Complex structure on the smooth dual of GL(n)

Brodzki, Jacek and Plymen, Roger

2002

MIMS EPrint: **2006.105**

ISSN 1749-9097

Complex structure on the smooth dual of $GL(n)$

Jacek Brodzki[1] and Roger Plymen

Abstract. Let G denote the p-adic group $GL(n)$, let $\Pi(G)$ denote the smooth dual of G, let $\Pi(\Omega)$ denote a Bernstein component of $\Pi(G)$ and let $H(\Omega)$ denote a Bernstein ideal in the Hecke algebra $H(G)$. With the aid of Langlands parameters, we equip $\Pi(\Omega)$ with the structure of complex algebraic variety, and prove that the periodic cyclic homology of $\mathcal{H}(\Omega)$ is isomorphic to the de Rham cohomology of the Bernstein component $\Pi(\Omega)$. We show how the structure of the variety $\Pi(\Omega)$ is related to Xi's affirmation of a conjecture of Lusztig for $GL(n, \mathbb{C})$. The smooth dual $\Pi(G)$ admits a deformation retraction onto the tempered dual $\Pi^t(G)$.

2000 Mathematics Subject Classification: 46L80 22E50 46L87 11S37
Keywords and Phrases: Langlands correspondence, p-adic $GL(n)$, Baum-Connes map, smooth dual, tempered dual.

Introduction

The use of unramified quasicharacters to create a complex structure is well established in number theory. The group of unramified quasicharacters of the idele class group of a global field admits a complex structure: this complex structure provides the background for the functional equation of the zeta integral $Z(\omega, \Phi)$, see [39, Theorem 2, p. 121].

Let now G be a reductive p-adic group and let M be a Levi subgroup of G. Let $\Pi^{sc}(M)$ denote the set of equivalence classes of irreducible supercuspidal representations of M. Harish-Chandra

[1]Partially supported by a Leverhulme Trust Fellowship

creates a complex structure on the set $\Pi^{sc}(M)$ by using unramified quasicharacters of M [16, p.84]. This complex structure provides the background for the Harish-Chandra functional equations [16, p. 91].

Bernstein considered the set $\Omega(G)$ of all conjugacy classes of pairs (M, σ) where M is a Levi subgroup of G and σ is an irreducible supercuspidal representation of M. Making use of unramified quasicharacters of M, Bernstein gave the set $\Omega(G)$ the structure of a complex algebraic variety. Each irreducible component Ω of $\Omega(G)$ has the structure of a complex affine algebraic variety [5].

Let $\Pi(G)$ denote the set of equivalence classes of irreducible smooth representations of G. We will call $\Pi(G)$ the smooth dual of G. Bernstein defines the *infinitesimal character* from $\Pi(G)$ to $\Omega(G)$:

$$inf.ch. : \Pi(G) \to \Omega(G).$$

The infinitesimal character is a finite-to-one map from the set $\Pi(G)$ to the variety $\Omega(G)$.

Let F be a nonarchimedean local field and from now on let $G = GL(n) = GL(n, F)$. Let now W_F be the Weil group of the local field F, then W_F admits unramified quasicharacters, namely those which are trivial on the inertia subgroup I_F. Making use of the unramified quasicharacters of W_F, we introduced in [8] a complex structure on the set of Langlands parameters for $GL(n)$. In view of the local Langlands correspondence for $GL(n)$ this creates, by transport of structure, a complex structure on the smooth dual of $GL(n)$.

In Section 1 of this article, we describe in detail the complex structure on the set of L-parameters for $GL(n)$. We prove that the smooth dual $\Pi(GL(n))$ has the structure of complex manifold. The local L-factors $L(s, \pi)$ then appear as complex valued functions of several complex variables. We illustrate this with the local L-factors attached to the unramified principal series of $GL(n)$.

The complex structure on $\Pi(GL(n))$ is well adapted to the periodic cyclic homology of the Hecke algebra $\mathcal{H}(GL(n))$. The identical structure arises in the work of Xi on Lusztig's conjecture [40]. Let W be the extended affine Weyl group associated to $GL(n, \mathbb{C})$, and let J be the associated based ring (asymptotic algebra) [27, 40]. Xi confirms Lusztig's conjecture and proves that $J \otimes_{\mathbb{Z}} \mathbb{C}$ is Morita equivalent to the coordinate ring of the complex algebraic variety

$\widetilde{(\mathbb{C}^\times)^n/S_n}$, the *extended* quotient by the symmetric group S_n of the complex n-dimensional torus $(\mathbb{C}^\times)^n$. In Section 2 we describe the theorem of Xi on the structure of the based ring J.
So the structure of extended quotient, which runs through our work, occurs in the work of Xi *at the level of algebras*. The link with our work is now provided by the theorem of Baum and Nistor [3, 4]

$$\mathrm{HP}_*(\mathcal{H}(n,q)) \simeq \mathrm{HP}_*(J)$$

where $\mathcal{H}(n,q)$ is the associated extended affine Hecke algebra.
Let Ω be a component in the Bernstein variety $\Omega(GL(n))$, and let $\mathcal{H}(G) = \bigoplus \mathcal{H}(\Omega)$ be the Bernstein decomposition of the Hecke algebra.
Let

$$\Pi(\Omega) = (inf.ch.)^{-1}\Omega.$$

Then $\Pi(\Omega)$ is a smooth complex algebraic variety with finitely many irreducible components. We have the following Bernstein decomposition of $\Pi(G)$:

$$\Pi(G) = \bigsqcup \Pi(\Omega).$$

Let M be a compact C^∞ manifold. Then $C^\infty(M)$ is a Fréchet algebra, and we have Connes' fundamental theorem [14, Theorem 2, p. 208]:

$$\mathrm{HP}_*(C^\infty(M)) \cong \mathrm{H}^*(M;\mathbb{C}).$$

Now the ideal $\mathcal{H}(\Omega)$ is a purely algebraic object, and, in computing its periodic cyclic homology, we would hope to find an algebraic variety to play the role of the manifold M. This algebraic variety is $\Pi(\Omega)$.

THEOREM 0.1. *Let Ω be a component in the Bernstein variety $\Omega(G)$. Then the periodic cyclic homology of $\mathcal{H}(G)$ is isomorphic to the periodised de Rham cohomology of $\Pi(\Omega)$:*

$$\mathrm{HP}_*(\mathcal{H}(\Omega)) \cong \mathrm{H}^*(\Pi(\Omega);\mathbb{C}).$$

This theorem constitutes the main result of Section 3, which is then used to show that the periodic cyclic homology of the Hecke algebra of $GL(n)$ is isomorphic to the periodic cyclic homology of the Schwartz algebra of $GL(n)$. We also provide an explicit numerical formula for the dimension of the periodic cyclic homology of $\mathcal{H}(\Omega)$ in terms of certain natural number invariants attached to Ω.

The smooth dual $\Pi(GL(n))$ has a natural stratification-by-dimension. We compare this stratification with the Schneider-Zink stratification [34]. Stratification-by-dimension is finer than the Schneider-Zink stratification, see Section 3.

A *scheme* X is a topological space, called the *support* of X and denoted $|X|$, together with a sheaf $\mathcal{O}_X$ of rings on X, such that the pair $(|X|, \mathcal{O}_X)$ is locally affine, see [15, p. 21]. The smooth dual $\Pi(G)$ determines a reduced scheme, see [18, Prop. 2.6]. If S is the reduced scheme determined by the Bernstein variety $\Omega(G)$, then $\Pi(G)$ is a *scheme over* S, i.e. a scheme together with a morphism $\Pi(G) \to S$. This morphism is the q-projection introduced in [8]:

$$\pi_q : \Pi(G) \to S.$$

In Section 4 we give a detailed description of the q-projection and prove that the q-projection is a finite morphism.

¿From the point of view of noncommutative geometry it is natural to seek the spaces which underlie the noncommutative algebras $\mathcal{H}(G)$ and $\mathcal{S}(G)$. The space which underlies the Hecke algebra $\mathcal{H}(G)$ is the complex manifold $\Pi(G)$. The space which underlies the Schwartz algebra is the Harish-Chandra parameter space, which is a disjoint union of compact orbifolds. In Section 5 we construct a deformation retraction of the smooth dual onto the tempered dual. We view this deformation retraction as a geometric counterpart of the Baum-Connes assembly map for $GL(n)$.

In Section 6 we track the fate of supercuspidal representations of G through the diagram which appears in Section 5. In particular, the index map μ manifests itself as an example of Ahn reciprocity.

We would like to thank Paul Baum, Alain Connes, Jean-Francois Dat and Nigel Higson for many valuable conversations. Jacek Brodzki was supported in part by a Leverhulme Trust Fellowship. This article was completed while Roger Plymen was at IHES, France.

1. The complex structure on the smooth dual of $GL(n)$

The field F is a nonarchimedean local field, so that F is a finite extension of $\mathbb{Q}_p$, for some prime p or F is a finite extension of the function field $\mathbb{F}_p((x))$. The residue field k_F of F is the quotient

$\mathbf{o}_F/\mathbf{m}_F$ of the ring of integers $\mathbf{o}_F$ by its unique maximal ideal $\mathbf{m}_F$. Let q be the cardinality of k_F.
The essence of local class field theory, see [29, p.300], is a pair of maps

$$(d : G \longrightarrow \widehat{\mathbb{Z}}, v : F^\times \longrightarrow \mathbb{Z})$$

where G is a profinite group, $\widehat{\mathbb{Z}}$ is the profinite completion of $\mathbb{Z}$, and v is the valuation.
Let $\overline{F}$ be a separable algebraic closure of F. Then the absolute Galois group $G(\overline{F}|F)$ is the projective limit of the finite Galois groups $G(E|F)$ taken over the finite extensions E of F in $\overline{F}$. Let $\widetilde{F}$ be the maximal unramified extension of F. The map d is in this case the projection map

$$d : G(\overline{F}|F) \longrightarrow G(\tilde{F}|F) \cong \hat{\mathbb{Z}}$$

The group $G(\tilde{F}|F)$ is procyclic. It has a single topological generator: the Frobenius automorphism ϕ_F of $\tilde{F}|F$. The Weil group W_F is by definition the pre-image of $< \phi_F >$ in $G(\overline{F} \mid F)$. We thus have the surjective map

$$d : W_F \longrightarrow \mathbb{Z}$$

The pre-image of 0 is the inertia group I_F. In other words we have the following short exact sequence

$$1 \to I_F \to W_F \to \mathbb{Z} \to 0$$

The group I_F is given the profinite topology induced by $G(\overline{F}|F)$. The topology on the Weil group W_F is dictated by the above short exact sequence. The Weil group W_F is a locally compact group with maximal compact subgroup I_F. The map

$$W_F \longrightarrow G(\tilde{F}|F)$$

is a continuous homomorphism with dense image.
A detailed account of the Weil group for local fields may be found in [37]. For a topological group G we denote by G^{ab} the quotient $G^{\mathrm{ab}} = G/G^c$ of G by the closure G^c of the commutator subgroup of G. Thus G^{ab} is the maximal abelian Hausdorff quotient of G. The local reciprocity laws [29, p.320]

$$r_{E|F} : G(E|F)^{ab} \cong F^\times/N_{E|F}E^\times$$

now create an isomorphism [30, p.69]:

$$r_F : W_F^{ab} \cong F^\times$$

We have $W_F = \sqcup \Phi^n I_F, n \in \mathbb{Z}$. The Weil group is a locally compact, totally disconnected group, whose maximal compact subgroup is I_F. This subgroup is also open. There are three models for the Weil-Deligne group.

One model is the crossed product $W_F \ltimes \mathbb{C}$, where the Weil group acts on $\mathbb{C}$ by $w \cdot x = \|w\|x$, for all $w \in W_F$ and $x \in \mathbb{C}$.

The action of W_F on $\mathbb{C}$ extends to an action of W_F on $SL(2, \mathbb{C})$. The semidirect product $W_F \ltimes SL(2, \mathbb{C})$ is then isomorphic to the direct product $W_F \times SL(2, \mathbb{C})$, see [22, p.278]. Then a complex representation of $W_F \times SL(2, \mathbb{C})$ is determined by its restriction to $W_F \times SU(2)$, where $SU(2)$ is the standard compact Lie group.

¿From now on, we shall use this model for the Weil-Deligne group:

$$\mathcal{L}_F = W_F \times SU(2).$$

DEFINITION 1.1. An L-parameter is a continuous homomorphism

$$\phi : \mathcal{L}_F \to GL(n, \mathbb{C})$$

such that $\phi(w)$ is semisimple for all $w \in W_F$. Two L-parameters are equivalent if they are conjugate under $GL(n, \mathbb{C})$. The set of equivalence classes of L-parameters is denoted $\Phi(G)$.

DEFINITION 1.2. A representation of G on a complex vector space V is *smooth* if the stabilizer of each vector in V is an open subgroup of G. The set of equivalence classes of irreducible smooth representations of G is the *smooth dual* $\Pi(G)$ of G.

THEOREM 1.3. *Local Langlands Correspondence for $GL(n)$. There is a natural bijection between $\Phi(GL(n))$ and $\Pi(GL(n))$.*

The naturality of the bijection involves compatibility of the L-factors and ϵ-factors attached to the two types of objects.

The local Langlands conjecture for $GL(n)$ was proved by Laumon, Rapoport and Stuhler [25] when F has positive characteristic and by Harris-Taylor [17] and Henniart [19] when F has characteristic zero.

We recall that a *matrix coefficient* of a representation ρ of a group G on a vector space V is a function on G of the form $f(g) = \langle \rho(g)v, w \rangle$, where $v \in V$, $w \in V^*$, and V^* denotes the dual space of

V. The inner product is given by the duality between V and V^*. A representation ρ of G is called *supercuspidal* if and only if the support of every matrix coefficient is compact modulo the centre of G.

Let $\tau_j = \text{spin}(j)$ denote the $(2j+1)$-dimensional complex irreducible representation of the compact Lie group $SU(2)$, $j = 0, 1/2, 1, 3/2, 2, \ldots$.

For $GL(n)$ the local Langlands correspondence works in the following way.

- Let ρ be an irreducible representation of the Weil group W_F. Then $\pi_F(\rho \otimes 1)$ is an irreducible supercuspidal representation of $GL(n)$, and every irreducible supercuspidal representation of $GL(n)$ arises in this way. If $\det(\rho)$ is a unitary character, then $\pi_F(\rho \otimes 1)$ has unitary central character, and so is pre-unitary.
- We have $\pi_F(\rho \otimes \text{spin}(j)) = Q(\Delta)$, the Langlands quotient associated to the segment $\{|\ |^{-(j-1)/2}\pi_F(\rho), \ldots, |\ |^{(j-1)/2}\pi_F(\rho)\}$. If $\det(\rho)$ is unitary, then $Q(\Delta)$ is in the discrete series. In particular, if $\rho = 1$ then $\pi_F(1 \otimes \text{spin}(j))$ is the Steinberg representation $St(2j+1)$ of $GL(2j+1)$.
- If ϕ is an L-parameter for $GL(n)$ then $\phi = \phi_1 \oplus \ldots \oplus \phi_m$ where $\phi_j = \rho_j \otimes \text{spin}(j)$. Then $\pi_F(\rho)$ is the Langlands quotient $Q(\Delta_1, \ldots, \Delta_m)$. If $\det(\rho_j)$ is a unitary character for each j, then $\pi_F(\phi)$ is a tempered representation of $GL(n)$.

This correspondence creates, as in [23, p. 381], a natural bijection

$$\pi_F : \Phi(GL(n)) \to \Pi(GL(n)).$$

A quasi-character $\psi : W_F \to \mathbb{C}^\times$ is *unramified* if ψ is trivial on the inertia group I_F. Recall the short exact sequence

$$0 \to I_F \to W_F \xrightarrow{d} \mathbb{Z} \to 0$$

Then $\psi(w) = z^{d(w)}$ for some $z \in \mathbb{C}^\times$. Note that ψ is not a *Galois* representation unless z has finite order in the complex torus $\mathbb{C}^\times$, see [37]. Let $\Psi(W_F)$ denote the group of all unramified quasi-characters of W_F. Then

$$\begin{array}{rcl} \Psi(W_F) & \simeq & \mathbb{C}^\times \\ \psi & \mapsto & z \end{array}$$

Each L-parameter $\phi : \mathcal{L}_F \to GL(n, \mathbb{C})$ is of the form $\phi_1 \oplus \cdots \oplus \phi_m$ with each ϕ_j irreducible. Each irreducible L-parameter is of the form $\rho \otimes \mathrm{spin}(j)$ with ρ an irreducible representation of the Weil group W_F.

DEFINITION 1.4. The orbit $\mathcal{O}(\phi) \subset \Phi_F(G)$ is defined as follows

$$\mathcal{O}(\phi) = \{\bigoplus_{r=1}^{m} \psi_r \phi_r \mid \psi_r \in \Psi(W_F), 1 \le r \le m\}$$

where each ψ_r is an unramified quasi-character of W_F.

DEFINITION 1.5. Let $\det \phi_r$ be a unitary character, $1 \le r \le m$ and let $\phi = \phi_1 \oplus \ldots \oplus \phi_m$. The compact orbit $\mathcal{O}^t(\phi) \subset \Phi^t(G)$ is defined as follows:

$$\mathcal{O}^t(\phi) = \{\bigoplus_{r=1}^{m} \psi_r \phi_r \mid \psi_r \in \Psi(W_F), 1 \le r \le m\}$$

where each ψ_r is an unramified unitary character of W_F.

We note that $I_F \times SU(2) \subset W_F \times SU(2)$ and in fact $I_F \times SU(2)$ is the maximal compact subgroup of $\mathcal{L}_F$. Now let ϕ be an L-parameter. Moving (if necessary) to another point in the orbit $\mathcal{O}(\phi)$ we can write ϕ in the canonical form

$$\phi = \phi_1 \oplus \ldots \oplus \phi_1 \oplus \ldots \oplus \phi_k \oplus \ldots \oplus \phi_k$$

where ϕ_1 is repeated l_1 times, ..., ϕ_k is repeated l_k times, and the representations

$$\phi_j|_{I_F \times SU(2)}$$

are irreducible and pairwise inequivalent, $1 \le j \le k$. We will now write $k = k(\phi)$. This natural number is an invariant of the orbit $\mathcal{O}(\phi)$. We have

$$\mathcal{O}(\phi) = \mathrm{Sym}^{l_1}\mathbb{C}^\times \times \ldots \times \mathrm{Sym}^{l_k}\mathbb{C}^\times$$

the product of symmetric products of $\mathbb{C}^\times$.

THEOREM 1.6. *The set $\Phi(GL(n))$ has the structure of complex algebraic variety. Each irreducible component $\mathcal{O}(\phi)$ is isomorphic to the product of a complex affine space and a complex torus*

$$\mathcal{O}(\phi) = \mathbb{A}^l \times (\mathbb{C}^\times)^k$$

where $k = k(\phi)$.

Proof. Let $Y = \mathbb{V}(x_1y_1 - 1, \ldots, x_ny_n - 1) \subset \mathbb{C}^{2n}$. Then Y is a Zariski-closed set in $\mathbb{C}^{2n}$, and so is an affine complex algebraic variety. Let $X = (\mathbb{C}^\times)^n$. Set $\alpha : Y \to X, \alpha(x_1, y_1, \ldots, x_n, y_n) = (x_1, \ldots, x_n)$ and $\beta : X \to Y, \beta(x_1, \ldots, x_n) = (x_1, x_1^{-1}, \ldots, x_n, x_n^{-1})$. So X can be embedded in affine space $\mathbb{C}^{2n}$ as a Zariski-closed subset. Therefore X is an affine algebraic variety, as in [36, p.50].

Let $A = \mathbb{C}[X]$ be the coordinate ring of X. This is the restriction to X of polynomials on $\mathbb{C}^{2n}$, and so $A = \mathbb{C}[X] = \mathbb{C}[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$, the ring of Laurent polynomials in n variables $x_1, \ldots, x_n$. Let S_n be the symmetric group, and let Z denote the quotient variety X/S_n. The variety Z is an affine complex algebraic variety.

The coordinate ring of Z is

$$\mathbb{C}[Z] \simeq \mathbb{C}[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]^{S_n}.$$

Let σ_i, $i = 1, \ldots, n$ be the elementary symmetric polynomials in n variables. Then from the last isomorphism we have

$$\begin{aligned}
\mathbb{C}[Z] &\simeq \mathbb{C}[x_1, \ldots, x_n]^{S_n} \otimes \mathbb{C}[\sigma_n^{-1}] \\
&\simeq \mathbb{C}[\sigma_1, \ldots, \sigma_n] \otimes \mathbb{C}[\sigma_n^{-1}] \\
&\simeq \mathbb{C}[\sigma_1, \ldots, \sigma_{n-1}] \otimes \mathbb{C}[\sigma_n, \sigma_n^{-1}] \\
&\simeq \mathbb{C}[\mathbb{A}^{n-1}] \otimes \mathbb{C}[\mathbb{A} - \{0\}] \\
&\simeq \mathbb{C}[\mathbb{A}^{n-1} \times (\mathbb{A} - \{0\})]
\end{aligned}$$

where $\mathbb{A}^n$ denotes complex affine n-space. The coordinate ring of the quotient variety $\mathbb{C}^{\times n}/S_n$ is isomorphic to the coordinate ring of $\mathbb{A}^{n-1} \times (\mathbb{A} - \{0\})$. Now the categories of affine algebraic varieties and of finitely generated reduced $\mathbb{C}$-algebras are equivalent, see [36, p.26]. Therefore the variety $\mathbb{C}^{\times n}/S_n$ is isomorphic to the variety $\mathbb{A}^{n-1} \times (\mathbb{A} - \{0\})$.

Consider $\mathbb{A} - \{0\} = \mathbb{V}(f)$ where $f(x) = x_1x_2 - 1$. Then $\partial f/\partial x_1 = x_2 \neq 0$ and $\partial f/\partial x_2 = x_1 \neq 0$ on the variety $\mathbb{V}(f)$. So $\mathbb{A} - \{0\}$ is smooth. Then $\mathbb{A}^{n-1} \times (\mathbb{A} - \{0\})$ is smooth. Therefore the quotient variety $\mathbb{C}^{\times n}/S_n$ is a smooth complex affine algebraic variety of dimension n. Now each orbit $\mathcal{O}(\phi)$ is a product of symmetric products of $\mathbb{C}^\times$. Therefore each orbit $\mathcal{O}(\phi)$ is a smooth complex affine algebraic variety. We have

$$\mathcal{O}(\phi) = \text{Sym}^{l_1}\mathbb{C}^\times \times \ldots \times \text{Sym}^{l_k}\mathbb{C}^\times = \mathbb{A}^l \times (\mathbb{C}^\times)^k$$

where $l = l_1 + \ldots + l_k - k$ and $k = k(\phi)$. □

We now transport the complex structure from $\Phi(GL(n))$ to $\Pi(GL(n))$ via the local Langlands correspondence. This leads to the next result.

Theorem 1.7. *The smooth dual $\Pi(GL(n))$ has a natural complex structure. Each irreducible component is a smooth complex affine algebraic variety.*

The smooth dual $\Pi(GL(n))$ has countably many irreducible components of each dimension d with $1 \leq d \leq n$. The irreducible supercuspidal representations of $GL(n)$ arrange themselves into the 1-dimensional tori.
It follows from Theorems 1.6 and 1.7 that the smooth dual $\Pi(GL(n))$ is a complex manifold. Then $\mathbb{C} \times \Pi(GL(n))$ is a complex manifold. So the local L-factor $L(s, \pi_v)$ and the local ϵ-factor $\epsilon(s, \pi_v)$ are functions of *several complex variables*:

$$L : \mathbb{C} \times \Pi(GL(n)) \longrightarrow \mathbb{C}$$

$$\epsilon : \mathbb{C} \times \Pi(GL(n)) \longrightarrow \mathbb{C}.$$

Example 1.8. Unramified representations. Let $\psi_1, \ldots, \psi_n$ be unramified quasicharacters of the Weil group W_F. Then we have

$$\psi_j(w) = z_j^{d(w)}$$

with $z_j \in \mathbb{C}^\times$ for all $1 \leq j \leq n$. Let ϕ be the L-parameter given by $\psi_1 \oplus \ldots \oplus \psi_n$. Then the image $\pi_F(\phi)$ of ϕ under the local Langlands correspondence π_F is an unramified principal series representation. For the local L-factors $L(s, \pi)$ see [23, p. 377]. The local L-factor attached to such an unramified representation of $GL(n)$ is given by

$$L(s, \pi_F(\phi)) = \prod_{j=1}^{n} (1 - z_j q^{-s})^{-1}.$$

This exhibits the local L-factor as a function on the complex manifold $\mathbb{C} \times \mathrm{Sym}^n \mathbb{C}^\times$.

2. The structure of the based ring J

Let W be the extended affine Weyl group associated to $GL(n,\mathbb{C})$. For each two-sided cell $\mathbf{c}$ of W we have a corresponding partition λ of n. Let μ be the dual partition of λ. Let u be a unipotent element in $GL(n,\mathbb{C})$ whose Jordan blocks are determined by the partition μ. Let the distinct parts of the dual partition μ be $\mu_1,\ldots,\mu_p$ with μ_r repeated n_r times, $1\le r\le p$.

Let $C_G(u)$ be the centralizer of u in $G = GL(n,\mathbb{C})$. Then the maximal reductive subgroup $F_{\mathbf{c}}$ of $C_G(u)$ is isomorphic to $GL(n_1,\mathbb{C})\times GL(n_2,\mathbb{C})\times\cdots\times GL(n_p,\mathbb{C})$.

Following Lusztig [27] and Xi [40, 1.5] let J be the free $\mathbb{Z}$-module with basis $\{t_w \mid w\in W\}$. The multiplication $t_w t_u = \sum_{v\in W}\gamma_{w,u,v}t_v$ defines an associative ring structure on J. The ring J is the based ring of W. For each two-sided cell $\mathbf{c}$ of W the $\mathbb{Z}$-submodule $J_{\mathbf{c}}$ of J, spanned by all t_w, $w\in\mathbf{c}$, is a two-sided ideal of J. The ring $J_{\mathbf{c}}$ is the based ring of the two-sided cell $\mathbf{c}$. Let $|Y|$ be the number of left cells contained in $\mathbf{c}$. The Lusztig conjecture says that there is a ring isomorphism

$$J_{\mathbf{c}} \simeq M_{|Y|}(R_{F_{\mathbf{c}}}), \quad t_w \mapsto \pi(w)$$

where $R_{F_{\mathbf{c}}}$ is the rational representation ring of $F_{\mathbf{c}}$. This conjecture for $GL(n,\mathbb{C})$ has been proved by Xi [40, 1.5, 4.1, 8.2].

Since $F_{\mathbf{c}}$ is isomorphic to a direct product of the general linear groups $GL(n_i,\mathbb{C})$ $(1\le i\le p)$ we see that $R_{F_{\mathbf{c}}}$ is isomorphic to the tensor product over $\mathbb{Z}$ of the representation rings $R_{GL(n_i,\mathbb{C})}$, $1\le i\le p$. For the ring $R_{GL(n,\mathbb{C})}$ we have

$$R_{GL(n,\mathbb{C})} \simeq \mathbb{Z}[X_1,X_2,\ldots,X_n][X_n^{-1}]$$

where the elements $X_1,X_2,\ldots,X_n,X_n^{-1}$ are described in [40, 4.2][6, IX.125]. Then

$$\begin{aligned} R_{GL(n,\mathbb{C})} &\simeq \mathbb{Z}[\sigma_1,\ldots,\sigma_n,\sigma_n^{-1}] \\ &\simeq \mathbb{Z}[x_1,\ldots,x_n,x_1^{-1},\ldots,x_n^{-1}]^{S_n} \end{aligned}$$

We have

$$R_{GL(n,\mathbb{C})}\otimes_{\mathbb{Z}}\mathbb{C} \simeq \mathbb{C}[\mathrm{Sym}^n\mathbb{C}^\times]$$

and

$$R_{F_{\mathbf{c}}}\otimes_{\mathbb{Z}}\mathbb{C} \simeq \mathbb{C}[\mathrm{Sym}^{n_1}\mathbb{C}^\times\times\cdots\times\mathrm{Sym}^{n_p}\mathbb{C}^\times]$$

We recall the *extended quotient*. Let the finite group Γ act on the space X. Let $\widetilde{X} = \{(x,\gamma) : \gamma x = x\}$, let Γ act on $\widehat{X}$ by $\gamma_1(x,\gamma) = (\gamma_1 x, \gamma_1\gamma\gamma_1^{-1})$. Then $\widetilde{X}/\Gamma$ is the extended quotient of X by Γ, and we have

$$\widetilde{X}/\Gamma = \bigsqcup X^\gamma/Z(\gamma)$$

where one γ is chosen in each Γ-conjugacy class.
There is a canonical projection $\widetilde{X}/\Gamma \to X/\Gamma$.
Let $\gamma \in S_n$ have cycle type μ, let $X = (\mathbb{C}^\times)^n$. Then

$$\begin{array}{rcl} X^\gamma & \simeq & (C^\times)^{n_1} \times \cdots \times (C^\times)^{n_p} \\ Z(\gamma) & \simeq & (\mathbb{Z}/\mu_1\mathbb{Z}) \wr S_{n_1} \times \cdots \times (\mathbb{Z}/\mu_p\mathbb{Z}) \wr S_{n_p} \\ X^\gamma/Z(\gamma) & \simeq & \mathrm{Sym}^{n_1}\mathbb{C}^\times \times \cdots \times \mathrm{Sym}^{n_p}\mathbb{C}^\times \end{array}$$

and so

$$R_{F_{\mathbf{c}}} \otimes_{\mathbb{Z}} \mathbb{C} \simeq \mathbb{C}[X^\gamma/Z(\gamma)]$$

Then

$$J \otimes_{\mathbb{Z}} \mathbb{C} = \oplus_{\mathbf{c}}(J_{\mathbf{c}} \otimes_{\mathbb{Z}} \mathbb{C}) \sim \oplus_{\mathbf{c}}(R_{F_{\mathbf{c}}} \otimes_{\mathbb{Z}} \mathbb{C}) \simeq \mathbb{C}[\widetilde{X}/S_n]$$

The algebra $J \otimes_{\mathbb{Z}} \mathbb{C}$ is Morita equivalent to a reduced, finitely generated, commutative unital $\mathbb{C}$-algebra, namely the coordinate ring of the extended quotient $\widetilde{X}/S_n$.

3. Periodic cyclic homology of the Hecke algebra

The Bernstein variety $\Omega(G)$ of G is the set of G-conjugacy classes of pairs (M,σ), where M is a Levi (i.e. block-diagonal) subgroup of G, and σ is an irreducible supercuspidal representation of M. Each irreducible smooth representation of G is a subquotient of an induced representation $i_{GM}\sigma$. The pair (M,σ) is unique up to conjugacy. This creates a finite-to-one map, the infinitesimal character, from $\Pi(G)$ onto $\Omega(G)$.
Let $\Omega(G)$ be the Bernstein variety of G. Each point in $\Omega(G)$ is a conjugacy class of cuspidal pairs (M,σ). A quasicharacter $\psi : M \longrightarrow \mathbb{C}^\times$ is *unramified* if ψ is trivial on M°. The group of unramified quasicharacters of M is denoted $\Psi(M)$. We have $\Psi(M) \cong (\mathbb{C}^\times)^\ell$ where ℓ is the parabolic rank of M. The group $\Psi(M)$ now creates orbits: the orbit of (M,σ) is $\{(M,\psi\otimes\sigma) : \psi \in \Psi(M)\}$. Denote this orbit by D, and set $\Omega = D/W(M,D)$, where $W(M)$ is the Weyl group of M and $W(M,D)$ is the subgroup of $W(M)$ which leaves D globally invariant. The orbit D has the structure of a complex

torus, and so Ω is a complex algebraic variety. We view Ω as a component in the algebraic variety $\Omega(G)$.
The Bernstein variety $\Omega(G)$ is the disjoint union of ordinary quotients. We now replace the ordinary quotient by the extended quotient to create a new variety $\Omega^+(G)$. So we have

$$\Omega(G) = \bigsqcup D/W(M,D) \text{ and } \Omega^+(G) = \bigsqcup \widetilde{D}/W(M,D)$$

Let Ω be a component in the Bernstein variety $\Omega(GL(n))$, and let $\mathcal{H}(G) = \bigoplus \mathcal{H}(\Omega)$ be the Bernstein decomposition of the Hecke algebra.
Let

$$\Pi(\Omega) = (inf.ch.)^{-1}\Omega.$$

Then $\Pi(\Omega)$ is a smooth complex algebraic variety with finitely many irreducible components. We have the following Bernstein decomposition of $\Pi(G)$:

$$\Pi(G) = \bigsqcup \Pi(\Omega).$$

Let M be a compact C^∞ manifold. Then $C^\infty(M)$ is a Fréchet algebra, and we have Connes' fundamental theorem [14, Theorem 2, p. 208]:

$$\mathrm{HP}_*(C^\infty(M)) \cong \mathrm{H}^*(M;\mathbb{C}).$$

Now the ideal $\mathcal{H}(\Omega)$ is a purely algebraic object, and, in computing its periodic cyclic homology, we would hope to find an algebraic variety to play the role of the manifold M. This algebraic variety is $\Pi(\Omega)$.

Theorem 3.1. *Let Ω be a component in the Bernstein variety $\Omega(G)$. Then the periodic cyclic homology of $\mathcal{H}(G)$ is isomorphic to the periodised de Rham cohomology of $\Pi(\Omega)$:*

$$\mathrm{HP}_*(\mathcal{H}(\Omega)) \cong \mathrm{H}^*(\Pi(\Omega);\mathbb{C}).$$

Proof. We can think of Ω as a vector $(\tau_1,\ldots,\tau_r)$ of irreducible supercuspidal representations of smaller general linear groups, the entries of this vector being only determined up to tensoring with unramified quasicharacters and permutation. If the vector is equivalent to $(\sigma_1,\ldots,\sigma_1,\ldots,\sigma_r,\ldots,\sigma_r)$ with σ_j repeated e_j times, $1 \le j \le r$, and $\sigma_1,\ldots,\sigma_r$ are pairwise distinct, then we say that Ω has *exponents* $e_1,\ldots,e_r$.

Then there is a Morita equivalence

$$\mathcal{H}(\Omega) \sim \mathcal{H}(e_1, q_1) \otimes \ldots \otimes \mathcal{H}(e_r, q_r)$$

where $q_1, \ldots, q_r$ are natural number invariants attached to Ω.
This result is due to Bushnell-Kutzko [11, 12, 13]. We describe the steps in the proof. Let (ρ, W) be an irreducible smooth representation of the compact open subgroup K of G. As in [12, 4.2], the pair (K, ρ) is an Ω-type in G if and only if, for $(\pi, V) \in \Pi(G)$, we have inf.ch.$(\pi) \in \Omega$ if and only if π contains ρ. The existence of an Ω-type in $GL(n)$, for each component Ω in $\Omega(GL(n))$, is established in [13, 1.1]. So let (K, ρ) be an Ω-type in $GL(n)$. As in [12, 2.9], let

$$e_\rho(x) = (\mathrm{vol}K)^{-1}(\dim\, \rho)\ \mathrm{Trace}_W(\rho(x^{-1}))$$

for $x \in K$ and 0 otherwise.
Then e_ρ is an idempotent in the Hecke algebra $\mathcal{H}(G)$. Then we have

$$\mathcal{H}(\Omega) \cong \mathcal{H}(G) * e_\rho * \mathcal{H}(G)$$

as in [12, 4.3] and the two-sided ideal $\mathcal{H}(G) * e_\rho * \mathcal{H}(G)$ is Morita equivalent to $e_\rho * \mathcal{H}(G) * e_\rho$. Now let $\mathcal{H}(K, \rho)$ be the endomorphism-valued Hecke algebra attached to the semisimple type (K, ρ). By [12, 2.12] we have a canonical isomorphism of unital $\mathbb{C}$-algebras :

$$\mathcal{H}(G, \rho) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}\, W \cong e_\rho * \mathcal{H}(G) * e_\rho$$

so that $e_\rho * \mathcal{H}(G) * e_\rho$ is Morita equivalent to $\mathcal{H}(G, \rho)$. Now we quote the main theorem for semisimple types in $GL(n)$ [13, 1.5]: there is an isomorphism of unital $\mathbb{C}$-algebras

$$\mathcal{H}(G, \rho) \cong \mathcal{H}(G_1, \rho_1) \otimes \ldots \otimes \mathcal{H}(G_r, \rho_r)$$

The factors $\mathcal{H}(G_i, \rho_i)$ are (extended) affine Hecke algebras whose structure is given explicitly in [11, 5.6.6]. This structure is in terms of generators and relations [11, 5.4.6]. So let $\mathcal{H}(e, q)$ denote the affine Hecke algebra associated to the affine Weyl group $\mathbb{Z}^e \rtimes S_e$. Putting all this together we obtain a Morita equivalence

$$\mathcal{H}(\Omega) \sim \mathcal{H}(e_1, q_1) \otimes \ldots \otimes \mathcal{H}(e_r, q_r)$$

The natural numbers $q_1, \ldots, q_r$ are specified in [11, 5.6.6]. They are the cardinalities of the residue fields of certain extension fields $E_1/F, \ldots, E_r/F$.

Using the Künneth formula the calculation of $\mathrm{HP}_*(\mathcal{H}(\Omega))$ is reduced to that of the affine Hecke algebra $\mathcal{H}(e, q)$. Baum and Nistor demonstrate the spectral invariance of periodic cyclic homology in the class of finite type algebras [3, 4]. Now $\mathcal{H}(e, q)$ is the Iwahori-Hecke algebra associated to the extended affine Weyl group $\mathbb{Z}^e \rtimes S_e$, and let J be the asymptotic Hecke algebra (based ring) associated to $\mathbb{Z}^e \rtimes S_e$. According to [3, 4], Lusztig's morphisms $\phi_q : \mathcal{H}(e, q) \to J$ induce isomorphisms

$$(\phi_q)_* : \mathrm{HP}_*(\mathcal{H}(e, q)) \to \mathrm{HP}_*(J)$$

for all $q \in \mathbb{C}^\times$ that are not proper roots of unity. At this point we can back track and deduce that

$$\mathrm{HP}_*(\mathcal{H}(e, q)) \simeq \mathrm{HP}_*(J) \simeq \mathrm{HP}_*(\mathcal{H}_1)$$

and use the fact that $\mathcal{H}(e, 1) \simeq \mathbb{C}[\mathbb{Z}^e \rtimes S_e]$. It is much more illuminating to quote Xi's proof of the Lusztig conjecture for the based ring J, see Section 2. Then we have

$$\mathrm{HP}_*(\mathcal{H}(e, q)) \simeq \mathrm{HP}_*(J) \simeq \mathrm{HP}_*(\mathbb{C}[\widetilde{(\mathbb{C}^\times)^e/S_e}]) \simeq \mathrm{H}^*(\widetilde{(\mathbb{C}^\times)^e/S_e}; \mathbb{C}).$$

If Ω has exponents $e_1, \ldots, e_r$ then $e_1 + \ldots + e_r = d(\Omega) = \dim_{\mathbb{C}} \Omega$, and $W(\Omega)$ is a product of symmetric groups:

$$W(\Omega) = S_{e_1} \times \ldots \times S_{e_r}$$

We have

$$\begin{aligned} \mathrm{HP}_*(\mathcal{H}(\Omega)) &\simeq \mathrm{HP}_*(\mathcal{H}(e_1, q_1) \otimes \cdots \otimes \mathcal{H}(e_r, q_r)) \\ &\simeq \mathrm{HP}_*(\mathcal{H}(e_1, q_1)) \otimes \cdots \otimes \mathrm{HP}_*(\mathcal{H}(e_r, q_r)) \\ &\simeq \mathrm{H}^*(\widetilde{(\mathbb{C}^\times)^{e_1}/S_{e_1}}; \mathbb{C}) \otimes \cdots \otimes \mathrm{H}^*(\widetilde{(\mathbb{C}^\times)^{e_r}/S_{e_r}}; \mathbb{C}) \end{aligned}$$

Now the extended quotient is multiplicative, i.e.

$$\widetilde{(\mathbb{C}^\times)^{d(\Omega)}/W(\Omega)} = \widetilde{(\mathbb{C}^\times)^{e_1}/S_{e_1}} \times \cdots \times \widetilde{(\mathbb{C}^\times)^{e_r}/S_{e_r}}$$

which implies that

$$\mathrm{HP}_*(\mathcal{H}(\Omega)) = \mathrm{H}^*(\widetilde{(\mathbb{C}^\times)^{d(\Omega)}/W(\Omega)}; \mathbb{C})$$

Recall that

$$\begin{aligned} \Omega &= (\mathbb{C}^\times)^{d(\Omega)}/W(\Omega) \\ \Omega^+ &= \widetilde{(\mathbb{C}^\times)^{d(\Omega)}}/W(\Omega) \end{aligned}$$

and by [8, p. 217] we have $\Pi(\Omega) \simeq \Omega^+$. It now follows that

$$\mathrm{HP}_*(\mathcal{H}(\Omega)) \simeq \mathrm{H}^*(\Pi(\Omega); \mathbb{C})$$

□

LEMMA 3.2. *Let Ω be a component in the variety $\Omega(G)$ and let Ω have exponents $\{e_1, \dots, e_r\}$. Then for $j = 0, 1$ we have*

$$\dim_{\mathbb{C}} \mathrm{HP}_j\, \mathcal{H}(\Omega) = 2^{r-1}\beta(e_1)\cdots\beta(e_r)$$

where

$$\beta(e) = \sum_{|\lambda|=e} 2^{\alpha(\lambda)-1}$$

and where $\alpha(\lambda)$ is the number of unequal parts of λ. Here $|\lambda|$ is the weight of λ, i.e. the sum of the parts of λ so that λ is a partition of e.

Proof. Suppose first that Ω has the single exponent e. By Theorem 3.1 the periodic cyclic homology of $\mathcal{H}(\Omega)$ is isomorphic to the periodised de Rham cohomology of the extended quotient of $(\mathbb{C}^\times)^e$ by the symmetric group S_e. The components in this extended quotient correspond to the partitions of e. In fact, if $\alpha(\lambda)$ is the number of unequal parts in the partition λ then the corresponding component is homotopy equivalent to the compact torus of dimension $\alpha(\lambda)$. We now proceed by induction, using the fact that the extended quotient is multiplicative and the Künneth formula. □

Theorem 3.1, combined with the calculation in [7], now leads to the next result.

THEOREM 3.3. *The inclusion $\mathcal{H}(G) \longrightarrow \mathcal{S}(G)$ induces an isomorphism at the level of periodic cyclic homology:*

$$\mathrm{HP}_*(\mathcal{H}(G)) \simeq \mathrm{HP}_*(\mathcal{S}(G)).$$

Remark 3.4. We now consider further the disjoint union

$$\Phi(\Omega) = \mathcal{O}(\phi_1) \sqcup \cdots \sqcup \mathcal{O}(\phi_r) \simeq \Omega^+$$

If we apply the local Langlands correspondence π_F then we obtain

$$\Pi(\Omega) = \pi_F(\mathcal{O}(\phi_1)) \sqcup \cdots \sqcup \pi_F(\mathcal{O}(\phi_r)) \simeq \Omega^+$$

This partition of $\Pi(\Omega)$ is *identical* to that in Schneider-Zink [34, p. 198], modulo notational differences. In their notation, for each $\mathcal{P} \in \mathcal{B}$ there is a natural map

$$Q_{\mathcal{P}} : X_{nr}(N_{\mathcal{P}}) \to \mathrm{Irr}(\Omega)$$

such that

$$\mathrm{Irr}(\Omega) = \bigsqcup_{\mathcal{P}\in\mathcal{B}} \mathrm{im}(Q_{\mathcal{P}}).$$

In fact this is a special stratification of $\mathrm{Irr}(\Omega)$ in the precise sense of their article [34, p.198].
Let

$$Z_{\mathcal{P}} = \bigcup_{\mathcal{P}'\leq\mathcal{P}} \mathrm{im}(Q_{\mathcal{P}})$$

Then $Z_{\mathcal{P}}$ is a Jacobson closed set, in fact $Z_{\mathcal{P}} = V(J_{\mathcal{P}})$, where $J_{\mathcal{P}}$ is a certain 2-sided ideal [34, p.198]. We note that the set $Z_{\mathcal{P}}$ is also closed in the topology of the present article: each component in Ω^+ is equipped with the classical (analytic) topology.
Issues of stratification play a dominant role in [34]. The stratification of the tempered dual $\Pi^t(GL(n))$ arises from their construction of *tempered* K-types, see [34, p. 162, p. 189]. In the context of the present article, there is a natural stratification-by-dimension as follows. Let $1 \leq k \leq n$ and define

$$k\text{-stratum} = \{\mathcal{O}(\phi) \mid \dim_{\mathbb{C}} \mathcal{O}(\phi) \leq k\}$$

If $\pi_F(\mathcal{O}(\phi))$ is the complexification of the component $\Theta \subset \Pi^t(G)$ then we have

$$\dim_{\mathbb{R}} \Theta = \dim_{\mathbb{C}} \mathcal{O}(\phi).$$

The partial order in [34] on the components Θ transfers to a partial order on complex orbits $\mathcal{O}(\phi)$. This partial order originates in the opposite of the natural partial order on partitions, and the partitions manifest themselves in terms of Langlands parameters. For example, let

$$\begin{aligned} \phi &= \rho \otimes \mathrm{spin}(j_1) \oplus \cdots \oplus \rho \otimes \mathrm{spin}(j_r) \\ \phi' &= \rho \otimes \mathrm{spin}(j_1') \oplus \cdots \oplus \rho \otimes \mathrm{spin}(j_r') \end{aligned}$$

Let $\lambda_1 = 2j_1 + 1, \ldots, \lambda_r = 2j_r + 1$, $\mu_1 = 2j_1' + 1, \ldots, \mu_r = 2j_r' + 1$ and define partitions as follows

$$\begin{aligned} \lambda &= (\lambda_1, \ldots, \lambda_r), \quad \lambda_1 \geq \lambda_2 \geq \ldots \\ \mu &= (\mu_1, \ldots, \mu_r), \quad \mu_1 \geq \mu_2 \geq \ldots \end{aligned}$$

The natural partial order on partitions is: $\lambda \le \mu$ if and only if

$$\lambda_1 + \cdots + \lambda_i \le \mu_1 + \cdots + \mu_i$$

for all $i \ge 1$, see [28, p.6]. Let $l(\lambda)$ be the length of λ, that is the number of parts in λ. Then $\dim_{\mathbb{C}} \mathcal{O}(\phi) = l(\lambda)$. Let λ', μ' be the dual partitions as in [28]. Then we have [28, 1.11] $\lambda \ge \mu$ if and only if $\mu' \ge \lambda'$. Note that $l(\lambda) = \lambda'_1$, $l(\mu) = \mu'_1$. Then

$$\Theta_\lambda \le \Theta_\mu \Leftrightarrow \lambda \ge \mu \Leftrightarrow \mu' \ge \lambda' \Rightarrow \lambda'_1 \le \mu'_1$$

So if $\Theta_\lambda \le \Theta_\mu$ then $\dim_{\mathbb{R}} \Theta_\lambda \le \dim_{\mathbb{R}} \Theta_\mu$, similarly $\mathcal{O}(\phi) \le \mathcal{O}(\phi')$ implies $\dim_{\mathbb{C}} \mathcal{O}(\phi) \le \dim_{\mathbb{C}} \mathcal{O}(\phi')$. Stratification-by-dimension is finer than the Schneider-Zink stratification [34].

Let now R denote the ring of all regular functions on $\Pi(G)$. The ring R is a commutative, reduced, unital ring over $\mathbb{C}$ which is not finitely generated. We will call R the *extended centre* of G. It is natural to believe that the extended centre R of G is the centre of an 'extended category' made from smooth G-modules. The work of Schneider-Zink [34, p. 201] contains various results in this direction.

4. THE q-PROJECTION

Let Ω be a component in the Bernstein variety. This component is an ordinary quotient D/Γ. We now consider the extended quotient $\widetilde{D/\Gamma} = \bigsqcup D^\gamma/Z_\gamma$, where D is the complex torus $\mathbb{C}^{\times m}$. Let γ be a permutation of n letters with cycle type

$$\gamma = (1 \ldots \alpha_1) \cdots (1 \ldots \alpha_r)$$

where $\alpha_1 + \cdots + \alpha_r = m$. On the fixed set D^γ the map π_q, by definition, sends the element $(z_1, \ldots, z_1, \ldots, z_r, \ldots, z_r)$ where z_j is repeated α_j times, $1 \le j \le r$, to the element

$$(q^{(\alpha_1-1)/2} z_1, \ldots, q^{(1-\alpha_1)/2} z_1, \ldots, q^{(\alpha_r-1)/2} z_r, \ldots, q^{(1-\alpha_r)/2} z_r)$$

The map π_q induces a map from D^γ/Z_γ to D/Γ, and so a map, still denoted π_q, from the extended quotient $\widetilde{D/\Gamma}$ to the ordinary quotient D/Γ. This creates a map π_q from the extended Bernstein variety to the Bernstein variety:

$$\pi_q : \Omega^+(G) \longrightarrow \Omega(G).$$

DEFINITION 4.1. The map π_q is called the *q-projection*.

The q-projection π_q occurs in the following commutative diagram [8]:

$$\begin{array}{ccc} \Phi(G) & \longrightarrow & \Pi(G) \\ {\scriptstyle\alpha}\Big\downarrow & & \Big\downarrow{\scriptstyle \text{inf. ch.}} \\ \Omega^+(G) & \xrightarrow{\pi_q} & \Omega(G) \end{array}$$

Let A, B be commutative rings with $A \subset B, 1 \in A$. Then the element $x \in B$ is *integral* over A if there exist $a_1, \ldots, a_n \in A$ such that

$$x^n + a_1 x^{n-1} + \ldots + a_n = 0.$$

Then B is *integral* over A if each $x \in B$ is integral over A. Let X, Y be affine varieties, $f : X \longrightarrow Y$ a regular map such that $f(X)$ is dense in Y. Then the pull-back $f^\#$ defines an isomorphic inclusion $\mathbb{C}[Y] \longrightarrow \mathbb{C}[X]$. We view $\mathbb{C}[Y]$ as a subring of $\mathbb{C}[X]$ by means of $f^\#$. Then f is a *finite* map if $\mathbb{C}[X]$ is integral over $\mathbb{C}[Y]$, see [35]. This implies that the pre-image $F^{-1}(y)$ of each point $y \in Y$ is a finite set, and that, as y moves in Y, the points in $F^{-1}(y)$ may merge together but not disappear. The map $\mathbb{A}^1 - \{0\} \longrightarrow \mathbb{A}^1$ is the classic example of a map which is *not* finite.

Lemma 4.2. *Let X be a component in the extended variety $\Omega^+(G)$. Then the q-projection π_q is a finite map from X onto its image $\pi_q(X)$.*

Proof. Note that the fixed-point set D^γ is a complex torus of dimension r, that $\pi_q(D^\gamma)$ is a torus of dimension r and that we have an isomorphism of affine varieties $D^\gamma \cong \pi_q(D^\gamma)$. Let $X = D^\gamma/Z_\gamma, Y = \pi_q(D^\gamma)/\Gamma$ where Z_γ is the Γ-centralizer of γ. Now each of X and Y is a quotient of the variety D^γ by a finite group, hence X, Y are affine varieties [35, p.31]. We have $D^\gamma \longrightarrow X \longrightarrow Y$ and $\mathbb{C}[Y] \longrightarrow \mathbb{C}[X] \longrightarrow \mathbb{C}[D^\gamma]$. According to [35, p.61], $\mathbb{C}[D^\gamma]$ is integral over $\mathbb{C}[Y]$ since $Y = D^\gamma/\Gamma$. Therefore the subring $\mathbb{C}[X]$ is integral over $\mathbb{C}[Y]$. So the map $\pi_q : X \longrightarrow Y$ is finite. □

Example 4.3. $GL(2)$. Let T be the diagonal subgroup of $G = GL(2)$ and let Ω be the component in $\Omega(G)$ containing the cuspidal pair $(T, 1)$. Then $\sigma \in \Pi(GL(2))$ is *arithmetically unramified* if $inf.ch.\sigma \in \Omega$. If $\pi_F(\phi) = \sigma$ then ϕ is a 2-dimensional representation of $\mathcal{L}_F$ and there are two possibilities:

ϕ is reducible, $\phi = \psi_1 \oplus \psi_2$ with ψ_1, ψ_2 unramified quasicharacters of W_F. So $\psi_j(w) = z_j^{d(w)}, z_j \in \mathbb{C}^\times, j = 1, 2$. We have $\pi_F(\phi) = Q(\psi_1, \psi_2)$ where ψ_1 does not precede ψ_2. In particular we obtain the 1-dimensional representations of G as follows:

$$\pi_F(|\;|^{1/2}\psi \oplus |\;|^{-1/2}\psi) = Q(|\;|^{1/2}\psi, |\;|^{-1/2}\psi) = \psi \circ det.$$

ϕ is irreducible, $\phi = \psi \otimes \text{spin}(1/2)$. Then $\pi_F(\phi) = Q(\Delta)$ with $\Delta = \{|\;|^{-1/2}\psi, |\;|^{1/2}\psi\}$ so $\pi_F(\phi) = \psi \otimes St(2)$ where $St(2)$ is the Steinberg representation of $GL(2)$.
The orbit of $(T, 1)$ is $D = (\mathbb{C}^\times)^2$, and $W(T, D) = \mathbb{Z}/2\mathbb{Z}$. Then $\Omega \cong (\mathbb{C}^\times)^2/\mathbb{Z}/2\mathbb{Z} \cong \text{Sym}^2\,\mathbb{C}^\times$. The extended quotient is $\Omega^+ = \text{Sym}^2\,\mathbb{C}^\times \sqcup \mathbb{C}^\times$. The $q-$projection works as follows:

$$\pi_q : \{z_1, z_2\} \mapsto \{z_1, z_2\}$$

$$\pi_q : z \mapsto \{q^{1/2}z, q^{-1/2}z\}$$

where q is the cardinality of the residue field of F.

Let $A = \mathcal{H}(GL(2)//I)$ be the Iwahori-Hecke algebra of $GL(2)$. This is a finite type algebra. Following [21, p. 327], denote by $\text{Prim}_n(A) \subset \text{Prim}(A)$ the set of primitive ideals $B \subset A$ which are kernels of irreducible representations of A of dimension n. Set $X_1 = \text{Prim}_1(A)$, $X_2 = \text{Prim}_1(A) \sqcup \text{Prim}_2(A) = \text{Prim}(A)$. Then X_1 and X_2 are closed sets in $\text{Prim}(A)$ defining an increasing filtration of $\text{Prim}(A)$. Now A is Morita equivalent to the Bernstein ideal $\mathcal{H}(\Omega)$, and $\Pi(\Omega) \simeq \text{Prim}(A)$.
Let $\phi_1 = 1 \otimes \text{spin}(1/2)$, $\phi_2 = 1 \otimes 1 \oplus 1 \otimes 1$. The 1-dimensional representations of $GL(2)$ determine 1-dimensional representations of $\mathcal{H}(G//I)$ and so lie in X_1. The L-parameters of the 1-dimensional representations of $GL(2)$ do *not* lie in the 1-dimensional orbit $\mathcal{O}(\phi_1)$: they lie in the 2-dimensional orbit $\mathcal{O}(\phi_2)$. The Kazhdan-Nistor-Schneider stratification [21] does *not* coincide with stratification-by-dimension.

EXAMPLE 4.4. $GL(3)$. In the above example, the q-projection is stratified-injective, i.e. injective on each orbit type. This is not so in general, as shown by the next example. Let T be the diagonal subgroup of $GL(3)$ and let Ω be the component containing the cuspidal pair $(T, 1)$. Then $\Omega = \mathrm{Sym}^3\,\mathbb{C}^\times$ and

$$\Omega^+ = \mathrm{Sym}^3\,\mathbb{C}^\times \sqcup (\mathbb{C}^\times)^2 \sqcup \mathbb{C}^\times$$

.
The map π_q works as follows:

$$\{z_1, z_2, z_3\} \mapsto \{z_1, z_2, z_3\}$$

$$(z, w, w) \mapsto \{z, q^{1/2}w, q^{-1/2}w\}$$

$$(z, z, z) \mapsto \{qz, z, q^{-1}z\}.$$

Consider the L-parameter

$$\phi = \psi_1 \otimes 1 \oplus \psi_2 \otimes \mathrm{spin}(1/2) \in \Phi(GL(3)).$$

If $\psi(w) = z^{d(w)}$ then we will write $\psi = z$. With this understood, let

$$\phi_1 = q \otimes 1 \oplus q^{-1/2} \otimes \mathrm{spin}(1/2)$$

$$\phi_2 = q^{-1} \otimes 1 \oplus q^{1/2} \otimes \mathrm{spin}(1/2).$$

Then $\alpha(\phi_1), \alpha(\phi_2)$ are distinct points in the same stratum of the extended quotient, but their image under the q-projection π_q is the single point $\{q^{-1}, 1, q\} \in \mathrm{Sym}^3\,\mathbb{C}^\times$.
Let

$$\phi_3 = 1 \otimes \mathrm{spin}(3/2)$$

$$\phi_4 = q^{-1} \otimes 1 \oplus 1 \otimes 1 \oplus q \otimes 1.$$

Then the distinct L-parameters $\phi_1, \phi_2, \phi_3, \phi_4$ all have the same image under the q-projection π_q.

5. The diagram

In this section we create a diagram which incorporates several major results. The following diagram serves as a framework for the whole article:

$$\begin{array}{ccccc}
K_*^{\mathrm{top}}(G) & & \xrightarrow{\quad\quad\quad\mu\quad\quad\quad} & & K_*(C_r^*(G)) \\
\downarrow{\scriptstyle \mathrm{ch}} & & & & \downarrow{\scriptstyle \mathrm{ch}} \\
\mathrm{H}_*(G;\beta G) & \longrightarrow & \mathrm{HP}_*(\mathcal{H}(G)) & \xrightarrow{\imath_*} & \mathrm{HP}_*(\mathcal{S}(G)) \\
\vdots & & \downarrow & & \downarrow \\
\mathrm{H}_c^*(\Phi(G);\mathbb{C}) & \longrightarrow & \mathrm{H}_c^*(\Pi(G);\mathbb{C}) & \longrightarrow & \mathrm{H}_c^*(\Pi^t(G);\mathbb{C})
\end{array}$$

An outline of the construction of the Chern character on the left hand side of the diagram is given in [1]. The Baum-Connes assembly map μ is an isomorphism [1, 24]. The map

$$\mathrm{H}_*(G;\beta G) \to \mathrm{HP}_*(\mathcal{H}(G))$$

is an isomorphism [20, 33]. The map $\imath_*$ is an isomorphism by Theorem 3.3. The right hand Chern character is constructed in [9] and is an isomorphism after tensoring over $\mathbb{Z}$ with $\mathbb{C}$ [9, Theorem 3]. We believe that the top half of the diagram is commutative but at present there is no proof of this. We emphasize that the results in this papers do not depend on commutativity of this part of the diagram.

In the diagram, $\mathrm{H}_c^*(\Pi^t(G);\mathbb{C})$ denotes the (periodised) compactly supported de Rham cohomology of the tempered dual $\Pi^t(G)$, and $\mathrm{H}_c^*(\Pi(G);\mathbb{C})$ denotes the (periodised) de Rham cohomology supported on finitely many components of the smooth dual $\Pi(G)$. The map

$$\mathrm{HP}_*(\mathcal{S}(G)) \to \mathrm{H}_c^*(\Pi^t(G);\mathbb{C})$$

is constructed in [7] and is an isomorphism [7, Theorem 7].

The map

$$\mathrm{H}_c^*(\Pi(G);\mathbb{C}) \to \mathrm{H}_c^*(\Pi^t(G);\mathbb{C})$$

is constructed in the following way. Given an L-parameter $\phi : \mathcal{L}_F \to GL(n,\mathbb{C})$ we have

$$\phi = \phi_1 \oplus \ldots \oplus \phi_m$$

with each ϕ_j an irreducible representation. We have $\phi_j = \rho_j \otimes \mathrm{spin}(j)$ where each ρ_j is an irreducible representation of the Weil group W_F. We shall assume that $\det \rho_j$ is a unitary character. Let

$\mathcal{O}(\phi)$ be the orbit of ϕ as in Definition 1.4. The map $\mathcal{O}(\phi) \to \mathcal{O}^t(\phi)$ is now defined as follows

$$\psi_1\phi_1 \oplus \ldots \oplus \psi_m\phi_m \mapsto |\psi_1|^{-1} \cdot \psi_1\phi_1 \oplus \ldots \oplus |\psi_m|^{-1} \cdot \psi_m\phi_m.$$

This map is a deformation retraction of the complex orbit $\mathcal{O}(\phi)$ onto the compact orbit $\mathcal{O}^t(\phi)$. Since $\Phi(G)$ is a disjoint union of such complex orbits this formula determines, via the local Langlands correspondence for $GL(n)$, a deformation retraction of $\Pi(G)$ onto the tempered dual $\Pi^t(GL(n))$, which implies that the induced map on cohomology is an isomorphism.

The map

$$\mathrm{H}_c^*(\Phi(G);\mathbb{C}) \to \mathrm{H}_c^*(\Pi(G);\mathbb{C})$$

is an isomorphism, induced by the local Langlands correspondence π_F.

The map

$$\mathrm{HP}_*(\mathcal{H}(G)) \to \mathrm{H}_c^*(\Pi(G);\mathbb{C})$$

is an isomorphism by Theorem 3.1.

There is at present no direct definition of the map

$$\mathrm{H}_*(G;\beta G) \to \mathrm{H}_c^*(\Phi(G);\mathbb{C}).$$

Suppose for the moment that F has characteristic 0 and has residue field of characteristic p. An irreducible representation ρ of the Weil group W_F is called wildly ramified if $\dim \rho$ is a power of p and $\rho \not\simeq \rho \otimes \psi$ for any unramified quasicharacter $\psi \neq 1$ of W_F. We write $\Phi_m^{wr}(F)$ for the set of equivalence classes of such representations of dimension p^m. An irreducible supercuspidal representation π of $GL(n)$ is wildly ramified if n is a power of p and $\pi \not\simeq \pi \otimes (\psi \circ \det)$ for any unramified quasicharacter $\psi \neq 1$ of $F^\times$. We write $\Pi_m^{wr}(F)$ for the set of equivalence classes of such representations of $GL(p^m, F)$. In this case Bushnell-Henniart [10] construct, for each m, a canonical bijection

$$\pi_{F,m} : \Phi_m^{wr}(F) \to \Pi_m^{wr}(F).$$

Now the maximal simple type (J, λ) of an irreducible supercuspidal representation determines an element in the chamber homology of the affine building [2, 6.7]. The construction of Bushnell-Henniart therefore determines a map from a *subspace* of $\mathrm{H}_c^{\mathrm{even}}(\Phi(G);\mathbb{C})$ to a *subspace* of $\mathrm{H}_0(G;\beta G)$.

In the context of the above diagram the Baum-Connes map has a geometric counterpart: it is induced by the deformation retraction of $\Pi(GL(n))$ onto the tempered dual $\Pi^t(GL(n))$.

6. SUPERCUSPIDAL REPRESENTATIONS OF $GL(n)$

In this section we track the fate of supercuspidal representations of $GL(n)$ through the diagram constructed in the previous Section. Let ρ be an irreducible n-dimensional complex representation of the Weil group W_F such that $\det \rho$ is a unitary character and let $\phi = \rho \otimes 1$. Then ϕ is the L-parameter for a pre-unitary supercuspidal representation ω of $GL(n)$. Let $\mathcal{O}(\phi)$ be the orbit of ϕ and $\mathcal{O}^t(\phi)$ be the compact orbit of ϕ. Then $\mathcal{O}(\phi)$ is a component in the Bernstein variety isomorphic to $\mathbb{C}^\times$ and $\mathcal{O}^t(\phi)$ is a component in the tempered dual, isomorphic to $\mathbb{T}$. The L-parameter ϕ now determines the following data.

6.1. Let (J, λ) be a maximal simple type for ω in the sense of Bushnell and Kutzko [11, chapter 6]. Then J is a compact open subgroup of G and λ is a smooth irreducible complex representation of J. We will write

$$\mathbb{T} = \{\psi \otimes \omega : \psi \in \Psi^t(G)\}$$

where $\Psi^t(G)$ denotes the group of unramified unitary characters of G.

THEOREM 6.1. *Let K be a maximal compact subgroup of G containing J and form the induced representation $W = Ind_J^K(\lambda)$. We then have*

$$\ell^2(G \times_K W) \simeq Ind_K^G(W) \simeq Ind_J^G(\lambda) \simeq \int_{\mathbb{T}} \pi d\pi.$$

Proof. The supercuspidal representation ω contains λ and, modulo unramified unitary twist, is the only irreducible unitary representation with this property [11, 6.2.3]. Now the Ahn reciprocity theorem expresses Ind_J^G as a direct integral [26, p.58]:

$$\mathrm{Ind}_J^G(\lambda) = \int n(\pi, \lambda) \pi d\pi$$

where $d\pi$ is Plancherel measure and $n(\pi, \lambda)$ is the multiplicity of λ in $\pi|_J$. But the Hecke algebra of a maximal simple type is commutative (a Laurent polynomial ring). Therefore $\omega|_J$ contains λ with multiplicity 1 (thanks to C. Bushnell for this remark). We then

have $n(\psi \otimes \omega, \lambda) = 1$ for all $\psi \in \Psi^t(G)$. We note that Plancherel measure induces Haar measure on $\mathbb{T}$, see [31].
The affine building of G is defined as follows [38, p. 49]:

$$\beta G = \mathbb{R} \times \beta SL(n)$$

where $g \in G$ acts on the affine line $\mathbb{R}$ via $t \mapsto t + val(det(g))$. Let $G^\circ = \{g \in G : val(det(g)) = 0\}$. We use the standard model for $\beta SL(n)$ in terms of equivalence classes of $\mathbf{o}_F$ - lattices in the n-dimensional F-vector space V. Then the vertices of $\beta SL(n)$ are in bijection with the maximal compact subgroups of G°, see [32, 9.3]. Let $P \in \beta G$ be the vertex for which the isotropy subgroup is $K = GL(n, \mathbf{o}_F)$. Then the G-orbit of P is the set of all vertices in βG and the discrete space G/K can be identified with the set of vertices in the affine building βG. Now the base space of the associated vector bundle $G \times_K W$ is the discrete coset space G/K, and the Hilbert space of ℓ^2-sections of this homogeneous vector bundle is a realization of the induced representation $\mathrm{Ind}_K^G(W)$. $\square$

The $C_0(\beta G)$-module structure is defined as follows. Let $f \in C_0(\beta G)$, $s \in \ell^2(G \times_K W)$ and define

$$(fs)(v) = f(v)s(v)$$

for each vertex $v \in \beta G$. We proceed to construct a K-cycle in degree 0. This K-cycle is

$$(C_0(\beta G), \ell^2(G \times_K W) \oplus 0, 0)$$

interpreted as a $\mathbb{Z}/2\mathbb{Z}$-graded module. This triple satisfies the properties of a (pre)-Fredholm module [14, IV] and so creates an element in $K_0^{\mathrm{top}}(G)$. By Theorem 5.1 this generator creates a free $C(\mathbb{T})$-module of rank 1, and so provides a generator in $K_0(C_r^*(G))$.

6.2. The Hecke algebra of the maximal simple type (J, λ) is commutative (the Laurent polynomials in one complex variable). The periodic cyclic homology of this algebra is generated by 1 in degree zero and dz/z in degree 1.
The corresponding summand of the Schwartz algebra $\mathcal{S}(G)$ is Morita equivalent to the Fréchet algebra $C^\infty(\mathbb{T})$. By an elementary application of Connes' theorem [14, Theorem 2, p. 208], the periodic cyclic homology of this Fréchet algebra is generated by 1 in degree 0 and $d\theta$ in degree 1.

6.3. The corresponding component in the Bernstein variety is a copy of $\mathbb{C}^\times$. The cohomology of $\mathbb{C}^\times$ is generated by 1 in degree 0 and $d\theta$ in degree 1.
The corresponding component in the tempered dual is the circle $\mathbb{T}$. The cohomology of $\mathbb{T}$ is generated by 1 in degree 0 and $d\theta$ in degree 1.

References

[1] P. Baum, N. Higson, R. J. Plymen, A proof of the Baum-Connes conjecture for p-adic $GL(n)$, C. R. Acad. Sci. Paris 325 (1997) 171-176.
[2] P. Baum, N. Higson, R. J. Plymen, Representation theory of p-adic groups: a view from operator algebras, Proc. Symp. Pure Math. 68 (2000) 111 – 149.
[3] P. Baum, V. Nistor, The periodic cyclic homology of Iwahori-Hecke algebras, C. R. Acad. Sci. Paris 332 (2001) 1 –6.
[4] P. Baum, V. Nistor, The periodic cyclic homology of Iwahori-Hecke algebras, preprint 2001.
[5] J. Bernstein, Representations of p-adic groups, Notes by K.E. Rumelhart, Harvard University, 1992.
[6] N. Bourbaki, Groupes et algèbres de Lie, Chapitre 9, Masson, Paris 1982.
[7] J. Brodzki, R.J. Plymen, Periodic cyclic homology of certain nuclear algebras, C. R. Acad. Sci. Paris., 329 (1999), 671–676.
[8] J. Brodzki, R. J. Plymen, Geometry of the smooth dual of $GL(n)$, C. R. Acad. Sci. Paris., 331 (2000), 213–218.
[9] J. Brodzki, R.J. Plymen, Chern character for the Schwartz algebra of p-adic $GL(n)$, Bull. London. Math. Soc. 34 (2002), 219-228
[10] C.J. Bushnell and G. Henniart, Local tame lifting for $GL(n)$ II: wildly ramified supercuspidals, Astérisque 254, SMF 1999.
[11] C.J. Bushnell and P.C. Kutzko, The admissible dual of $GL(N)$ via compact open subgroups, Annals of Math. Studies 129, Princeton University Press, Princeton, 1993.
[12] C.J. Bushnell and P.C. Kutzko, Smooth representations of p-adic reductive groups: Structure theory via types, Proc. London Math. Soc 77(1998) 582 – 634.
[13] C.J. Bushnell and P.C. Kutzko, Semisimple types for $GL(N)$, Compositio Math. 119 (1999) 53 –97.
[14] A. Connes, Noncommutative Geometry, Academic Press, New York 1994.
[15] D. Eisenbud, J. Harris, The geometry of schemes, Graduate Text 197, Springer, Berlin 2000.
[16] Harish-Chandra, Collected papers Volume IV, Springer 1984.
[17] M. Harris, R. Taylor, On the geometry and cohomology of some simple Shimura varieties, Ann. Math. Study 151, Princeton University Press 2001.
[18] R. Hartshorne, Algebraic geometry, Springer, Berlin 1977.

[19] G. Henniart, Une preuve simple des conjectures de Langlands pour $GL(n)$ sur un corps p-adique, Invent. Math. 139 (2000) 439 –455.
[20] N. Higson, V. Nistor, Cyclic homology of totally disconnected groups acting on buildings, J. Functional Analysis 141 (1996), 466–485.
[21] D. Kazhdan, V. Nistor, P. Schneider, Hochschild and cyclic homology of finite type algebras, Sel. Math. 4 (1998), 321-359.
[22] A. W. Knapp, Introduction to the Langlands program, Proc. Symp. Pure Math. 61 (1997) 245–302.
[23] S. S. Kudla, The local Langlands correspondence, Proc. Symp. Pure Math. 55 (1994) 365 –391.
[24] V. Lafforgue, Une démonstration de la conjecture de Baum-Connes pour les groupes réductifs sur un corps p-adique et pour certains groupes discrets possédant la proriété (T), C. R. Acad. Sci. Paris 327 (1998), 439–444.
[25] G. Laumon, M. Rapoport, U. Stuhler, $\mathcal{D}$-elliptic sheaves and the Langlands correspondence, Invent. Math. 113 (1993) 217 –338.
[26] R.L. Lipsman, Group representations, Lecture Notes in Math. 388, Springer, 1974.
[27] G. Lusztig, Cells in affine Weyl groups II, J. Algebra 109 (1987), 536-548.
[28] I. G. Macdonald, Symmetric functions and Hall polynomials, Oxford Math. Monograph, Oxford 1979.
[29] J. Neukirch, Algebraic Number Theory, Springer, Berlin, 1999.
[30] J. Neukirch, Local class field theory, Springer, Berlin 1986.
[31] R.J. Plymen, Reduced C^*-algebra of the p-adic group $GL(n)\, II$, J. Functional Analysis, to appear.
[32] M. Ronan, Lectures on buildings, Perspectives in Math. 7, Academic Press, 1989.
[33] P. Schneider, The cyclic homology of p-adic reductive groups, J. reine angew. Math. 475 (1996), 39-54.
[34] P. Schneider, E.-W. Zink, K-types for the tempered components of a p-adic general linear group, J. reine angew. Math., 517 (1999) 161-208.
[35] I. R. Shafarevich, Basic algebraic geometry 1, Springer, Berlin, 1994.
[36] K. E. Smith, L. Kahanpää, P. Kekäläinen, W. Traves, An invitation to algebraic geometry, Universitext, Springer, Berlin, 2000.
[37] J. Tate, Number theoretic background, Proc. Symp. Pure Math. 33 (1979) part 2, 3 - 26.
[38] J. Tits, Reductive groups over local fields, Proc. Symp. Pure Math. 33 (1979) part 1, 29 – 69.
[39] A. Weil, Basic number theory, Classics in Math., Springer 1995.
[40] N. Xi, The based ring of two-sided cells of affine Weyl groups of type $\tilde{A}_{n-1}$, AMS Memoir, July 2002, to appear.

Faculty of Mathematical Studies,
University of Southampton,
Southampton SO17 1BJ, U.K.,
j.brodzki@maths.soton.ac.uk

Department of Mathematics
University of Manchester, Manchester
M13 9PL, U.K.,
roger@maths.man.ac.uk