle positions, where $\mathbb{R}^{n+1}$ is equipped with the standard Euclidean metric.

The initial position and velocity vectors of the two particles span at most a 4-dimensional linear subspace (for $n > 2$). The intersection of this with the sphere is an equatorial 3-sphere. Reflection in this subspace is a symmetry of the dynamics and therefore, the motion must be forever contained to this 3-sphere. Consequently, it suffices to consider the case $n = 3$. This case also encompasses those for $n = 1$ and $n = 2$.

The space $\mathbb{R}^4$ may be identified with the algebra $\mathbb{H} = \mathrm{Span}\{1, i, j, k\}$ of real quaternions. The standard inner product is written in terms of quaternionic multiplication by

$$\langle p, q\rangle = \frac{1}{2}\left(pq^\dagger + qp^\dagger\right) \tag{3.1}$$

where $q^\dagger$ is the complex conjugate of the quaternion q. We will denote the unit sphere by the letter G to highlight that it forms a group with respect to quaternionic multiplication. The Hamiltonian formulation of the problem has phase space $T^*(G_1 \times G_2)$. Strictly speaking one should subtract the collision set from this space, but we will not concern ourselves with this for now. By identifying tangent spaces with their duals using the inner product in $\mathbb{H}$, the phase space may be identified with the set

$$M = \left\{(g_1, p_1, g_2, p_2) \in \mathbb{H}^4 \mid g_1, g_2 \in G,\ \langle p_1, g_1\rangle = \langle p_2, g_2\rangle = 0\right\}. \tag{3.2}$$

The position vectors for both particles are g_1 and g_2, and the linear momenta p_1 and p_2 are dynamically given by $m_1\dot{g}_1$ and $m_2\dot{g}_2$ respectively. The dynamics are determined by the Hamiltonian

$$H(g_1, p_1, g_2, p_2) = \frac{|p_1|^2}{2m_1} + \frac{|p_2|^2}{2m_2} + V(g_1, g_2). \tag{3.3}$$

Here $|p|^2 = \langle p, p \rangle$, and $V(g_1, g_2)$ is a function of the distance $|g_2 - g_1|$.

3.2.2 Symmetries and one-parameter subgroups

Owing to the reformulation of the problem in terms of quaternions, the $SO(4)$-symmetry on the configuration space may be realised by the action of its well-known double cover, $G \times G$. Explicitly this is given by $\Phi\colon G \times G \to SO(4)$, where

$$\Phi(l, r) \cdot q = lqr^{-1} \tag{3.4}$$

for $q \in \mathbb{H} \cong \mathbb{R}^4$. The cotangent lift of this action to M acts diagonally on each component, and a quick check confirms that it indeed preserves the Hamiltonian. We will from now on write the symmetry group as $G_L \times G_R$ to distinguish it from the configuration space $G_1 \times G_2$.

The Lie algebra $\mathfrak{g}$ of G is the space $\operatorname{Im}\mathbb{H}$ of purely imaginary quaternions. The adjoint action is given by $\Phi(g, g)$, and the infinitesimal adjoint action obtained by differentiating g is

$$\operatorname{ad}_\omega q = \omega q - q\omega = [\omega, q] \tag{3.5}$$

for $q, \omega \in \operatorname{Im}\mathbb{H}$. By identifying $\operatorname{Im}\mathbb{H}$ with $\mathbb{R}^3$ in the obvious sense, the adjoint action is related to the cross-product by $[\omega, q] = 2(\omega \times q)$.

The adjoint action of G acts transitively on each sphere of imaginary quaternions of a given length. It follows that every one-parameter subgroup of $G_L \times G_R$ is conjugate to one of the form $\{(e^{it\eta}, e^{it\xi}), t \in \mathbb{R}\}$ for some $\eta, \xi \geq 0$. With the aid of (3.4) one sees that the action of this subgroup on $\mathbb{H}$ preserves the mutually orthogonal, oriented planes $\mathbb{C} = \operatorname{Span}\{1, i\}$ and $\mathbb{C}j = \operatorname{Span}\{j, k\}$. This action rotates $\mathbb{C}$ and $\mathbb{C}j$ through an angle of $\xi - \eta$ and $\xi + \eta$ respectively with each unit of time.

Given any one-parameter subgroup conjugate to that given above, we categorise it into one of the following four types: *trivial* for when $\xi = \eta = 0$; a *simple rotation* for $\xi = \eta \neq 0$; an *isoclinic rotation* for when precisely one of either ξ or η is equal to zero; and finally, the generic subgroup is called a *double rotation* for when $\xi, \eta \neq 0$ and $\xi \neq \eta$.

3.2.3 Reduction and relative equilibria

It is curious that the symmetry group and configuration space are both the same. This is identical to the familiar situation of cotangent bundle reduction of a group under left or right multiplication. However, for our example the essential difference is that the group action is given by simultaneous left and right diagonal multiplication. This complicates the picture somewhat; in particular, this group action is not free. The points at which the action is not free may be characterised with the following argument: g_1 and g_2 belong to some plane and thus, the isotropy group fixing these two points includes rotations in the orthogonal plane. The action therefore fails to be free if and only if the momenta p_1 and p_2 have no component in this orthogonal plane, and thus, all vectors are coplanar. We will call such points *cocircular* as the resulting motion remains inside a great circle on the sphere.

Nonetheless, the actions of left and right multiplication, given by restriction to one of the G_L or G_R subgroups is free. For a group acting on its cotangent bundle by left or right cotangent lift, the momentum map is given by right or left translation back to the origin respectively [Arn13]. As the G_L- and G_R-actions are the product of two copies of left and right multiplication respectively, the left momentum map is given by

$$J_L(g_1, p_1, g_2, p_2) = \underbrace{p_1 g_1^{-1}}_{L_1} + \underbrace{p_2 g_2^{-1}}_{L_2} = \lambda \in \mathfrak{g}_L^*, \tag{3.6}$$

and the right momentum map by

$$J_R(g_1, p_1, g_2, p_2) = \underbrace{g_1^{-1} p_1}_{R_1} + \underbrace{g_2^{-1} p_2}_{R_2} = \rho \in \mathfrak{g}_R^*. \tag{3.7}$$

We write $L_i = p_i g_i^{-1}$ and $R_i = g_i^{-1} p_i$ to denote the *left* and *right momentum* of the i^{th}-particle respectively. The *total left* and *total right momenta* λ and ρ are both first integrals. As the actions of left and right multiplication are each free and proper, we may safely define the symplectic *left* and *right reduced spaces* M_λ and M_ρ respectively.

The left and right reduced spaces both inherit a group action from the residual right and left symmetry. These reduced spaces can therefore be reduced again. From the Commuting Reduction Theorem [MMO$^+$07], the momentum map for the full symmetry group is $J_{L,R} = J_L \times J_R$, and for when the action is free and proper, the staged reduced spaces $(M_\lambda)_\rho$ and $(M_\rho)_\lambda$ are both symplectomorphic to the 'one-shot' *full reduced space* $M_{\lambda,\rho}$. We would therefore like to understand the set of critical values and points of $J_{L,R}$.

Proposition 3.2.1. *The set of critical values of $J_{L,R}$ are those (λ, ρ) with $|\lambda| = |\rho|$. The pre-image of this set consists of all (g_1, p_1, g_2, p_2) belonging to a common 3-dimensional subspace, and thus correspond to solutions contained within an equatorial 2-sphere. For this reason we will refer to such critical points as being* cospherical.

Proof. The momentum map has critical values on points at which the action is not locally free [AMM81]. As we have seen, these are the cocircular points. One may suppose the momenta and positions of such a point are contained to the complex plane in $\mathbb{H}$, from which it follows from the definitions that $\lambda = \rho$. As the momentum map is equivariant, taking the orbit through these values gives us the desired set of critical values.

From the definition of λ and ρ, we may write

$$|\lambda|^2 - |\rho|^2 = 2\langle L_1, L_2\rangle - 2\langle R_1, R_2\rangle. \tag{3.8}$$

We may suppose $g_1 = 1$ and p_1 is purely imaginary. By writing the imaginary part of g_2 as $\overline{g}_2$ the expression above may be rewritten as

$$4\langle p_1, \overline{g}_2 \times p_2\rangle.$$

This is equal to zero if and only if p_1, p_2 and $\overline{g}_2$ span a common plane in $\operatorname{Im}\mathbb{H}$. However, this is equivalent to g_1, p_1, g_2 and p_2 belonging to a common 3-dimensional subspace given by the span of this plane together with the real line. Hence, by equivariance, the set of cospherical points is exactly the set of critical points of the momentum map. □

A relative equilibria (RE) in a symplectic manifold with a Hamiltonian group action is a solution which is also the orbit under the action of a one-parameter subgroup of the group of symmetries [Mar92]. Equivalently, it is a solution which projects to a point in the reduced space. For our problem, the right and left multiplication naturally descend to give well-defined actions on the left and right

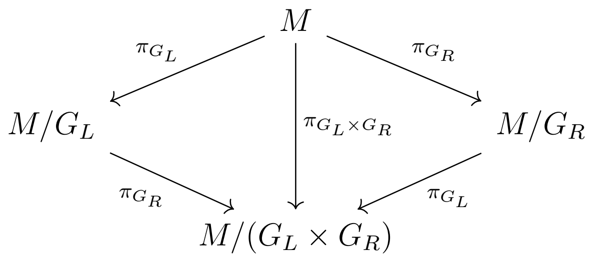


Figure 3.1: Commuting reduction

reduced spaces respectively. It follows that RE in M with respect to the $G_L \times G_R$-action project into RE in both the left and right reduced spaces. In fact, the converse is also true.

Proposition 3.2.2. *Any RE in M projects to RE in both the left and right reduced spaces. Conversely, any RE in any of the left or right reduced spaces is the projection of a RE in M.*

Proof. The proof follows from commutativity of the diagram in Figure 3.1 which consists of canonical projection maps onto orbit quotients, and the definition of a RE as a fixed point in a reduced space. □

The task of classifying RE in M is therefore equivalent to that of finding all RE in any one of the left or right reduced spaces. This will turn out to be more tractable than trying to equivalently classify all of the fixed points on the full reduced space.

3.2.4 The Lagrange top

The double cover in (3.4) is a local diffeomorphism and so lifts to a double cover of cotangent bundles which is a local symplectomorphism. Furthermore, as the Hamiltonian factors through this double cover, the dynamics factor through as well. One may also see from (3.4) that the left and right G-symmetry descends through the double cover to give the left and right multiplication of $SO(3)$ on $SO(4)$, where $SO(3)$ is the subgroup fixing the real line in $\mathbb{H} \cong \mathbb{R}^4$. We thus have a dynamical system on $SO(4)$ with a left and right $SO(3)$-symmetry. This situation should feel familiar: it is the exact same situation we have for a symmetric spinning top in 4 dimensions, as we now recall.

The configuration of a rigid body fixed to move about the origin in $\mathbb{R}^4$ may be determined by an element in $SO(4)$, that element being the transformation which sends the body from a given initial state to its current one. For when the body is under the influence of a potential which is a function of 'height' in $\mathbb{R}^4$, for which the direction of increasing height we will call the *vertical*, the dynamical system is that of the *heavy top*. This system is invariant under left multiplication of the $SO(3)$ subgroup which fixes the vertical. If the body is also invariant about rotations through a line within the body and through the origin, which we shall call *the body axis*, then we have the 4-dimensional generalisation of the *Lagrange top*. We may suppose that the body axis and the vertical are aligned when the body is in its initial identity configuration. In this way, the full symmetry of the system is both left and right multiplication of the $SO(3)$ subgroup fixing the vertical.

The study of the ordinary Lagrange top in 3 dimensions is old and well understood. We recommend the modern accounts of the problem given in [CB97, Rv82, LRSM92]. There has been some attention given to the higher dimensional generalisations of the spinning top [DG16]. The higher dimensional version of the Lagrange top, as we have defined it, was studied by Beljaev in [Bel81] and shown to be integrable. We note that an alternative generalisation is given in [Rat82] which is also shown to be integrable.

The aim now is to find the Hamiltonian on M whose dynamics project through the double cover to give the Lagrange top dynamics on $SO(4)$. To do this, we describe the Hamiltonian on $T^*SO(4)$ and pull it back under the double cover. When the potential is linear in height the Hamiltonian is given by

$$\frac{1}{2}\langle L, \mathbb{I}^{-1}(L)\rangle + \gamma\langle ac_0, v_0\rangle. \tag{3.9}$$

Here $a \in SO(4)$ is the configuration of the body, c_0 and v_0 are the initial centre of mass and vertical vectors, $\gamma > 0$ a constant, and $\mathbb{I}\colon \mathfrak{so}(4) \to \mathfrak{so}(4)^*$ is the inertia tensor of the body, where L is the angular momentum in the body frame. Identify $\mathbb{R}^4$ with $\mathbb{H}$ by associating the vector (x, y, z, t) with the quaternion $t+ix+yj+zk$. We will suppose that both the initial vertical and centre of mass vectors coincide with the real unit 1. By identifying $\mathfrak{so}(4)$ with its dual using the standard trace form allows us to write the inertia tensor as

$$\mathbb{I}(\omega) = A\omega + \omega A \tag{3.10}$$

for A a diagonal matrix which, in an appropriate choice of units, has the form $\mathrm{diag}(1,1,1,I_4)$. Differentiating Φ in (3.4) at the identify gives the well-known isomorphism $\mathfrak{g}_1 \times \mathfrak{g}_2 \to \mathfrak{so}(4)$. The identification of $\mathbb{R}^4$ with $\mathbb{H}$ allows us to write the pullback of this isomorphism as

$$\Phi^*; L = \begin{pmatrix} \widehat{\Omega} & \eta \\ -\eta^T & 0 \end{pmatrix} \longmapsto (\Omega + \eta, \Omega - \eta). \tag{3.11}$$

Here $\widehat{\Omega}$ denotes the element of $\mathfrak{so}(3)$ with $\widehat{\Omega} v = \Omega \times v$ for all $v \in \mathbb{R}^3$. It is a routine exercise to show that the right momenta (R_1, R_2) and the body angular momentum L, both obtained by left translation to the identity, are related through the double cover by $(R_1, R_2) = \Phi^*(L)$. Using this identity along with $a = \Phi(g_1, g_2)$ allows us to pull back the Hamiltonian in (3.9) to obtain

$$\frac{1+\alpha}{4}(|p_1|^2 + |p_2|^2) + \frac{1-\alpha}{2}\langle R_1, R_2\rangle + \gamma\langle g_1, g_2\rangle \tag{3.12}$$

where $\alpha = 2(1+I_4)^{-1}$. Note that this is in the same form of the 2-body Hamiltonian in (3.3) except for the presence of the $\langle R_1, R_2\rangle$ term. We will see later that, although these Hamiltonians are different, on the full reduced space they differ by a Casimir, and thus, give the same flow.

3.3 Reduction

3.3.1 The left and right reduced spaces

Motivated by the connection between the symmetry of the problem with that of the Lagrange top, we will emulate a method used for reducing the Lagrange top by the left or right symmetry by using the Semidirect Product Reduction by Stages Theorem as demonstrated in [MRW84b]. In standard treatments of the Lagrange top this theorem identifies the reduced spaces with coadjoint orbits of the special Euclidean group.

Theorem 3.3.1 (Semidirect Product Reduction by Stages, [MRW84b])**.** *Let V be a representation of H and consider the semidirect product $S = H \ltimes V$ with Lie algebra $\mathfrak{s}$. For a given $p \in V^*$ let H_p denote the stabiliser of this element with respect to the contragredient representation and consider the action of H_p on T^*H by cotangent-lift on the left or right. There is a Poisson immersion of*

T^*H/H_p into $\mathfrak{s}^*_\mp = \mathfrak{h}^* \times V^*$ given by sending the orbit through $\eta \in T^*_a H$ to

$$\left(\mathcal{L}^*_{a^{-1}}\eta, a^{-1}p\right) \in \mathfrak{s}^*_- \quad \text{or} \quad \left(\mathcal{R}^*_{a^{-1}}\eta, ap\right) \in \mathfrak{s}^*_+ \tag{3.13}$$

for the left and right case respectively. Here the $\mp$ sign indicates that the Poisson structure differs by a sign between the spaces, and the $\mathcal{L}$ and $\mathcal{R}$ denote the left and right cotangent lifts on T^*H. Moreover, if we let $\mathcal{O}$ denote a coadjoint orbit through $\mu \in \mathfrak{h}^*_p$ and $J\colon T^*H \to \mathfrak{h}^*_p$ the momentum map for the action of H_p on T^*H, then the immersion restricted to $J^{-1}(\mathcal{O})/H_p$ establishes a symplectomorphism between this symplectic orbit-reduced space and a coadjoint orbit in $\mathfrak{s}^*$.

We apply this theorem directly to the task of reducing $T^*(G_1 \times G_2)$ by the diagonal subgroup G acting on either the left or right. To do this, we set H in the theorem to $G_1 \times G_2$ with the representation in (3.4) on $V = \mathbb{H}$. This semidirect product S is the simply-connected double cover over the special Euclidean group $SE(4)$. Thanks to the inner product on $\mathbb{H}$ we are free to identify spaces with their duals, and verify that the isotropy subgroup of $1 \in \mathbb{H}^*$ is indeed the diagonal subgroup G. Implementing (3.13) demonstrates that the left and right Poisson reduced spaces consist of the elements

$$(R_1, R_2, g_L) \quad \text{and} \quad (L_1, L_2, g_R) \tag{3.14}$$

inside $\mathfrak{s}^*_-$ and $\mathfrak{s}^*_+$ respectively. Here we have introduced the respective left- and right-invariant quantities $g_L = g_1^{-1}g_2$ and $g_R = g_1 g_2^{-1}$. The left reduced space M_λ written as an orbit-reduced space is $J_L^{-1}(\mathcal{O})/G_L$, where $\mathcal{O}$ is the coadjoint orbit in $\mathfrak{g}^*_L$ through λ. Therefore, this reduced space is equal to the set of (R_1, R_2, g_L) in $\mathfrak{s}^*_-$ with $|L_1 + L_2|^2 = |\lambda|^2$. This may be rewritten in terms of the left-invariant variables as

$$|L_1+L_2|^2 = |p_1 g_1^{-1} + p_2 g_2^{-1}|^2 = |(g_1^{-1}p_1)(g_1^{-1}g_2) + (g_1^{-1}g_2)(g_2^{-1}p_2)|^2 = |R_1 g_L + g_L R_2|^2. \tag{3.15}$$

The symplectic reduced space M_λ is thus the subset of (R_1, R_2, g_L) in $\mathfrak{s}^*_-$ with $|g_L|^2 = 1$ and $|R_1 g_L + g_L R_2|^2 = |\lambda|^2$. These two functions are the only Casimirs of $\mathfrak{s}^*_-$. The geometry of the orbits is made clearer by applying the following transformation on $\mathfrak{s}^*_-$ for $g_L \neq 0$

$$(R_1, R_2, g_L) \longmapsto \left((R_1 g_L + g_L R_2) g_L^{-1}, -R_1 + g_L R_2 g_L^{-1}, g_L\right). \tag{3.16}$$

One may now see that the reduced space is diffeomorphic to $\mathcal{O} \times \mathfrak{g}^* \times G$. An entirely similar argument can be made for the right reduced space as that above.

Proposition 3.3.2. *Let typical elements in $\mathfrak{s}^*_\mp = \mathfrak{g}^*_1 \times \mathfrak{g}^*_2 \times \mathbb{H}^*$ be denoted by (A_1, A_2, g_D). There are two Casimir functions given by*

$$\begin{aligned} C_1 &= |g_D|^2, \\ C_2 &= |A_1 g_D + g_D A_2|^2. \end{aligned}$$

The left and right reduced spaces M_λ and M_ρ are symplectomorphic to the coadjoint orbits in $\mathfrak{s}^*_\mp$ given by setting C_1 to 1, and C_2 to $|\lambda|^2$ and $|\rho|^2$ respectively. In each case, the typical elements of the orbit may be identified with the left- or right-invariant dynamic variables by setting the ambidextrous dummy variables (A, D) to either (R, L) or (L, R) for the left and right reduced spaces respectively. The coadjoint orbits in question are generically diffeomorphic to $S^2 \times \mathbb{R}^3 \times S^3$ for $C_2 \neq 0$, and to $\mathbb{R}^3 \times S^3$ for $C_2 = 0$.

In both the left and right reduced spaces the Hamiltonian in (3.3) descends through the reduction procedure to give the reduced Hamiltonian on $\mathfrak{s}^*_\mp$

$$H(A_1, A_2, g_D) = \frac{|A_1|^2}{2m_1} + \frac{|A_2|^2}{2m_2} + V(g_D). \tag{3.17}$$

The function $V(g_D)$ is the *reduced potential* defined by $V(g_D) = V(1, g_D)$, which for both the left and right-reduced spaces is equal to $V(1, g_L) = V(g_R, 1) = V(g_1, g_2)$ using the left- and right-invariance of V. This function is currently only defined on G, and so for the reduced Hamiltonian above to make sense we must extend its definition to all of $\mathbb{H}$. For $g_D \in G$, the potential $V(1, g_D)$ is a function of the distance from 1 to g_D, and so it is only a function of the real part r of g_D. We will therefore choose an extension of V to $\mathbb{H}$ such that it remains a function $V(r)$ of the real part alone.

We wish to highlight an interesting feature for when one of the left or right momenta is zero. The corresponding reduced space $\mathbb{R}^3 \times S^3$ is in fact symplectomorphic to T^*S^3 with the canonical symplectic form [MMO+07][Chapter 4].

Furthermore, for when the masses are equal, the reduced Hamiltonian on T^*S^3 gives the same dynamical system as the Kepler one-body problem with the second body fixed at $1 \in S^3$. This should be contrasted with what was said at the beginning: in Euclidean space the 2-body problem may be reduced to the Kepler problem by transforming into a centre of mass frame. Here we have a kind of analogue to this, that when one of the left or right momenta is zero, the corresponding reduced space gives the standard Kepler problem on the sphere.

3.3.2 The full reduced space

Now we consider the task of reducing the left/right reduced space by the residual right/left symmetry. Without any loss of generality, we focus on reducing the left Poisson reduced space M/G_L by the group G_R of right translations. From the definitions of the left-invariant variables this group action descends to $\mathfrak{s}^*_-$ as

$$a \cdot (R_1, R_2, g_L) = (aR_1a^{-1}, aR_2a^{-1}, ag_La^{-1}) \tag{3.18}$$

for $r \in G_R$. From the Commuting Reduction Theorem the momentum map for this action is also given by the total right momentum $\rho = R_1 + R_2$.

We must now confront the issue that this action is not free. By writing an element as $(R_1, R_2, \overline{g}_L, r)$, where we have decomposed g_L into its imaginary and real parts, the G_R-action on $\mathfrak{s}^*_-$ decomposes into the irreducible pieces $\mathfrak{g}_1 \times \mathfrak{g}_2 \times \mathfrak{g}_3 \times \mathbb{R}$; that is, three copies of the adjoint representation of G, and the trivial representation. The adjoint representation of G factors through the double cover $G \to SO(3)$ to give the standard vector representation of $SO(3)$ on $\mathfrak{g} = \operatorname{Im} \mathbb{H} \cong \mathbb{R}^3$. It follows that this action fails to be free whenever R_1, R_2 and $\overline{g}_L$ are colinear. In order to handle these singular cases we will employ the method of universal reduction from [ACG91] via the use of invariant theory to obtain the orbit quotient as a semialgebraic variety. This technique is demonstrated in, for example [LMS93], and similarly in [CB97] for the second stage of reduction for the ordinary Lagrange top.

Temporarily denote elements in $\mathfrak{g}_1 \times \mathfrak{g}_2 \times \mathfrak{g}_3$ by (v_1, v_2, v_3). The First Fundamental Theorem of Invariant Theory for the special orthogonal group [KP00] tells us that the invariant ring is generated by the pairwise inner products $k_{ij} = \langle v_i, v_j \rangle$ for $i \leq j$, and the determinant $\delta = \langle v_1 \times v_2, v_3 \rangle$. These are not independent as they satisfy the algebraic relation $\delta^2 = \det(k_{ij})$. As the group G_R is compact, the orbits are separated by the values taken by the generators of the invariant ring, and hence, the quotient $\mathfrak{s}_-^*/G_R$ may be identified with the image of the Hilbert map

$$\sigma\colon \mathfrak{s}_-^* \longrightarrow \mathbb{R}^8; \quad (v_1, v_2, v_3, r) \longmapsto (\{k_{ij}\}_{i\le j}, \delta, r)\,.$$

The image of this map is the semialgebraic variety defined by those points satisfying $\delta^2 = \det(k_{ij})$, and the inequalities $k_{ii} \geq 0$, and $k_{ij}^2 \leq k_{ii}k_{jj}$ for each (i,j)-pair.

Since the momentum $\rho = R_1 + R_2$ is conserved, the G_R-invariant quantity $|\rho|^2$ descends to $\mathfrak{s}_-^*/G_R$ to give an additional Casimir

$$C_3 = |R_1 + R_2|^2 = k_{11} + 2k_{12} + k_{22}.$$

The two Casimirs C_1 and C_2 in Proposition 3.3.2 also descend to the full reduced space. The first of these is easily seen to be

$$C_1 = |g_L|^2 = k_{33} + r^2.$$

The second however, requires a special effort to express in terms of the invariant generators.

Lemma 3.3.3. *The Casimir $C_2 = |R_1 g_L + g_L R_2|^2$ may be expressed in terms of the generators of the G_R-invariant ring on $\mathfrak{s}_-^*$ as*

$$C_2 = (k_{33} + r^2)(k_{11} + k_{22}) + 2k_{12}(r^2 - k_{33}) + 4k_{13}k_{23} - 4r\delta.$$

Proof. Expanding C_2 gives

$$|R_1 g_L|^2 + |g_L R_2|^2 + 2\langle R_1 g_L, g_L R_2\rangle = (k_{33} + r^2)(k_{11} + k_{22}) + 2\langle R_1 g_L, g_L R_2\rangle.$$

Rewriting g_L as $r + \overline{g}_L$, the final term above may be written as

$$2r^2\langle R_1, R_2\rangle + 2r\left(\langle R_1\overline{g}_L, R_2\rangle + \langle R_1, \overline{g}_L R_2\rangle\right) + 2\langle R_1\overline{g}_L, \overline{g}_L R_2\rangle \\ = 2r^2 k_{12} - 4r\delta + 2\langle R_1\overline{g}_L, \overline{g}_L R_2\rangle.$$

By multiplying out the cross-product terms, it is possible to establish the following identity

$$2\langle R_1\overline{g}_L, \overline{g}_L R_2\rangle = 4\langle R_1 \times \overline{g}_L, \overline{g}_L \times R_2\rangle + 2\langle \overline{g}_L, \overline{g}_L\rangle\langle R_1, R_2\rangle.$$

Finally, using the vector quadruple product $\langle a\times b, c\times d\rangle = \langle a, c\rangle\langle b, d\rangle - \langle a, d\rangle\langle b, c\rangle$ in the expression above gives the desired result. □

Theorem 3.3.4. *The Poisson reduced space $\mathfrak{s}^*_-/G_R$ is the semialgebraic variety given by coordinates $(\{k_{ij}\}_{i\leq j}, \delta, r)$ in $\mathbb{R}^8$ satisfying $\delta^2 = \det(k_{ij})$, $k_{ii} \geq 0$, and $k_{ij}^2 \leq k_{ii}k_{jj}$. There are three Casimirs,*

$$C_1 = k_{33} + r^2,$$
$$C_2 = (k_{33} + r^2)(k_{11} + k_{22}) + 2k_{12}(r^2 - k_{33}) + 4k_{13}k_{23} - 4r\delta,$$
$$C_3 = k_{11} + k_{22} + 2k_{12}.$$

The full reduced space $(M_\lambda)_\rho$ is obtained by setting $C_1 = 1$, $C_2 = |\lambda|^2$, and $C_3 = |\rho|^2$.

For $|\lambda|, |\rho| \neq 0$, these typical reduced spaces are 4-dimensional. The algebraic awkwardness of the Casimir C_2 together with the relation $\delta^2 = \det(k_{ij})$ makes it difficult to grasp the geometry of these reduced spaces. Indeed, it would be of considerable interest to be able to say more about them. Nonetheless, below we describe the degenerate 2-dimensional reduced spaces.

Consider the full reduced space $(M_\lambda)_\rho$ for $\rho = 0$. By applying the algebraic inequalities in Theorem 3.3.4 for $C_3 = 0$ we obtain an additional two constraints: $k_{11} = k_{22}$ and $k_{13} = -k_{23}$. After eliminating variables the reduced space is found to be homeomorphic to the set of points (k_{11}, k_{13}, θ) satisfying

$$4k_{11} = \frac{|\lambda|^2 + 4k_{13}^2}{\sin^2\theta}, \tag{3.19}$$

where we write $k_{33} = \sin^2\theta$. When $|\lambda|$ is non-zero this leaf is homeomorphic to $\mathbb{R}^2$. On the other hand, if $\lambda = 0$ this leaf degenerates into the singular canoe shown in Figure 3.2. As we have previously remarked, the reduced spaces M_0 are symplectomorphic to T^*S^3, and for when the masses are equal the resulting dynamical system is the Kepler problem on the sphere. These reduced spaces we have described therefore coincide with those for the Kepler problem.

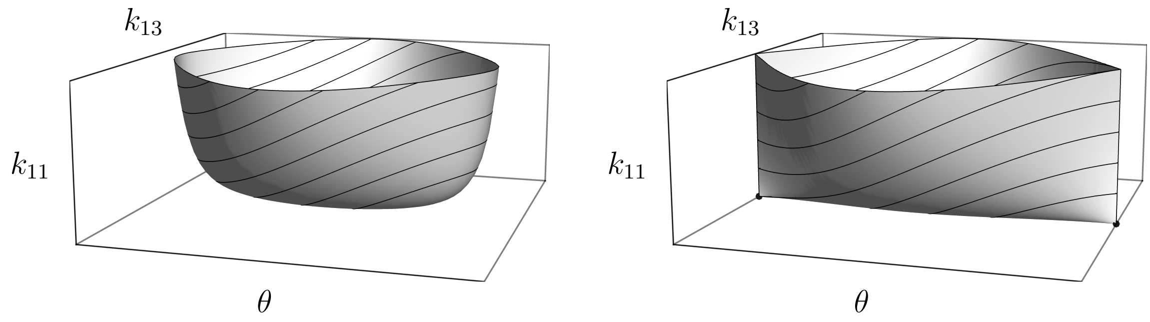


Figure 3.2: The full reduced spaces $(M_\lambda)_\rho$ when $\rho = 0$. These spaces are 2-dimensional and homeomorphic to the plane for $\lambda \neq 0$ as shown in the figure on the left. The leaf degenerates at $\lambda = 0$ into the canoe as shown on the right. The contours are the level sets of the altered Hamiltonian in (3.25) for the Lagrange top.

3.3.3 The singular strata

An advantage of using invariants to describe the reduced space as a semialgebraic variety is that it includes those points at which the action is not free. Consequently, these reduced spaces are not smooth in general, but stratified symplectic spaces. The theory of such stratified spaces is detailed in [SL91]. It is shown that the strata of a reduced space, which are invariant under the dynamics, correspond to the different possible isotropy subgroups of the action. We now discuss each of the possible isotropy subgroups for the G_R-action in (3.18) and their corresponding strata in turn.

From Proposition 3.2.1, the action is free whenever $|\lambda| \neq |\rho|$ and not free precisely on those points which are cocircular. It follows that the reduced spaces $M_{\lambda,\rho}$ for $|\lambda| \neq |\rho|$ consist of a single stratum and are thus bonafide smooth manifolds. For when $|\lambda| = |\rho|$, the reduced space contains an open dense stratum corresponding to the non-cocircular points at which the action is free.

The stratum corresponding to when the isotropy subgroup is all of G_R is for when $R_1 = R_2 = \overline{g}_L = 0$, and thus consists solely of the two points $r = \pm 1$, where the other generators are zero. These two points are the corners of the canoe in $M_{0,0}$ in Figure 3.2, and correspond to the states where the two particles are motionless and either antipodal, or in the same position.

The strata corresponding to when the isotropy subgroup is $SO(2)$ are for those points where $(R_1, R_2, \overline{g}_L)$ are colinear and not all zero. It follows that the generators of the invariant ring satisfy three further relations, given by changing the inequalities in Theorem 3.3.4 into equalities. In fact, as $\delta = 0$, and since $|\lambda| = |\rho|$ at this point, the relation $k_{11}k_{33} = k_{13}^2$ is not independent and so we have 6 constraints in total. Indeed, after eliminating variables one may show that there are two degrees of freedom given by $k_{11} + k_{22}$ and θ, where we are writing $r = \cos\theta$. For when $|\lambda| = |\rho| \neq 0$ this stratum is homeomorphic to a cylinder and degenerates into the canoe, minus the corners, when the momentum is zero. These strata correspond to the cocircular configurations of the particles.

3.3.4 The equations of motion and the Poisson structure

We will now derive Hamilton's equations, first on the left and right reduced spaces, and then on the full reduced space. To begin with, we need the Poisson structure between two functions f and g on $\mathfrak{s}^*_\mp$. The definition of the Lie-Poisson structure, $\{f, g\}$ evaluated at (A_1, A_2, g_D) in $\mathfrak{s}^*_\mp$ is given in [HMR98] by

$$\mp \langle A_1, [\nabla_1 f, \nabla_1 g]\rangle \mp \langle A_2, [\nabla_2 f, \nabla_2 g]\rangle \\ \mp \langle g_D, (\nabla_1 f \nabla_3 g - \nabla_3 g \nabla_2 f) - (\nabla_1 g \nabla_3 f - \nabla_3 f \nabla_2 g)\rangle. \quad (3.20)$$

Here we have written ∇_i for $i = 1, 2, 3$ to mean the gradient of a function on $\mathfrak{s}^* = \mathfrak{g}_1^* \times \mathfrak{g}_2^* \times \mathbb{H}^*$ with respect to the three component spaces. The equations of motion may be derived from the Poisson bracket, where the convention we use is $\dot{\varphi} = \{\varphi, H\}$. Before doing this, we pause to consider the term $\nabla_3 V$ from the Hamiltonian in (3.17). In the definition for the reduced potential we chose an extension of the function $V(1, g_L)$ from G to $\mathbb{H}$ which remained a function of the real part alone. The gradient $\nabla_3 V$ is therefore the purely real quaternion dV/dr. Armed with this foresight, we may proceed to write down Hamilton's equations on each of $\mathfrak{s}^*_-$ and $\mathfrak{s}^*_+$.

$$\begin{aligned} \dot{R}_1 &= +f(r)\overline{g}_L & \dot{L}_1 &= -f(r)\overline{g}_R \\ \dot{R}_2 &= -f(r)\overline{g}_L & \dot{L}_2 &= +f(r)\overline{g}_R \\ \dot{g}_L &= -\frac{R_1}{m_1} g_L + g_L \frac{R_2}{m_2} & \dot{g}_R &= +\frac{L_1}{m_1} g_R - g_R \frac{L_2}{m_2} \end{aligned} \quad (3.21)$$

Here we are writing $\overline{g}_D$ to mean the imaginary part of g_D, and suggestively abbreviating $-dV/dr$ with $f(r)$ for *force*. Using these equations together with the definitions of the generators of the invariant ring, we can write the full reduced equations of motion in $\mathfrak{s}_-^*/G_R$, which are given below.

$$
\begin{aligned}
\dot{k}_{11} &= 2fk_{13} \\
\dot{k}_{12} &= f(k_{23} - k_{13}) \\
\dot{k}_{13} &= fk_{33} - r\left(\frac{k_{11}}{m_1} - \frac{k_{12}}{m_2}\right) - \frac{\delta}{m_2} \\
\dot{k}_{22} &= -2fk_{23} \\
\dot{k}_{23} &= -fk_{33} - r\left(\frac{k_{12}}{m_1} - \frac{k_{22}}{m_2}\right) + \frac{\delta}{m_1} \\
\dot{k}_{33} &= 2r\left(\frac{k_{23}}{m_2} - \frac{k_{13}}{m_1}\right) \\
\dot{r} &= \frac{k_{13}}{m_1} - \frac{k_{23}}{m_2} \\
\dot{\delta} &= \left(\frac{k_{12}k_{13} - k_{11}k_{23}}{m_1}\right) + \left(\frac{k_{13}k_{22} - k_{12}k_{23}}{m_2}\right)
\end{aligned}
\tag{3.22}
$$

These equations give the flow generated by the Hamiltonian in (3.17), which descends to $\mathfrak{s}_-^*/G_R$ as

$$
\frac{k_{11}}{2m_1} + \frac{k_{22}}{2m_2} + V(r). \tag{3.23}
$$

It is important to appreciate that although these reduced spaces are considerably smaller than the original phase space, the dimension dropping from 12 to 4, there is a trade-off: the Poisson bracket on $\mathfrak{s}_-^*/G_R$ between the generators is extremely cumbersome. By definition, this Poisson bracket descends from the bracket on $\mathfrak{s}_-^*$ as given in (3.20). The difficulty lies in the fact that the invariants are typically quadratic in $\mathfrak{s}_-^*$ and thus the bracket between them is cubic. This makes deriving a general formula for the Poisson bracket on $\mathfrak{s}_-^*/G_R$ a rather Herculean task, and one that this author has failed to complete. As consolation, we offer instead the structure matrix in Table 3.1 listing the Poisson bracket between the generators of the invariant ring. It should be emphasised that this Poisson bracket should only be expected to satisfy the Jacobi identity and form a Lie algebra when the generators satisfy the algebraic relations in Theorem 3.3.4.

$\{\ ,\ \}$	k_{11}	k_{12}	k_{13}	k_{22}	k_{23}	k_{33}
k_{11}	$\cdot$	0	$2rk_{11}$	0	$2rk_{12} - 2\delta$	$4rk_{13}$
k_{12}	$\cdot$	$\cdot$	$r(k_{12} - k_{11})$	0	$r(k_{22} - k_{12})$	$2r(k_{23} - k_{13})$
k_{13}	$\cdot$	$\cdot$	$\cdot$	$2rk_{12} - 2\delta$	$r(k_{13} + k_{23})$	$2rk_{33}$
k_{22}	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$-2rk_{22}$	$-4rk_{23}$
k_{23}	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$-2rk_{33}$
k_{23}	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$

$\{k_{ij}, r\}$	1	2	3
1	$-2k_{13}$	$k_{13} - k_{23}$	$-k_{33}$
2	$\cdot$	$2k_{23}$	k_{33}
3	$\cdot$	$\cdot$	0

$\{k_{ij}, \delta\}$	1	2	3
1	$2(k_{11}k_{23} - k_{12}k_{13})$	$(k_{12} + k_{22})k_{13} - (k_{11} + k_{12})k_{23}$	$(k_{13} + k_{23})k_{13} - (k_{11} + k_{12})k_{33}$
2	$\cdot$	$2(k_{12}k_{23} - k_{13}k_{22})$	$(k_{12} + k_{22})k_{33} - (k_{13} + k_{23})k_{23}$
3	$\cdot$	$\cdot$	0

$\{\ ,\ \}$	δ
r	0

Table 3.1: The Poisson bracket between generators of the G_R-invariant ring on $\mathfrak{s}_-^*$ and thus the Poisson structure on the full reduced space $\mathfrak{s}_-^*/G_R$.

3.3.5 A reprise of the Lagrange top

The Hamiltonian given in (3.12) for the Lagrange top, descends through this whole reduction procedure to give

$$H = \frac{1+\alpha}{4}(k_{11} + k_{22}) + \frac{1-\alpha}{2}k_{12} + \gamma r. \tag{3.24}$$

This is of course, as we have already remarked, different to the 2-body Hamiltonian in (3.23). However, the two Hamiltonians differ by a constant multiple of C_3. Since a Casimir trivially generates a stationary flow, the flow on the full reduced space for the Lagrange top is equivalently the flow generated by the *altered Hamiltonian*

$$\widetilde{H} = H + \frac{(\alpha-1)}{4}C_3 = \frac{\alpha}{2}(k_{11} + k_{22}) + \gamma r. \tag{3.25}$$

Rather remarkably, one sees that although the flows on the left and right reduced spaces for the Lagrange top and the 2-body problem are different, they are identical on the full reduced space for when $m_1 = m_2 = \alpha^{-1}$, and $V(r) = \gamma r$. We can therefore continue in generality to consider a Hamiltonian of the form in (3.23), and in doing so, will simultaneously treat both the 2-body problem and the Lagrange top.

Finally we highlight one further remarkable feature of the Lagrange top: that it is completely integrable. Indeed, in [Bel81] it is shown that the n-dimensional generalisation of the Lagrange top admits a complete set of independent integrals which commute with the left and right $SO(n-1)$ symmetry. For the case $n = 4$, this single extra integral is defined on the left reduced space $\mathfrak{s}^*_-$ by

$$\eta^T \widehat{\Omega}^2 \eta + \left(\frac{2\gamma}{\alpha}\right) \overline{g}_L^T \widehat{\Omega} \eta. \tag{3.26}$$

Here we are using the isomorphism in (3.4) to identify $\mathfrak{s}^*_-$ with $\mathfrak{se}(4)^*$ and using the notation for the body angular momentum in (3.11). After a scaling, this may be expressed in terms of the invariant generators as

$$I = \alpha(k_{12}^2 - k_{11}k_{22}) - 2\gamma\delta. \tag{3.27}$$

One may verify directly from the reduced equations in (3.22) that this is constant for the flow generated by the altered Hamiltonian in (3.25). Conversely, for the 2-body problem on a sphere, in the case of equal masses and for a potential proportional to $\cos\theta$, the function I is an additional first integral.

3.4 Relative equilibria

3.4.1 A classification of the relative equilibria on the left reduced space

To classify the RE we could of course find the fixed points of the system of full reduced equations in (3.22). However, this is easier said than done, and following Proposition 3.2.2 we may equivalently classify the solutions in, say the left reduced space M/G_L which are the orbits under a one-parameter subgroup of G_R. From the G_R-action in (3.18) we see that such an orbit in $\mathfrak{g}^*$ is of the form $q(t) = e^{t\eta}q(0)e^{-t\eta}$ for the generator $\eta \in \mathfrak{g}_R$. By differentiating this to find the velocity vector, and comparing with the left reduced equations in (3.21), one sees that a solution through (R_1, R_2, g_L) in $\mathfrak{s}^*_-$ is the orbit of a one-parameter subgroup $e^{t\eta}$ of G_R if and only if the following holds

$$2(\eta \times R_1) = +f(r)\overline{g}_L, \tag{3.28}$$

$$2(\eta \times R_2) = -f(r)\overline{g}_L, \tag{3.29}$$

$$2(\eta \times \overline{g}_L) = -R_1 g_L/m_1 + g_L R_2/m_2. \tag{3.30}$$

We suppose without any loss of generality that η is of the form $|\eta|j$, recalling that j is an imaginary quaternion in $\operatorname{Im}\mathbb{H} = \mathfrak{g}$. The real part of g_L is fixed by the action of G_R. We write this real part as $r = \cos\theta$ for $\theta \in [0, \pi]$ and classify the solutions of the form above according to the angle θ. Furthermore, from now on we will suppose that the potential is such that the force $f(r)$ is never zero.

Case 1: $\theta = 0, \pi$

In this case, g_L is equal to ± 1, and consequently $\overline{g}_L = 0$. It follows from the equations above that $|\eta| \geq 0$ is arbitrary, and that R_1 and R_2 are scalar multiples of j which satisfy $m_2 R_1 = m_1 R_2$.

Case 2: $0 < \theta < \pi,\ \theta \neq \pi/2$

Equations (3.28) and (3.29) imply that R_1, R_2 and η are orthogonal to $\overline{g}_L$. We may also suppose without any loss of generality that $\overline{g}_L = i\sin\theta$, and thus, that

$g_L = e^{i\theta}$. It then follows that R_1 and R_2 must be of the form

$$R_1 = x_1 j + yk \quad \text{and} \quad R_2 = x_2 j - yk \tag{3.31}$$

for x_1 and x_2 to be determined, and

$$y = \frac{f \sin\theta}{2|\eta|}. \tag{3.32}$$

It now remains to solve Equation (3.30), which can be expanded using quaternionic multiplication to give

$$\begin{aligned} -2|\eta|\sin\theta\ k &= -\frac{1}{m_1}(x_1 j + yk)e^{i\theta} + \frac{1}{m_2}e^{i\theta}(x_2 j - yk), \\ &= -\frac{1}{m_1}x_1 e^{-i\theta} j - \frac{1}{m_1} y e^{-i\theta} k + \frac{1}{m_2} x_2 e^{i\theta} j - \frac{1}{m_2} y e^{i\theta} k. \end{aligned}$$

Multiplying both sides of this equation on the right by k gives an equation purely in terms of complex numbers. Separating this into real and imaginary parts results in a linear equation in x_1 and x_2 which may be written as

$$-\frac{1}{m_1 m_2}\begin{pmatrix} \sin\theta & \sin\theta \\ \cos\theta & -\cos\theta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2|\eta|\sin\theta - y\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\cos\theta \\ y\left(\frac{1}{m_1} - \frac{1}{m_2}\right)\sin\theta \end{pmatrix}. \tag{3.33}$$

This linear system is non-degenerate for $\theta \neq \pi/2$ and has unique solutions

$$\begin{aligned} x_1 &= y\left(\cot 2\theta + \tfrac{m_1}{m_2}\csc 2\theta\right) - m_1|\eta|, \\ x_2 &= y\left(\cot 2\theta + \tfrac{m_2}{m_1}\csc 2\theta\right) - m_2|\eta|. \end{aligned} \tag{3.34}$$

Case 3: $\theta = \pi/2$

For $\theta = \pi/2$ the linear system in (3.33) becomes degenerate. Solutions of the system can only exist when m_1 and m_2 are both equal to say m. There is then an entire line's worth of solutions given by

$$x_1 + x_2 = -2m^2|\eta|. \tag{3.35}$$

3.4.2 Reconstruction and the full relative equilibria classification

In accordance with Proposition 3.2.2, having classified all RE solutions in M/G_L, it remains to reconstruct the corresponding solutions in M. Clearly, the action of G_L on M/G_L is trivial, and thus we may safely suppose that the corresponding one-parameter subgroup of G_L is generated by $\xi = |\xi| j \in \mathfrak{g}_L$.

From the definitions, the real part of g_L and g_R must each be equal to $r = \cos\theta$, where θ is the angular separation between the two particles

$$\cos\theta = \langle g_1, g_2 \rangle = \langle g_1 g_2^{-1}, 1 \rangle = \langle 1, g_1^{-1} g_2 \rangle = r.$$

A similar set of equations to (3.28), (3.29) and (3.30) also hold in the right reduced space, and it follows from these that $\overline{g}_R$ must also be orthogonal to ξ and hence j. Therefore, we may suppose that $\overline{g}_R = -i\sin\theta$, and hence, that $g_R = e^{-i\theta}$ and $g_L = e^{i\theta}$. The definitions of g_L and g_R give $g_2 = g_1 e^{i\theta} = e^{i\theta} g_1$. For when θ is not equal to 0 or π, this implies that g_1 commutes with i. This is only the case for when g_1 belongs to the complex plane $\mathbb{C} \subset \mathbb{H}$. This forces g_1 and g_2 to be of the form

$$g_1 = e^{-i\phi_1}, \qquad \text{and} \qquad g_2 = e^{i\phi_2} \tag{3.36}$$

where $\phi_1 + \phi_2 = \theta$. We may additionally suppose that $\phi_2 \in [0, \pi]$, since $(g_1, g_2) \mapsto (-g_1, -g_2)$ is an element of $SO(4)$. The orbit of a point q in $\mathbb{H}$ under the one-parameter subgroup of $G_L \times G_R$ is $e^{t\xi} q e^{-t\eta}$. For each particle g_i we can differentiate this motion to obtain the momentum $p_i = m_i \dot{g}_i$, and then find the right momentum $R_i = g_i^{-1} p_i$ as below

$$\begin{aligned} R_1 &= m_1(|\xi| \cos 2\phi_1 - |\eta|) j + m_1(|\xi| \sin 2\phi_1) k, \\ R_2 &= m_2(|\xi| \cos 2\phi_2 - |\eta|) j - m_2(|\xi| \sin 2\phi_2) k. \end{aligned} \tag{3.37}$$

These expressions must agree with those in (3.31) for the forms of R_1 and R_2, and therefore the following must hold

$$m_1 \sin 2\phi_1 = m_2 \sin 2\phi_2 = \frac{y}{|\xi|}.$$

For when $\theta \neq \pi/2$, this equation uniquely determines ϕ_1 and ϕ_2. For when $\theta = \pi/2$, and therefore $m_1 = m_2$, the angles are not unique: for when y is positive the solutions are for all $\phi_1 \in (0, \pi/2)$, and all $\phi_1 \in (-\pi/2, 0)$ for y negative. Finally, for the exceptional cases, we have $g_1 = g_2$ for $\theta = 0$, and $g_1 = -g_2$ for $\theta = \pi$. As θ is constant, it follows from consideration of Case 1 above, that these motions correspond to the two particles moving together around a great circle arbitrarily quickly, either occupying the same position, or antipodal to each other.

Theorem 3.4.1. *For the 2-body problem on the 3-sphere with an either strictly attractive or repulsive potential, all relative equilibria solutions are completely classified, up to conjugacy, according to the angle $\theta \in [0, \pi]$ subtended by both particles in the sense listed below. In each of these cases we suppose, without loss of generality, that the corresponding one-parameter subgroup of $G_L \times G_R$ is generated by $(|\xi|j, |\eta|j) \in \mathfrak{g}_L \times \mathfrak{g}_R$.*

- *For $\theta = 0$ and π, we will call these solutions* singular*. For $\theta = 0$ we may take the initial positions to be $g_1 = g_2 = 1$, and $g_1 = -g_2 = 1$ for $\theta = \pi$. Both $|\xi|$ and $|\eta|$ are arbitrary.*

- *For $\theta < \pi/2$ we will call the solutions* acute*, and* obtuse *when $\pi/2 < \theta < \pi$. In both cases we may take the initial positions of the particles to be at $g_1 = e^{-i\phi_1}$ and $g_2 = e^{i\phi_2}$ where the angles ϕ_1 and ϕ_2 are determined by $\phi_1 + \phi_2 = \theta$ and by*
$$m_1 \sin 2\phi_1 = m_2 \sin 2\phi_2. \tag{3.38}$$

- *Solutions for $\theta = \pi/2$ will be called* right-angled *and only exist for $m_1 = m_2$. In this case, ϕ_1 and ϕ_2 are not uniquely determined by (3.38) and may be any angles satisfying $\phi_1 + \phi_2 = \pi/2$ where $\phi_1 \in (0, \pi/2)$ for a strictly attractive potential ($f > 0$), or $\phi_1 \in (-\pi/2, 0)$ for a strictly repulsive potential ($f < 0$). In the case of equal masses, we will call the RE for any θ* isosceles *if $\phi_1 = \phi_2$.*

For all of the non-singular cases, the angular velocities satisfy

$$2|\xi||\eta| = f(\theta) \sin\theta/\zeta \tag{3.39}$$

where $\zeta = m_1 \sin 2\phi_1 = m_2 \sin 2\phi_2$, and where $f(\theta) = -dV/dr$ for $V = V(1, e^{i\theta})$ and $r = \cos\theta$.

To understand these rigid motions, one can show that a one-parameter subgroup generated by $(|\xi|j, |\eta|j)$ in $\mathfrak{g}_L \times \mathfrak{g}_R$ acts by rotating the oriented planes $\mathrm{Span}\{1, j\}$ and $\mathrm{Span}\{k, i\}$ through the angle $|\xi| - |\eta|$ and $|\xi| + |\eta|$ with each unit of time respectively. For when $|\xi| = |\eta|$ this gives a simple rotation, and the two particles carry out a cospherical motion contained to the 2-sphere in $\mathrm{Span}\{1, i, k\}$ rotated about the real line with angular velocity $\omega = 2|\xi| = 2|\eta|$. In this case, the theorem above coincides with the RE classification given in [BGNMM18] for the situation on the 2-sphere.

We conclude this classification by remarking that, if we use the expressions in (3.34) and (3.35) for x_1 and x_2, one can express (although this is a fairly unremitting calculation) the explicit values of the generators of the invariant ring in $\mathfrak{s}_-^*/G_R$ at the given RE. It is then possible to verify that the full reduced equations in (3.22) do indeed yield a fixed point for these values, as expected.

3.4.3 Linearisation and the energy-momentum map

Although the RE were classified by first finding the solutions in the left reduced space, the stability results will be derived by directly working in the full reduced space. As a RE is precisely a fixed point of the full reduced equations of motion, we may linearise the system in (3.22) at such a point. Noting that R_1 and R_2 are orthogonal with $\overline{g}_L$ at a RE, and thus, that $k_{13} = k_{23} = 0$, this linear system may be written as the following 8×8 matrix in $\mathbb{R}^8$ with coordinates ordered as $(k_{11}, k_{12}, k_{13}, k_{22}, k_{23}, k_{33}, r, \delta)$,

$$\begin{pmatrix} 0 & 0 & 2f & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -f & 0 & f & 0 & 0 & 0 \\ -r/m_1 & r/m_2 & 2f & 0 & 0 & f & A_1 & -1/m_2 \\ 0 & 0 & 0 & 0 & -2f & 0 & 0 & 0 \\ 0 & -r/m_1 & 0 & r/m_2 & 0 & -f & A_2 & 1/m_1 \\ 0 & 0 & -2r/m_1 & 0 & 2r/m_2 & 0 & 0 & 0 \\ 0 & 0 & 1/m_1 & 0 & -1/m_2 & 0 & 0 & 0 \\ 0 & 0 & A_3 & 0 & A_4 & 0 & 0 & 0 \end{pmatrix}. \qquad (3.40)$$

We have used the abbreviations

$$
\begin{aligned}
A_1 &= +\frac{df}{dr}k_{33} - \left(\frac{k_{11}}{m_1} - \frac{k_{12}}{m_2}\right), \\
A_2 &= -\frac{df}{dr}k_{33} - \left(\frac{k_{12}}{m_1} - \frac{k_{22}}{m_2}\right), \\
A_3 &= +\frac{k_{12}}{m_1} + \frac{k_{22}}{m_2}, \\
A_4 &= -\frac{k_{11}}{m_1} - \frac{k_{12}}{m_2}.
\end{aligned}
$$

As one might expect, the characteristic polynomial of such a large symbolic matrix is fairly horrendous. For this reason, we are forced to make a choice for the force f. The two potentials we will consider are the *gravitational potential*

$$V = -m_1 m_2 \cot\theta \quad \text{with} \quad f = m_1 m_2 \csc^3\theta \tag{3.41}$$

for the planetary 2-body problem, and

$$V = \gamma\cos\theta \quad \text{with} \quad f = -\gamma \tag{3.42}$$

for the potential of the 4-dimensional Lagrange top as in (3.24). With these choices of f computing the characteristic polynomial becomes feasible, although we have enjoyed the aid of software which can handle such symbolic calculations. Before presenting these polynomials, we remark that we should expect at least four of the eigenvalues to be zero. This is because the generic symplectic leaves are 4-dimensional, and the fixed points of the flow should vary continuously as we move through the leaves.

For the 2-body potential the characteristic polynomial in the variable t is given by

$$t^4(c_0 + c_2 t^2 + t^4) \tag{3.43}$$

where

$$c_2 = 2\left(\frac{k_{11}}{m_1^2} + \frac{k_{22}}{m_2^2} + (m_1+m_2)\cot\theta\csc^2\theta\right)$$

and

$$c_0 = \left(\frac{k_{11}}{m_1^2} - \frac{k_{22}}{m_2^2}\right)^2 + 2\cot\theta\csc^2\theta\left[\frac{k_{11}}{m_1}\left(1+\frac{m_2}{m_1}\right) + \frac{k_{22}}{m_2}\left(1+\frac{m_1}{m_2}\right)\right] + \left[(m_1+m_2)\cot\theta\csc^2\theta\right]^2 .$$

Before proceeding to give the characteristic polynomial for the Lagrange top, we note that it follows from (3.34) that $|R_1| = |R_2|$ for when $\theta \neq \pi/2$. We will therefore first give the polynomial for when $\theta \neq \pi/2$ and where we set $k_{11} = k_{22} = |R|^2$. This characteristic polynomial is then

$$t^4\left(t^2 - 2\alpha\gamma\cos(\theta)\right)\left(t^2 + 4\alpha^2|R|^2 - 8\alpha\gamma\cos(\theta)\right). \tag{3.44}$$

For when $\theta = \pi/2$ it is no longer the case that $k_{11} = k_{22}$. The resulting polynomial is then

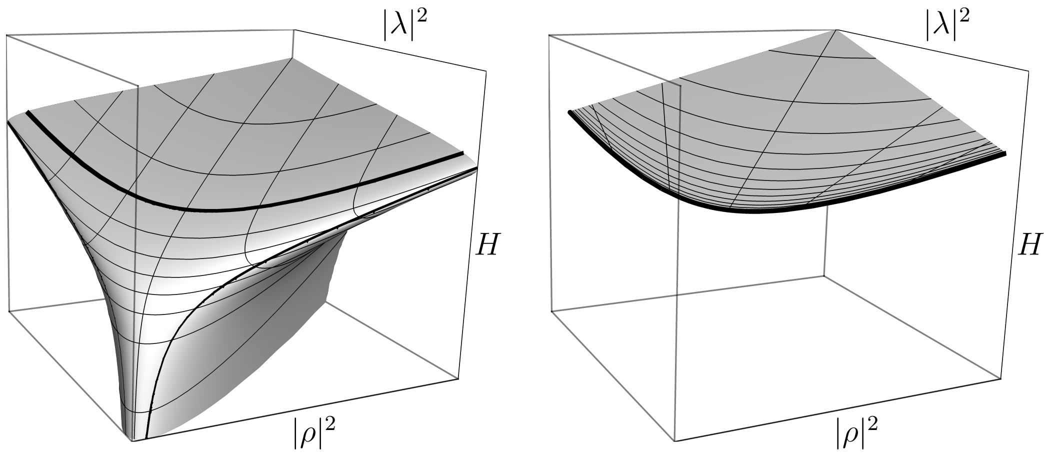
$$t^4\left(t^4 + 2\alpha^2(k_{11}+k_{22})t^2 + \alpha^4(k_{11}-k_{22})^2\right). \tag{3.45}$$

In the next two subsections we analyse the roots of these characteristic polynomials accompanied by images of the energy-momentum map. The RE are equivalently defined to be critical points of this map, and the set of critical values helps illuminate the study of the RE by understanding how they bifurcate and change in nature.

We have opted to omit the full details of how the images of the energy-momentum (actually the energy-Casimir) map are obtained. Ultimately this task is pure calculation, but it does deserve some comments on how to obtain it in practice. Following Theorem 3.4.1 one can parameterise the families of RE by θ and $|\eta|$ alone, or ϕ_1 and $|\eta|$ for the right-angled RE. One can then acquire explicit formulae for H, $|\lambda|^2$ and $|\rho|^2$ in terms of these. This statement conceals an implicit exercise in elementary geometry to express ζ and relate the angles ϕ_1, ϕ_2 to θ in (3.39). The energy-Casimir map simplifies considerably upon introducing the reparameterisation

$$2e^\tau|\eta|^2 = f\sin\theta/\zeta. \tag{3.46}$$

For our purposes it will be enough to state that the image of each family of RE under the energy-Casimir map can be parameterised by θ and τ alone, or ϕ_1 and τ for the right-angled RE.



(a) $0 < \theta < \pi$, isosceles RE

(b) $\theta = \pi/2$, right-angled RE

Figure 3.3: The energy-Casimir bifurcation diagram for the case of equal masses. The axes for both diagrams are to the same scale. For the isosceles component, the coordinate lines running from left to right are of constant θ, and those transversal to them are for constant τ. The line $\theta = \pi/2$ is thickened, and it is along this line that the component for the right-angled RE is attached.

3.4.4 Stability of relative equilibria for the 2-body problem

Surprisingly, despite the complicated appearance of the coefficients in (3.43), all four roots may be solved and compactly written as

$$z_{1,2} = \pm\sqrt{-\left(\frac{\sqrt{k_{11}}}{m_1} + \frac{\sqrt{k_{22}}}{m_2}\right)^2 - (m_1 + m_2)\cot\theta\csc^2\theta}, \tag{3.47}$$

$$w_{1,2} = \pm\sqrt{-\left(\frac{\sqrt{k_{11}}}{m_1} - \frac{\sqrt{k_{22}}}{m_2}\right)^2 - (m_1 + m_2)\cot\theta\csc^2\theta}. \tag{3.48}$$

It is clear that these eigenvalues are all purely imaginary when θ is acute. Furthermore, by writing k_{11} and k_{22} in terms of θ and $|\eta|$ using the forms in (3.31) and (3.34), one can show that

$$\frac{k_{11}}{m_1^2} + \frac{k_{22}}{m_2^2} + (m_1 + m_2)\cot\theta\csc\theta$$
$$= \frac{1}{8|\eta|^2}\left(16|\eta|^4\cos^2\theta\sin^6\theta + (m_1^2 + m_2^2 + 2m_1m_2\cos 2\theta)\right).$$

This expression is always greater than zero, and so the eigenvalue pair $z_{1,2}$ is always purely imaginary and non-zero. On the other hand, the eigenvalue pair $w_{1,2}$ does undergo a transition from imaginary to real. For when the masses are equal, it follows from (3.34) that $k_{11} = k_{22}$ for the isosceles RE, and therefore, that $w_{1,2}$ is a non-zero real pair for θ obtuse, and zero for $\theta = \pi/2$. For the remaining right-angled RE which are not isosceles, as $k_{11} \neq k_{22}$ the $w_{1,2}$ roots are a non-zero imaginary pair.

For non-equal masses it becomes more difficult to describe the transition in reality of the $w_{1,2}$ pair. Unlike the case for motion on the 2-sphere in [BGNMM18], this transition is not determined solely by a critical angle. With reference to the energy-Casimir diagram in Figure 3.4, we argue that this transition occurs along the *fold* in the obtuse component. A rigorous proof of this requires extremely lengthy calculations and so we will merely sketch it here. The image of the energy-Casimir map is given as a surface parameterised by θ and τ. As this surface is folded, the curves of constant $|\lambda|^2$ and $|\rho|^2$ in (θ, τ)-space generically intersect in two points. Along the fold where such a pair of $|\lambda|^2$ and $|\rho|^2$ occurs only once, these two curves must intersect tangentially at a point. The fold may then be characterised by the condition that the Jacobian of $|\lambda|^2$ and $|\rho|^2$ with respect to θ and τ vanishes. This condition turns out to give a quadratic in $\cosh \tau$. As the pair $z_{1,2}$ is always imaginary, the reality of $w_{1,2}$ is determined by the sign of c_0 in (3.43): the pair is imaginary when $c_0 \geq 0$, and real for $c \leq 0$. The expression for c_0 may be written in terms of θ and τ using (3.34) and the reparameterisation in (3.46). Setting c_0 to zero then gives an expression for $\cosh \tau$ in terms of θ, and upon substituting this into the Jacobian, yields zero as desired. It is the portion of the surface above the fold for which $w_{1,2}$ is imaginary, and after crossing the fold, the portion below for which $w_{1,2}$ is real.

Now that we have a picture for the eigenvalues of the linearisation at the RE, we have sufficient conditions for instability but not stability. A helpful result in this direction would be the signature of the Hessian for the Hamiltonian at the RE restricted to a leaf. A definite Hessian implies Lyapunov stability, however we should be pessimistic about this prospect since the Hessian obtained in [BGNMM18] for the linearly stable RE on the 2-sphere have mixed signature. In this situation, stability is again ambiguous. We encourage the reader to observe that the energy-momentum diagrams given in [BGNMM18] coincide with our figures for when $|\lambda| = |\rho|$. This is not a coincidence, but instead a consequence of Sjamaar's Principle [SL91] which, roughly speaking, establishes a symplectomorphism between the reduced spaces with the reduced spaces of isotropy submanifolds equipped with the isotropy action. We can be more explicit about the implementation of this principle for our example.

Proposition 3.4.2. *Let the 2-sphere be given by the unit imaginary quaternions* $S^2 \subset \operatorname{Im}\mathbb{H}$ *and consider the action of* G *on* $N = T^*S^2 \times T^*S^2$ *with momentum map* J*. For any* $\Omega \in \mathfrak{g}^*$ *the reduced space* N_Ω *is symplectomorphic to* $M_{\Omega,-\Omega}$*.*

Proof. For (g_1, p_1, g_2, p_2) in N the value of J is the angular momentum

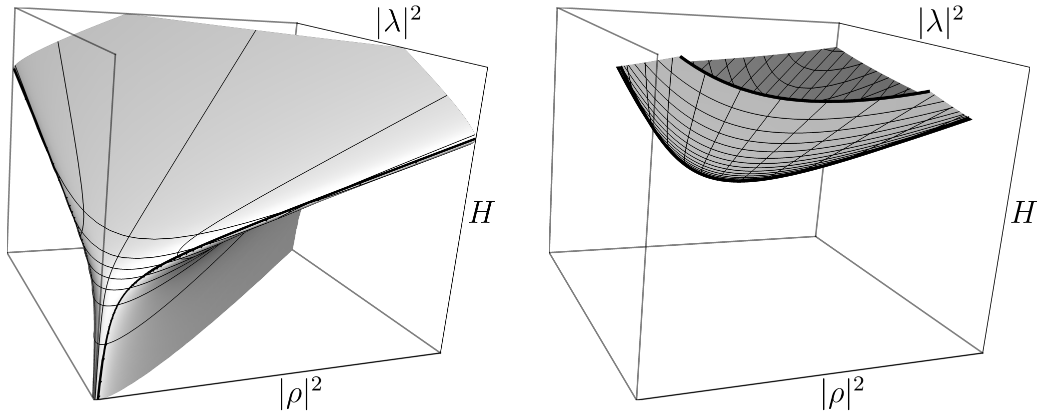
$$\Omega = (g_1 \times p_1) + (g_2 \times p_2).$$

As these quaternions are imaginary, we have $g_i^{-1} = -g_i$. By expanding the cross product as multiplication between quaternions and taking the complex conjugate of the R_i, one can show that $2\Omega = \lambda - \rho$ and $\lambda = -\rho$. As the isotropy subgroup fixing a 2-sphere in $\mathbb{H}$ is trivial, the symplectic submanifold $N \hookrightarrow M$ descends to the orbit spaces to give an inclusion $N/G \hookrightarrow M/(G_L \times G_R)$. From Proposition 3.2.1, any point with $|\lambda| = |\rho|$ is cospherical, and thus, at the level of orbit spaces there is a bijection between the orbit reduced spaces $J^{-1}(\mathcal{O}_\Omega)/G \to J_{L,R}^{-1}(\mathcal{O}_\Omega \times \mathcal{O}_{-\Omega})/G_L \times G_R$ induced by the canonical inclusion. As all of the maps here are Poisson, it follows that this bijection between symplectic leaves is a symplectomorphism. □

One sees from this proof that the reduced Hamiltonian on $M_{\Omega,-\Omega}$ pulls back to give the Hamiltonian on N_Ω. It follows that the Hessian at the RE must be the same, and therefore, that the Hessian of the RE in the energy-Casimir diagrams in Figures 3.3 and 3.4 agrees with that in [BGNMM18] for when $|\lambda| = |\rho|$. As the Hessian varies continuously with the relative equilibria, and that the signature can only change when it crosses a zero eigenvalue, we may apply a continuity argument to extend the Hessian over all of the RE with non-zero eigenvalues. This observation, combined with the above discussion concerning the linearisation provides the proof to the following theorem.

Theorem 3.4.3. *For the 2-body problem on the 3-sphere with the gravitational potential, we have the following stability results for the RE:*

- *All acute RE are linearly stable, with the signature of the Hessian of the Hamiltonian being* $(+, +, -, -)$*.*



(a) $0 < \theta < \pi/2$, acute RE (b) $\pi/2 < \theta < \pi$, obtuse RE

Figure 3.4: The energy-Casimir bifurcation diagram for non-equal masses, specifically for $m_1 = 3$ and $m_2 = 2$. The axes for both diagrams are to the same scale. The coordinate lines running from left to right are of constant θ, and those transversal to them are for constant τ. We have deliberately removed an upper section of the obtuse component to demonstrate that the surface is folded along a cusp. The lines of constant θ above the fold are for θ smaller than those lines below.

– *For when the masses are equal, all right-angled RE which are not isosceles are linearly stable with signature* $(+, +, -, -)$.

– *Obtuse RE which are above the fold in Figure 3.4(b) are linearly stable with signature* $(+, +, -, -)$. *Obtuse RE under the fold are linearly unstable with signature* $(+, +, +, -)$. *For when the masses are equal there is no such fold, and all obtuse RE are linearly unstable with signature* $(+, +, +, -)$.

3.4.5 Stability of the relative equilibria for the Lagrange top

Before jumping straight into a stability analysis, we pause to review some of our terminology for the Lagrange top RE. As the angle θ corresponds to the angle the body axis makes with the vertical, we will alternatively refer to acute and obtuse RE by *upright* and *downward* respectively. Furthermore, right-angled RE with $\theta = \pi/2$ might more fittingly be described as being *horizontal*. Alternatively of course, the potential in (3.42) applies equally well to the case of two particles on a sphere which are repelled by each other. In this case, the old terminology is perfectly applicable.

From the characteristic polynomial in (3.44), one can immediately see from the root $t^2 = 2\alpha\gamma\cos\theta$ that RE which are upright are unstable. This is in marked contrast to the ordinary Lagrange top where a sufficiently quickly spinning top is stable when standing vertical (the sleeping top). The second pair of roots may be analysed by again using (3.34) to write $|R|^2 = |R_1|^2 = |R_2|^2$ in terms of θ and $|\eta|$ which gives

$$4\alpha^2|R|^2 - 8\alpha\gamma\cos\theta = 4|\eta|^2 + \frac{\alpha^2\gamma^2}{|\eta|^2} - 4\alpha\gamma\cos\theta. \tag{3.49}$$

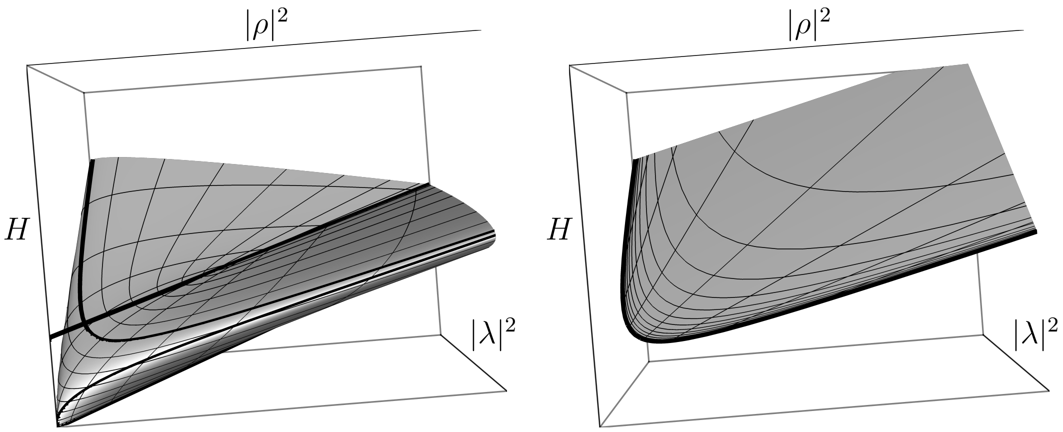
This is always strictly positive for θ greater than zero, and therefore always corresponds to imaginary eigenvalues. For the right-angled RE we have the characteristic polynomial in (3.45), for which the roots of the quartic factor are equal to

$$\pm\sqrt{-\alpha^2(\sqrt{k_{11}} \pm \sqrt{k_{22}})^2}. \tag{3.50}$$

Consequently, we see that these eigenvalues are all non-zero and imaginary away from the isosceles $k_{11} = k_{22}$ branch.

Unlike for the 2-body problem, where we ignored the singular RE since the Hamiltonian is not defined for $\theta = 0$ or π, the energy-Casimir diagram for the Lagrange top contains two 'threads' corresponding to these families of RE. From Figure 3.5 for the 'isosceles' RE, one sees that the lines of constant θ converge to a single thread on top of the surface as $\theta \to 0$. This thread corresponds to those configurations of the top held vertically upright and spinning about its axis. As the spin decreases, that is, as $k_{11} = k_{22} = |R|^2$ decreases below $2\gamma/\alpha$, the second root pair in (3.44) transitions from imaginary to real, and the thread detaches from the surface and extends outwards as an isolated thread until $|R| = 0$, where the top is motionless. This is entirely analogous to the case for the ordinary Lagrange top as shown in [CB97] and is a mathematical realisation of 'gyroscopic stabilisation'. It cannot be seen in our picture, however as θ tends to π, the lines of constant θ converge underneath the surface to a thread which is not isolated, corresponding to those motions for when the body is hanging vertically downwards.

As with the 2-body problem, we remark that taking the slice through the energy-Casimir diagram for $|\rho| = |\lambda|$ in Figure 3.5 gives the same diagram as in [BMK04], where they also consider the 2-body problem on the 2-sphere for the same potential.



(a) $0 < \theta < \pi$, 'isosceles' (b) $\theta = \pi/2$, horizontal RE

Figure 3.5: The energy-Casimir bifurcation diagram for the 4-dimensional Lagrange top, specifically for $\alpha = 2$. The axes for both diagrams are to the same scale. For the component on the left, the coordinate lines emanating away from the origin are of constant τ, and those transversal to them are of constant θ. The thickened line is that for $\theta = \pi/2$, and it is along this branch that the additional component of horizontal RE is attached.

A linearised stability analysis is sufficient for deducing instability, but not conclusive for stability. Fortunately, the existence of the additional integral I in (3.27) on the full reduced space will allow us to obtain the strongest possible results for stability. We begin by claiming that all downward RE are not only critical points of the Hamiltonian, but also of I. To see this, observe that for $\theta > \pi/2$, the linearisation admits two distinct non-zero imaginary eigenvalue pairs. If such a point were not a critical point of I, then in a neighbourhood of this point we could introduce coordinates which include I as a coordinate function. As $\{H, I\} = 0$, the Hamiltonian in these coordinates is independent of I, and therefore a change in the I coordinate away from a RE would also give a RE. This is not compatible with the non-zero eigenvalues.

Therefore, at such RE the differentials dH and dI are both zero. We also claim that their Hessians d^2H and d^2I are linearly independent. This is equivalent to showing that the linearisations of the flows generated by H and I at the fixed point are independent. With help from Table 3.1 one can find the flow generated by the integral I. In particular, one can show

$$\dot{k}_{11} = \{k_{11}, I\} = 4\gamma(k_{13}k_{12} - k_{11}k_{23}),$$
$$\dot{k}_{22} = \{k_{22}, I\} = 4\gamma(k_{13}k_{22} - k_{12}k_{23}).$$

By linearising these at a RE, and comparing with the first and fourth rows of the matrix in (3.40), we see that the two linearisations are indeed linearly independent for when $k_{11}, k_{12}, k_{22} \neq 0$, and therefore so too are d^2H and d^2I.

The quadratic forms d^2H and d^2I are both well defined on the tangent space of the symplectic leaf at a RE since it is a critical point for each of them. The Lie algebra of such quadratic forms with respect to the Poisson bracket is isomorphic to the symplectic Lie algebra Symp$(4;\mathbb{R})$. Furthermore, as $\{H, I\} = 0$, the quadratic forms also commute, and as d^2H has distinct eigenvalues, and is linearly independent from d^2I, it follows that they span a Cartan subalgebra of Symp$(4;\mathbb{R})$. Up to conjugacy by canonical transformations there are only four such Cartan subalgebras: center-center, saddle-centre, saddle-saddle, and focus-focus [BM99]. As the eigenvalues of the linearisation are all purely imaginary, this forces it to be of center-center type. It follows from a normal-form result in [LU94] that there exist Darboux coordinates (q_1, p_1, q_2, p_2) in a neighbourhood of the RE (which may be taken to be the origin) where

$$H = a(q_1^2 + p_1^2) + b(q_2^2 + p_2^2) + \dots$$
$$I = c(q_1^2 + p_1^2) + d(q_2^2 + p_2^2) + \dots$$

Here the dots denote terms of cubic order in the coordinates, and where we have further supposed that I and H are zero at the origin. As the quadratic forms are linearly independent, we can find $x, y \in \mathbb{R}$ such that the function $F = xH + yI$ has d^2F positive definite at the RE. The function F therefore has a minimum at this point and thus, for any small $\delta > 0$ there exists an ε for which $F^{-1}(\epsilon)$ is contained to a ball of radius δ. It follows then from

$$H^{-1}(\varepsilon_1) \cap I^{-1}(\varepsilon_2) \subset F^{-1}(x\varepsilon_1 + y\varepsilon_2)$$

that the level sets of H and I are also contained to arbitrarily small neighbourhoods around the RE. As the flow is contained to these level sets, small perturbations away from these RE result in motions contained to tori which remain close to the RE.

To be completely watertight, the single RE which has evaded our argument so far is the one corresponding to the body hanging vertically downward and motionless as $k_{11} = k_{12} = k_{22} = 0$, and thus the two Hessians may not be independent. For this point, Lyapunov stability is a consequence of it being a global minimum of the Hamiltonian. We can now pull all of this together into a theorem.

Theorem 3.4.4. *For the 4-dimensional Lagrange top the following stability results apply to the relative equilibria in the full reduced space.*

- *All upright relative equilibria, that is, those with $\theta < \pi/2$, are linearly unstable.*
- *All hanging relative equilibria with $\theta > \pi/2$, and all horizontal relative equilibria with $\theta = \pi/2$, excluding those which are isosceles, are Lyapunov stable.*

3.5 Conclusions

It is natural to ask how we might extend these results. In this regard it is crucial to note that the fundamental idea upon which this work rests is the ‘accidental’ isomorphism between $\mathfrak{g} \times \mathfrak{g}$ and $\mathfrak{so}(4)$. It is thanks to this that we have the double cover over $SO(4)$ and the connection with the Lagrange top, and the commuting left and right actions which allow us to reduce in stages. It is because of this ‘accident’ that our work does not generalise to more bodies, or to the negative curvature case of the 2-body problem on hyperbolic 3-space. For this space, the symmetry group $SO(1,3)$ is double covered by $SL(2;\mathbb{C})$. It is clear that a different approach must be taken to resolve this case. It is however true that hyperbolic space may be equipped with a group structure which is compatible with the metric with respect to left translations alone [GNMPCRO16]. In light of this, it would still be possible to obtain an analogous left reduced space for the case of negative curvature.

We do however obtain a fairly straightforward generalisation of our work by replacing the algebra of real quaternions with the split quaternions. This alteration results in the 2-body problem on $SL(2;\mathbb{R})$ which is the unit ‘sphere’ in this algebra. Everything then proceeds almost exactly the same, the full symmetry group is $SO(2,2)$ and this is double covered by two 'spheres' and we have again two commuting left and right actions. This alteration is entirely due to another 'accidental' isomorphism between $\mathfrak{sl}(2;\mathbb{R}) \times \mathfrak{sl}(2;\mathbb{R})$ and $\mathfrak{so}(2,2)$. In a similar fashion we could even push this idea further and replace the quaternions with the biquaternions and consider the 2-body problem on $SL(2;\mathbb{C})$.

Another route of study concerns the more general problem of dynamics on a cotangent bundle of a group G which is symmetric with respect to both the left and right translations by a given subgroup H. In addition to the example we have dealt with, a famous example of such a system are the Riemannian ellipsoids [Rie61]. This system concerns the motion of a self-gravitating distribution of mass whose configuration is given by an element of $G = SL(3;\mathbb{R})$ which is symmetric by the left and right actions of $H = SO(3)$. This system is comprehensively treated in the work of Chandrasekhar in [Cha67], and a more modern Hamiltonian account may be found in [RS99]. The famous work of Riemann concerns the classification of the relative equilibria, and one wonders whether our use of reduction by stages could be applied as a possible alternative approach.

It would be interesting to study the limit as one particle's mass dominates the other. This should be expected to approach what is referred to as the restricted 2-body problem on the sphere [CM99]. In particular, it would be interesting to see what the flow on the full reduced space limits to. Furthermore, given that we have the Poisson structure on the full reduced space, it would be nice to see if this offers any use in demonstrating the non-integrability for the 2-body problem (see [Shc06] and [MP03]) or whether additional integrable systems can be found for different potentials. One might hope that this would connect with the substantial literature that exists for integrable systems on $SO(4)$.

Finally, our work concerning the 2-body problem on the sphere cannot be considered complete, and there remain interesting unresolved questions. The nature of the full reduced spaces is one such question. One would like to say more about their geometry, to describe the fibres of the energy-momentum map, and the invariant integral manifolds. Moreover, as the Lagrange top is integrable, one can also ask questions about the foliation of the reduced spaces into invariant tori, the image of the momentum map, and the monodromy of these tori. The stability of the RE for the 2-body problem also remains an open question, and as in [BGNMM18], we leave the door open for the use of sophisticated KAM methods to strengthen the stability results.

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